Experiment 1 - Kirchhoff's laws ad Ohm's law

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1 Introduction

This report is about experiment 1, it had two parts. The aim is to understand and explain the theory that is used in the experiments, show our ability to describe an experimental setup, and gain insight into the fundamentals of physical models through our data analysis.

1.1 Ohm's law

$$U = RI \tag{1}$$

Here, U is the Voltage drop, unit Volt (V), I is the electrical current, unit Ampere (A) and R is the resistance, unit Ohm (Ω).

1.2 Kirchhoff's laws

For defining Kirchhoff's laws we first need to define what nodes and meshes are. A node is a point on an electrical circuit, usually defined to be on a junction (since those are the points which may carry information that is of use to us). A mesh is a closed path in a circuit that starts and ends on the same node.

Node law (or current law) describes the flow of current through a given node - whatever enters the node also has to exit the node. For a more concrete definition: **The sum of all currents towards a node is 0.**

$$\sum I_{incoming} - \sum I_{outgoing} = 0 \tag{2}$$

Mesh law: **The sum of the electromotive forces** (ϵ) **are equal to the sum of voltage drops around a mesh.**

$$\sum \epsilon = \sum U \tag{3}$$

1.2.1 Resistances in series and in parallel connection

With the Kirchhoff's laws established, we can derive the resistances for different kinds of circuits.

Suppose we have a **series** connection. According to the node law, since there is no branching in the circuit and the current just "flows" in and out of the elements of the circuit, it always stays the same.

$$I = I_1 = I_2 = ... = I_n \tag{4}$$

If we look at the mesh law, however, we discover that voltages drop with every circuit element added to the series. Since only one mesh can be drawn, the sum of voltage drops is the electromotive force.

$$\epsilon = U_1 + U_2 + ... + U_n \tag{5}$$

By combining Equation 4, Equation 5 and Ohm's law, we get that

$$\epsilon = I_1 R_1 + I_2 R_2 + ... + I_n R_n = I(R_1 + R_2 + ... + R_n) \tag{6}$$

from which we conclude that the total resistance in a series circuit is the sum of individual resistances.

$$R = R_1 + R_2 + ... + R_n \tag{7}$$

Things get switched around when we look at a **parallel** connection. In that case, the number of meshes depends on how many circuit elements there are. Since one path is has 2 elements at most, with one (usually) being the *source* and the other one being the *user*, this leads to the following mesh equation:

$$\epsilon = U_1 = U_2 = ... = U_n \tag{8}$$

The currents, on the other hand, have to split up on the junctions, leading us to

$$I = I_1 + I_2 + ... + I_n \tag{9}$$

With the help of Ohm's law and Equation 8 we can rewrite Equation 9 as

$$I = \frac{U_1}{R_1} + \frac{U_2}{R_2} + ... + \frac{U_n}{R_n} = (\frac{1}{R_1} + \frac{1}{R_2} + ... + \frac{1}{R_n})\epsilon \tag{10}$$

From that we can infer that the inverse of the total resistance in a parallel circuit is the sum of the inverses of the individual resistances.

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} + ... + \frac{1}{R_n} \tag{11}$$

2 A non-trivial circuit

The given circuit is depicted in Figure 1, along with the directions of the branch currents, the nodes and the meshes used in Kirchhoff's law.

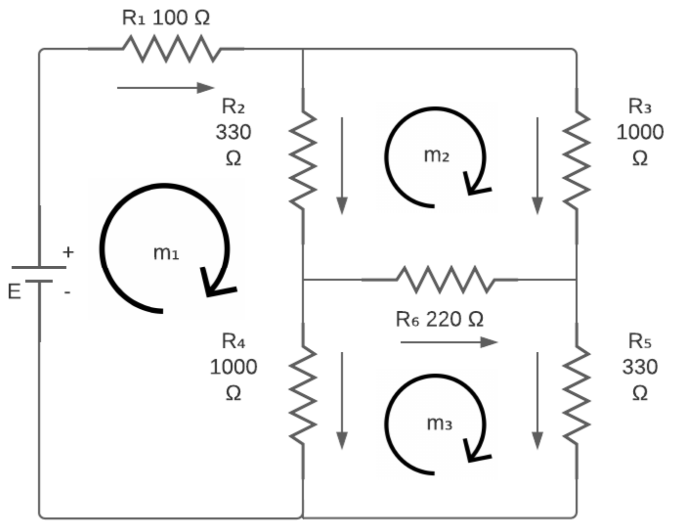

Figure 1: Example figure.

The first task was to read off the value of the resistors by the colour code and compare to direct measurements. As seen in Table 1, the values read off the resistors were the same as what was measured. Afterwards, the voltage was adjusted and measured to be $U = 10.01$ V. Then, the current going through each branch was measured, where the numbering corresponds to the numbering of the resistors, i.e. I_1 is the current going through R_1 etc. The branch current measurements can be found in Table 2.

Table 1: Table of resistors

Resistor	Marked resistance	Measured resistance
R_1	100 Ω	100 Ω
R_2	330 Ω	330 Ω
R_3	1000 Ω	1000 Ω
R_4	1000 Ω	1000 Ω
R_5	330 Ω	330 Ω
R_6	220 Ω	220 Ω

Table 2: Table of measured branch currents

Branch	measured current
I_1	15.49 mA
I_2	10.67 mA
I_3	4.84 mA
I_4	4.85 mA
I_5	10.66 mA
I_6	5.84 mA
I	15.49 mA

2.1 Data Analysis

From the data obtained in the experiment the branch currents can be calculated by solving the Kirchhoff equations of the circuit.

The defined meshes lead to the following mesh equations:

$$
\begin{aligned}
m_1 : \quad && \epsilon = U_1 + U_2 + U_4, && (12)\\
m_2 : \quad && 0 = U_3 - U_6 - U_2, && (13)\\
m_3 : \quad && 0 = U_6 + U_5 - U_4. && (14)
\end{aligned}
$$

By applying Ohm's law $U = RI$, the mesh equations can be rewritten as

$$
\begin{aligned}
m_1 : \quad && \epsilon = U_1 + U_2 + U_4 = R_1 I_1 + R_2 I_2 + R_4 I_4, && (15)\\
m_2 : \quad && 0 = U_3 - U_6 - U_2 = R_3 I_3 - R_6 I_6 - R_1 I_1, && (16)\\
m_3 : \quad && 0 = U_6 + U_5 - U_4 = R_6 I_6 + R_5 I_5 - R_4 I_4. && (17)
\end{aligned}
$$

The current going through the resistor R_1 must be equal to the total current I, so $I_1 = I$. Using this and the defined nodes we get the following node equations:

$$
\begin{aligned}
k_1 : \quad && I_1 - I_2 - I_3 = 0, && (18)\\
k_2 : \quad && I_2 - I_6 - I_4 = 0, && (19)\\
k_3 : \quad && I_3 + I_6 - I_5 = 0, && (20)\\
k_4 : \quad && I_4 + I_5 - I_1 = 0. && (21)
\end{aligned}
$$

In order to calculate the branch currents I_1 through I_6, a system of linear equations is made, consisting of the node and mesh equations. It can be represented in matrix form

$$
\begin{pmatrix}
R_1 & R_2 & 0 & R_4 & 0 & 0\\
0 & -R_2 & R_3 & 0 & 0 & -R_6\\
0 & 0 & 0 & -R_4 & R_5 & R_6\\
1 & -1 & -1 & 0 & 0 & 0\\
0 & 1 & 0 & -1 & 0 & -1\\
0 & 0 & 1 & 0 & -1 & 1\\
-1 & 0 & 0 & 1 & 1 & 0
\end{pmatrix}
\begin{pmatrix} I_1\\ I_2\\ I_3\\ I_4\\ I_5\\ I_6 \end{pmatrix}
=
\begin{pmatrix} \epsilon\\ 0\\ 0\\ 0\\ 0\\ 0\\ 0 \end{pmatrix}. \tag{22}
$$

Substituting $\epsilon = 10$ V, as well as the values for R_1 through R_6 (Table 1) leads to the coefficient matrix

$$
\begin{pmatrix}
R_1 & R_2 & 0 & R_4 & 0 & 0 & \epsilon\\
0 & -R_2 & R_3 & 0 & 0 & -R_6 & 0\\
0 & 0 & 0 & -R_4 & R_5 & R_6 & 0\\
1 & -1 & -1 & 0 & 0 & 0 & 0\\
0 & 1 & 0 & -1 & 0 & -1 & 0\\
0 & 0 & 1 & 0 & -1 & 1 & 0\\
-1 & 0 & 0 & 1 & 1 & 0 & 0
\end{pmatrix}
=
\begin{pmatrix}
100 & 330 & 0 & 1000 & 0 & 0 & 10\\
0 & -330 & 1000 & 0 & 0 & -220 & 0\\
0 & 0 & 0 & -1000 & 330 & 220 & 0\\
1 & -1 & -1 & 0 & 0 & 0 & 0\\
0 & 1 & 0 & -1 & 0 & -1 & 0\\
0 & 0 & 1 & 0 & -1 & 1 & 0\\
-1 & 0 & 0 & 1 & 1 & 0 & 0
\end{pmatrix} \tag{23}
$$

The Python script for solving this system of linear equations can be found at our GitHub repository or the url: `https://github.com/stine-fohrmann/Physical-Modelling/blob/main/Experiment_1_Kirchhoff_Ohm.ipynb`. The values it gave are in Table 3.

Table 3: Table of measured branch currents

Branch	measured current	calculated current
I_1	15.49 mA	15.67 mA
I_2	10.67 mA	10.80 mA
I_3	4.84 mA	4.87 mA
I_4	4.85 mA	4.87 mA
I_5	10.66 mA	10.80 mA
I_6	5.84 mA	5.93 mA
I	15.49 mA	15.67 mA

3 The line of resistors - A model of heat conduction

For the second part a chain of resistors is used for measuring total resistance. The resistors are set up as shown in section 3, where $R_1 = 100\ \Omega$ and $R_2 = 1000\ \Omega$. These resistances were verified by measuring.

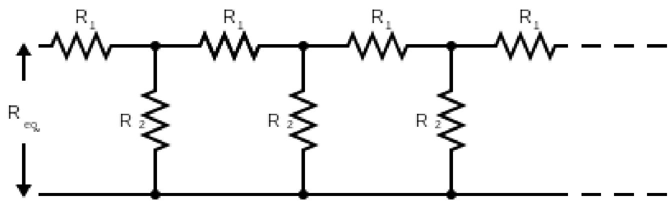

Figure 2: Chain of resistors.

For a chain consisting of one link the resistance is given by $R_{eq,1} = R_1 + R_2$, while the resistance for a chain consisting of two links is given by

$$R_{eq,2} = R_1 + \frac{1}{\frac{1}{R_2} + \frac{1}{R_{eq,1}}}, \tag{24}$$

leading to the general formula

$$R_{eq,N} = R_1 + \frac{1}{\frac{1}{R_2} + \frac{1}{R_{eq,N-1}}}. \tag{25}$$

For chains consisting of one through four links we thus expect the values depicted in Table 4. Figure 3 shows the total resistance dependent on chain length for chains of up to ten links (see our GitHub repository or the url: `https://github.com/stine-fohrmann/Physical-Modelling/blob/main/Experiment_1_Kirchhoff_Ohm.ipynb` for the code).

Table 4: Table of expected resistance.

Number of links	Resistance
1	1100 Ω
2	623,81 Ω
3	484.164 Ω
4	426.22 Ω

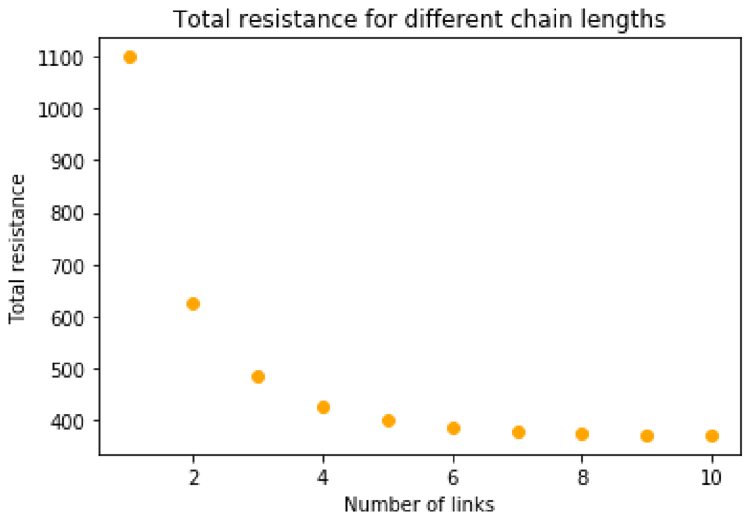

Figure 3: Total resistance dependent on chain length.

3.1 Chain of infinite length

The general formula for the total resistance $R_{eq,N}$ of a chain with N links is

$$R_{eq,N} = R_1 + \frac{1}{\frac{1}{R_2} + \frac{1}{R_{eq,N-1}}}. \tag{26}$$

For a chain of an infinite amount of links, N approaches ∞. As N approaches ∞, $R_{eq,N-1}$ approaches $R_{eq,N}$. Thus, we need to solve $R_{eq,N} = R_1 + \frac{1}{\frac{1}{R_2}+\frac{1}{R_{eq,N}}}$.

$$\begin{aligned}
R_{eq,N} &= R_1 + \frac{1}{\frac{1}{R_2} + \frac{1}{R_{eq,N}}} && (27)\\
&= R_1 + \frac{1}{\frac{1}{R_2} + \frac{1}{R_{eq,N}}} \cdot \frac{R_2 R_{eq,N}}{R_2 R_{eq,N}} && (28)\\
&= R_1 + \frac{R_2 R_{eq,N}}{R_{eq,N} + R_2} && (29)\\
\Rightarrow R_{eq,N} - R_1 &= \frac{R_2 R_{eq,N}}{R_{eq,N} + R_2} && (30)\\
\Rightarrow (R_{eq,N} - R_1)(R_{eq,N} + R_2) &= R_2 R_{eq,N} && (31)\\
= R_{eq,N}^2 - R_1 R_2 + R_2 R_{eq,N} - R_1 R_{eq,N} &= R_2 R_{eq,N} && (32)\\
\Rightarrow R_{eq,N}^2 - R_1 R_2 - R_1 R_{eq,N} &= 0 && (33)
\end{aligned}$$

Using the quadratic formula, $x_{1,2} = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$, where $x_{1,2} = R_{eq,N}$, $a = 1$, $b = -R_1$ and $c = -R_1 R_2$, we get:

$$R_{eq,N} = \frac{R_1 \pm \sqrt{(-R_1)^2 + 4R_1 R_2}}{2} \tag{34}$$

Inserting the values $R_1 = 100k\Omega$ and $R_2 = 1000k\Omega$, we get:

$$
\begin{aligned}
R_{eq,N} &= \frac{100 \pm \sqrt{100^2 + 4 \cdot 100 \cdot 1000}}{2} k\Omega && (35)\\
&= 50 \pm \frac{\sqrt{410000}}{2} k\Omega && (36)\\
&= 50 \pm 50\sqrt{41} k\Omega && (37)\\
&\approx 370.16 k\Omega \quad \vee \quad -270.16 k\Omega. && (38)
\end{aligned}
$$

The resistance must be positive because all the involved components are resistors with positive resistance. Thus, the resistance of an infinite chain is $R_{eq,N} \approx 370.16k\Omega$.

3.2 Experiment

The total resistance of chains with up to seven links was measured, along with the current at the first horizontal resistor and the last vertical resistor of each chain. See Table 5 for these measurements.

Table 5: Table of measurements ($U = 2.99$ V)

Number links	Resistance	Current(First horizontal)	Current(Last vertical)
1	1100 Ω	2.69 mA	2.69 mA
2	623 Ω	4.73 mA	2.26 mA
3	483 Ω	6.08 mA	1.79 mA
4	425 Ω	6.9 mA	1.37 mA
5	397 Ω	7.37 mA	1.03 mA
6	383 Ω	7.63 mA	0.76 mA
7	377 Ω	7.77 mA	0.55 mA

3.3 Model

A direct analogue of electrical resistance is thermal resistance:

$$R_{thermal} = r\frac{\Delta x}{A} \tag{39}$$

which is derived from Fourier's Law for heat conduction. Δx, the thickness of the object, is then the direct equivalent of l, the length of the wire, r, the thermal resistivity (which is more commonly denoted as $1/\lambda$, the inverse of the conductivity) is the equivalent of resistivity ρ. A is the cross sectional area in both cases.