ACCELERATING CONVERGENCE OF ITERATIVE PROCESSES

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ABSTRACT

A technique is discussed which, when applied to an iterative procedure for the solution of an equation, accelerates the rate of convergence if the iteration converges and induces convergence if the iteration diverges. An illustrative example is given.

1. *Introduction*

This paper discusses a technique for accelerating the convergence of iterative procedures employed in the solution of non-linear algebraic and transcendental equations. Iterative procedures in problems of this kind are of considerable importance since, with the aid of high speed computers, they very often provide the only effective means of solving such equations. The methods described here are of interest because they are capable of producing solutions even in those cases where the iteration algorithm may be divergent.

2. *The Iteration Algorithm*

Consider the problem of finding a root of the equation

$$F(x)=0. \tag{1}$$

If this equation can be expressed in the form

$$x=f(x), \tag{2}$$

an algorithm for the iterative solution is

$$x_{n+1}=f(x_n). \tag{3}$$

If (1) cannot be written in the form (2), the algorithm

$$x_{n+1}=x_n+\Gamma F(x_n) \tag{4}$$

can be used where $\Gamma \neq 0$ is some suitably chosen constant. It is assumed that (1) and (2) have solutions.

The usual iteration procedure is started by substituting an initial value $x_n=x_o$ in (3) or (4) whereby a new approximate value x_1 is obtained. This value is then used in the right hand side of (3) or (4) to obtain another approximation. The procedure may not converge, but when it does converge, it is continued until $|x_n-x_{n+1}|$ is smaller than some preassigned number. The sequence of values x_n will display any one of the following behavior patterns:

Case 1. The values of x_n oscillate and converge
Case 2. The values of x_n oscillate and diverge.
Case 3. The values of x_n converge monotonically.
Case 4. The values of x_n diverge monotonically.

Numerical illustrations will be given for each of these cases along with the behavior when the procedure is accelerated.

3. *Modification of the Iterative Procedure*

In order to change the rate of convergence or divergence of the iterative procedure, the successive values of x_n must somehow be changed. For example, after x_{n+1} is obtained in (3) or (4), let it be replaced by $\bar{x}_{n+1}$ where

$$\bar{x}_{n+1}=q\,x_n+(1-q)\,x_{n+1}. \tag{5}$$

Then, in the next application of (3) or (4), x_n is replaced by $\bar{x}_{n+1}$. A particular value for q can be found which will make each of the divergent cases above convergent, and those which are already convergent can be made to converge more rapidly. In solving for the roots of equations it is often sufficient to experimentally choose a constant value for q which will always assure convergence. However, it is much more desirable to have a technique for computing the optimum value of q for each step of the iteration, and such a technique will now be described.

4. *Derivation of a Formula for q*

Geometrically the solution of (2) amounts to the problem of finding the point of intersection P of the curve $y=x$ and $y=f(x)$.

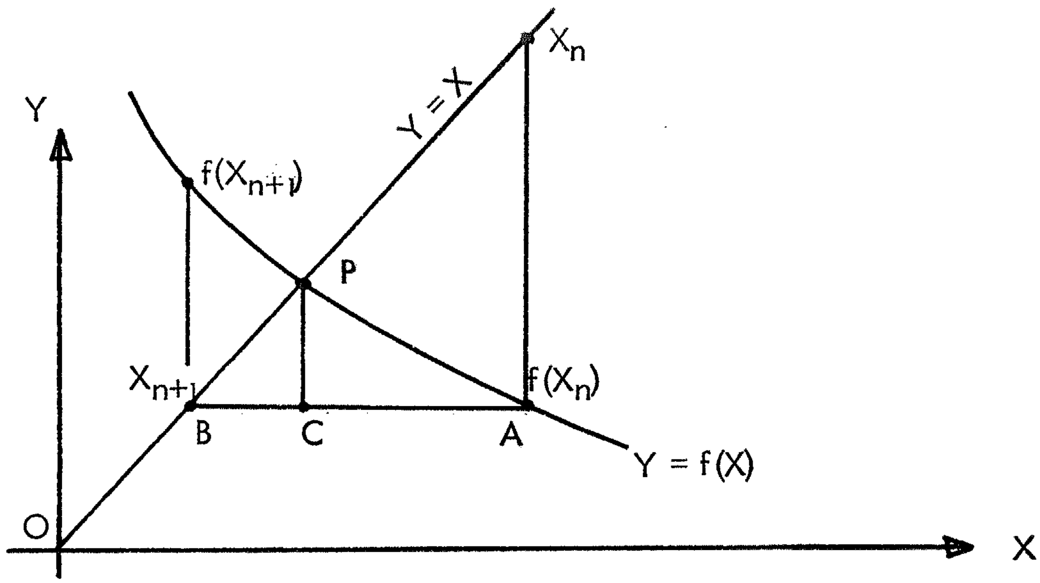

Figure 1

The iteration (3) can be represented graphically as follows: Pass a vertical line through a point (x_n, x_n) on $y=x$ so that it intersects the curve $y=f(x)$ at some point A with the coordinates $(x_n, f(x_n))$. To find x_{n+1}, draw a horizontal line through A so that it intersects the curve $y=x$. The point of intersection B has the coordinates (x_{n+1}, x_{n+1}).

The ideal location for $\bar{x}_{n+1}$ of (5) on $\overline{AB}$ would of course be the intersection point C with the normal to $\overline{AB}$ drawn through P. Thus q should be chosen such that

$$\frac{q}{1-q}=\frac{\overline{BC}}{\overline{AC}}$$

To determine q approximately, observe that $\overline{PC}=\overline{BC}$ and $\overline{PC}/\overline{AC}=-a$, where a is a value of $f'(x)$ between A and P. Thus

$$\frac{q}{1-q}=-a \quad \text{or} \quad q=\frac{a}{a-1}. \tag{6}$$

Since a more convenient expression is lacking, a can be approximated by a suitable difference quotient

$$a \cong \frac{f(x_n)-f(x_{n-1})}{x_n-x_{n-1}}=\frac{x_{n+1}-x_n}{x_n-x_{n-1}} \tag{7}$$

The above argument can be made analytically rigorous for all *sufficiently smooth curves* $y=f(x)$ with $f'(x)\neq 1$ in the neighborhood of the root. It can then be shown that the convergence of the method is quadratic, i.e. that asymptotically, the number of correct decimal places is doubled at each step.

The author is aware that this method is contained implicitly in earlier work on iteration. In particular, it is related to Aitken's δ^2-method and to a modified form of Newton's method[1]. However, the fact that convergence can be forced even in otherwise divergent cases does not seem to have been sufficiently emphasized.

5. *Use of the Formula for q*

Instead of using the algorithm (3) repeatedly, one can utilize a variation of (7) and compute a new value for q after each iterative cycle as shown in flow diagram form in Fig. 2.

[1]F. A. Willers, Methoden ner paraktischen Analysis, W. de Grunter Co., Berlin 1950, p. 256–262.

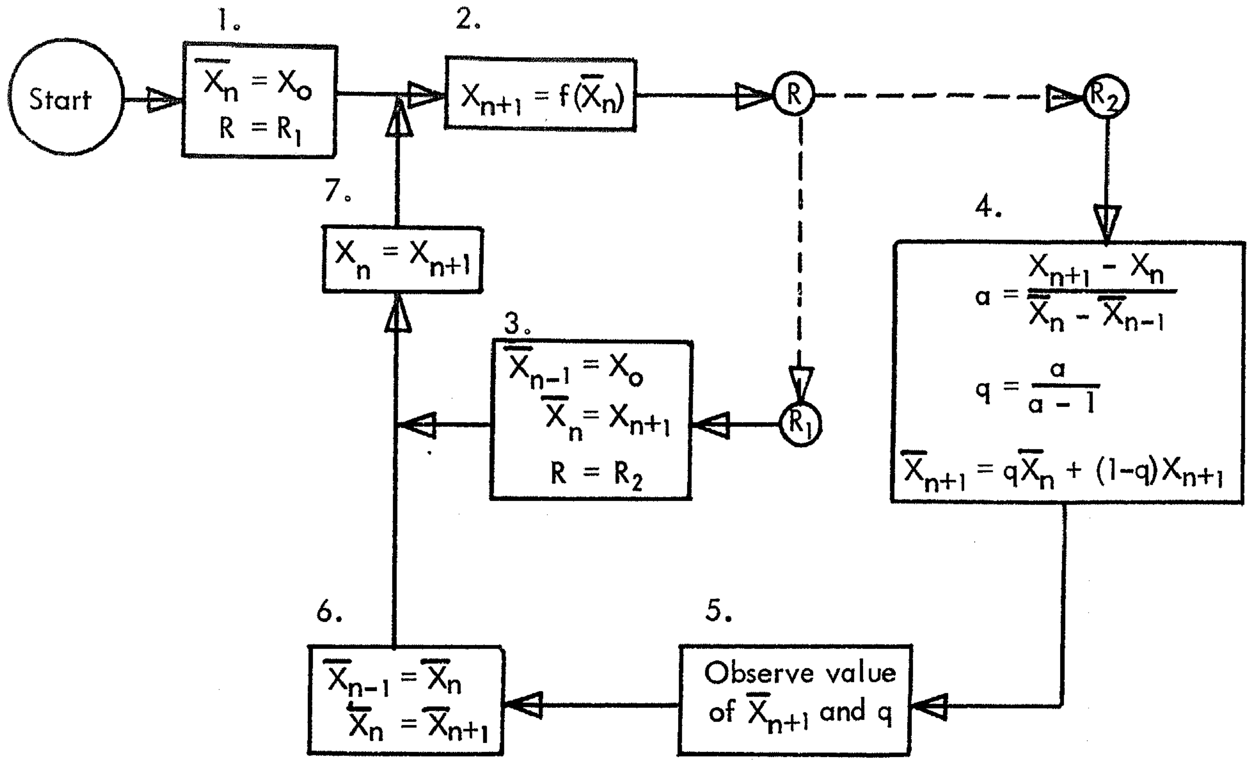

Figure 2

Starting with $\bar{x}_n = x_o$ as the initial trial value for the root, an improved value x_{n+1} is obtained in box 2 with the algorithm. At first, with $R = R_1$, the computation passes through box 3 where a second iteration is made to obtain starting values. Thereafter, with $R = R_2$, the process cycles through the loop of boxes 4, 5, 6, 7, 2, 4, 5, 6, 7, 2, 4. . . . In box 4, q is computed and an improved value $\bar{x}_{n+1}$ is then obtained. If $\bar{x}_{n+1}$ is observed each cycle, it is found to rapidly converge to the solution of the equation. With this method, convergence is even found to occur in those iterative processes which were formerly divergent. The value of q is found to approach a constant which depends on whether the simple iterative process would have been convergent or divergent and monotonic or oscillatory.

Editor's Note

If the successive values of q are not looked at or operated upon specifically, computing time may be saved (1 add and 1 divide) by using the form:

$$\overline{X}_{n+1} = \frac{X_{n+1}\,\overline{X}_{n-1} - X_n\,\overline{X}_n}{X_{n+1} + \overline{X}_{n-1} - X_n - \overline{X}_n}$$

6. *Illustrative Example*

Consider the equation $$x = \sinh \alpha\, x = \tfrac{1}{2}(e^{\alpha x} - e^{-\alpha x}). \qquad (8)$$

Using the algorithm (3), it is to be iterated for its root $x = 0$ using four different values for the parameter α so chosen as to demonstrate the four cases described in section 2. In each case, the equation is also solved using the method of section 5. As an initial starting value, let $x_o = 1$.

Note that, in each case, q seems to tend toward a particular constant value. It is quite feasible to obtain a solution in each of the above cases by simply using (5) with a suitably chosen constant value for q. Furthermore, experience with other equations has shown that convergence can similarly be obtained and that the optimum value of q falls into certain ranges which are associated with the iterative behavior patterns as follows:

Case	Range of Optimum q
1. Oscillatory Convergence	$0 < q < .5$
2. Oscillatory Divergence	$.5 < q < 1$
3. Monotonic Convergence	$q < 0$
4. Monotonic Divergence	$1 < q$

7. *Acknowledgment*

The author is indebted to Dr. Peter Henrici, formerly of American University and the National Bureau of Standards, for suggesting the method for computing q and contributing Section 4 of this paper.

CASE I $\alpha = -0.5$ (Oscillatory-convergent)

n	X_n (eq. 3)	X_n (sect. 5)	q
0	1.000 →		
1	−.521 →		
2	.263	−.00348	.340
3	−.132	-1.32×10^{-5}	.335
4	.066	-1.65×10^{-11}	.333
5	−.033		
6	.017		
7	−.008		
8	.004		

CASE II $\alpha = -1.2$ (Oscillatory-divergent)

n	X_n (eq. 3)	X_n (sect. 5)	q
0	1.000 →		
1	−1.509 →		
2	2.978	.100	.641
3	−17.801	.0247	.658
4	9.45×10^{8}	4.02×10^{-5}	.546
5	—	3.19×10^{-9}	.545
6	—	2.10×10^{-11}	.545

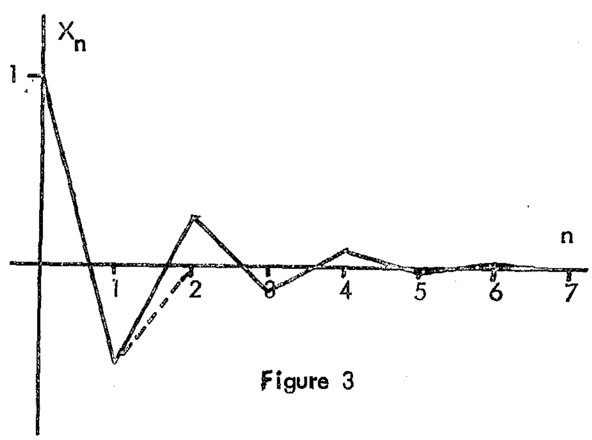

Figure 3

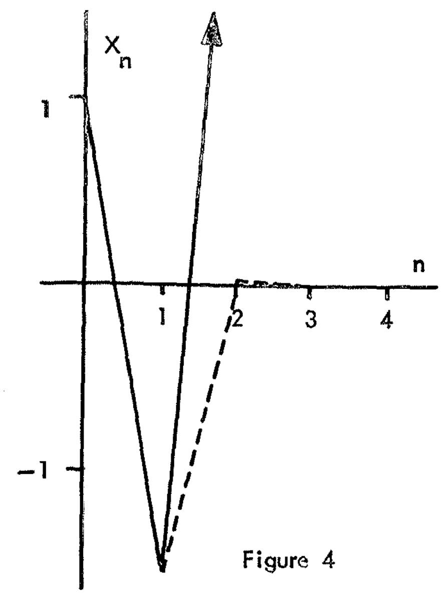

Figure 4

CASE III $\alpha=0.5$ (Monotonic-convergent)

n	X_n (eq. 3)	X_n (sect. 5)	q
0	1.000	→	
1	.521	→	
2	.263	−.0363	−1.164
3	.132	3.9×10^{-4}	−1.021
4	.066	2.1×10^{-8}	−1.000
5	.033	—	
6	.017	—	
7	.008	—	

CASE IV $\alpha=1.2$ (Monotonic-divergent)

n	X_n (eq. 3)	X_n (sect. 5)	q
0	1.000	→	
1	1.509	→	
2	2.978	.729	1.53
3	17.801	.560	1.64
4	9.45×10^{-8}	.278	2.72
5	—	.107	3.77
6	—	.014	5.27
7	—	2.57×10^{-4}	5.90
8	—	7.49×10^{-8}	5.99

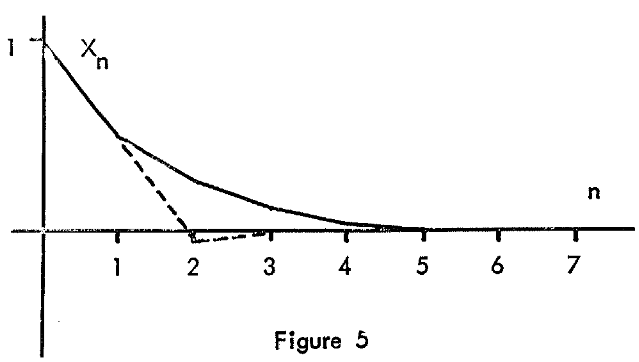

Figure 5

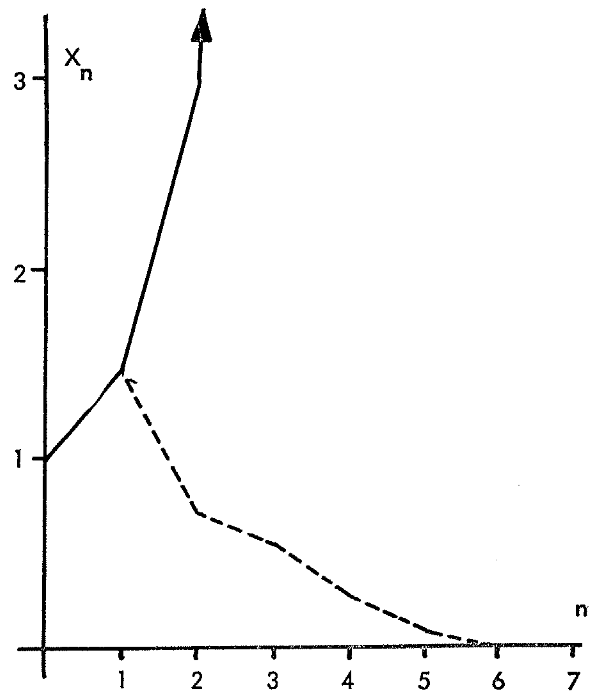

Figure 6