A Quasi-linear Theory of Impurity Transport in Circular, Axisymmetric Tokamaks

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1 Introduction

1.1 Impurity Transport

Impurity transport is an important issue for fusion plasmas as it can strongly affect the plasma performance. In order to achieve continuous operation, particle control is essential.[1] For D-T burning plasmas, helium ash exhaust is an important requirement; the helium density as well as impurity ion density must be kept sufficiently low in order to minimize the dilution of fuel ions. The requirement that the fraction of helium ash remains acceptable ($\lesssim 10\%$) can only be met if the outward transport of helium is sufficiently rapid.[2] Impurity accumulation also leads to radiative losses and radiative instabilities, which will further lower confinement times and fusion power output. The neoclassical theory of impurity transport is highly developed, and it provides a model for the transport of particles, heat, and momentum due to Coulomb collisions; however, predictions by this theory are rarely well-matched by experiment. A simple representation of particle flux can be expressed as the sum of a diffusive and convective term,

$$\Gamma = -D\frac{dn}{dr} - vn, \tag{1.1.1}$$

where D is the diffusivity, and the second term describes an inward pinch with velocity v. Diffusivities are generally observed to be much larger than the neoclassical values predicted, and experiments on JET, for instance, indicate a sharp transition from approximately neoclassical values of D in the core to very anomalous values in the outer region.[3] Turbulent transport is considered a plausible candidate for explaining this anomalous transport.

1.2 Distribution Function

Perhaps the most important piece of information used here is the particle distribution function - the function that describes the probability of finding particles at a certain position with a certain velocity in 6-D phase space. The distribution function is taken to be the standard adiabatic form for now, with plans to update the perturbed distribution function with gyrokinetic effects as well as with trapped or passing particle dynamics as necessary. We start by assuming a Maxwellian distribution modified by a potential term and with T_j represented in terms of energy, i.e. $k_B T_j \Rightarrow T_j$,

$$f_{0,j} = n_j \left(\frac{m_j}{2\pi T_j}\right)^{3/2} e^{-\frac{m_j \epsilon_j^*}{T_j}} = f_{M,j} e^{-\frac{q_j \delta\Phi}{T_j}} \simeq f_{M,j}(1 - \frac{q_j \delta\Phi}{T_j}), \tag{1.2.1}$$

$$\Phi = \delta\Phi \tag{1.2.2}$$

with $\epsilon_j^* \equiv \frac{v^2}{2} + \frac{q_j\Phi}{m_j}$, the particle energy per unit mass, and the approximation that that $\frac{q\delta\Phi}{T_j} \ll 1$. (1.2.1) makes clear the following definition for the total distribution function in the case of a Maxwellian,

$$f_j = f_{0,j} + \delta f_j = f_{M,j} - \frac{q_j\delta\Phi}{T_j} f_{M,j}, \tag{1.2.3}$$

or, more generally as given by Frieman and Chen,[4]

$$\delta f_j = -\frac{q_j}{m_j}\delta\Phi\partial_{\epsilon^*} f_{0,j}\ . \tag{1.2.4}$$

1.3 Tokamak Geometry

For the tokamak geometry, we consider, for simplicity, an axisymmetric, large aspect-ratio, circular tokamak. Using the Grad-Shafranov equation, the following definition for the equilibrium magnetic field can be found,

$$\boldsymbol{B} = B_\theta\hat{\boldsymbol{e}}_\theta + B_\zeta\hat{\boldsymbol{e}}_\zeta = B_\theta\hat{\boldsymbol{e}}_\theta + B_0(1-\epsilon\cos\theta)\hat{\boldsymbol{e}}_\zeta = B_0[\frac{\epsilon}{q}\hat{\boldsymbol{e}}_\theta + (1-\epsilon\cos\theta)\hat{\boldsymbol{e}}_\zeta], \tag{1.3.1}$$

where $\epsilon = \frac{r}{R_0} \ll 1$ is the inverse aspect ratio, with $R = R_0 + r\cos\theta$, for r the minor radius, and R_0 the major radius, and q $\simeq \frac{rB_\zeta}{R_0B_\theta} \sim 1$ is the safety factor - the number of toroidal turns required for one poloidal turn of magnetic field lines. The term $\epsilon\cos\theta$ in R takes into account the change in toroidal radius along the tokamak midplane. Working to $\mathcal{O}(\epsilon)$, the magnetic field magnitude, magnetic field unit vector, and toroidal gradient terms can be written as,

$$B \equiv \sqrt{\boldsymbol{B}\cdot\boldsymbol{B}} = \sqrt{B_0^2[(1-\epsilon\cos\theta)^2 + (\frac{\epsilon}{q})^2]} = B_0\sqrt{1-2\epsilon\cos\theta} \simeq B_0(1-\epsilon\cos\theta), \tag{1.3.2}$$

$$\hat{\boldsymbol{b}} \equiv \frac{\boldsymbol{B}}{B} = \frac{\frac{\epsilon}{q}\hat{\boldsymbol{e}}_\theta + (1-\epsilon\cos\theta)\hat{\boldsymbol{e}}_\zeta}{1-\epsilon\cos\theta} \simeq \frac{\epsilon}{q}(1+\epsilon\cos\theta)\hat{\boldsymbol{e}}_\theta + \hat{\boldsymbol{e}}_\zeta \simeq \frac{\epsilon}{q}\hat{\boldsymbol{e}}_\theta + \hat{\boldsymbol{e}}_\zeta, \tag{1.3.3}$$

$$\nabla = \partial_r\hat{\boldsymbol{e}}_r + \frac{1}{r}\partial_\theta\hat{\boldsymbol{e}}_\theta + \frac{1}{R}\partial_\zeta\hat{\boldsymbol{e}}_\zeta = \partial_r\hat{\boldsymbol{e}}_r + \frac{1}{r}\partial_\theta\hat{\boldsymbol{e}}_\theta + \frac{1}{R_0 + r\cos\theta}\partial_\zeta\hat{\boldsymbol{e}}_\zeta\ . \tag{1.3.4}$$

2 Quasi-linear Theory

The theoretical picture of turbulent transport is that the free energy released by an instability drives a steady level of fluctuations in associated perturbed quantities, which results in radial transport of particles and energy. Precise relationships between the fluctuations and the corresponding transport can be obtained by quasi-linear theory.[5] In quasi-linear theory, it is assumed that the plasma is weakly unstable, and that the instability leads to a broad spectrum of waves that modify the background plasma in a self-consistent way via nonlinear interactions.[6] Generally, quantities of interest are taken to be a main spatially-averaged part which varies slowly with time compared to the frequency of perturbations, summed with a fluctuation quantity. Within our plasma description, the density and velocity contributing to the flux are written as,

$$n = \langle n\rangle + \delta n, \tag{2.0.1}$$

$$\boldsymbol{v} = \delta\boldsymbol{v}, \tag{2.0.2}$$

where $\langle ... \rangle$ represents a flux-surface average, and $\delta \boldsymbol{v}$ represents velocity fluctuations. We are interested in the effects of linear perturbations and so assume the form,

$$A(x,t) = A_0 e^{-i\omega t + i\boldsymbol{k}\cdot\boldsymbol{x}} \tag{2.0.3}$$

2.1 Anomalous Transport

(2.0.1) and (2.0.2) together give a possibility for the anomalous transport, namely,

$$\Gamma_r = \langle \delta v_r \delta n \rangle_\theta, \tag{2.1.1}$$

noting that we are considering flow in the radial direction, so averaging over the poloidal angle when considering the tokamak geometry described in the following section. The lowest order term is second-order in perturbed quantities because $\langle \langle n \rangle \delta v_r \rangle = \langle n \rangle \langle \delta v_r \rangle = 0$, since the average of an averaged quantity is itself, and the average of truly random fluctuations is zero. The density perturbation can be written in terms of a perturbed distribution function integrated over all velocity space, and the perturbed velocity as an $\boldsymbol{E} \times \boldsymbol{B}$ drift where the perturbed $\boldsymbol{E}$ field and the background $\boldsymbol{B}$ field are used. Taking the poloidal average and bringing the drift velocity into the velocity-space integral, we get the following result for the quasi-linear particle flux,

$$\Gamma_r = \langle \int_{-\infty}^{\infty} \delta f \frac{\delta \boldsymbol{E} \times \hat{\boldsymbol{b}}}{B} d^3 v \rangle_\theta = \frac{1}{2\pi} \int_0^{2\pi} \int_{-\infty}^{\infty} \delta f \frac{\delta E_\theta}{B} d\theta d^3 v \ . \tag{2.1.2}$$

Note there is a subtle point that only the θ-component of δE remains because we have dropped the term of $\mathcal{O}(\epsilon)$ in the cross product with (1.9). For quasi-linear theory, equations will be linearized and a Fourier transform can be taken to give a flux for each mode,

$$\Gamma_{r,k} = \frac{1}{2\pi} \int_0^{2\pi} \int_{-\infty}^{\infty} \delta \widetilde{f} \frac{(-\nabla_\theta (\delta \widetilde{\Phi}))}{B} d\theta d^3 v = -\frac{i k_\theta}{2\pi B} \int_0^{2\pi} \int_{-\infty}^{\infty} \delta \widetilde{f} \delta \widetilde{\Phi} d\theta d^3 v, \tag{2.1.3}$$

which can be made more general by allowing $\delta \widetilde{f}$ and $\delta \widetilde{\Phi}$ to be complex when calculating a real flux value, i.e.,

$$\Gamma_{r,k} = -\frac{i k_\theta}{2\pi B} \int_0^{2\pi} \int_{-\infty}^{\infty} (\delta \widetilde{f}^* \delta \widetilde{\Phi} + \delta \widetilde{f} \delta \widetilde{\Phi}^*) d\theta d^3 v \ . \tag{2.1.4}$$

3 Gyrokinetics

Gyrokinetics provides a framework to study plasma behavior on perpendicular scales comparable to that of the particle gyroradius ρ and frequencies much lower than the particle cyclotron frequencies ω_c. This model assumes the following: $\rho \ll L$, with L the characteristic macroscopic plasma scale; $\omega \ll \omega_{ci} \ll \omega_{ce}$; and $k_\perp \rho_i \sim 1$.[7] The trajectory of these particles is decomposed into a slow guiding center motion along the field line and a fast circular motion around the field line. The gyrokinetic Vlasov equation is found by converting the standard Vlasov equation to guiding center coordinates $\boldsymbol{X}$ and $\boldsymbol{V}$ and averaging over the gyrophase angle $\alpha \equiv \omega_c t$, to account for the fast circular motion, where,[4]

$$\boldsymbol{X} = \boldsymbol{x} + \frac{\boldsymbol{v} \times \hat{\boldsymbol{b}}}{\omega_c} = \boldsymbol{x} + \frac{\boldsymbol{v}_\perp}{\omega_c} = \boldsymbol{x} + \boldsymbol{\rho}, \tag{3.0.1}$$

$$\boldsymbol{V} = \boldsymbol{V}(\epsilon^*, \mu, \alpha) = \dot{\boldsymbol{X}}, \tag{3.0.2}$$

where ϵ^* is defined in section 1.2, $\mu \equiv \frac{v_\perp^2}{2B}$, and $\boldsymbol{v}_\perp = v_\perp [\cos\alpha \hat{\boldsymbol{e}}_1 + \sin\alpha \hat{\boldsymbol{e}}_2]$,[4] with $\hat{\boldsymbol{e}}_1$ and $\hat{\boldsymbol{e}}_2$ being local orthogonal unit vectors. We now note that the gradient defined in (1.3.4) is the gradient in guiding center coordinates because the geometry of the field lines is that of the guiding center - i.e. $\nabla = \nabla_X$ from now on.

3.1 Gyrokinetic Vlasov Equation

The perturbed, gyrokinetic distribution function is given as a combination of adiabatic and non-adiabatic terms,[4]

$$\delta F = \frac{q}{m}\delta F_a + \delta G, \tag{3.1.1}$$

where,

$$\delta F_a = \delta\Phi[\frac{\partial}{\partial\epsilon^*} + \frac{\partial}{B\partial\mu}]F_0, \tag{3.1.2}$$

$$\delta G_0 = -\frac{q}{m}\langle\delta\Phi\rangle_\alpha \frac{\partial F_0}{B\partial\mu} + \delta H_0, \tag{3.1.3}$$

$$\langle\ldots\rangle_\alpha = \frac{1}{2\pi}\int_0^{2\pi}(\ldots)d\alpha\ . \tag{3.1.4}$$

Higher order terms in δG have been dropped, as well as magnetic fluctuation terms - i.e. we choose to work in the electrostatic limit. We can simplify things further by choosing for F_0 a Maxwellian equilibrium distribution function, f_M, so that it only depends on ϵ^* and not μ or α. This gives us a final distribution function,

$$\delta F = \frac{q}{m}\delta\Phi\frac{\partial}{\partial\epsilon^*}f_M + \delta H_0\ . \tag{3.1.5}$$

This distribution function can be plugged into the Vlasov equation and gyrophase-averaged to give the standard gyrokinetic Vlasov equation for a species j,[4]

$$\begin{aligned}\partial_t\delta H_0 + v_\parallel\nabla_{X_\parallel}\delta H_0 + (\boldsymbol{v}_d + \frac{\hat{\boldsymbol{b}}\times\nabla_X\langle\delta\Phi\rangle_\alpha}{B})\cdot\nabla_X\delta H_0 \\ = -\frac{q_j}{m_j}[\partial_t\langle\delta\Phi\rangle_\alpha\partial_{\epsilon^*}f_M - \frac{1}{\omega_{cj}}(\nabla_X\langle\delta\Phi\rangle_\alpha\times\hat{\boldsymbol{b}})\cdot\nabla_X f_M],\end{aligned} \tag{3.1.6}$$

where $\boldsymbol{v}_d$, the sum of magnetic curvature and gradient drift terms, is defined as,

$$\boldsymbol{v}_d = \frac{v_\parallel^2 + \frac{1}{2}v_\perp^2}{\omega_{cj}}\frac{\boldsymbol{B}\times\nabla B}{B^2}, \tag{3.1.7}$$

with, simplifying to lowest order in ϵ,

$$\begin{aligned}\frac{\boldsymbol{B}\times\nabla B}{B^2} &= \frac{B_0[(1-\epsilon\cos\theta)\hat{\boldsymbol{e}}_\zeta + \frac{\epsilon}{q}\hat{\boldsymbol{e}}_\theta]\times(\partial_r\hat{\boldsymbol{e}}_r + \frac{1}{r}\partial_\theta\hat{\boldsymbol{e}}_\theta + \frac{1}{R}\partial_\zeta\hat{\boldsymbol{e}}_\zeta)B_0(1-\epsilon\cos\theta)}{B_0^2(1-\epsilon\cos\theta)^2} \\ &= \frac{[1-\epsilon\cos\theta)\hat{\boldsymbol{e}}_\zeta + \frac{\epsilon}{q}\hat{\boldsymbol{e}}_\theta]\times[-\frac{1}{R_0}\cos\theta\hat{\boldsymbol{e}}_r + \frac{r}{rR_0}\sin\theta\hat{\boldsymbol{e}}_\theta]}{(1-\epsilon\cos\theta)^2} \\ &= \frac{1}{(1-\epsilon\cos\theta)^2}[-\frac{(1-\epsilon\cos\theta)\cos\theta}{R_0}(\hat{\boldsymbol{e}}_\zeta\times\hat{\boldsymbol{e}}_r) - \frac{(1-\epsilon\cos\theta)\cos\theta}{R_0}(\hat{\boldsymbol{e}}_\zeta\times\hat{\boldsymbol{e}}_\theta) \\ &\qquad - \frac{\epsilon}{qR_0}\cos\theta(\hat{\boldsymbol{e}}_\theta\times\hat{\boldsymbol{e}}_r)] \\ &\simeq (1+2\epsilon\cos\theta)[-\frac{\cos\theta}{R_0}\hat{\boldsymbol{e}}_\theta - \frac{\sin\theta}{R_0}\hat{\boldsymbol{e}}_r] \simeq -\frac{1}{R_0}(\sin\theta\hat{\boldsymbol{e}}_r + \cos\theta\hat{\boldsymbol{e}}_\theta)\ .\end{aligned} \tag{3.1.8}$$

The second and third terms on the left-hand side of (3.1.6) can be simplified to lowest order in ϵ using (1.3.1)-(1.3.4) and (3.1.7)-(3.1.8) as,

$$\begin{aligned}v_\parallel\nabla_{X_\parallel} &= v_\parallel(\hat{\boldsymbol{b}}\cdot\nabla) = v_\parallel(\frac{\epsilon}{q}\hat{\boldsymbol{e}}_\theta + \hat{\boldsymbol{e}}_\zeta)\cdot(\partial_r\hat{\boldsymbol{e}}_r + \frac{1}{r}\partial_\theta\hat{\boldsymbol{e}}_\theta + \frac{1}{R_0 + r\cos\theta}\partial_\zeta\hat{\boldsymbol{e}}_\zeta) \\ &= v_\parallel(\frac{\epsilon}{qr}\partial_\theta + \frac{1}{R}\partial_\zeta) = v_\parallel(\frac{1}{qR_0}\partial_\theta + \frac{1}{R}\partial_\zeta) = \frac{v_\parallel}{qR}(\frac{R}{R_0}\partial_\theta + q\partial_\zeta) \\ &= \frac{v_\parallel}{qR}((1+\epsilon\cos\theta)\partial_\theta + q\partial_\zeta) \simeq \frac{v_\parallel}{qR}(\partial_\theta + q\partial_\zeta) = v_\parallel\frac{\partial}{\partial l},\end{aligned} \tag{3.1.9}$$

$$
\begin{aligned}
\boldsymbol{v}_d \cdot \nabla_X = \boldsymbol{v}_d \cdot \nabla &= -\frac{v_\parallel^2 + \frac{1}{2}v_\perp^2}{\omega_{cj}}(\sin\theta \hat{\boldsymbol{e}}_r + \cos\theta \hat{\boldsymbol{e}}_\theta) \cdot (\partial_r \hat{\boldsymbol{e}}_r + \frac{1}{r}\partial_\theta \hat{\boldsymbol{e}}_\theta) \\
&= -\frac{v_\parallel^2 + \frac{1}{2}v_\perp^2}{\omega_{cj} R_0}(\sin\theta \partial_r + \frac{\cos\theta}{r}\partial_\theta),
\end{aligned} \tag{3.1.10}
$$

with l being the length along the field lines. Note that we have dropped the non-linear $\boldsymbol{E} \times \boldsymbol{B}$ drift term on the left-hand side of (3.1.6) because we are interested in linearizing this equation.

Since the gyrokinetic analysis employs two spatial scales, Frieman and Chen rewrite functions of $\boldsymbol{x}$ in terms of $\boldsymbol{X}$ using a WKB approximation,[4]

$$
A(\boldsymbol{x}, \boldsymbol{v}) = \sum_{\boldsymbol{k}_\perp} \bar{A}(\boldsymbol{x}, \boldsymbol{v}, k_\perp) e^{i\int_{\boldsymbol{x}_{\perp 0}} \boldsymbol{k} \cdot d\boldsymbol{x}_\perp} . \tag{3.1.11}
$$

where $\bar{A}$ and $\boldsymbol{k}_\perp$ contain slow spatial variations. Ignoring nonlinear terms and dropping the sum, then integrating-by-parts with the definition from (3.0.1), we can rewrite (3.1.11) as,[4]

$$
\begin{aligned}
A(\boldsymbol{x}, \boldsymbol{v}, k_\perp) &= \bar{A}(\boldsymbol{X}, \boldsymbol{V}, k_\perp) e^{i(\int_{\boldsymbol{X}_{\perp 0}} \boldsymbol{k}_\perp \cdot \boldsymbol{X} - \boldsymbol{k}_\perp \cdot \boldsymbol{\rho})} \\
&= A(\boldsymbol{X}, \boldsymbol{V}, k_\perp) e^{-i\boldsymbol{k}_\perp \cdot \boldsymbol{\rho}}.
\end{aligned} \tag{3.1.12}
$$

Note that for macroscopic quantities $\rho/L \ll 1$, meaning $A(\boldsymbol{x}) \simeq A(\boldsymbol{X})$. It is also worth noting that the choice of direction of $\boldsymbol{k}_\perp$ is arbitrary when gyrophase-averaging. To see this, we can choose an arbitrary angle, ϕ, for $\boldsymbol{k}_\perp$ to give,

$$
e^{-i\boldsymbol{k}_\perp \cdot \boldsymbol{\rho}} = e^{-i\frac{k_\perp v_\perp}{\omega_c}\cos(\alpha - \phi)} = e^{-iz\cos(\alpha-\phi)}, \tag{3.1.13}
$$

where $z = \frac{k_\perp v_\perp}{\omega_c}$. Then the gyrophase-average gives a final result independent of ϕ,

$$
\frac{1}{2\pi}\int_0^{2\pi} e^{-iz\cos(\alpha-\phi)} d\alpha = J_0(z) . \tag{3.1.14}
$$

Because of this arbitariness, the standard choice is to pick $\boldsymbol{k}_\perp$ in the direction $\hat{\boldsymbol{e}}_2$, as defined in $\boldsymbol{v}_\perp$ in the gyrokinetic introduction above. This leads to the following form for gyrophase-averages,

$$
\begin{aligned}
\langle A(\boldsymbol{x}) \rangle_\alpha &= \frac{1}{2\pi}\int_0^{2\pi} \bar{A}(\boldsymbol{X}, \boldsymbol{V}, k_\perp) e^{-ik_\perp \rho \sin\alpha} d\alpha \\
&= J_0(k_\perp \rho) A(\boldsymbol{X}) = J_0(\frac{k_\perp v_\perp}{\omega_c}) A(\boldsymbol{X}) \\
&= J_0(z) A(\boldsymbol{X}),
\end{aligned} \tag{3.1.15}
$$

with the following definition for the zeroth-order Bessel function,

$$
J_0(z) = \frac{1}{2\pi}\int_0^{2\pi} e^{-iz\sin\alpha} d\alpha . \tag{3.1.16}
$$

3.2 Gyrokinetic Distribution Function

We can simplify (3.1.6) further by plugging in (3.1.9), (3.1.10), and (3.1.15) and then taking the Fourier transform, giving,

$$
\begin{aligned}
(-i\omega\delta + v_\parallel \frac{\partial}{\partial l} + i\boldsymbol{v}_d \cdot \boldsymbol{k}_X)\delta\widetilde{H}_0 &= -\frac{q_j}{m_j}[-i\omega J_0(z_j)\delta\widetilde{\Phi}\partial_{\epsilon^*} f_M \\
-\frac{i}{\omega_{cj}} J_0(z_j)\delta\widetilde{\Phi}(\boldsymbol{k} \times \hat{\boldsymbol{b}}) \cdot \frac{d}{dr} f_M \hat{\boldsymbol{e}}_r] &= i\frac{q_j}{m_j} J_0(z_j)\delta\widetilde{\Phi}[\omega\partial_{\epsilon^*} + \frac{k_\theta}{\omega_{cj}}\frac{d}{dr}] f_M,
\end{aligned} \tag{3.2.1}
$$

$$\Rightarrow \; (v_\parallel \partial_l - i(\omega - \bar{\omega}_{dj}))\delta \widetilde{H}_0 = i\frac{q_j}{m_j} J_0(z_j)\delta\widetilde{\Phi}[\omega\partial_{\epsilon^*} + \frac{k_\theta}{\omega_{cj}}\frac{d}{dr}]f_M, \tag{3.2.2}$$

with the following definitions,

$$\begin{aligned}\bar{\omega}_{dj} \equiv -\boldsymbol{v}_d \cdot \boldsymbol{k} &= \frac{k_\theta(v_\parallel^2 + \frac{1}{2}v_\perp^2)}{\omega_{cj}R_0}(\cos\theta + \frac{k_r}{k_\theta}\sin\theta) \\ &= \frac{\omega_{dj}}{2}(\left(\frac{v_\parallel}{v_{Tj}}\right)^2 + \frac{1}{2}\left(\frac{v_\perp}{v_{Tj}}\right)^2)(\cos\theta + \hat{s}\theta\sin\theta),\end{aligned} \tag{3.2.3}$$

$$\boldsymbol{k}\times\hat{\boldsymbol{b}} = (k_r\hat{\boldsymbol{e}}_r + k_\theta\hat{\boldsymbol{e}}_\theta + k_\zeta\hat{\boldsymbol{e}}_\zeta)\times(\frac{\epsilon}{q}\hat{\boldsymbol{e}}_\theta + \hat{\boldsymbol{e}}_\zeta) \simeq -k_r\hat{\boldsymbol{e}}_\theta + k_\theta\hat{\boldsymbol{e}}_r, \tag{3.2.4}$$

where $\partial_r = ik_\theta\hat{s}\theta \Rightarrow k_r = k_\theta\hat{s}\theta$, for magnetic shear $\hat{s}$.[8] ω_{dj} represents the magnetic curvature drift frequency,

$$\omega_{dj} \equiv 2\frac{n}{dn/dr}\frac{\omega_{*j}}{R_0}, \tag{3.2.5}$$

and ω_{*j} the diamagnetic drift frequency,

$$\omega_{*j} \equiv \frac{k_\theta T_j}{q_j B}\frac{1}{n}\frac{dn}{dr}\;. \tag{3.2.6}$$

Note that f_M depends only on r due to it being a function of $n_j(r)$ and $T_j(r)$ which are only changing across the circular flux-surfaces - therefore only functions of radius. Next, we can simplify further by plugging in the definition of the Maxwellian, giving the following values for derivatives,

$$\partial_{\epsilon^*} f_M = -\frac{m_j}{T_j}n_j\left(\frac{m_j}{2\pi T_j}\right)^{3/2} e^{-\frac{m_j\epsilon^*}{T_j(r)}} = -\frac{m_j}{T_j}f_M, \tag{3.2.7}$$

$$\begin{aligned}\frac{d}{dr}f_M &= \frac{dn}{dr}\frac{d}{dn}f_M + \frac{dT}{dr}\frac{d}{dT}f_M = \frac{dn}{dr}e^{-\frac{m_j\epsilon^*}{T_j(r)}} + \frac{dT}{dr}\frac{du}{dT}\frac{d}{du}[n_j\left(\frac{m_j u(T(r))}{2\pi}\right)^{3/2} e^{-m_j\epsilon^* u(T_j(r))}] \\ &= \frac{1}{n}\frac{dn}{dr}f_M + \frac{dT}{dr}\frac{du}{dT}n_j[\frac{3}{2}\left(\frac{m_j u}{2\pi}\right)^{1/2}\left(\frac{m_j}{2\pi}\right) - m_j\epsilon^*\left(\frac{m_j u}{2\pi}\right)^{3/2}]e^{-m_j\epsilon^* u(T_j(r))} \\ &= \frac{1}{n}\frac{dn}{dr}f_M + \frac{dT}{dr}\frac{du}{dT}[\frac{3}{2}u^{-1} - m_j\epsilon^*]f_M = [\frac{1}{n}\frac{dn}{dr} + \frac{dT}{dr}(-\frac{1}{T^2})(\frac{3}{2}u^{-1} - m_j\epsilon^*)]f_M \\ &= [\frac{1}{n}\frac{dn}{dr} - \frac{1}{T}\frac{dT}{dr}(\frac{3}{2} - \frac{m_j v^2}{2T_j})]f_M = [\frac{1}{n}\frac{dn}{dr} - \frac{1}{T}\frac{dT}{dr}(\frac{3}{2} - \frac{v^2}{2v_{Tj}^2})]f_M \\ &= \frac{1}{n}\frac{dn}{dr}[1 + (\frac{1}{2}\left(\frac{v}{v_{Tj}}\right)^2 - \frac{3}{2})\eta_j]f_M,\end{aligned} \tag{3.2.8}$$

given that $u = T^{-1}$, $du = -T^{-2}dT$, $v_{Tj} \equiv \sqrt{\frac{T_j}{m_j}}$, and $\eta_j \equiv \frac{n}{T}\frac{dT}{dn}$. Putting these derivatives into (3.2.2) then gives,

$$
\begin{aligned}
&(v_\parallel \partial_l - i(\omega - \bar{\omega}_{dj}))\delta\widetilde{H}_0 = \\
&- i\frac{q_j}{m_j} J_0(z_j)\delta\widetilde{\Phi}[-\frac{m_j}{T_j}\omega + \frac{k_\theta}{\omega_{cj}}\frac{1}{n}\frac{dn}{dr}[1 + (\frac{1}{2}\left(\frac{v}{v_{Tj}}\right)^2 - \frac{3}{2})\eta_j]] \\
\Rightarrow\ &(v_\parallel \partial_l - i(\omega - \bar{\omega}_{dj}))\delta\widetilde{H}_0 = \\
&- i\frac{q_j}{T_j} J_0(z_j)\delta\widetilde{\Phi}[\omega - \frac{k_\theta T_j m_j}{q_j B m_j}\frac{1}{n}\frac{dn}{dr}[1 + (\frac{1}{2}\left(\frac{v}{v_{Tj}}\right)^2 - \frac{3}{2})\eta_j]]f_M \\
\Rightarrow\ &(-v_\parallel \partial_l + i(\omega - \bar{\omega}_{dj}))\delta\widetilde{H}_0 = \\
&i\frac{q_j}{T_j} J_0(z_j)\delta\widetilde{\Phi}[\omega - \omega_{*j}[1 + (\frac{1}{2}\left(\frac{v}{v_{Tj}}\right)^2 - \frac{3}{2})\eta_j]]f_M
\end{aligned}
\tag{3.2.9}
$$

$$
\Rightarrow\ (-v_\parallel \partial_l + i(\omega - \bar{\omega}_{dj}))\delta\widetilde{H}_0 = i\frac{q_j}{T_j} J_0(z_j)\delta\widetilde{\Phi}[\omega - \omega^T_{*j}]f_M, \tag{3.2.10}
$$

with the following definition,

$$
\omega^T_{*j} \equiv \omega_{*j}[1 + (\frac{1}{2}\left(\frac{v}{v_{Tj}}\right)^2 - \frac{3}{2})\eta_j]\ . \tag{3.2.11}
$$

Then (3.2.10) can be simplified further and solved for $\delta\widetilde{H}$ if a few physical assumptions are made for electron drift and ITG modes. First off, trapped particle effects are neglected, i.e.,

$$
\omega \gg \omega_{bj}, \tag{3.2.12}
$$

for ω_{bj} as the bounce frequency. We also assume electrons move rapidly in response to the electrostatic potential,

$$
k_\parallel v_{Te} \gg \omega \gg k_\parallel v_{Ti}, \tag{3.2.13}
$$

noting that $v_{Tj} \simeq v_{\parallel j}$ since the majority of the velocity is in the parallel direction. Finally we assume that the frequency associated with magnetic drifts of ions is much smaller than that of the perturbed modes,

$$
\omega \gg \omega_{di}\ . \tag{3.2.14}
$$

Now, (3.2.13) and (3.2.14) allow the left-hand side of (3.2.10) to be rewritten for ions as,

$$
i(\omega - \bar{\omega}_{di})(1 + i\frac{v_\parallel}{\omega - \omega_{di}}\frac{\partial}{\partial l}) \simeq i(\omega - \bar{\omega}_{di})(1 - \frac{k_\parallel v_{\parallel i}}{\omega}) \simeq i(\omega - \bar{\omega}_{di})\ . \tag{3.2.15}
$$

Finally, we can solve for $\delta\widetilde{H}$ explicitly by replacing the left-hand side of (3.2.10) with (3.2.15), giving the perturbed, non-adiabatic distribution function for ions,

$$
\delta\widetilde{H}_0 = \frac{q_i}{T_i} J_0(z_i)\delta\widetilde{\Phi} f_M \frac{\omega - \omega^T_{*i}}{\omega - \bar{\omega}_{di}}, \tag{3.2.16}
$$

or, converting back to $\boldsymbol{x}$ coordinates using (3.1.12) for the velocity-space integration to get n_j,

$$
\delta\widetilde{H}_0(\boldsymbol{x}) = \frac{q_i}{T_i} J_0(z_i)\delta\widetilde{\Phi}(\boldsymbol{x}) f_M \frac{\omega - \omega^T_{*i}}{\omega - \bar{\omega}_{di}} e^{iz\sin\alpha}\ . \tag{3.2.17}
$$

Note that again the arbitrary choice in the direction of $\boldsymbol{k}_\perp$ is invokable because α must be integrated over again when the distribution function is integrated over velocity-space to give the flux.

3.3 Quasi-linear Anomalous Flux

We can now plug our perturbed distribution functions into the quasi-linear estimate for radial flux, (2.1.4). For a system of ions and electrons with no impurities, the total perturbed distribution function is given as,

$$\delta\widetilde{f}_i = -\frac{q_i}{T_i}\delta\widetilde{\Phi} f_M + \delta\widetilde{H}_0 \tag{3.3.1}$$

4 Comparison to Simulation

4.1 Impurities

4.2 GENE Normalization

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