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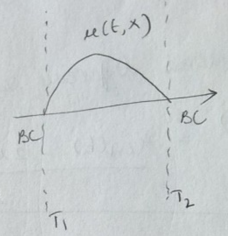
PROBLEM SET - 13

1. Conceptual abstraction

1D heat problem with temperature distribution - $u(t,x)$ as function of time & space (x) has to be found with given initial (IC) and boundary condition (BC)



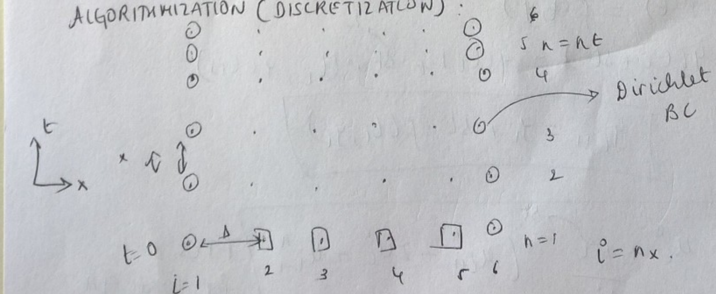
MATHEMATICAL MODEL :

We have to solve 1D heat equation

$$\boxed{\frac{\partial u}{\partial t}(t,x) = k \frac{\partial^2 u(t,x)}{\partial x^2}}$$

k = Thermal conductivity $\left[\frac{M^2}{s}\right]$

ALGORITHMIZATION (DISCRETIZATION) :



Discretize space : $x_i = \Delta \cdot i$; $i = 1 \dots nx$

Discretize time : $t_n = \tau . n$; $n = 1 \dots nt$

=> $u_i^n = u(t_n, x_i)$

Initial condition : u_i^{1}

Boundary condition : u_1^n , u_{nx}^n } given.

we apply centered difference scheme for SPACE

$$\frac{\partial^2 u_i(t)}{\partial x^2} = \frac{u_{i+1}(t) - 2u_i(t) + u_{i-1}(t)}{\Delta^2}$$

$$\Rightarrow \boxed{\frac{\partial u_i(t)}{\partial t} = \frac{k}{\Delta^2}\left[u_{i+1}(t) - 2u_i(t) + u_{i-1}(t)\right]} \quad **$$

Eq (**) can be solved by using forward Euler scheme ~~that~~ give us forward time centred scheme (FTCS)

$$f'(t) = \frac{f(t+\Delta t) - f(t)}{\Delta t} \Rightarrow f(t+\Delta t) = f(t) + \Delta t\, f'(t)$$

forward Euler scheme

$$y'(t) = f(t, y)$$

$$y(t_i + \Delta t) = y(t_i) + \overset{h}{\Delta t}\, y'(t_i, y(t_i))$$

$$\boxed{y_{i+1} = y_i + \Delta t \cdot f(t_i, y_i)}$$

$$\boxed{u_i^{n+1} = u_i^n + \frac{k \overset{\Delta t}{\tau}}{\Delta^2}\left(u_{i+1}^n - 2u_i^n + u_{i-1}^n\right)}$$ FTCS for heat eqn.

* Calculate the static distribution of temperature

$$\frac{\partial u(t,x)}{\partial t} = \frac{\partial^2 u(t,x)}{\partial x^2}$$

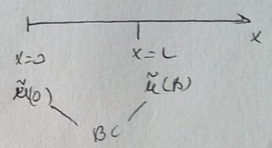
$t \to \infty$

$$\Rightarrow \boxed{\frac{\partial^2 \tilde{u}(t,x)}{\partial x^2}} = 0 \quad \to \text{for static case}$$

$$\Rightarrow \frac{\partial \tilde{u}(t,x)}{\partial t} = 0$$

Analytically

$$\frac{\partial \tilde{u}(x)}{\partial x} = c \Rightarrow \tilde{u}(x) = cx + B.$$



$\tilde{u}(0) = B \qquad c = 1/L\left[\tilde{u}(L) - \tilde{u}(0)\right]$

$\tilde{u}(L) = c \cdot L + \tilde{u}(0)$

$$\Rightarrow \boxed{\tilde{u}(x) = \tilde{u}(0) + \frac{x}{L}\left[\tilde{u}(L) - \tilde{u}(0)\right]}$$

Numerically using finite Difference scheme:

$$\frac{\partial^2 \tilde{u}_i}{\partial x^2} = \frac{\tilde{u}_{i+1} - 2\tilde{u}_i + \tilde{u}_{i-1}}{\Delta^2} = 0$$

$$\boxed{\tilde{u}_{i+1} - 2\tilde{u}_i + \tilde{u}_{i-1} = 0}$$

$BC = \tilde{u}_1 = \tilde{u}(0) \quad ; \quad \tilde{u}_{xx} = \tilde{u}(L)$

$$\underbrace{\begin{pmatrix} 1 & & & & & \\ -1 & 2 & 1 & & & \\ & -1 & 2 & -1 & & \\ & & -1 & 2 & 1 & \\ & & & -1 & 2 & 1 \\ & & & & & 1 \end{pmatrix}}_{M} \underbrace{\begin{pmatrix} \tilde{u}_1 \\ \tilde{u}_2 \\ \tilde{u}_3 \\ \tilde{u}_4 \\ \tilde{u}_5 \\ \tilde{u}_L \end{pmatrix}}_{\tilde{u}} = \underbrace{\begin{pmatrix} \tilde{u}(0) \\ 0 \\ 0 \\ 0 \\ 0 \\ \tilde{u}(L) \end{pmatrix}}_{B}$$

$$M\tilde{u} = B.$$