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Chapter 1

Complex Numbers

1.1 The Algebra of Complex Numbers

To achieve a proper perspective for studying the system of complex numbers, let us begin by briefly reviewing the construction of the various numbers used in computation.

We start with the rational numbers. These are ratios of integers and are written in the form m/n, $n \neq 0$, with the stipulation that all rationals of the form n/n are equal to 1 (so we can cancel common factors). The arithmetic operations of addition, subtraction, multiplication, and division with these numbers can always be performed in a finite number of steps, and the results are, again, rational numbers. Furthermore, there are certain simple rules concerning the order in which the computations can proceed. These are the familiar commutative, associative, and distributive laws:

Commutative Law of Addition

$$a + b = b + a$$

Commutative Law of Multiplication

$$ab = ba$$

Associative Law of Addition

$$a + (b + c) = (a + b) + c$$

Associative Law of Multiplication

$$a(bc) = (ab)c$$

Distributive Law

$$(a + b)c = ac + bc,$$

for any rationals a, b, and c.

Notice that the rationals are the only numbers we would ever need, to solve equations of the form

$$ax + b = 0.$$

The solution, for nonzero a, is $x = -b/a$, and since this is the ratio of two rationals, it is itself rational.

However, if we try to solve quadratic equations in the rational system, we find that some of them have no solution; for example, the simple equation

$$x^2 = 2 \tag{1}$$

cannot be satisfied by any rational number (see Prob. 29 at the end of this section). Therefore, to get a more satisfactory number system, we extend the concept of "number" by appending to the rationals a new symbol, mnemonically written as $\sqrt{2}$, which is defined to be a solution of Eq. (1)). Our revised concept of a number is now an expression in the standard form

$$a + b\sqrt{2}, \tag{2}$$

where a and b are rationals. Addition and subtraction are performed according to

$$(a + b\sqrt{2}) \pm (c + d\sqrt{2}) = (a \pm c) + (b \pm d)\sqrt{2}. \tag{3}$$

Multiplication is defined via the distributive law with the proviso that the square of the symbol $\sqrt{2}$ can always be replaced by the rational number 2. Thus we have

$$(a + b\sqrt{2})(c + d\sqrt{2}) = (ac + 2bd) + (bc + ad)\sqrt{2}. \tag{4}$$

Finally, using the well-known process of *rationalizing the denominator*, we can put the quotient of any two of these new numbers into the standard form

$$\frac{a + b\sqrt{2}}{c + d\sqrt{2}} = \frac{a + b\sqrt{2}}{c + d\sqrt{2}}\,\frac{c - d\sqrt{2}}{c - d\sqrt{2}} = \frac{ac - 2bd}{c^2 - 2d^2} + \frac{bc - ad}{c^2 - 2d^2}\sqrt{2}. \tag{5}$$

This procedure of "calculating with radicals" should be very familiar to the reader, and the resulting arithmetic system can easily be shown to satisfy the commutative, associative, and distributive laws. However, observe that the symbol $\sqrt{2}$ has not been absorbed by the rational numbers painlessly. Indeed, in the standard form (2) and in the algorithms (3), (4), and (5) its presence stands out like a sore thumb. Actually, we are only using the symbol $\sqrt{2}$ to "hold a place" while we compute around it using the rational components, except for those occasional opportunities when it occurs squared and we are temporarily relieved of having to carry it. So the inclusion of $\sqrt{2}$ as a number is a somewhat artificial process, devised solely so that we might have a richer system in which we can solve the equation $x^2 = 2$.

With this in mind, let us jump to the stage where we have appended all the real numbers to our system. Some of them, such as $\sqrt[4]{17}$, arise as solutions of more complicated equations, while others, such as π and e, come from certain limit processes.

Each irrational is absorbed in a somewhat artificial manner, but once again the resulting conglomerate of numbers and arithmetic operations satisfies the commutative, associative, and distributive laws.[†]

At this point we observe that we still cannot solve the equation

$$x^2 = -1. \tag{6}$$

But now our experience suggests that we can expand our number system once again by appending a symbol for a solution to Eq. (6); instead of $\sqrt{-1}$, it is customary to use the symbol i. (Engineers often use the letter j.) Next we imitate the model of expressions (2) through (5) (pertaining to $\sqrt{2}$) and thereby generalize our concept of number as follows:[‡]

> **Definition 1.** A **complex number** is an expression of the form $a + bi$, where a and b are real numbers. Two complex numbers $a + bi$ and $c + di$ are said to be equal ($a + bi = c + di$) if and only if $a = c$ and $b = d$.

The operations of addition and subtraction of complex numbers are given by

$$(a + bi) \pm (c + di) := (a \pm c) + (b \pm d)i,$$

where the symbol := means "is defined to be."

In accordance with the distributive law and the proviso that $i^2 = -1$, we postulate the following:

The multiplication of two complex numbers is defined by

$$(a + bi)(c + di) := (ac - bd) + (bc + ad)i.$$

To compute the quotient of two complex numbers, we again "rationalize the denominator":

$$\frac{a + bi}{c + di} = \frac{a + bi}{c + di}\,\frac{c - di}{c - di} = \frac{ac + bd}{c^2 + d^2} + \frac{bc - ad}{c^2 + d^2}\,i.$$

Thus we formally postulate the following:

The division of complex numbers is given by

$$\frac{a + bi}{c + di} := \frac{ac + bd}{c^2 + d^2} + \frac{bc - ad}{c^2 + d^2}\,i \qquad (\text{if } c^2 + d^2 \neq 0).$$

These are rules for computing in the complex number system. The usual algebraic properties (commutativity, associativity, etc.) are easy to verify and appear as exercises.

[†] The algebraic aspects of extending a number field are discussed in Ref. 5 at the end of this chapter.

[‡] Karl Friedrich Gauss (1777–1855) was the first mathematician to use complex numbers freely and give them full acceptance as genuine mathematical objects.

Example 1
Find the quotient

$$\frac{(6+2i)-(1+3i)}{(-1+i)-2}.$$

Solution.

$$\frac{(6+2i)-(1+3i)}{(-1+i)-2} = \frac{5-i}{-3+i} = \frac{(5-i)}{(-3+i)}\frac{(-3-i)}{(-3-i)}$$
$$= \frac{-15-1-5i+3i}{9+1} \tag{7}$$
$$= -\frac{8}{5}-\frac{1}{5}i. \quad \blacksquare$$

(A slug marks the end of solutions or proofs throughout the text.)

Historically, i was considered as an "imaginary" number because of the blatant impossibility of solving Eq. (6) with any of the numbers at hand. With the perspective we have developed, we can see that this label could also be applied to the numbers $\sqrt{2}$ or $\sqrt[4]{17}$; like them, i is simply one more symbol appended to a given number system to create a richer system. Nonetheless, tradition dictates the following designations:†

Definition 2. The **real part** of the complex number $a+bi$ is the (real) number a; its **imaginary part** is the (real) number b. If a is zero, the number is said to be a **pure imaginary number**.

For convenience we customarily use a single letter, usually z, to denote a complex number. Its real and imaginary parts are then written $\operatorname{Re} z$ and $\operatorname{Im} z$, respectively. With this notation we have $z = \operatorname{Re} z + i \operatorname{Im} z$.

Observe that the equation $z_1 = z_2$ holds if and only if $\operatorname{Re} z_1 = \operatorname{Re} z_2$ and $\operatorname{Im} z_1 = \operatorname{Im} z_2$. Thus any equation involving complex numbers can be interpreted as a pair of real equations.

The set of all complex numbers is sometimes denoted as **C**. Unlike the real number system, there is no natural ordering for the elements of **C**; it is meaningless, for example, to ask whether $2+3i$ is greater than or less than $3+2i$. (See Prob. 30.)

EXERCISES 1.1

1. Verify that $-i$ is also a root of Eq. (6).

2. Verify the commutative, associative, and distributive laws for complex numbers.

†René Descartes introduced the terminology "real" and "imaginary" in 1637. W. R. Hamilton referred to a number's "imaginary part" in 1843.

3. Notice that 0 and 1 retain their "identity" properties as complex numbers; that is, $0+z=z$ and $1\cdot z = z$ when z is complex.

(a) Verify that complex subtraction is the inverse of complex addition (that is, $z_3 = z_2 - z_1$ if and only if $z_3 + z_1 = z_2$).

(b) Verify that complex division, as given in the text, is the inverse of complex multiplication (that is, if $z_2 \neq 0$, then $z_3 = z_1/z_2$ if and only if $z_3 z_2 = z_1$).

4. Prove that if $z_1 z_2 = 0$, then $z_1 = 0$ or $z_2 = 0$.

In Problems 5–13, write the number in the form $a+bi$.

5. **(a)** $-3\left(\frac{i}{2}\right)$ **(b)** $(8+i)-(5+i)$ **(c)** $\frac{2}{i}$

6. **(a)** $(-1+i)^2$ **(b)** $\frac{2-i}{\frac{1}{3}}$ **(c)** $i(\pi-4i)$

7. **(a)** $\frac{8i-1}{i}$ **(b)** $\frac{-1+5i}{2+3i}$ **(c)** $\frac{3}{i}+\frac{i}{3}$

8. $\frac{(8+2i)-(1-i)}{(2+i)^2}$

9. $\frac{2+3i}{1+2i}-\frac{8+i}{6-i}$

10. $\left[\frac{2+i}{6i-(1-2i)}\right]^2$

11. $i^3(i+1)^2$

12. $(2+i)(-1-i)(3-2i)$

13. $((3-i)^2-3)i$

14. Show that $\operatorname{Re}(iz) = -\operatorname{Im} z$ for every complex number z.

15. Let k be an integer. Show that

$$i^{4k}=1, \qquad i^{4k+1}=i, \qquad i^{4k+2}=-1, \qquad i^{4k+3}=-i.$$

16. Use the result of Problem 15 to find

(a) i^7 **(b)** i^{62} **(c)** i^{-202} **(d)** i^{-4321}

17. Use the result of Problem 15 to evaluate

$$3i^{11}+6i^3+\frac{8}{i^{20}}+i^{-1}.$$

18. Show that the complex number $z=-1+i$ satisfies the equation

$$z^2+2z+2=0.$$

19. Write the complex equation $z^3 + 5z^2 = z + 3i$ as two real equations.

20. Solve each of the following equations for z.

(a) $iz = 4 - zi$ (b) $\dfrac{z}{1-z} = 1 - 5i$

(c) $(2 - i)z + 8z^2 = 0$ (d) $z^2 + 16 = 0$

21. The complex numbers z_1, z_2 satisfy the system of equations

$$(1 - i)z_1 + 3z_2 = 2 - 3i,$$
$$iz_1 + (1 + 2i)z_2 = 1.$$

Find z_1, z_2.

22. Find all solutions to the equation $z^4 - 16 = 0$.

23. Let z be a complex number such that $\text{Re } z > 0$. Prove that $\text{Re}(1/z) > 0$.

24. Let z be a complex number such that $\text{Im } z > 0$. Prove that $\text{Im}(1/z) < 0$.

25. Let z_1, z_2 be two complex numbers such that $z_1 + z_2$ and $z_1 z_2$ are each negative real numbers. Prove that z_1 and z_2 must be real numbers.

26. Verify that

$$\text{Re}\Big(\sum_{j=1}^{n} z_j\Big) = \sum_{j=1}^{n} \text{Re } z_j$$

and that

$$\text{Im}\Big(\sum_{j=1}^{n} z_j\Big) = \sum_{j=1}^{n} \text{Im } z_j.$$

[The real (imaginary) part of the sum is the sum of the real (imaginary) parts.] Formulate, and then *disprove*, the corresponding conjectures for multiplication.

27. Prove the *binomial formula* for complex numbers:

$$(z_1 + z_2)^n = z_1^n + \binom{n}{1} z_1^{n-1} z_2 + \cdots + \binom{n}{k} z_1^{n-k} z_2^k + \cdots + z_2^n,$$

where n is a positive integer, and the *binomial coefficients* are given by

$$\binom{n}{k} := \frac{n!}{k!(n-k)!}.$$

28. Use the binomial formula (Prob. 27) to compute $(2 - i)^5$.

29. Prove that there is no rational number x that satisfies $x^2 = 2$. [HINT: Show that if p/q were a solution, where p and q are integers, then 2 would have to divide both p and q. This contradicts the fact that such a ratio can always be written without common divisors.]

30. The definition of the order relation denoted by $>$ in the real number system is based upon the existence of a subset $\mathcal{P}$ (the positive reals) having the following properties:

(i) For any number $\alpha \neq 0$, either α or $-\alpha$ (but not both) belongs to $\mathcal{P}$.

(ii) If α and β belong to $\mathcal{P}$, so does $\alpha + \beta$.

(iii) If α and β belong to $\mathcal{P}$, so does $\alpha \cdot \beta$.

When such a set $\mathcal{P}$ exists we write $\alpha > \beta$ if and only if $\alpha - \beta$ belongs to $\mathcal{P}$.† Prove that the *complex* number system does not possess a nonempty subset $\mathcal{P}$ having properties (i), (ii), and (iii). [HINT: Argue that neither i nor $-i$ could belong to such a set $\mathcal{P}$.]

31. Write a computer program for calculating sums, differences, products, and quotients of complex numbers. The input and output parameters should be the corresponding real and imaginary parts.

32. The straightforward method of computing the product $(a + bi)(c + di) = (ac - bd) + i(bc + ad)$ requires four (real) multiplications (and two signed additions). On most computers multiplication is far more time-consuming than addition. Devise an algorithm for computing $(a + bi)(c + di)$ with only three multiplications (at the cost of extra additions). [HINT: Start with $(a + b)(c + d)$.]

1.2 Point Representation of Complex Numbers

It is presumed that the reader is familiar with the Cartesian coordinate system (Fig. 1.1) which establishes a one-to-one correspondence between points in the xy-plane and ordered pairs of real numbers. The ordered pair $(-2, 3)$, for example, corresponds to that point P that lies two units to the left of the y-axis and three units above the x-axis.

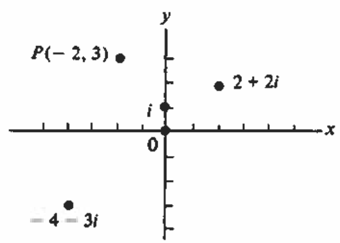

Figure 1.1 Cartesian coordinate system.

†On computers this is, in fact, the method by which the statement $\alpha > \beta$ is tested.

The Cartesian coordinate system suggests a convenient way to represent complex numbers as points in the xy-plane; namely, to each complex number $a + bi$ we associate that point in the xy-plane that has the coordinates (a, b). The complex number $-2 + 3i$ is therefore represented by the point P in Fig. 1.1. Also shown in Fig. 1.1 are the points that represent the complex numbers $0, i, 2 + 2i$, and $-4 - 3i$.

When the xy-plane is used for the purpose of describing complex numbers it is referred to as the *complex plane* or z-plane. (The term *Argand diagram* is sometimes used; the representation of complex numbers in the plane was proposed independently by Caspar Wessel in 1797 and Jean Pierre Argand in 1806.) Since each point on the x-axis represents a real number, this axis is called the *real axis*. Analogously, the y-axis is called the *imaginary axis* for it represents the pure imaginary numbers.

Hereafter, we shall refer to the point that represents the complex number z as simply *the point* z; that is, the point $z = a + bi$ is the point with coordinates (a, b).

Example 1

Suppose that n particles with masses $m_1, m_2, \ldots, m_n$ are located at the respective points $z_1, z_2, \ldots, z_n$ in the complex plane. Show that the center of mass of the system is the point

$$\widehat{z} = \frac{m_1 z_1 + m_2 z_2 + \cdots + m_n z_n}{m_1 + m_2 + \cdots + m_n}.$$

Solution. Write $z_1 = x_1 + y_1 i, z_2 = x_2 + y_2 i, \ldots, z_n = x_n + y_n i$, and let M be the total mass $\sum_{k=1}^{n} m_k$. Presumably the reader will recall that the center of mass of the given system is the point with coordinates $(\widehat{x}, \widehat{y})$, where

$$\widehat{x} = \frac{\sum\limits_{k=1}^{n} m_k x_k}{M}, \qquad \widehat{y} = \frac{\sum\limits_{k=1}^{n} m_k y_k}{M}.$$

But clearly $\widehat{x}$ and $\widehat{y}$ are, respectively, the real and imaginary parts of the complex number $(\sum_{k=1}^{n} m_k z_k)/M = \widehat{z}$. ■

Absolute Value. By the Pythagorean theorem, the distance from the point $z = a + bi$ to the origin is given by $\sqrt{a^2 + b^2}$. Special notation for this distance is given in

Definition 3. The **absolute value** or **modulus** of the number $z = a + bi$ is denoted by $|z|$ and is given by

$$|z| := \sqrt{a^2 + b^2}.$$

In particular,

$$|0| = 0, \qquad \left|\frac{i}{2}\right| = \frac{1}{2}, \qquad |3 - 4i| = \sqrt{9 + 16} = 5.$$

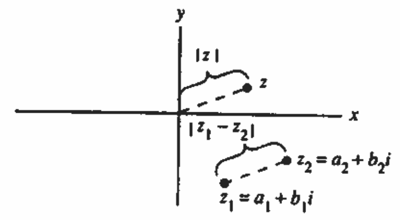

Figure 1.2 Distance between points.

The reader should note that $|z|$ is always a nonnegative real number and that the *only* complex number whose modulus is zero is the number 0.

Let $z_1 = a_1 + b_1 i$ and $z_2 = a_2 + b_2 i$. Then

$$|z_1 - z_2| = |(a_1 - a_2) + (b_1 - b_2)i| = \sqrt{(a_1 - a_2)^2 + (b_1 - b_2)^2},$$

which is the distance between the points with coordinates (a_1, b_1) and (a_2, b_2) (see Fig. 1.2). Hence *the distance between the points z_1 and z_2 is given by $|z_1 - z_2|$*. This fact is useful in describing certain curves in the plane. Consider, for example, the set of all numbers z that satisfy the equation

$$|z - z_0| = r, \tag{1}$$

where z_0 is a fixed complex number and r is a fixed positive real number. This set consists of all points z whose distance from z_0 is r. Consequently Eq. (1) is the equation of a circle.

Example 2

Describe the set of points z that satisfy the equations

(a) $|z + 2| = |z - 1|$, **(b)** $|z - 1| = \operatorname{Re} z + 1$.

Solution. **(a)** A point z satisfies Eq. (a) if and only if it is equidistant from the points -2 and 1. Hence Eq. (a) is the equation of the perpendicular bisector of the line segment joining -2 and 1; that is, Eq. (a) describes the line $x = -\frac{1}{2}$.

A more routine method for solving Eq. (a) is to set $z = x + iy$ in the equation and perform the algebra:

$$\begin{aligned} |z + 2| &= |z - 1|, \\ |x + iy + 2| &= |x + iy - 1|, \\ (x + 2)^2 + y^2 &= (x - 1)^2 + y^2, \\ 4x + 4 &= -2x + 1, \\ x &= -\frac{1}{2}. \end{aligned}$$

(b) The geometric interpretation of Eq. (b) is less obvious, so we proceed directly with the mechanical approach and derive $\sqrt{(x - 1)^2 + y^2} = x + 1$, or $y^2 = 4x$, which describes a parabola (see Fig. 1.3). ■

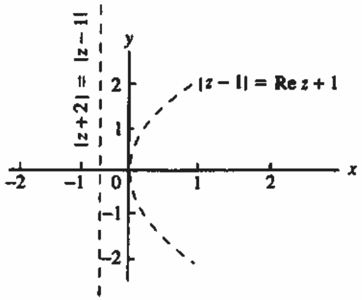

Figure 1.3 Graphs for Example 2.

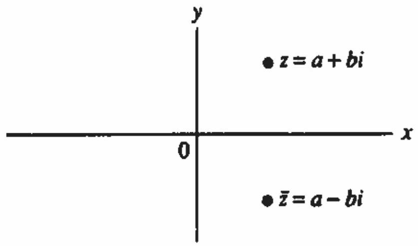

Figure 1.4 Complex conjugates.

Complex Conjugates. The reflection of the point $z = a + bi$ in the real axis is the point $a - bi$ (see Fig. 1.4). As we shall see, the relationship between $a + bi$ and $a - bi$ will play a significant role in the theory of complex variables. We introduce special notation for this concept in the next definition.

Definition 4. The **complex conjugate** of the number $z = a + bi$ is denoted by $\bar{z}$ and is given by

$$\bar{z} := a - bi.$$

Thus,

$$\overline{-1 + 5i} = -1 - 5i, \qquad \overline{\pi - i} = \pi + i, \qquad \bar{8} = 8.$$

Some authors use the asterisk, z^*, to denote the complex conjugate.

It follows from Definition 4 that $z = \bar{z}$ if and only if z is a real number. Also it is clear that the conjugate of the sum (difference) of two complex numbers is equal to the sum (difference) of their conjugates; that is,

$$\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}, \quad \overline{z_1 - z_2} = \overline{z_1} - \overline{z_2}.$$

Perhaps not so obvious is the analogous property for multiplication.

Example 3

Prove that the conjugate of the product of two complex numbers is equal to the product of the conjugates of these numbers.

Solution. It is required to verify that

$$\overline{(z_1 z_2)} = \overline{z_1}\,\overline{z_2}. \tag{2}$$

Write $z_1 = a_1 + b_1 i$, $z_2 = a_2 + b_2 i$. Then

$$\begin{aligned}\overline{(z_1 z_2)} &= \overline{a_1a_2 - b_1b_2 + (a_1b_2 + a_2b_1)i}\\ &= a_1a_2 - b_1b_2 - (a_1b_2 + a_2b_1)i.\end{aligned}$$

On the other hand,

$$\begin{aligned}\overline{z_1}\,\overline{z_2} = (a_1 - b_1 i)(a_2 - b_2 i) &= a_1a_2 - b_1b_2 - a_1b_2 i - a_2b_1 i\\ &= a_1a_2 - b_1b_2 - (a_1b_2 + a_2b_1)i.\end{aligned}$$

Thus Eq. (2) holds. ■

In addition to Eq. (2) the following properties can be seen:

$$\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\overline{z_1}}{\overline{z_2}} \quad (z_2 \neq 0); \tag{3}$$

$$\operatorname{Re} z = \frac{z + \bar{z}}{2}; \tag{4}$$

$$\operatorname{Im} z = \frac{z - \bar{z}}{2i}; \tag{5}$$

Property (4) demonstrates that the sum of a complex number and its conjugate is real, whereas (5) shows that the difference is (pure) imaginary. The conjugate of the conjugate of a complex number is, of course, the number itself:

$$\overline{(\bar{z})} = z. \tag{6}$$

It is clear from Definition 4 that

$$|z| = |\bar{z}|;$$

that is, the points z and $\bar{z}$ are equidistant from the origin. Furthermore, since

$$z\bar{z} = (a + bi)(a - bi) = a^2 + b^2,$$

we have

$$z\bar{z} = |z|^2. \tag{7}$$

This is a useful fact to remember: *The square of the modulus of a complex number equals the number times its conjugate.*

Actually we have already employed complex conjugates in Sec. 1.1, in the process of rationalizing the denominator for the division algorithm. Thus, for instance, if z_1 and z_2 are complex numbers, then we rewrite z_1/z_2 as a ratio with a real denominator by using $\overline{z_2}$:

$$\frac{z_1}{z_2} = \frac{z_1\overline{z_2}}{z_2\overline{z_2}} = \frac{z_1\overline{z_2}}{|z_2|^2}. \tag{8}$$

In particular,

$$\frac{1}{z} = \frac{\bar{z}}{|z|^2}. \tag{9}$$

In closing we would like to mention that there is another, possibly more enlightening, way to see Eq. (2). Notice that when we represent a complex number in terms of two real numbers and the symbol i, as in $z = a + bi$, then the action of conjugation is equivalent to changing the sign of the i term. Now recall the role that i plays in computations; it merely holds a place while we compute around it, replacing its square by -1 whenever it arises. Except for these occurrences i is never really absorbed into the computations; we could just as well call it j, λ, $\sqrt{-1}$, or any other symbol whose square we agree to replace by -1. *In fact, without affecting the validity of the calculation, we could replace it throughout by the symbol $(-i)$, since the square of the latter is also -1.* Thus, for instance, if in the expression $(a_1 + b_1 i)(a_2 + b_2 i)$ we replace i by $-i$ and then multiply, the only thing different about the product will be the appearance of $-i$ instead of i. But expressed in terms of conjugation, this is precisely the statement of Example 3.†

EXERCISES 1.2

1. Show that the point $(z_1 + z_2)/2$ is the midpoint of the line segment joining z_1 and z_2.
2. Given four particles of masses 2, 1, 3, and 5 located at the respective points $1 + i$, $-3i$, $1 - 2i$, and -6, find the center of mass of this system.
3. Which of the points i, $2 - i$, and -3 is farthest from the origin?
4. Let $z = 3 - 2i$. Plot the points z, $-z$, $\bar{z}$, $-\bar{z}$, and $1/z$ in the complex plane. Do the same for $z = 2 + 3i$ and $z = -2i$.
5. Show that the points 1, $-1/2 + i\sqrt{3}/2$, and $-1/2 - i\sqrt{3}/2$ are the vertices of an equilateral triangle.
6. Show that the points $3 + i$, 6, and $4 + 4i$ are the vertices of a right triangle.

†By the same token we should be able to replace $\sqrt{2}$ by $-\sqrt{2}$ in $(3 + 2\sqrt{2})(4 - 3\sqrt{2})$ *either before or after multiplying* and obtain the same result. (Try it.)

7. Describe the set of points z in the complex plane that satisfies each of the following.

   (a) $\operatorname{Im} z = -2$ **(b)** $|z - 1 + i| = 3$

   (c) $|2z - i| = 4$ **(d)** $|z - 1| = |z + i|$

   (e) $|z| = \operatorname{Re} z + 2$ **(f)** $|z - 1| + |z + 1| = 7$

   (g) $|z| = 3|z - 1|$ **(h)** $\operatorname{Re} z \geq 4$

   (i) $|z - i| < 2$ **(j)** $|z| > 6$

8. Show, both analytically and graphically, that $|z - 1| = |\bar{z} - 1|$.
9. Show that if r is a nonnegative real number, then $|rz| = r|z|$.
10. Prove that $|\operatorname{Re} z| \leq |z|$ and $|\operatorname{Im} z| \leq |z|$.
11. Prove that if $|z| = \operatorname{Re} z$, then z is a nonnegative real number.
12. Verify properties (3), (4), and (5).
13. Prove that if $(\bar{z})^2 = z^2$, then z is either real or pure imaginary.
14. Prove that $|z_1 z_2| = |z_1||z_2|$. [HINT: Use Eqs. (7) and (2) to show that $|z_1 z_2|^2 = |z_1|^2|z_2|^2$.]
15. Prove that $(\bar{z})^k = \overline{(z^k)}$ for every integer k (provided $z \neq 0$ when k is negative).
16. Prove that if $|z| = 1$ $(z \neq 1)$, then $\operatorname{Re}[1/(1 - z)] = \frac{1}{2}$.
17. Let $a_1, a_2, \ldots, a_n$ be real constants. Show that if z_0 is a root of the polynomial equation $z^n + a_1 z^{n-1} + a_2 z^{n-2} + \cdots + a_n = 0$, then so is $\overline{z_0}$.
18. Use the familiar formula for the roots of a quadratic polynomial to give another proof of the statement in Prob. 17 for the case $n = 2$.
19. We have noted that the conjugate $(\bar{z})$ is the reflection of the point z in the real axis (the horizontal line $y = 0$). Show that the reflection of z in the line $ax + by = c$ $(a, b, c$ real) is given by
$$\frac{2ic + (b - ai)\bar{z}}{b + ai}.$$
20. (*Matrices with Complex Entries*) Let **B** be an m by n matrix whose entries are complex numbers. Then by $\mathbf{B}^\dagger$ we denote the n by m matrix that is obtained by forming the transpose (interchanging rows and columns) of **B** followed by taking the conjugate of each entry. In other words, if $\mathbf{B} = [b_{ij}]$, then $\mathbf{B}^\dagger = [\bar{b}_{ji}]$. For example:
$$\begin{bmatrix} i & 3 \\ 4 - i & -2i \end{bmatrix}^\dagger = \begin{bmatrix} -i & 4 + i \\ 3 & 2i \end{bmatrix}, \qquad \begin{bmatrix} 1 + i \\ 3 \end{bmatrix}^\dagger = \begin{bmatrix} 1 - i & 3 \end{bmatrix}.$$

For an n by n matrix $\mathbf{A} = [a_{ij}]$ with complex entries, prove the following:

(a) If $\mathbf{u}^\dagger \mathbf{A} \mathbf{u} = \mathbf{0}$ for all n by 1 column vectors $\mathbf{u}$ with complex entries, then $\mathbf{A}$ is the zero matrix (that is, $a_{ij} = 0$ for all i, j). [HINT: To show $a_{ij} = 0$, take $\mathbf{u}$ to be a column vector with all zeros except for its i^{th} and j^{th} entries.]

(b) Show by example that the conclusion ("$\mathbf{A}$ is the zero matrix") can fail if the hypothesis for part (a) only holds for vectors $\mathbf{u}$ with *real* number entries. [HINT: Try to find a 2 by 2 real *nonzero* matrix $\mathbf{A}$ such that $\mathbf{u}^\dagger \mathbf{A} \mathbf{u} = 0$ for all real 2 by 1 vectors $\mathbf{u}$.]

21. Let $\mathbf{A}$ be an n by n matrix with complex entries. We say that $\mathbf{A}$ is *Hermitean* if $\mathbf{A}^\dagger = \mathbf{A}$ (see Prob. 20).

(a) Show that if $\mathbf{A}$ is Hermitean, then $\mathbf{u}^\dagger \mathbf{A} \mathbf{u}$ is real for any n by 1 column vector $\mathbf{u}$ with complex entries.

(b) Show that if $\mathbf{B}$ is any m by n matrix with complex entries, then $\mathbf{B}^\dagger \mathbf{B}$ is Hermitean.

(c) Show that if $\mathbf{B}$ is any n by n matrix and $\mathbf{u}$ is any n by 1 column vector (each with complex entries), then $\mathbf{u}^\dagger \mathbf{B}^\dagger \mathbf{B} \mathbf{u}$ must be a nonnegative real number.

1.3 Vectors and Polar Forms

With each point z in the complex plane we can associate a *vector*, namely, the directed line segment from the origin to the point z. Recall that vectors are characterized by length and direction, and that a given vector remains unchanged under translation. Thus the vector determined by $z = 1 + i$ is the same as the vector from the point $2 + i$ to the point $3 + 2i$ (see Fig. 1.5). Note that every vector parallel to the real axis corresponds to a real number, while those parallel to the imaginary axis represent pure imaginary numbers. Observe, also, that the length of the vector associated with z is $|z|$.

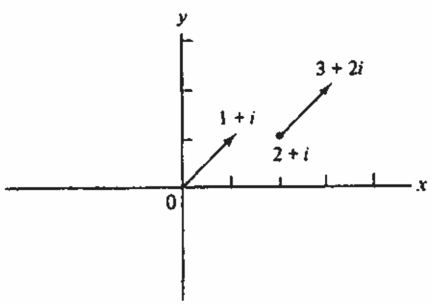

Figure 1.5 Complex numbers as vectors.

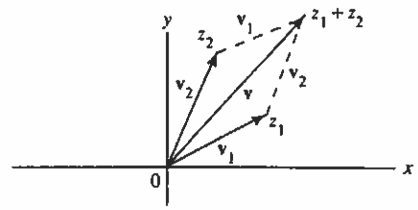

Figure 1.6 Vector addition.

Let $\mathbf{v}_1$ and $\mathbf{v}_2$ denote the vectors determined by the points z_1 and z_2, respectively. The vector sum $\mathbf{v} = \mathbf{v}_1 + \mathbf{v}_2$ is given by the parallelogram law, which is illustrated in Fig. 1.6. If $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$, then the terminal point of the vector $\mathbf{v}$ in Fig. 1.6 has the coordinates $(x_1 + x_2, y_1 + y_2)$; that is, it corresponds to the point $z_1 + z_2$. Thus we see that *the correspondence between complex numbers and planar vectors carries over to the operation of addition.*

Hereafter, the vector determined by the point z will be simply called *the vector* z.

Recall the geometric fact that the length of any side of a triangle is less than or equal to the sum of the lengths of the other two sides. If we apply this theorem to the triangle in Fig. 1.6 with vertices $0, z_1$, and $z_1 + z_2$, we deduce a very important law relating the magnitudes of complex numbers and their sum:

Triangle Inequality. For any two complex numbers z_1 and z_2, we have

$$|z_1 + z_2| \le |z_1| + |z_2|. \tag{1}$$

The triangle inequality can easily be extended to more than two complex numbers, as requested in Prob. 22.

The vector $z_2 - z_1$, when added to the vector z_1, obviously yields the vector z_2. Thus $z_2 - z_1$ can be represented as the directed line segment from z_1 to z_2 (see Fig. 1.7). Applying the geometric theorem to the triangle in Fig. 1.7, we deduce another form of the triangle inequality:

$$|z_2| \le |z_1| + |z_2 - z_1|$$

or

$$|z_2| - |z_1| \le |z_2 - z_1|. \tag{2}$$

Inequality (2) states that the difference in the lengths of any two sides of a triangle is no greater than the length of the third side.

Example 1

Prove that the three distinct points z_1, z_2, and z_3 lie on the same straight line if and only if $z_3 - z_2 = c(z_2 - z_1)$ for some real number c.

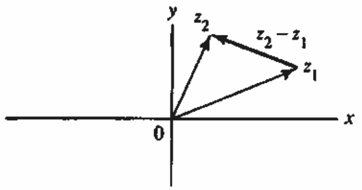

Figure 1.7 Vector subtraction.

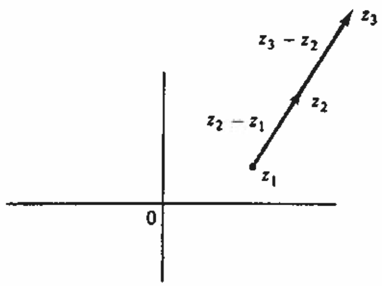

Figure 1.8 Collinear points.

Solution. Recall that two vectors are parallel if and only if one is a (real) scalar multiple of the other. In the language of complex numbers, this says that z is parallel to w if and only if $z = cw$, where c is real. From Fig. 1.8 we see that the condition that the points z_1, z_2, and z_3 be collinear is equivalent to the statement that the vector $z_3 - z_2$ is parallel to the vector $z_2 - z_1$. Using our characterization of parallelism, the conclusion follows immediately. ■

There is another set of parameters that characterize the vector from the origin to the point z (other, that is, than the real and imaginary parts of z), which more intimately reflects its interpretation as an object with magnitude and direction. These are the polar coordinates, r and θ, of the point z. The coordinate r is the distance from the origin to z, and θ is the angle of inclination of the vector z, measured positively in a counterclockwise sense from the positive real axis (and thus negative when measured clockwise) (see Fig. 1.9). *We shall always measure angles in radians in this book*; the use of degree measure is fine for visualization purposes, but it becomes quite treacherous in any discipline where calculus is involved. Notice that r is the modulus, or absolute value, of z and is never negative: $r = |z|$.

From Fig. 1.9 we readily derive the equations expressing the rectangular (or *Cartesian*) coordinates (x, y) in terms of the polar coordinates (r, θ):

$$x = r\cos\theta, \qquad y = r\sin\theta. \tag{3}$$

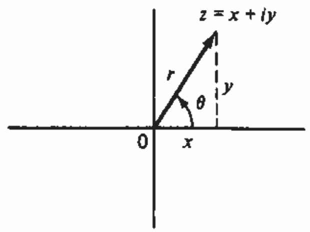

Figure 1.9 Polar coordinates.

On the other hand, the expressions for (r, θ) in terms of (x, y) contain some minor but troublesome complications. Indeed the coordinate r is given, unambiguously, by

$$r = \sqrt{x^2 + y^2} = |z|. \tag{4}$$

However, observe that although it is certainly true that $\tan\theta = y/x$, the natural conclusion

$$\theta = \tan^{-1}\left(\frac{y}{x}\right)$$

is *invalid* for points z in the second and third quadrants (since the standard interpretation of the arctangent function places its range in the first and fourth quadrants). Since an angle *is* fixed by its sine *and* cosine, θ is uniquely determined by the pair of equations

$$\cos\theta = \frac{x}{|z|}, \qquad \sin\theta = \frac{y}{|z|}, \tag{5}$$

but in practice we usually compute $\tan^{-1}(y/x)$ and adjust for the quadrant problem by adding or subtracting π (radians) when appropriate (see Prob. 14).

The nuisance aspects of θ do not end here, however. Even using Eqs. (5) one can, because of its identification as an angle, determine θ only up to an integer multiple of 2π. To accommodate this feature we shall call the value of any of these angles an *argument*, or *phase*, of z, denoted

$$\arg z.$$

Thus if θ_0 qualifies as a value of $\arg z$, then so do

$$\theta_0 \pm 2\pi,\ \theta_0 \pm 4\pi,\ \theta_0 \pm 6\pi, \ldots,$$

and every value of $\arg z$ must be one of these.† In particular, the values of $\arg i$ are

$$\frac{\pi}{2}, \frac{\pi}{2} \pm 2\pi, \frac{\pi}{2} \pm 4\pi, \ldots$$

†An alternative way to express $\arg z$ is to write it as the *set*

$$\arg z = \{\theta_0 + 2k\pi : k = 0, \pm1, \pm2, \ldots\}.$$

and we write

$$\arg i = \frac{\pi}{2} + 2k\pi \qquad (k = 0, \pm 1, \pm 2, \ldots).$$

It is convenient to have a notation for some *definite* value of $\arg z$. Notice that any half-open interval of length 2π will contain one and only one value of the argument. By specifying such an interval we say that we have selected a particular *branch* of $\arg z$. Figure 1.10 illustrates three possible branch selections. The first diagram (Fig. 1.10(a)) depicts the branch that selects the value of $\arg z$ from the interval $(-\pi, \pi]$; it is known as the *principal value of the argument* and is denoted $\operatorname{Arg} z$ (with capital A). The principal value is most commonly used in complex arithmetic computer codes; it is inherently discontinous, jumping by 2π as z crosses the negative real axis. This line of discontinuities is known as the *branch cut.*

Of course, *any* branch of $\arg z$ must have a jump of 2π somewhere. The branch depicted in Fig. 1.10(b) is discontinuous on the *positive* real axis, taking values from the interval $(0, 2\pi]$. The branch in Fig. 1.10(c) has the same branch cut but selects values from the interval $(2\pi, 4\pi]$.

The notation $\arg_\tau z$ is used for the branch of $\arg z$ taking values from the interval $(\tau, \tau + 2\pi]$. Thus $\arg_{-\pi} z$ is the principal value $\operatorname{Arg} z$, and the branches depicted in Fig. 1.10(b) and 1.10(c), respectively, are $\arg_0 z$ and $\arg_{2\pi} z$. Note that $\arg 0$ cannot be sensibly defined for any branch.

With these conventions in hand, one can now write $z = x + iy$ in the *polar form* [recall Eq. (3)]

$$z = x + iy = r(\cos\theta + i\sin\theta) = r \operatorname{cis}\theta, \tag{6}$$

where we abbreviate the "cosine plus i sine" operator as cis.

Example 2

Find $\arg(1 + \sqrt{3}i)$ and write $1 + \sqrt{3}i$ in polar form.

Solution. Note that $r = |1 + \sqrt{3}i| = 2$ and that the equations $\cos\theta = 1/2$, $\sin\theta = \sqrt{3}/2$ are satisfied by $\theta = \pi/3$. Hence $\arg(1 + \sqrt{3}i) = \pi/3 + 2k\pi$, $k = 0, \pm 1, \pm 2, \ldots$ [in particular, $\operatorname{Arg}(1 + \sqrt{3}i) = \pi/3$]. The polar form of $1 + \sqrt{3}i$ is $2(\cos\pi/3 + i\sin\pi/3) = 2\operatorname{cis}\pi/3$. ■

In many circumstances one of the forms $x + iy$ or $r\operatorname{cis}\theta$ may be more suitable than the other. The rectangular form, for example, is very convenient for addition or subtraction, whereas the polar form can be a monstrosity (see Prob. 21). On the other hand, the polar form lends a very interesting geometric interpretation to the process of multiplication. If we let

$$z_1 = r_1(\cos\theta_1 + i\sin\theta_1), \qquad z_2 = r_2(\cos\theta_2 + i\sin\theta_2),$$

then we compute

$$z_1 z_2 = r_1 r_2 [(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2)],$$

and so

$$z_1 z_2 = r_1 r_2 [\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)]. \tag{7}$$

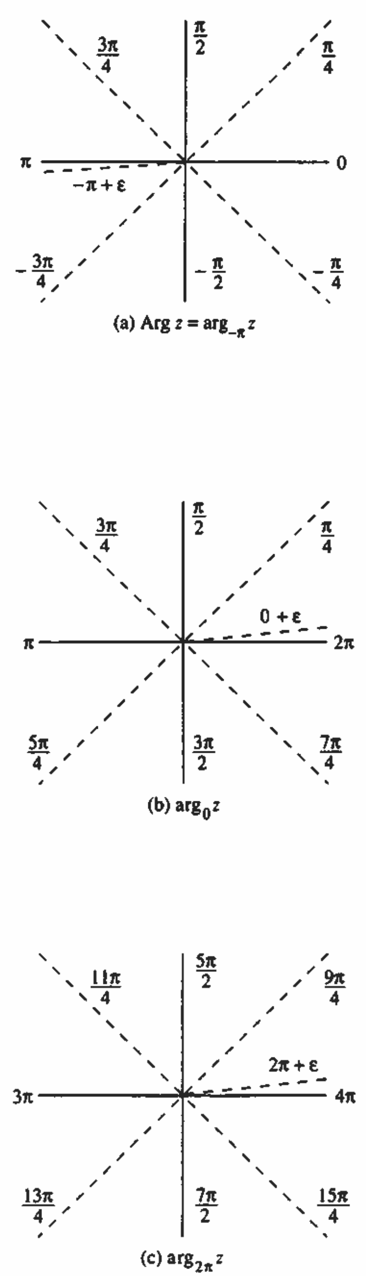

Figure 1.10 Branches of arg z.

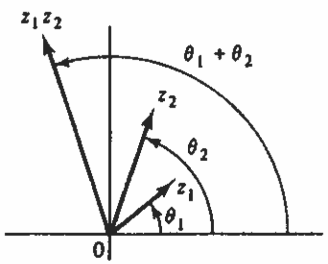

Figure 1.11 Geometric interpretation of the product.

The abbreviated version of Eq. (7) reads as follows:

$$z_1 z_2 = (r_1 \operatorname{cis} \theta_1)(r_2 \operatorname{cis} \theta_2) = (r_1 r_2) \operatorname{cis}(\theta_1 + \theta_2)$$

and we see that

The modulus of the product is the product of the moduli:

$$|z_1 z_2| = |z_1|\,|z_2| \quad (= r_1 r_2)\,; \tag{8}$$

The argument of the product is the sum of the arguments:

$$\arg z_1 z_2 = \arg z_1 + \arg z_2 \quad (= \theta_1 + \theta_2)\,. \tag{9}$$

(To be precise, the ambiguous Eq. (9) is to be interpreted as saying that if particular values are assigned to any pair of terms therein, then one can find a value for the third term that satisfies the identity.)

Geometrically, the vector $z_1 z_2$ has length equal to the product of the lengths of the vectors z_1 and z_2 and has angle equal to the sum of the angles of the vectors z_1 and z_2 (see Fig. 1.11). For instance, since the vector i has length 1 and angle $\pi/2$, it follows that the vector iz can be obtained by rotating the vector z through a right angle in the counterclockwise direction.

Observing that division is the inverse operation to multiplication, we are led to the following equations:

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}[\cos(\theta_1 - \theta_2) + i \sin(\theta_1 - \theta_2)] = \frac{r_1}{r_2}\operatorname{cis}(\theta_1 - \theta_2)\,, \tag{10}$$

$$\arg\left(\frac{z_1}{z_2}\right) = \arg z_1 - \arg z_2, \tag{11}$$

and

$$\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}. \tag{12}$$

Equation (10) can be proved in a manner similar to Eq. (7), and Eqs. (11) and (12) follow immediately. Geometrically, the vector z_1/z_2 has length equal to the quotient of the lengths of the vectors z_1 and z_2 and has angle equal to the difference of the angles of the vectors z_1 and z_2.

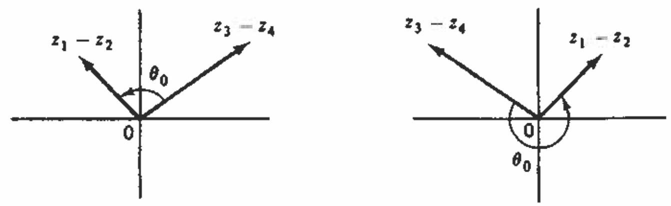

Figure 1.12 Perpendicular vectors.

Example 3

Write the quotient $(1+i)/(\sqrt{3}-i)$ in polar form.

Solution. The polar forms for $(1+i)$ and $(\sqrt{3}-i)$ are

$$1 + i = |1+i| \operatorname{cis}(\arg(1+i)) = \sqrt{2}\operatorname{cis}(\pi/4),$$
$$\sqrt{3} - i = 2\operatorname{cis}(-\pi/6).$$

Hence, from Eq. (10), we have

$$\frac{1+i}{\sqrt{3}-i} = \frac{\sqrt{2}}{2}\operatorname{cis}\left[\frac{\pi}{4} - \left(-\frac{\pi}{6}\right)\right] = \frac{\sqrt{2}}{2}\operatorname{cis}\frac{5\pi}{12}. \quad \blacksquare$$

Example 4

Prove that the line l through the points z_1 and z_2 is perpendicular to the line L through the points z_3 and z_4 if and only if

$$\operatorname{Arg}\frac{z_1 - z_2}{z_3 - z_4} = \pm\frac{\pi}{2}\,. \tag{13}$$

Solution. Note that the lines l and L are perpendicular if and only if the vectors $z_1 - z_2$ and $z_3 - z_4$ are perpendicular (see Fig. 1.12). Since

$$\arg\frac{z_1 - z_2}{z_3 - z_4} = \arg(z_1 - z_2) - \arg(z_3 - z_4)$$

gives the angle from $z_3 - z_4$ to $z_1 - z_2$, orthogonality holds precisely when this angle (up to an integer multiple of 2π) is equal to $\pi/2$ or $-\pi/2$. But this is the same as saying that (13) holds. $\blacksquare$

Recall that, geometrically, the vector $\bar{z}$ is the reflection in the real axis of the vector z (see Fig. 1.13). Hence we see that *the argument of the conjugate of a complex number is the negative of the argument of the number*; that is,

$$\arg \bar{z} = -\arg z. \tag{14}$$

In fact, as a special case of Eq. (11) we also have

$$\arg\frac{1}{z} = -\arg z.$$

Thus $\bar{z}$ and z^{-1} have the same argument and represent parallel vectors (see Fig. 1.13).

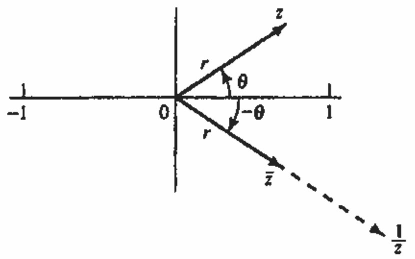

Figure 1.13 The argument of the conjugate and the reciprocal.

EXERCISES 1.3

1. Let $z_1 = 2 - i$ and $z_2 = 1 + i$. Use the parallelogram law to construct each of the following vectors.

(a) $z_1 + z_2$ (b) $z_1 - z_2$ (c) $2z_1 - 3z_2$

2. Show that $|z_1 z_2 z_3| = |z_1|\,|z_2|\,|z_3|$.

3. Translate the following geometric theorem into the language of complex numbers: The sum of the squares of the lengths of the diagonals of a parallelogram is equal to the sum of the squares of its sides. (See Fig. 1.6.)

4. Show that for any integer k, $|z^k| = |z|^k$ (provided $z \neq 0$ when k is negative).

5. Find the following.

(a) $\left|\dfrac{1+2i}{-2-i}\right|$ (b) $|\overline{(1+i)}(2-3i)(4i-3)|$ (c) $\left|\dfrac{i(2+i)^3}{(1-i)^2}\right|$ (d) $\left|\dfrac{(\pi+i)^{100}}{(\pi-i)^{100}}\right|$

6. Draw each of the following vectors.

(a) $7\,\text{cis}(3\pi/4)$ (b) $4\,\text{cis}(-\pi/6)$ (c) $\text{cis}(3\pi/4)$ (d) $3\,\text{cis}(27\pi/4)$

7. Find the argument of each of the following complex numbers and write each in polar form.

(a) $-1/2$ (b) $-3+3i$ (c) $-\pi i$
(d) $-2\sqrt{3}-2i$ (e) $(1-i)(-\sqrt{3}+i)$ (f) $(\sqrt{3}-i)^2$
(g) $\dfrac{-1+\sqrt{3}i}{2+2i}$ (h) $\dfrac{-\sqrt{7}(1+i)}{\sqrt{3}+i}$

8. Show geometrically that the nonzero complex numbers z_1 and z_2 satisfy $|z_1+z_2| = |z_1| + |z_2|$ if and only if they have the same argument.

9. Given the vector z, interpret geometrically the vector $(\cos\phi + i\sin\phi)z$.

10. Show the following:

(a) $\arg z_1 z_2 z_3 = \arg z_1 + \arg z_2 + \arg z_3$
(b) $\arg z_1 \overline{z_2} = \arg z_1 - \arg z_2$.

11. Using the complex product $(1+i)(5-i)^4$, derive

$$\pi/4 = 4\tan^{-1}(1/5) - \tan^{-1}(1/239).$$

12. Find the following.

(a) $\text{Arg}(-6-6i)$ (b) $\text{Arg}(-\pi)$ (c) $\text{Arg}(10i)$ (d) $\text{Arg}(\sqrt{3}-i)$

13. Decide which of the following statements are always true.

(a) $\text{Arg}\ z_1 z_2 = \text{Arg}\ z_1 + \text{Arg}\ z_2$ if $z_1 \neq 0,\ z_2 \neq 0$.
(b) $\text{Arg}\ \bar{z} = -\text{Arg}\ z$ if z is not a real number.
(c) $\text{Arg}(z_1/z_2) = \text{Arg}\ z_1 - \text{Arg}\ z_2$ if $z_1 \neq 0, z_2 \neq 0$.
(d) $\arg z = \text{Arg}\ z + 2\pi k, k = 0, \pm1, \pm2, \ldots$, if $z \neq 0$.

14. Show that a correct formula for $\arg(x+iy)$ can be computed using the form

$$\arg(x+iy) = \begin{cases} \tan^{-1}(y/x) + (\pi/2)[1 - \text{sgn}(x)] & \text{if } x \neq 0, \\ (\pi/2)\,\text{sgn}(y) & \text{if } x = 0 \text{ and } y \neq 0, \\ \text{undefined} & \text{if } x = y = 0, \end{cases}$$

where the "signum" function is specified by

$$\text{sgn}(t) := \begin{cases} +1 & \text{if } t > 0, \\ 0 & \text{if } t = 0, \\ -1 & \text{if } t < 0. \end{cases}$$

Show also that the expression $\text{sgn}(y)\cos^{-1}(x/\sqrt{x^2+y^2})$, at its points of continuity, equals $\text{Arg}(x+iy)$.

15. Prove that $|z_1 - z_2| \leq |z_1| + |z_2|$.

16. Prove that $||z_1| - |z_2|| \leq |z_1 - z_2|$.

17. Show that the vector z_1 is parallel to the vector z_2 if and only if $\text{Im}(z_1\bar{z}_2) = 0$.

18. Show that every point z on the line through the distinct points z_1 and z_2 is of the form $z = z_1 + c(z_2 - z_1)$, where c is a real number. What can be said about the value of c if z also lies strictly between z_1 and z_2?

19. Prove that $\arg z_1 = \arg z_2$ if and only if $z_1 = cz_2$, where c is a positive real number.

20. Let z_1, z_2, and z_3 be distinct points and let ϕ be a particular value of $\arg[(z_3 - z_1)/(z_2 - z_1)]$. Prove that

$$|z_3 - z_2|^2 = |z_3 - z_1|^2 + |z_2 - z_1|^2 - 2|z_3 - z_1||z_2 - z_1|\cos\phi.$$

[HINT: Consider the triangle with vertices z_1, z_2, z_3.]

21. If $r \operatorname{cis}\theta = r_1 \operatorname{cis}\theta_1 + r_2 \operatorname{cis}\theta_2$, determine r and θ in terms of r_1, r_2, θ_1, and θ_2. Check your answer by applying the law of cosines.

22. Use mathematical induction to prove the *generalized triangle inequality*:

$$\left|\sum_{k=1}^{n} z_k\right| \le \sum_{k=1}^{n} |z_k|.$$

23. Let m_1, m_2, and m_3 be three positive real numbers and let z_1, z_2, and z_3 be three complex numbers, each of modulus less than or equal to 1. Use the generalized triangle inequality (Prob. 22) to prove that

$$\left|\frac{m_1 z_1 + m_2 z_2 + m_3 z_3}{m_1 + m_2 + m_3}\right| \le 1,$$

and give a physical interpretation of the inequality.

24. Write computer programs for converting between rectangular and polar coordinates (using the principal value of the argument).

25. Recall that the dot (scalar) product of two planar vectors $\mathbf{v}_1 = (x_1, y_1)$ and $\mathbf{v}_2 = (x_2, y_2)$ is given by

$$\mathbf{v}_1 \cdot \mathbf{v}_2 = x_1 x_2 + y_1 y_2 .$$

Show that the dot product of the vectors represented by the complex numbers z_1 and z_2 is given by

$$z_1 \cdot z_2 = \operatorname{Re}(\bar{z}_1 z_2) .$$

26. Use the formula for the dot product in Prob. 25 to show that the vectors represented by the (nonzero) complex numbers z_1 and z_2 are orthogonal if and only if $z_1 \cdot z_2 = 0$. [HINT: Recall from the discussion following Eq. (9) that orthogonality holds precisely when $z_1 = icz_2$ for some real c.]

27. Recall that in three dimensions the cross (vector) product of two vectors $\mathbf{v}_1 = (x_1, y_1, 0)$ and $\mathbf{v}_2 = (x_2, y_2, 0)$ in the xy-plane is given by

$$\mathbf{v}_1 \times \mathbf{v}_2 = (0, 0, x_1 y_2 - x_2 y_1) .$$

(a) Show that the third component of the cross product of vectors in the xy-plane represented by the complex numbers z_1 and z_2 is given by $\operatorname{Im}(\bar{z}_1 z_2)$.

(b) Show that the vectors represented by the (nonzero) complex numbers z_1 and z_2 are parallel if and only if $\operatorname{Im}(\bar{z}_1 z_2) = 0$. [HINT: Observe that these vectors are parallel precisely when $z_1 = cz_2$ for some real c.]

28. This problem demonstrates how complex notation can simplify the kinematic analysis of planar mechanisms.

Consider the crank-and-piston linkage depicted in Fig. 1.14. The crank arm a rotates about the fixed point O while the piston arm c executes horizontal motion. (If this were a gasoline engine, combustion forces would drive the piston and the connecting arm b would transform this energy into a rotation of the crankshaft.)

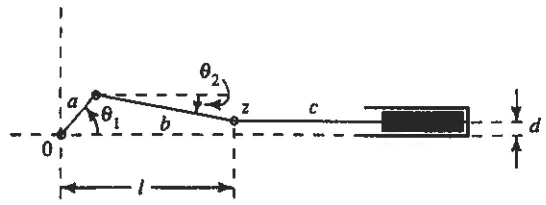

Figure 1.14 Crank-and-piston linkage.

For engineering analysis it is important to be able to relate the crankshaft's angular coordinates—position, velocity, and acceleration—to the corresponding linear coordinates for the piston. Although this calculation can be carried out using vector analysis, the following complex variable technique is more "automatic."

Let the crankshaft pivot O lie at the origin of the coordinate system, and let z be the complex number giving the location of the base of the piston rod, as depicted in Fig. 1.14,

$$z = l + id,$$

where l gives the piston's (linear) excursion and d is a fixed offset. The crank arm is described by $A = a(\cos\theta_1 + i\sin\theta_1)$ and the connecting arm by $B = b(\cos\theta_2 + i\sin\theta_2)$ (θ_2 is negative in Fig. 1.14). Exploit the obvious identity $A+B = z = l+id$ to derive the expression relating the piston position to the crankshaft angle:

$$l = a\cos\theta_1 + b\cos\left[\sin^{-1}\left(\frac{d - a\sin\theta_1}{b}\right)\right].$$

29. Suppose the mechanism in Prob. 28 has the dimensions

$$a = 0.1 \text{ m}, \qquad b = 0.2 \text{ m}, \qquad d = 0.1 \text{ m}$$

and the crankshaft rotates at a uniform velocity of 2 rad/s. Compute the position and velocity of the piston when $\theta_1 = \pi$.

30. For the linkage illustrated in Fig. 1.15, use complex variables to outline a scheme for expressing the angular position, velocity, and acceleration of arm c in terms of those of arm a. (You needn't work out the equations.)

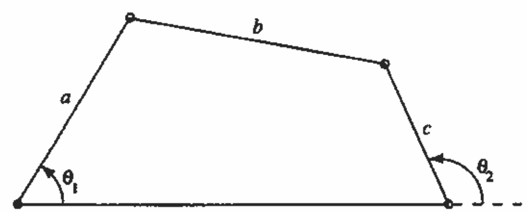

Figure 1.15 Linkage in Prob. 30.

1.4 The Complex Exponential

The familiar exponential function $f(x) = e^x$ has a natural and extremely useful extension to the complex plane. Indeed the complex function e^z provides a basic tool for the application of complex variables to electrical circuits, control systems, wave propagation, and time-invariant physical systems in general.

To find a suitable definition for e^z when $z = x + iy$, we want to preserve the basic identities satisfied by the real function e^x. So, first of all, we postulate that the multiplicative property should persist:

$$e^{z_1} e^{z_2} = e^{z_1+z_2}. \tag{1}$$

This simplifies matters considerably, since Eq. (1) enables the decomposition

$$e^z = e^{x+iy} = e^x e^{iy} \tag{2}$$

and we see that to define e^z, we need only specify e^{iy} (in other words we will be able to exponentiate complex numbers once we discover how to exponentiate pure imaginary ones).

Next we propose that the differentiation law

$$\frac{de^z}{dz} = e^z \tag{3}$$

be preserved. Differentiation with respect to a complex variable $z = x + iy$ is a very profound and, at this stage, ambiguous operation; indeed Chapter 2 is devoted to a painstaking study of this concept (and the rest of the book is dedicated to exploring its consequences). But thanks to the factorization displayed in Eq. (2) we need only consider (for the moment) a special case of Eq. (3)—namely,

$$\frac{de^{iy}}{d(iy)} = e^{iy}$$

or, equivalently (by the chain rule),

$$\frac{de^{iy}}{dy} = ie^{iy}. \tag{4}$$

The consequences of postulating Eq. (4) become more apparent if we differentiate again:

$$\begin{aligned}\frac{d^2e^{iy}}{dy^2} &= \frac{d}{dy}(ie^{iy})\\ &= i^2e^{iy}\\ &= -e^{iy};\end{aligned}$$

in other words, the function $g(y) := e^{iy}$ satisfies the differential equation

$$\frac{d^2g}{dy^2} = -g. \tag{5}$$

Now observe that any function of the form

$$A\cos y + B\sin y \qquad (A, B \text{ constants})$$

satisfies Eq. (5). In fact, from the theory of differential equations it is known that every solution of Eq. (5) must have this form. Hence we can write

$$g(y) = A\cos y + B\sin y. \tag{6}$$

To evaluate A and B we use the conditions that

$$g(0) = e^{i0} = e^0 = 1 = A\cos 0 + B\sin 0$$

and

$$\frac{dg}{dy}(0) = ig(0) = i = -A\sin 0 + B\cos 0.$$

Thus $A = 1$ and $B = i$, leading us to the identification

$$\boxed{e^{iy} = \cos y + i\sin y.} \tag{7}$$

Equation (7) is known as *Euler's equation*.† Combining Eqs. (7) and (2) we formulate the following.

> **Definition 5.** If $z = x + iy$, then e^z is defined to be the complex number
>
> $$e^z := e^x(\cos y + i\sin y). \tag{8}$$

It is not difficult to verify directly that e^z, as defined above, satisfies the usual algebraic properties of the exponential function—in particular, the multiplicative identity (1) and the associated division rule

$$\frac{e^{z_1}}{e^{z_2}} = e^{z_1-z_2} \tag{9}$$

(see Prob. 15a). In Sec. 2.5 we will obtain further confirmation that we have made the "right choice" by showing that Definition 5 produces a function that has the extremely desirable property of *analyticity*. Another confirmation is exhibited in the following example.

†Leonhard Euler (1707–1783).

Example 1
Show that Euler's equation is formally consistent with the usual Taylor series expansions

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots,$$
$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots,$$
$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots.$$

Solution. We shall study series representations of complex functions in full detail in Chapter 5. For now we ignore questions of convergence, etc., and simply substitute $x = iy$ into the exponential series:

$$\begin{aligned} e^{iy} &= 1 + iy + \frac{(iy)^2}{2!} + \frac{(iy)^3}{3!} + \frac{(iy)^4}{4!} + \frac{(iy)^5}{5!} + \cdots \\ &= \left(1 - \frac{y^2}{2!} + \frac{y^4}{4!} - \cdots\right) + i\left(y - \frac{y^3}{3!} + \frac{y^5}{5!} - \cdots\right) \\ &= \cos y + i \sin y. \quad \blacksquare \end{aligned}$$

Euler's equation (7) enables us to write the polar form (Sec. 1.3) of a complex number as

$$z = r \operatorname{cis} \theta = r(\cos\theta + i\sin\theta) = re^{i\theta}.$$

Thus we can (and do) drop the awkward "cis" artifice and use, as the standard polar representation,

$$z = re^{i\theta} = |z|e^{i \arg z}. \tag{10}$$

In particular, notice the following identities:

$$e^{i0} = e^{2\pi i} = e^{-2\pi i} = e^{4\pi i} = e^{-4\pi i} = \cdots = 1,$$
$$e^{(\pi/2)i} = i, \qquad e^{(-\pi/2)i} = -i, \qquad e^{\pi i} = -1.$$

(Students of mathematics, including Euler himself, have often marveled at the last identity. The constant e comes from calculus, π comes from geometry, and i comes from algebra—and the combination $e^{\pi i}$ gives -1, the basic unit for generating the arithmetic system from the counting numbers, or cardinals!)

Observe also that $|e^{i \arg z}| = 1$ and that Euler's equation leads to the following representations of the customary trigonometric functions:

$$\cos\theta = \operatorname{Re} e^{i\theta} = \frac{e^{i\theta} + e^{-i\theta}}{2}, \tag{11}$$

$$\sin\theta = \operatorname{Im} e^{i\theta} = \frac{e^{i\theta} - e^{-i\theta}}{2i}. \tag{12}$$

The rules derived in Sec. 1.3 for multiplying and dividing complex numbers in polar form now find very natural expressions:

$$z_1 z_2 = \left(r_1 e^{i\theta_1}\right)\left(r_2 e^{i\theta_2}\right) = (r_1 r_2)\, e^{i(\theta_1 + \theta_2)}, \tag{13}$$

$$\frac{z_1}{z_2} = \frac{r_1 e^{i\theta_1}}{r_2 e^{i\theta_2}} = \left(\frac{r_1}{r_2}\right) e^{i(\theta_1 - \theta_2)}, \tag{14}$$

and complex conjugation of $z = re^{i\theta}$ is accomplished by changing the sign of i in the exponent:

$$\bar{z} = re^{-i\theta}. \tag{15}$$

Example 2
Compute **(a)** $(1+i)/(\sqrt{3} - i)$ and **(b)** $(1+i)^{24}$.

Solution. **(a)** This quotient was evaluated using the cis operator in Example 1.11 of Sec. 1.3; using the exponential the calculations take the form

$$1 + i = \sqrt{2}\operatorname{cis}(\pi/4) = \sqrt{2}e^{i\pi/4}, \qquad \sqrt{3} - i = 2\operatorname{cis}(-\pi/6) = 2e^{-i\pi/6},$$

and, therefore,

$$\frac{1+i}{\sqrt{3}-i} = \frac{\sqrt{2}e^{i\pi/4}}{2e^{-i\pi/6}} = \frac{\sqrt{2}}{2}e^{i5\pi/12}.$$

(b) The exponential forms become

$$(1+i)^{24} = (\sqrt{2}e^{i\pi/4})^{24} = (\sqrt{2})^{24}e^{i24\pi/4} = 2^{12}e^{i6\pi} = 2^{12}. \quad \blacksquare$$

In the solution to part (b) above we glossed over the justification for the identity $(e^{i\pi/4})^{24} = e^{i24\pi/4}$. Actually, a careful scrutiny yields much more—a powerful formula involving trigonometric functions, which we describe in the next example.

Example 3
Prove *De Moivre's formula:*†

$$\boxed{(\cos\theta + i\sin\theta)^n = \cos n\theta + i \sin n\theta, \qquad n = 1, 2, 3, \ldots} \tag{16}$$

Solution. By the multiplicative property, Eq. (1),

$$(e^{i\theta})^n = \underbrace{e^{i\theta}e^{i\theta}\cdots e^{i\theta}}_{(n \text{ times})} = e^{i\theta + i\theta + \cdots + i\theta} = e^{in\theta}.$$

Now applying Euler's formula (7) to the first and last members of this equation string, we deduce (16). $\blacksquare$

De Moivre's formula can be a convenient tool for deducing multiple-angle trigonometric identities, as is illustrated by the following example. (See also Probs. 12 and 20.)

†Published by Abraham De Moivre in 1707.

Example 4

Express $\cos 3\theta$ in terms of $\cos\theta$ and $\sin\theta$.

Solution. By Eq. (16) (with $n = 3$) we have

$$\cos 3\theta = \operatorname{Re}(\cos 3\theta + i\sin 3\theta) = \operatorname{Re}(\cos\theta + i\sin\theta)^3. \tag{17}$$

According to the binomial formula,

$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3.$$

Thus, making the obvious identifications $a = \cos\theta$, $b = i\sin\theta$ in (17), we deduce

$$\begin{aligned}\cos 3\theta &= \operatorname{Re}\left[\cos^3\theta + 3\cos^2\theta\,(i\sin\theta) + 3\cos\theta\,(-\sin^2\theta) - i\sin^3\theta\right] \\ &= \cos^3\theta - 3\cos\theta\sin^2\theta. \quad ■\end{aligned} \tag{18}$$

Example 5

Compute the integral

$$\int_0^{2\pi} \cos^4\theta\, d\theta$$

by using the representation (11) together with the binomial formula (see Exercises 1.1, Prob. 27).

Solution. We can express the integrand as

$$\cos^4\theta = \left(\frac{e^{i\theta}+e^{-i\theta}}{2}\right)^4 = \frac{1}{2^4}\left(e^{i\theta}+e^{-i\theta}\right)^4,$$

and expanding via the binomial formula gives

$$\begin{aligned}\cos^4\theta &= \frac{1}{2^4}\left(e^{4i\theta} + 4e^{3i\theta}e^{-i\theta} + 6e^{2i\theta}e^{-2i\theta} + 4e^{i\theta}e^{-3i\theta} + e^{-4i\theta}\right) \\ &= \frac{1}{2^4}\left(e^{4i\theta} + 4e^{2i\theta} + 6 + 4e^{-2i\theta} + e^{-4i\theta}\right) \\ &= \frac{1}{2^4}(6 + 8\cos 2\theta + 2\cos 4\theta).\end{aligned}$$

Thus

$$\begin{aligned}\int_0^{2\pi}\cos^4\theta\, d\theta &= \int_0^{2\pi}\frac{1}{2^4}(6 + 8\cos 2\theta + 2\cos 4\theta)\, d\theta \\ &= \frac{1}{2^4}\left[6\theta + 4\sin 2\theta + \frac{1}{2}\sin 4\theta\right]\Big|_0^{2\pi} = \frac{6}{2^4}2\pi = \frac{3}{4}\pi. \quad ■\end{aligned}$$

EXERCISES 1.4

In Problems 1 and 2 write each of the given numbers in the form $a + bi$.

1. (a) $e^{-i\pi/4}$ (b) $\dfrac{e^{1+i3\pi}}{e^{-1+i\pi/2}}$ (c) e^{e^i}

2. (a) $\dfrac{e^{3i}-e^{-3i}}{2i}$ (b) $2e^{3+i\pi/6}$ (c) e^z, where $z = 4e^{i\pi/3}$

In Problems 3 and 4 write each of the given numbers in the polar form $re^{i\theta}$.

3. (a) $\dfrac{1-i}{3}$ (b) $-8\pi(1+\sqrt{3}i)$ (c) $(1+i)^6$

4. (a) $\left(\cos\dfrac{2\pi}{9} + i\sin\dfrac{2\pi}{9}\right)^3$ (b) $\dfrac{2+2i}{-\sqrt{3}+i}$ (c) $\dfrac{2i}{3e^{4+i}}$

5. Show that $|e^{x+iy}| = e^x$ and $\arg e^{x+iy} = y + 2k\pi$ $(k = 0, \pm1, \pm2, \ldots)$.

6. Show that, for real θ,

   (a) $\tan\theta = \dfrac{e^{i\theta}-e^{-i\theta}}{i(e^{i\theta}+e^{-i\theta})}$ (b) $\csc\theta = \dfrac{2}{e^{i(\theta-\pi/2)} - e^{-i(\theta+\pi/2)}}$

7. Show that $e^z = e^{z+2\pi i}$ for all z. (*The exponential function is periodic with period $2\pi i$.*)

8. Show that, for all z,

   (a) $e^{z+\pi i} = -e^z$ (b) $\overline{e^z} = e^{\bar{z}}$

9. Show that $(e^z)^n = e^{nz}$ for any integer n.

10. Show that $|e^z| \le 1$ if $\operatorname{Re} z \le 0$.

11. Determine which of the following properties of the real exponential function remain true for the complex exponential function (that is, for x replaced by z).

    (a) e^x is never zero. (b) e^x is a one-to-one function.
    (c) e^x is defined for all x. (d) $e^{-x} = 1/e^x$.

12. Use De Moivre's formula together with the binomial formula to derive the following identities.

    (a) $\sin 3\theta = 3\cos^2\theta\sin\theta - \sin^3\theta$

    (b) $\sin 4\theta = 4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta$

13. Show how the following trigonometric identities follow from Eqs. (11) and (12).

    (a) $\sin^2\theta + \cos^2\theta = 1$

    (b) $\cos(\theta_1+\theta_2) = \cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2$

14. Does De Moivre's formula hold for negative integers n?

15. (a) Show that the multiplicative law (1) follows from Definition 5.
(b) Show that the division rule (9) follows from Definition 5.

16. Let $z = re^{i\theta}$, $(z \neq 0)$. Show that $\exp(\ln r + i\theta) = z$.†

17. Show that the function $z(t) = e^{it}$, $0 \le t \le 2\pi$, describes the unit circle $|z| = 1$ traversed in the counterclockwise direction (as t increases from 0 to 2π). Then describe each of the following curves.

(a) $z(t) = 3e^{it}$, $0 \le t \le 2\pi$ (b) $z(t) = 2e^{it} + i$, $0 \le t \le 2\pi$
(c) $z(t) = 2e^{i2\pi t}$, $0 \le t \le 1/2$ (d) $z(t) = 3e^{-it} + 2 - i$, $0 \le t \le 2\pi$

18. Sketch the curves that are given for $0 \le t \le 2\pi$ by

(a) $z(t) = e^{(1+i)t}$ (b) $z(t) = e^{(1-i)t}$
(c) $z(t) = e^{(-1+i)t}$ (d) $z(t) = e^{(-1-i)t}$

19. Let n be a positive integer greater than 2. Show that the points $e^{2\pi ik/n}$, $k = 0, 1, \ldots, n-1$, form the vertices of a regular polygon.

20. Prove that if $z \neq 1$, then

$$1 + z + z^2 + \cdots + z^n = \frac{z^{n+1} - 1}{z - 1}.$$

Use this result and De Moivre's formula to establish the following identities.

(a) $1 + \cos\theta + \cos 2\theta + \cdots + \cos n\theta = \dfrac{1}{2} + \dfrac{\sin[(n+\frac{1}{2})\theta]}{2\sin(\theta/2)}$

(b) $\sin\theta + \sin 2\theta + \cdots + \sin n\theta = \dfrac{\sin(n\theta/2)\sin((n+1)\theta/2)}{\sin(\theta/2)}$, where $0 < \theta < 2\pi$.

21. Prove that if n is a positive integer, then

$$\left|\frac{\sin(n\theta/2)}{\sin(\theta/2)}\right| \le n \qquad (\theta \neq 0, \pm 2\pi, \pm 4\pi, \ldots).$$

[HINT: Argue first that if $z = e^{i\theta}$, then the left-hand side equals $|(1 - z^n)/(1 - z)|$.]

22. Show that if n is an integer, then

$$\int_0^{2\pi} e^{in\theta}\,d\theta = \int_0^{2\pi} \cos(n\theta)\,d\theta + i\int_0^{2\pi} \sin(n\theta)\,d\theta = \begin{cases} 2\pi & \text{if } n = 0, \\ 0 & \text{if } n \neq 0. \end{cases}$$

23. Compute the following integrals by using the representations (11) or (12) together with the binomial formula.

(a) $\int_0^{2\pi} \cos^8\theta\,d\theta$ (b) $\int_0^{2\pi} \sin^6(2\theta)\,d\theta$.

†As a convenience in printing we sometimes write $\exp(z)$ instead of e^z.

1.5 Powers and Roots

In this section we shall derive formulas for the nth power and the mth roots of a complex number.

Let $z = re^{i\theta} = r(\cos\theta + i\sin\theta)$ be the polar form of the complex number z. By taking $z_1 = z_2 = z$ in Eq. (13) of Sec. 1.4, we obtain the formula

$$z^2 = r^2 e^{i2\theta}.$$

Since $z^3 = zz^2$, we can apply the identity a second time to deduce that

$$z^3 = r^3 e^{i3\theta}.$$

Continuing in this manner we arrive at the formula for the nth power of z:

$$z^n = r^n e^{in\theta} = r^n(\cos n\theta + i\sin n\theta). \tag{1}$$

Clearly this is just an extension of De Moivre's formula, discussed in Example 3 of Sec. 1.4.

Equation (1) is an appealing formula for raising a complex number to a positive integer power. It is easy to see that the identity is also valid for negative integers n (see Prob. 2). The question arises whether the formula will work for $n = 1/m$, so that $\zeta = z^{1/m}$ is an mth root of z satisfying

$$\zeta^m = z. \tag{2}$$

Certainly if we define

$$\zeta = \sqrt[m]{r}\, e^{i\theta/m} \tag{3}$$

(where $\sqrt[m]{r}$ denotes the customary, positive, mth root), we compute a complex number ζ satisfying Eq. (2) [as is easily seen by applying Eq. (1)]. But the matter is more complicated than this; the number 1, for instance, has *two* square roots: 1 and -1. And each of these has, in turn, two square roots—generating *four* fourth roots of 1, namely, $1, -1, i$, and $-i$.

To see how the additional roots fit into the scheme of things, let's work out the polar description of the equation $\zeta^4 = 1$ for each of these numbers:

$$\begin{aligned} 1^4 &= (1e^{i0})^4 = 1^4 e^{i0} = 1, \\ i^4 &= (1e^{i\pi/2})^4 = 1^4 e^{i2\pi} = 1, \\ (-1)^4 &= (1e^{i\pi})^4 = 1^4 e^{i4\pi} = 1, \\ (-i)^4 &= (1e^{i3\pi/2})^4 = 1^4 e^{i6\pi} = 1. \end{aligned}$$

It is instructive to trace the consecutive powers of these roots in the Argand diagram. Thus Fig. 1.16 shows that i, i^2, i^3, and i^4 complete one revolution before landing on 1; $(-1), (-1)^2, (-1)^3$, and $(-1)^4$ go around twice; the powers of $(-i)$ go around three times counterclockwise, and of course $1, 1^2, 1^3$, and 1^4 never move.

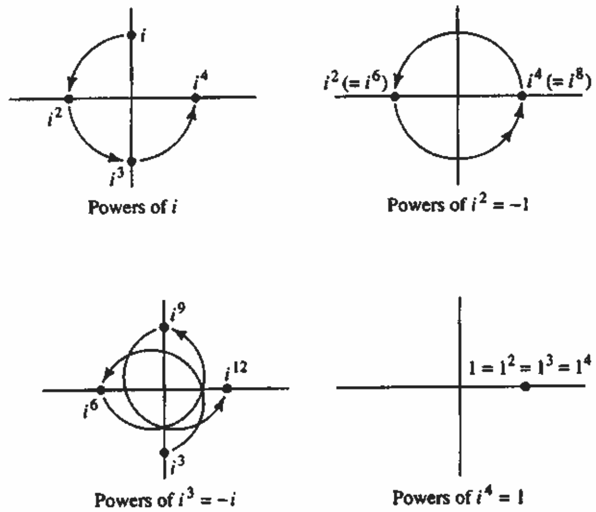

Figure 1.16 Successive powers of the fourth roots of unity.

Clearly, the multiplicity of roots is tied to the ambiguity in representing 1, in polar form, as $e^{i0}, e^{i2\pi}, e^{i4\pi}$, etc. Thus to compute *all* the mth roots of a number z, we must apply formula (3) to *every* polar representation of z. For the cube roots of unity, for example, we would compute as shown in the table opposite. Obviously the roots recur in sets of three, since $e^{i2\pi m_1/3} = e^{i2\pi m_2/3}$ whenever $m_1 - m_2 = 3$.

Generalizing, we can see that *there are exactly m distinct* mth *roots of unity, denoted by* $1^{1/m}$, *and they are given by*

$$\boxed{1^{1/m} = e^{i2k\pi/m} = \cos\frac{2k\pi}{m} + i\sin\frac{2k\pi}{m} \quad (k = 0, 1, 2, \ldots, m-1).} \tag{4}$$

The arguments of these roots are $2\pi/m$ radians apart, and the roots themselves form the vertices of a regular polygon (Fig. 1.17).

Taking $k = 1$ in Eq. (4) we obtain the root†

$$\omega_m := e^{i2\pi/m} = \cos\frac{2\pi}{m} + i\sin\frac{2\pi}{m},$$

and it is easy to see that the complete set of roots can be displayed as

$$1, \omega_m, \omega_m^2, \ldots, \omega_m^{m-1}.$$

†A number w is said to be a *primitive* mth *root of unity* if w^m equals 1 but $w^k \neq 1$ for $k = 1, 2, \ldots, m-1$. Clearly, ω_m is a primitive root.

Cube Roots of Unity

Polar representation of 1	Application of (3)
$\vdots$	$\vdots$
$1 = e^{-i6\pi}$	$1^{1/3} = e^{-i6\pi/3} = 1$
$1 = e^{-i4\pi}$	$1^{1/3} = e^{-i4\pi/3} = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$
$1 = e^{-i2\pi}$	$1^{1/3} = e^{-i2\pi/3} = -\frac{1}{2} - i\frac{\sqrt{3}}{2}$
$1 = e^{i0}$	$1^{1/3} = e^{i0/3} = 1$
$1 = e^{i2\pi}$	$1^{1/3} = e^{i2\pi/3} = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$
$1 = e^{i4\pi}$	$1^{1/3} = e^{i4\pi/3} = -\frac{1}{2} - i\frac{\sqrt{3}}{2}$
$1 = e^{i6\pi}$	$1^{1/3} = e^{i6\pi/3} = 1$
$\vdots$	$\vdots$

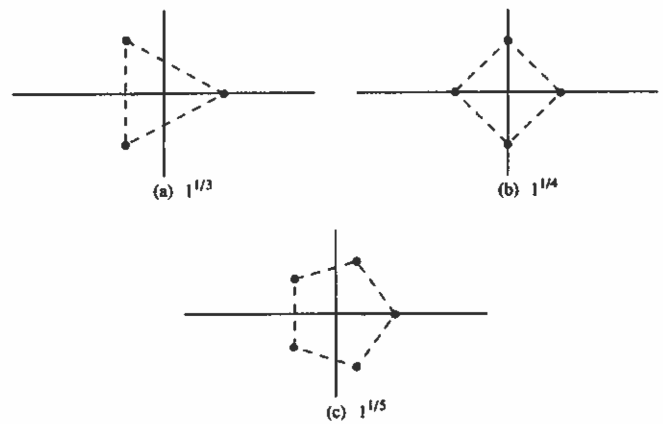

Figure 1.17 Regular polygons formed by the roots of unity.

Example 1

Prove that

$$1 + \omega_m + \omega_m^2 + \cdots + \omega_m^{m-1} = 0. \tag{5}$$

Solution. This result is obvious from a physical point of view, since, by symmetry, the center of mass $(1 + \omega_m + \omega_m^2 + \cdots + \omega_m^{m-1})/m$ of the system of m unit masses located at the mth roots of unity must be at the origin (see Fig. 1.17).

To give an algebraic proof we simply note that

$$(\omega_m - 1)(1 + \omega_m + \omega_m^2 + \cdots + \omega_m^{m-1}) = \omega_m^m - 1 = 0.$$

Since $\omega_m \neq 1$, Eq. (5) follows. ■

To obtain the mth roots of an *arbitrary* (nonzero) complex number $z = re^{i\theta}$, we generalize the idea displayed by Eq. (4) and, reasoning similarly, conclude that *the m distinct mth roots of z are given by*

$$z^{1/m} = \sqrt[m]{|z|}e^{i(\theta+2k\pi)/m} \qquad (k = 0, 1, 2, \ldots, m-1). \tag{6}$$

Equivalently, we can form these roots by taking any single one such as given in (3) and multiplying by the mth roots of unity.

Example 2

Find all the cube roots of $\sqrt{2} + i\sqrt{2}$.

Solution. The polar form for $\sqrt{2} + i\sqrt{2}$ is

$$\sqrt{2} + i\sqrt{2} = 2e^{i\pi/4}.$$

Putting $|z| = 2$, $\theta = \pi/4$, and $m = 3$ into Eq. (6), we obtain

$$(\sqrt{2} + i\sqrt{2})^{1/3} = \sqrt[3]{2}\, e^{i(\pi/12 + 2k\pi/3)} \qquad (k = 0, 1, 2).$$

Therefore, the three cube roots of $\sqrt{2} + i\sqrt{2}$ are $\sqrt[3]{2}(\cos \pi/12 + i \sin \pi/12)$, $\sqrt[3]{2}(\cos 3\pi/4 + i \sin 3\pi/4)$, and $\sqrt[3]{2}(\cos 17\pi/12 + i \sin 17\pi/12)$. ■

Example 3

Let a, b, and c be *complex* constants with $a \neq 0$. Prove that the solutions of the equation

$$az^2 + bz + c = 0 \tag{7}$$

are given by the (usual) quadratic formula

$$z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}, \tag{8}$$

where $\sqrt{b^2 - 4ac}$ denotes one of the values of $(b^2 - 4ac)^{1/2}$.

Solution. After multiplying Eq. (7) by $4a$, one can manipulate it into the form

$$4a^2z^2 + 4abz + b^2 = b^2 - 4ac.$$

The left hand side is $(2az + b)^2$, so

$$2az + b = (b^2 - 4ac)^{1/2} = \pm\sqrt{b^2 - 4ac},$$

which is equivalent to Eq. (8). ■

EXERCISES 1.5

1. Prove identity (1) by using induction.
2. Show that formula (1) also holds for negative integers n.
3. Let n be a positive integer. Prove that $\arg z^n = n \operatorname{Arg} z + 2k\pi$, $k = 0, \pm 1, \pm 2, \ldots$, for $z \neq 0$.
4. Use the identity (1) to show that

   (a) $(\sqrt{3} - i)^7 = -64\sqrt{3} + i64$ **(b)** $(1 + i)^{95} = 2^{47}(1 - i)$
5. Find all the values of the following.

   (a) $(-16)^{1/4}$ **(b)** $1^{1/5}$ **(c)** $i^{1/4}$

   (d) $(1 - \sqrt{3}i)^{1/3}$ **(e)** $(i - 1)^{1/2}$ **(f)** $\left(\dfrac{2i}{1+i}\right)^{1/6}$
6. Describe how to construct geometrically the fifth roots of z_0 if

   (a) $z_0 = -1$ **(b)** $z_0 = i$ **(c)** $z_0 = 1 + i$
7. Solve each of the following equations.

   (a) $2z^2 + z + 3 = 0$

   (b) $z^2 - (3 - 2i)z + 1 - 3i = 0$

   (c) $z^2 - 2z + i = 0$
8. Let a, b, and c be real numbers and let $a \neq 0$. Show that the equation $az^2 + bz + c = 0$ has

   (a) two real solutions if $b^2 - 4ac > 0$.

   (b) two nonreal conjugate solutions if $b^2 - 4ac < 0$.
9. Solve the equation $z^3 - 3z^2 + 6z - 4 = 0$.
10. Find all four roots of the equation $z^4 + 1 = 0$ and use them to deduce the factorization $z^4 + 1 = (z^2 - \sqrt{2}z + 1)(z^2 + \sqrt{2}z + 1)$.
11. Solve the equation $(z + 1)^5 = z^5$.