

# Quasilinear Theory of Anomalous Transport in Axisymmetric Tokamaks

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## 1 Introduction

## 2 Tokamak Geometry

For simplicity, we consider an axisymmetric, large aspect-ratio, circular tokamak. This gives the following definition for the equilibrium magnetic field,

$$\mathbf{B} = B_\theta \hat{\mathbf{e}}_\theta + B_\zeta \hat{\mathbf{e}}_\zeta = B_\theta \hat{\mathbf{e}}_\theta + B_0(1 - \epsilon \cos \theta) \hat{\mathbf{e}}_\zeta = B_0 \left[ \frac{\epsilon}{q} \hat{\mathbf{e}}_\theta + (1 - \epsilon \cos \theta) \hat{\mathbf{e}}_\zeta \right], \quad (1)$$

where  $\epsilon = \frac{r}{R_0} \ll 1$  is the inverse aspect ratio, with  $R = R_0 + r \cos \theta$ , for  $r$  the minor radius, and  $R_0$  the major radius, and  $q \simeq \frac{r B_\zeta}{R_0 B_\theta} \sim 1$  is the safety factor<sup>1</sup> - the number of toroidal turns required for one poloidal turn of magnetic field lines. The term  $\epsilon \cos \theta$  in  $R$  takes into account the change in toroidal radius along the tokamak midplane. Working to  $\mathcal{O}(\epsilon)$ , the magnetic field magnitude, magnetic field unit vector, and toroidal gradient terms can be written as,

$$B = \sqrt{\mathbf{B} \cdot \mathbf{B}} = \sqrt{B_0^2 [(1 - \epsilon \cos \theta)^2 + (\frac{\epsilon}{q})^2]} = B_0 \sqrt{1 - 2\epsilon \cos \theta} \simeq B_0(1 - \epsilon \cos \theta), \quad (2)$$

$$\hat{\mathbf{b}} = \frac{\mathbf{B}}{B} = \frac{\frac{\epsilon}{q} \hat{\mathbf{e}}_\theta + (1 - \epsilon \cos \theta) \hat{\mathbf{e}}_\zeta}{1 - \epsilon \cos \theta} \simeq \frac{\epsilon}{q} (1 + \epsilon \cos \theta) \hat{\mathbf{e}}_\theta + \hat{\mathbf{e}}_\zeta \simeq \frac{\epsilon}{q} \hat{\mathbf{e}}_\theta + \hat{\mathbf{e}}_\zeta, \quad (3)$$

$$\nabla = \partial_r \hat{\mathbf{e}}_r + \frac{1}{r} \partial_\theta \hat{\mathbf{e}}_\theta + \frac{1}{R} \partial_\zeta \hat{\mathbf{e}}_\zeta = \partial_r \hat{\mathbf{e}}_r + \frac{1}{r} \partial_\theta \hat{\mathbf{e}}_\theta + \frac{1}{R_0 + r \cos \theta} \partial_\zeta \hat{\mathbf{e}}_\zeta. \quad (4)$$

## 3 Gyrokinetics

*Talk about gyrophase – averaging and guiding center coordinates.*

### 3.1 Vlasov Equation

The perturbed, gyrokinetic distribution function is given as a combination of adiabatic and non-adiabatic terms,<sup>2</sup>

$$\delta F = \frac{q}{m} \delta F_a + \delta G, \quad (5)$$

where,

$$\delta F_a = [\delta\Phi \frac{\partial}{\partial\epsilon^*} + (\delta\Phi - \frac{v_{\parallel}\delta A_{\parallel}}{c}) \frac{\partial}{B\partial\mu}] F_0, \quad (6)$$

$$\delta G_0 = -\frac{q}{m} \langle \delta L \rangle_{\alpha} \frac{\partial}{B\partial\mu} + \delta H_0, \quad (7)$$

$$\langle \dots \rangle_{\alpha} = \frac{1}{2\pi} \int_0^{2\pi} (\dots) d\alpha, \quad (8)$$

with  $\alpha$  as the gyro-phase angle,  $\delta L = \delta\Phi - \frac{v \cdot \delta \mathbf{A}}{c}$ ,  $\epsilon^* = \frac{v_{\parallel}^2}{2} + \frac{q\Phi_0}{m}$ , and  $\mu = \frac{v_{\perp}^2}{2B}$ . Higher order terms in  $\delta G$ , the perturbed, non-adiabatic distribution function, are dropped. We can simplify things further by choosing for  $F_0$  a Maxwellian equilibrium distribution function,  $f_M$ , so that it only depends on  $\epsilon^*$  and not  $\mu$ . This gives us a final distribution function,

$$\delta F = \frac{q}{m} \delta\Phi \frac{\partial}{\partial\epsilon^*} f_M + \delta H_0. \quad (9)$$

This distribution function can be plugged into the Vlasov equation and gyrophase-averaged to give the standard gyrokinetic Vlasov equation for a species  $j$ ,<sup>2</sup>

$$\begin{aligned} & \partial_t \delta H_0 + v_{\parallel} \nabla_{\parallel} \delta H_0 + (\mathbf{v}_d + \frac{c \hat{\mathbf{b}} \times \nabla_X \langle \delta\Phi \rangle_{\alpha}}{B}) \cdot \nabla_X \delta H_0 \\ &= -\frac{e_j}{m_j} [\partial_t \langle \delta\Phi \rangle_{\alpha} \partial_{\epsilon^*} f_M - \frac{1}{\omega_{cj}} (\nabla_X \langle \delta\Phi \rangle_{\alpha} \times \hat{\mathbf{b}}) \cdot \nabla_X f_M], \end{aligned} \quad (10)$$

where  $\mathbf{v}_d$ , the sum of magnetic curvature and gradient drift terms, is defined as,

$$\mathbf{v}_d = \frac{v_{\parallel}^2 + \frac{1}{2}v_{\perp}^2}{\omega_{cj}} \frac{\mathbf{B} \times \nabla B}{B^2}, \quad (11)$$

with, simplifying to lowest order in  $\epsilon$ ,

$$\begin{aligned} \frac{\mathbf{B} \times \nabla B}{B^2} &= \frac{B_0 [(1 - \epsilon \cos \theta) \hat{\mathbf{e}}_{\zeta} + \frac{\epsilon}{q} \hat{\mathbf{e}}_{\theta}] \times (\partial_r \hat{\mathbf{e}}_r + \frac{1}{r} \partial_{\theta} \hat{\mathbf{e}}_{\theta} + \frac{1}{R} \partial_{\zeta} \hat{\mathbf{e}}_{\zeta}) B_0 (1 - \epsilon \cos \theta)}{B_0^2 (1 - \epsilon \cos \theta)^2} \\ &= \frac{[1 - \epsilon \cos \theta] \hat{\mathbf{e}}_{\zeta} + \frac{\epsilon}{q} \hat{\mathbf{e}}_{\theta} \times [-\frac{1}{R_0} \cos \theta \hat{\mathbf{e}}_r + \frac{r}{r R_0} \sin \theta \hat{\mathbf{e}}_{\theta}]}{(1 - \epsilon \cos \theta)^2} \\ &= \frac{1}{(1 - \epsilon \cos \theta)^2} \left[ -\frac{(1 - \epsilon \cos \theta) \cos \theta}{R_0} (\hat{\mathbf{e}}_{\zeta} \times \hat{\mathbf{e}}_r) - \frac{(1 - \epsilon \cos \theta) \cos \theta}{R_0} (\hat{\mathbf{e}}_{\zeta} \times \hat{\mathbf{e}}_{\theta}) \right. \\ &\quad \left. - \frac{\epsilon}{q R_0} \cos \theta (\hat{\mathbf{e}}_{\theta} \times \hat{\mathbf{e}}_r) \right] \\ &\simeq (1 + 2\epsilon \cos \theta) \left[ -\frac{\cos \theta}{R_0} \hat{\mathbf{e}}_{\theta} - \frac{\sin \theta}{R_0} \hat{\mathbf{e}}_r \right] \simeq -\frac{1}{R_0} (\sin \theta \hat{\mathbf{e}}_r + \cos \theta \hat{\mathbf{e}}_{\theta}) \end{aligned} \quad (12)$$

The second and third terms on the left-hand side of (10) can be simplified to lowest order in  $\epsilon$  using (1)-(4) and (11)-(12) as,

$$\begin{aligned} v_{\parallel} \nabla_{\parallel} &= v_{\parallel} (\hat{\mathbf{b}} \cdot \nabla) = v_{\parallel} \left( \frac{\epsilon}{q} \hat{\mathbf{e}}_{\theta} + \hat{\mathbf{e}}_{\zeta} \right) \cdot \left( \partial_r \hat{\mathbf{e}}_r + \frac{1}{r} \partial_{\theta} \hat{\mathbf{e}}_{\theta} + \frac{1}{R_0 + r \cos \theta} \partial_{\zeta} \hat{\mathbf{e}}_{\zeta} \right) \\ &= v_{\parallel} \left( \frac{\epsilon}{qr} \partial_{\theta} + \frac{1}{R} \partial_{\zeta} \right) = v_{\parallel} \left( \frac{1}{q R_0} \partial_{\theta} + \frac{1}{R} \partial_{\zeta} \right) = \frac{v_{\parallel}}{q R} \left( \frac{R}{R_0} \partial_{\theta} + q \partial_{\zeta} \right) \\ &= \frac{v_{\parallel}}{q R} ((1 + \epsilon \cos \theta) \partial_{\theta} + q \partial_{\zeta}) \simeq \frac{v_{\parallel}}{q R} (\partial_{\theta} + q \partial_{\zeta}), \end{aligned} \quad (13)$$

$$\begin{aligned}
\mathbf{v}_d \cdot \nabla_X &= -\frac{v_{\parallel}^2 + \frac{1}{2}v_{\perp}^2}{\omega_{cj}} (\sin \theta \hat{\mathbf{e}}_r + \cos \theta \hat{\mathbf{e}}_{\theta}) \cdot (\partial_r \hat{\mathbf{e}}_r + \frac{1}{r} \partial_{\theta} \hat{\mathbf{e}}_{\theta}) \\
&= -\frac{v_{\parallel}^2 + \frac{1}{2}v_{\perp}^2}{\omega_{cj}} (\sin \theta \partial_r + \frac{\cos \theta}{r} \partial_{\theta}) .
\end{aligned} \tag{14}$$

Note that we have dropped the non-linear  $\mathbf{E} \times \mathbf{B}$  drift term on the left-hand side of (10) because we are interested in linearizing this equation.

### 3.2 Non-Adiabatic Distribution Function

Using a WKB ansatz the gyrophase-averaged terms can be simplified as,

$$\langle A(\mathbf{x}) \rangle_{\alpha} = J_0 \left( \frac{k_{\perp} v_{\perp}}{\omega_{cj}} \right) A(\mathbf{X}) \tag{15}$$

## References

<sup>1</sup> Wesson 2004

<sup>2</sup> Frieman, Chen 1982