Normal Equation

Let $x^{(i)} \in \mathbf{R}^3$ be i-th data and $y^{(i)} \in \mathbf{R}$ be i-th label for $i = 1, 2, \cdots, m$. We want to find unknown weight, $w \in \mathbf{R}^3$.
y, X, w are defined as

$$y = \begin{bmatrix} y^{(1)} \\ y^{(2)} \\ \vdots \\ y^{(m)} \end{bmatrix}, \tag{1}$$

$$X = \begin{bmatrix} x_1^{(1)} & x_2^{(1)} & x_3^{(1)} \\ x_1^{(2)} & x_2^{(2)} & x_3^{(2)} \\ \vdots & \vdots & \vdots \\ x_1^{(m)} & x_2^{(m)} & x_3^{(m)} \end{bmatrix}, \tag{2}$$

$$w = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}. \tag{3}$$

Derive a normal equation, $X^T X w = X^T y$, to solve the following least squares problem.

$$\min_w \frac{1}{m} \sum_{i=1}^{m} |y^{(i)} - f(w; x^{(i)})|^2 \tag{4}$$

where

$$f(w; x) = w_1 x_1 + w_2 x_2 + w_3 x_3 \tag{5}$$

Proof. For $k = 1, 2, 3,$

$$\frac{\partial J}{\partial w_k} = \frac{\partial}{\partial w_k} \left[\frac{1}{m} \sum_{i=1}^{m} |y^{(i)} - f(w; x^{(i)})|^2 \right] \tag{6}$$

$$= \tag{7}$$

$$= -\frac{2}{m} \sum_{i=1}^{m} \left(y^{(i)} - f(w; x^{(i)}) \right) \frac{\partial}{\partial w_k} (w_1 x_1^{(i)} + w_2 x_2^{(i)} + w_3 x_3^{(i)}) \tag{8}$$

$$= -\frac{2}{m} \sum_{i=1}^{m} \left(y^{(i)} - f(w; x^{(i)}) \right) x_k^{(i)} \tag{9}$$

$$= -\frac{2}{m} \sum_{i=1}^{m} x_k^{(i)} \left(y^{(i)} - (w_1 x_1^{(i)} + w_2 x_2^{(i)} + w_3 x_3^{(i)}) \right) \tag{10}$$

$$\nabla J(w) = \begin{bmatrix} \frac{\partial J}{\partial w_1} & \frac{\partial J}{\partial w_2} & \frac{\partial J}{\partial w_3} \end{bmatrix}^T \tag{11}$$

$$= -\frac{2}{m} X^T (y - Xw) = 0 \tag{12}$$

Therefore,

$$-\frac{2}{m} X^T (y - Xw) = 0 \tag{13}$$

$$-X^T (y - Xw) = 0 \tag{14}$$

$$X^T X w = X^T y \tag{15}$$

□

Proof. For $k = 1, 2, 3$,

$$
\begin{aligned}
\nabla J(w) &= \nabla \left[\frac{1}{m} ||y - f(w; x)||^2 \right] && (16) \\
&= \nabla \left[\frac{1}{m} ||y - Xw||^2 \right] && (17) \\
&= \nabla \left[\frac{1}{m} (y - Xw)^T (y - Xw) \right] && (18) \\
&= && (19) \\
&= \frac{1}{m} \nabla \left(y^T y - 2w^T X^T y + w^T X^T X w \right) && (20) \\
&= \frac{1}{m} \left(\nabla(y^T y) - 2\nabla(w^T X^T y) + \nabla(w^T X^T X w) \right) && (21) \\
&= && (22)
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
-X^T y + X^T X w &= 0 && (23) \\
X^T X w &= X^T y && (24)
\end{aligned}
$$

□