Toroidal Magnetic Geometry: a Hamiltonian System

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Outline

A Brief Review of Hamiltonian Mechanics

Euler-Lagrange Equation

A Classical Variational Problem

Given a function $F(t, \vec{z}, \dot{\vec{z}})$ with twice continuous derivatives with respect to all of its arguments, we look for a twice continuously differentiable function $\vec{z}(t)$ on the interval $[t_0, t_1]$ with the specified boundary $\vec{z}(t_0) = const.$ and $\vec{z}(t_1) = const.$, so that the functional defined by

$$J[\vec{z}] = \int_{t_0}^{t_1} F(t, \vec{z}, \dot{\vec{z}}) dt \tag{1}$$

is extremized.

A functional is a function on a space of functions, i.e., it is a mapping which assigns a definite number to each function in the space.

Euler-Lagrange Equation

A Classical Variational Problem

To solve this problem, we consider a family of twice continuously differentiable functions on $[t_0, t_1]$ given by $\vec{z}(t) + \epsilon\vec{h}(t)$, with ϵ being a small number and $\vec{h}(t_0) = \vec{h}(t_1) = 0$.

If the functional J has a local extremum at the function $\vec{z}$, then, as a necessary condition, we must have

$$\begin{aligned} \delta J[\epsilon\vec{h}] &= \int_{t_0}^{t_1} [F(t, \vec{z} + \epsilon\vec{h}, \dot{\vec{z}} + \epsilon\dot{\vec{h}}) - F(t, \vec{z}, \dot{\vec{z}})]dt \\ &= \epsilon(\frac{\partial F}{\partial \dot{\vec{z}}} \cdot \vec{h})_{t_0}^{t_1} + \epsilon \int_{t_0}^{t_1} [\frac{\partial F}{\partial \vec{z}} - \frac{d}{dt}(\frac{\partial F}{\partial \dot{\vec{z}}})] \cdot \vec{h} dt \\ &= 0 \end{aligned} \tag{2}$$

for every $\vec{h}(t)$.

Euler-Lagrange Equation

A Classical Variational Problem

The fundamental lemma of calculus of variations

If $\int_{t_0}^{t_1} f(t)h(t)dt = 0$ for all h with twice continuous derivatives, then $f(t) = 0$ on $[t_0, t_1]$.

Euler-Lagrange Equation

The function $\vec{z}(t)$ that extremizes the functional J of Eq.(1) necessarily satisfies:

$$\frac{\partial F}{\partial \vec{z}} - \frac{d}{dt}\left(\frac{\partial F}{\partial \dot{\vec{z}}}\right) = 0. \tag{3}$$

Remarks:

1. In general, the Euler-Lagrange equation is a nonlinear second order differential equation.
2. The extremal solution $\vec{z}$ does not depend on the choice of coordinate system[a], since Eq.(1) is a scalar equation.

[a] V. I. Arnold, Mathematical methods of classical mechanics, 2nd ed. (Springer-Verlag), p. 59.

Hamilton's Equations

The Hamilton Principle

In Hamiltonian theory, the dynamical system is described by N generalized coordinates $\vec{q} = (q_1, q_2, \cdots, q_N)$ and their conjugate momenta $\vec{p} = (p_1, p_2, \cdots, p_N)$. The state of the system is a point in the phase space with coordinates $\vec{z} = (\vec{q}, \vec{p})$.

Then, for the dynamical system with holonomic constraints and external forces derivable from a generalized scalar potential, the Hamilton principle requires the path of the system from t_0 to t_1 is such that the action functional[1]

$$J[\vec{z}] = \int_{t_0}^{t_1} L(t, \vec{z}, \dot{\vec{z}}) dt \tag{4}$$

is extremized, where $L = \vec{p} \cdot \dot{\vec{q}} - H(\vec{q}, \vec{p}, t)$ is the Lagrangian.

Holonomic constraints

The constraints that depend only on the configuration coordinates and time, i.e., $f(\vec{r}, t) = 0$.

[1] In fact, the Hamilton principle refers to no coordinates, all the information is in the action integral.

Hamilton's Equations

Hamilton's Equations

Using the Euler-Lagrange equation, Eq.(3), we find the Hamilton's canonical equations:

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \tag{5}$$

$$\dot{p}_i = -\frac{\partial H}{\partial q_i}. \tag{6}$$

E: Repeat the derivation step by step.
E: Demonstrate the Lagrangian gauge invariant, i.e., the same equations of motion will be obtained if we take a new Lagrangian $L' = L + d_t g(\vec{z}, t)$.

Hamilton's Equations

Phase space flow

A dynamical system can be represented by the family of solution curves of its equations of motion, these curves fill the phase space and define a phase flow:

$$\vec{q} = \vec{q}(t, \vec{q}_0, \vec{p}_0, t_0), \quad \vec{p} = \vec{p}(t, \vec{q}_0, \vec{p}_0, t_0). \tag{7}$$

At each point of phase space, a velocity $\vec{V}$ can be defined as $\vec{V} = \dot{\vec{z}} = (\dot{\vec{q}}, \dot{\vec{p}})$, the phase flow in Hamiltonian theory is thus incompressible:

$$\nabla_z \cdot \vec{V} = \frac{\partial}{\partial \vec{q}}(\frac{\partial H}{\partial \vec{p}}) - \frac{\partial}{\partial \vec{p}}(\frac{\partial H}{\partial \vec{q}}) = 0. \tag{8}$$

In the case of $H = H(\vec{q}, \vec{p})$, the phase flow is a steady flow:

$$\vec{q} = \vec{q}(t - t_0, \vec{q}_0, \vec{p}_0), \quad \vec{p} = \vec{p}(t - t_0, \vec{q}_0, \vec{p}_0). \tag{9}$$

Hamilton's Equations

Example: the pendulum

The Hamiltonian is $H = p^2/2 - \omega_0^2 \cos(q)$.

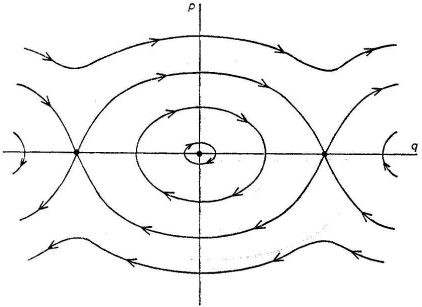


Figure 1: The phase flow for the pendulum[2], the O and X points are noticeable.

E: Solve the pendulum in terms of elliptic functions.

[2] K. J. Whiteman, Rep. Prog. Phys. 40, 1033, 1977.

Hamilton's Equations

Example: two uncoupled harmonic oscillators

The Hamiltonian is $H = \sum_{i=1}^{2}(p_i^2 + \omega_i^2 q_i^2)/2$. The general solution is

$$q_i = A_i \sin(\omega_i t + \phi_i), \quad p_i = A_i \omega_i \cos(\omega_i t + \phi_i), \tag{10}$$

where A_i and ϕ_i are invariants.

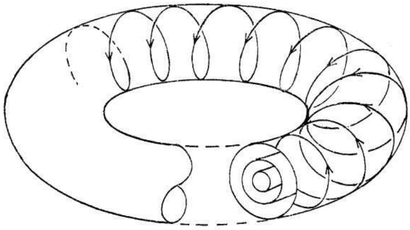


Figure 2: An orbit spiralling on the surface of an invariant torus[3].

The orbits lie on a $2D$ invariant torus. If ω_1/ω_2 is rational, the orbits are closed but, if ω_1/ω_2 is irrational, the orbits will cover the torus ergodically.

[3] K. J. Whiteman, Rep. Prog. Phys. 40, 1033, 1977.

Hamilton's Equations

Example: two uncoupled harmonic oscillators

E: Can any of the invariants be used to restrict the orbit in phase space?

The frequencies ω_i are constant on invariant tori for the uncoupled harmonics, however, the frequencies can vary from torus to torus for more complex systems. For example, consider a completely integrable periodic system $H(\vec{I}, \vec{\theta}) = H_0(\vec{I})$, where $\vec{I}$ and $\vec{\theta}$ are N-dimensional vectors of actions and angles, respectively. The Hamilton's equations are then written as

$$\dot{\vec{I}} = 0, \quad \dot{\vec{\theta}} = \nabla_I H_0 = \vec{\omega}_0(\vec{I}) \tag{11}$$

with $\vec{\omega}_0$ being the frequency vector.

The orbits lie on N-dimensional invariant tori $T_0(\vec{\omega}_0)$ in the $2N$-dimensional phase space.

E: Read about action-angle variables in your favorite book.

Hamilton's Equations

Kolmogorov-Arnold-Moser (KAM) Theorem

Let us assume that the integrable Hamiltonian H_0 is perturbed by a small term ϵH_1 such that $H = H_0(\vec{I}) + \epsilon H_1(\vec{I}, \vec{\theta})$ where H_1 is periodic in the angle variables. The Hamilton's equations are thus

$$\dot{\vec{I}} = -\epsilon \nabla_\theta H_1, \quad \dot{\vec{\theta}} = \vec{\omega}_0(\vec{I}) + \epsilon \nabla_I H_1. \tag{12}$$

Suppose the unperturbed motion is nondegenerate $(\det |\partial_{I_i} \partial_{I_j} H_0| \neq 0)$, the total Hamiltonian H is analytic in the phase space, and ϵ is small enough, then for almost all incommensurate frequency vector $\vec{\omega}_0$ ($\vec{\omega}_0 \cdot \vec{m} \neq 0$ for all integers m_i), there exists an invariant torus $T(\vec{\omega}_0)$ of the perturbed system such that $T(\vec{\omega}_0)$ is close to $T_0(\vec{\omega}_0)$.

Remarks

- If $\vec{\omega}_0$ is sufficiently irrational, the original invariant torus T_0 is modified in shape by the perturbation but not destroyed.
- The incommensurate frequency vectors that fail to satisfy the KAM theorem are a set of measure of order ϵ.

Toroidal Magnetic Geometry

Hamiltonian Form of the Magnetic-Field-Line Equation

Magnetic-Field-Line Equation

A magnetic field line is a curve whose tangent vector at any point is parallel to the magnetic field $\vec{B}$, i.e.,

$$\frac{d\vec{R}}{\vec{B}} = dt. \tag{13}$$

Given coorinates (u^1, u^2, u^3), the differential vector can be expressed as

$$d\vec{R} = \frac{\partial \vec{R}}{\partial u^1} du^1 + \frac{\partial \vec{R}}{\partial u^2} du^2 + \frac{\partial \vec{R}}{\partial u^3} du^3, \tag{14}$$

therefore, the magnetic-field-line equation is

$$\frac{du^1}{\vec{B} \cdot \nabla u^1} = \frac{du^2}{\vec{B} \cdot \nabla u^2} = \frac{du^3}{\vec{B} \cdot \nabla u^3}. \tag{15}$$

Hamiltonian Form of the Magnetic-Field-Line Equation

Canonical Representation

A divergence-free vector can be derived from a vector potential $\vec{A}(\vec{x})$. If we write $\vec{A}$ in terms of the coordinates (ρ, θ, ζ), where ρ is a radius-like variable, but not a flux-surface label, θ and ζ are, respectively, arbitrary poloidal and toroidal angles,

$$\vec{A} = A_\rho \nabla \rho + A_\theta \nabla \theta + A_\zeta \nabla \zeta. \tag{16}$$

Note that $\nabla \times \vec{A} = \nabla \times (\vec{A} + \nabla G)$, we can introduce a function G with $\partial_\rho G = A_\rho$, such that $\nabla G = A_\rho \nabla \rho + \partial_\theta G \nabla \theta + \partial_\zeta G \nabla \zeta$. We have[4]

$$\vec{A} = \nabla G + (A_\theta - \partial_\theta G)\nabla \theta + (A_\zeta - \partial_\zeta G)\nabla \zeta, \tag{17}$$

and thus $\vec{B}$ can be represented in the canonical form[5][6]

$$\vec{B} = \nabla \psi \times \nabla \theta + \nabla \zeta \times \nabla \psi_p, \tag{18}$$

where $\psi_p = \psi_p(\psi, \theta, \zeta)$, so that Eq.(18) is not a flux-coordinate representation.

[4] R. B. White, The theory of toroidally confined plasmas, 3rd ed., (Imperial College Press), p.10.

[5] A. H. Boozer, Phys. Fluids, 26, 1288, (1983).

[6] Z. Yoshida, Phys. Plasmas, 1, 208, (1994)

Hamiltonian Form of the Magnetic-Field-Line Equation

Canonical Representation

Using Eq.(15) and assuming $J^{-1} = \nabla\psi \cdot (\nabla\theta \times \nabla\zeta) \neq 0$, we obtain the equation of magnetic-field-line,

$$\frac{d\psi}{d\zeta} = \frac{\vec{B} \cdot \nabla\psi}{\vec{B} \cdot \nabla\zeta}, \quad \frac{d\theta}{d\zeta} = \frac{\vec{B} \cdot \nabla\theta}{\vec{B} \cdot \nabla\zeta}, \tag{19}$$

i.e.,

$$\frac{d\psi}{d\zeta} = -\frac{\partial\psi_p}{\partial\theta}, \quad \frac{d\theta}{d\zeta} = \frac{\partial\psi_p}{\partial\psi}. \tag{20}$$

which is of Hamiltonian form, with $\psi_p(\psi, \theta, \zeta)$ the field-line Hamiltonian, ψ and θ the canonical variables, and ζ the time.

Remarks

- The magnetic configuration in toroidal geometry is a Hamiltonian system with 1 degree of freedom.
- The magnetic-field-line flow is incompressible, due to $\nabla \cdot \vec{B} = 0$ or Liouville's theorem.

Hamiltonian Form of the Magnetic-Field-Line Equation

Remarks

- The topological structure of the magnetic field is determined by the field-line Hamiltonian $\psi_p(\psi, \theta, \zeta)$ alone.
- In general, the system is not integrable due to $\psi_p = \psi_p(\psi, \theta, \zeta)$. However, if there exists a nontrivial constant of motion (e.g., the axisymmetric field in tokamak), the magnetic field will have perfect magnetic surfaces, and we can choose flux, or action-angle, coordinates (ψ, θ, ϕ) so that $\psi_p = \psi_p(\psi)$. In this case, the equation of magnetic-field-line becomes

$$\frac{d\psi}{d\zeta} = 0, \quad \frac{d\theta}{d\zeta} = \frac{\partial \psi_p}{\partial \psi} = \frac{1}{q(\psi)}, \tag{21}$$

  where the flux function q is the safety factor, it plays the role of a frequency in the action-angle picture, and the magnetic field can be written as $\vec{B} = \nabla\psi \times \nabla(\theta - \zeta/q)$. Continuing further the analogy with the action-angle picture, one can define the rational and irrational surfaces according to q.

Hamiltonian Form of the Magnetic-Field-Line Equation

Remarks

If the field with perfect surfaces is slightly perturbed so that

$$\psi_p(\psi,\theta,\phi) = \psi_{p0}(\psi) + \epsilon\psi_{p1}(\psi,\theta,\zeta), \tag{22}$$

where ψ_{p1} is periodic in θ and ζ. Eq.(22) is often called a $3/2$ degrees of freedom Hamiltonian system. Then, by the KAM theorem, the magnetic-field-line has three different types of trajectories in toroiday geometry[a]:

- A field line can close on itself on the rational surface, this type of possibility is topologically unstable, an arbitraily small resonant perturbation can destroy it.
- A field line covers the irrational surface ergodically, this type of possibility is topologically stable for sufficiently small perturbations.
- A stochastic field line fills in a nonzero volume of space ergodically, implies the absence of magnetic confinement in that region.

The magnetic confinement depends on the formation of magnetic surface.

[a] A. H. Boozer, Rev. Mod. Phys., 76, 1071, (2005).

Hamiltonian Form of the Magnetic-Field-Line Equation

Remarks

- Note that, the behaviour of plasmas is not involved so far, the obtained results apply to any divergence-free vector field in toroidal geometry.
- However, the specific structure of equilirium magnetic field does require the solution of static equilibrium of a magnetoplasma, such as the Grad-Shafranov equation for axisymmetric toroidal plasmas.

E: Discuss the reason that q is called safety factor.

E: Read about the Grad-Shafranov equation in your favorite book.

E: Can you comment the importance of low-dimensional Hamiltonian systems for the magnetic confinement?

Evaluation of the Magnetic Coordinates

Evaluation of the Transformation[7]

Formulating the magnetic coordinates $(\psi(\vec{R}), \theta(\vec{R}), \zeta(\vec{R}))$ of a given integrable field $\vec{B}(\vec{R})$ is a difficult task in toroidal geometry.

1. The field line trajectory $\vec{R}(\zeta)$ is evaluated by Eq.(15)
$$\frac{d\vec{R}}{d\zeta} = \frac{\vec{B}(\vec{R})}{\vec{B}\cdot\nabla\zeta}, \tag{23}$$
with a definite choice of toroidal angle $\zeta(\vec{R})$, often the minus azimuthal angle of cylindrical coordinates.
2. Knowing $\vec{R}(\zeta)$ along the field lines, one can obtain the transformation $\vec{R}(\psi, \theta, \zeta)$ and $q(\psi)$ in Fourier form
$$\vec{R}(\zeta) = \vec{R}(\psi(\zeta), \theta(\zeta), \zeta) = \sum_{n,m} \vec{R}_{n,m}(\psi) e^{i(n\zeta - m\theta)} = \sum_{n,m} \vec{R}_{n,m}(\psi) e^{i(n-m/q)\zeta}. \tag{24}$$
A Fourier decomposition of $\vec{R}(\zeta)$ then gives the $\vec{R}_{n,m}$ and q on a magnetic surface.
3. The toroidal magnetic flux $2\pi\psi$ associated with the surface can be determined by an area integral $\iint \vec{B}\cdot d\vec{a}_t$.

[7] G. Kuo-Petrravis and A. H. Boozer, J. Comput. Phys., 73, 107, (1987).

Guiding-center Motion

Guiding-center Phase Space Lagrangian

Lagrangian in Phase Space

The phase space Lagrangian for a charged particle in an electromagnetic field:

$$L(\vec{x}, \vec{v}, \dot{\vec{x}}, t) = (m\vec{v} + \frac{e}{c}\vec{A}) \cdot \dot{\vec{x}} - H(\vec{v}, \vec{x}, t) \tag{25}$$

with the Hamiltonian

$$H = \frac{1}{2}mv^2 + e\phi(\vec{x}, t). \tag{26}$$

From Eq.(25), we recognize the canonical momentum for a charged particle in a electromagnetic field to be

$$\vec{p} = m\vec{v} + \frac{e}{c}\vec{A}. \tag{27}$$

Guiding-center Phase Space Lagrangian

Phase Space Coordinate Transformation

Now separate the perpendicular and parallel velocity components as

$$\vec{v} = v_{\parallel}\hat{b} + v_{\perp}\hat{c} \tag{28}$$

where unit vectors $\hat{b} = \vec{B}/B$ and $\hat{c} = -\sin\xi\hat{e}_1 - \cos\xi\hat{e}_2$ with $\hat{e}_1 \cdot \hat{e}_2 = 0$ and $\hat{e}_1 \times \hat{e}_2 = \hat{b}$. ξ the gyrophase. Note that, $\hat{b}$, $\hat{e}_1$ and $\hat{e}_2$ are functions of $(\vec{x}, t)$.
Then the guiding center can be defined by

$$\vec{x} = \vec{X} + \frac{v_{\perp}\hat{a}}{\Omega_c} \tag{29}$$

with $\hat{a} = \hat{b} \times \hat{c}$.
The Lagrangian becomes

$$L(\vec{X}, \vec{v}, \dot{\vec{X}}, t) = [mv_{\parallel}\hat{b} + mv_{\perp}\hat{c} + \frac{e}{c}\vec{A}(\vec{x}, t)] \cdot [\dot{\vec{X}} + \frac{d}{dt}(\frac{v_{\perp}\hat{a}}{\Omega_c})] - H(\vec{v}, \vec{x}, t) \tag{30}$$

All quantities on the right are evaluated at the guiding center $\vec{X}$.

Guiding-center Phase Space Lagrangian

Scale Separation

We here follow Littlejohn's work[8] and assume that the magnetic field $\vec{B}$ is so large that

$$\epsilon \sim \frac{\rho}{L} \sim \frac{\omega}{\Omega_c} \ll 1 \tag{31}$$

is well satisfied. Here ρ and Ω_c are the particle's gyroradius and gyrofrequency, respectively, while L and ω^{-1} are, respectively, the characteristic spatial and time scale of varations in the ambient electromagnetic fields.
Then ϵ is used as an ordering parameter for the perturbation expansion, the Lagrangian can be written as

$$L(\vec{X}, v_\parallel, v_\perp, \xi, \dot{\vec{X}}, t) = [mv_\parallel \hat{b} + mv_\perp \hat{c} + \frac{e}{c}\vec{A}(\vec{x}, t)] \cdot [\dot{\vec{X}} + \frac{d}{dt}(\frac{v_\perp \hat{a}}{\Omega_c})] - H(\vec{v}, \vec{x}, t), \tag{32}$$

where

$$\vec{x} = \vec{X} + \frac{v_\perp \hat{a}}{\Omega_c}. \tag{33}$$

We now regard the variables $(\vec{X}, v_\parallel, v_\perp, \xi)$ as the new six phase space coordinates.

[8] R. G. Littlejohn, J. Plasma Physics, 29, part 1, 111, (1983).

Guiding-center Phase Space Lagrangian

Phase Space Lagrangian to the Order $\mathcal{O}(\epsilon)$

The Lagrangian can be Taylor expanded up to the order $\mathcal{O}(\epsilon)$:

$$
\begin{aligned}
L \quad = \quad & \frac{e}{c}\vec{A}\cdot\dot{\vec{X}} + (mv_{\parallel}\hat{b} + mv_{\perp}\hat{c})\cdot\dot{\vec{X}} + \frac{e}{c}\vec{A}\cdot\frac{d}{dt}(\frac{v_{\perp}\hat{a}}{\Omega_c}) + \frac{v_{\perp}\hat{a}}{\Omega_c}\cdot\nabla(\frac{e}{c}\vec{A})\cdot\dot{\vec{X}} \\
& + \frac{e}{c}\frac{1}{2}[(\frac{v_{\perp}\hat{a}}{\Omega_c}\cdot\nabla)^2\vec{A}]\cdot\dot{\vec{X}} + (mv_{\parallel}\hat{b} + mv_{\perp}\hat{c} + \frac{e}{c}\frac{v_{\perp}\hat{a}}{\Omega_c}\cdot\nabla\vec{A})\cdot\frac{d}{dt}(\frac{v_{\perp}\hat{a}}{\Omega_c}) \\
& - \frac{1}{2}mv^2 - e\phi - (\frac{ev_{\perp}\hat{a}}{\Omega_c})\cdot\nabla\phi. \qquad (34)
\end{aligned}
$$

Before proceeding with this compicated Lagrangian, we recall the Lagrangian gauge invariant $L \to L + d_t S$ in phase space to simplify the phase space Lagrangian.

Guiding-center Phase Space Lagrangian

Phase Space Lagrangian to the Order $\mathcal{O}(\epsilon)$

In particular, by taking

$$S(\vec{X}, v_{\parallel}, v_{\perp}, \xi, t) = -\frac{v_{\perp}\hat{a}}{\Omega_c} \cdot (\frac{e}{c}\vec{A}), \tag{35}$$

the new phase space Lagrangian up to order $\mathcal{O}(\epsilon)$ becomes

$$\begin{aligned} L &= (\frac{e}{c}\vec{A} + mv_{\parallel}\hat{b} + mv_{\perp}\hat{c}) \cdot \dot{\vec{X}} - \frac{1}{2}mv^2 - e\phi - \frac{ev_{\perp}\hat{a}}{\Omega_c} \cdot [\frac{\partial \vec{A}}{c\partial t} + \nabla\phi] \\ &\quad + \frac{e}{c}\frac{1}{2}[(\frac{v_{\perp}\hat{a}}{\Omega_c} \cdot \nabla)^2 \vec{A}] \cdot \dot{\vec{X}} + (\frac{mv_{\perp}^2}{\Omega_c} + \frac{ev_{\perp}^2}{c\Omega_c^2}\hat{a} \cdot \nabla\vec{A} \cdot \hat{c})\dot{\xi} + \frac{ev_{\perp}\dot{v}_{\perp}}{c\Omega_c^2}\hat{a} \cdot \nabla\vec{A} \cdot \hat{a}. \end{aligned} \tag{36}$$

Now again add a perfect derivative to L with

$$S = -\frac{ev_{\perp}^2}{2c\Omega_c^2}\hat{a} \cdot \nabla\vec{A} \cdot \hat{a}, \tag{37}$$

we finally have the Lagrangian in phase space to the order $\mathcal{O}(\epsilon)$

$$L = (\frac{e}{c}\vec{A} + mv_{\parallel}\hat{b} + mv_{\perp}\hat{c}) \cdot \dot{\vec{X}} + \frac{mv_{\perp}^2}{2\Omega_c}\dot{\xi} - \frac{1}{2}mv^2 - e\phi - \frac{ev_{\perp}\hat{a}}{\Omega_c} \cdot [\frac{\partial \vec{A}}{c\partial t} + \nabla\phi]. \tag{38}$$

Guiding-center Phase Space Lagrangian

Phase Space Lagrangian to the Order $\mathcal{O}(\epsilon)$

E: Are Eqs. (34), (36) and (38) invariant under the electromagnetic gauge transformation?

Now average over the fast gyromotion time scale, we have the phase space Lagrangian in guiding-center phase space coordinates:

$$L(\vec{X}, \mu, U, \xi, t) = (\frac{e}{c}\vec{A} + mU\hat{b}) \cdot \dot{\vec{X}} + \frac{mc}{e}\mu\dot{\xi} - \frac{1}{2}mU^2 - \mu B - e\phi, \tag{39}$$

where U is the parallel velocity of guiding center and μ is the magnetic momentum.

Remarks

1. No assumptions were made about the topological structure of the magnetic field, this expression is valid independent of the existence of magnetic surfaces, equilibrium.
2. The derivation of the guiding center Lagrangian from the particle Lagrangian can be extended to arbitrarily high order, with the help of Lie transformation.

Guiding-center Motion

Euler-Lagrange Equation

The Euler-lagrange equation of μ, ξ and U yields $\dot{\xi} = \Omega_c$, $\dot{\mu} = 0$ and $U = \hat{b} \cdot \dot{\vec{X}}$, respectively.
The Euler-Lagrange equation of the coordinate $\vec{X}$ gives

$$\dot{U} = -\frac{\vec{B}^*}{mB_\parallel^*} \cdot (\mu \nabla B - e\vec{E}^*), \quad (40)$$

$$\dot{\vec{X}} = \frac{1}{B_\parallel^*}[U\vec{B}^* + c\hat{b} \times (\frac{\mu}{e}\nabla B - \vec{E}^*)], \quad (41)$$

where $\vec{\kappa} = \hat{b} \cdot \nabla \hat{b}$ is the curvature,

$$\vec{B}^* = B(\hat{b} + \underbrace{\frac{U}{\Omega_c}\nabla \times \hat{b}}_{\text{Banos correction}}), \quad \vec{E}^* = \vec{E} - \frac{m}{e}U\partial_t \hat{b}. \quad (42)$$

E: Derive the above results step by step.
E: Derive the expression of the evolution of the guiding center energy.
E: Where is the curvature drift term?
E: Without the electric field, Eq.(40) indicates that $\dot{U} \neq 0$, comment about the physical meaning of this fact.

Guiding-center Motion

Generalized Toroidal Angular Momentum

In an azimuthally symmetric system such as tokamak, the components of $\vec{B}$ are independent ot the azimuthal angle ζ with respect to the cylindrical coordinates (R, ζ, Z). The generalized momentum conjugating to ζ is thus defined by

$$P_\zeta = \frac{\partial L}{\partial \dot{\zeta}} = (\frac{e}{c}\vec{A} + mU\hat{b}) \cdot \frac{\partial \dot{\vec{X}}}{\partial \dot{\zeta}}. \tag{43}$$

Noting that $\vec{X} = R\hat{e}_R + Z\hat{e}_Z$ and

$$\dot{\vec{X}}(R, \zeta, Z) = \frac{\partial \vec{X}}{\partial R}\dot{R} + \frac{\partial \vec{X}}{\partial Z}\dot{Z} + \frac{\partial \vec{X}}{\partial \zeta}\dot{\zeta}, \tag{44}$$

we have

$$\frac{\partial \dot{\vec{X}}}{\partial \dot{\zeta}} = \frac{\partial \vec{X}}{\partial \zeta} = R\hat{e}_\zeta, \tag{45}$$

therefore, P_ζ can be expressed as

$$P_\zeta = \frac{e}{c}A_\zeta R + mU\frac{RB_\zeta}{B} = \frac{e}{c}\psi_p + mU\frac{RB_\zeta}{B}, \tag{46}$$

where we have introduced the poloidal flux $\psi_p = A_\zeta R$.

Guiding-center Motion

Generalized Toroidal Angular Momentum

The total time derivative of P_ζ can be given by the Euler-Lagrange equation of ζ,

$$\dot{P}_\zeta = \frac{\partial L}{\partial \zeta}. \tag{47}$$

In order to calculate the partial derivative of L with respect to ζ, we represent the vectors $\vec{A}$ and $\vec{b}$ in covariant form, and thus obtain

$$\frac{\partial L}{\partial \zeta} = [\frac{e}{c}\frac{\partial A_i}{\partial \zeta} + mU\frac{\partial b_i}{\partial \zeta}]\dot{X}^i - \mu\frac{\partial B}{\partial \zeta} - e\frac{\partial \phi}{\partial \zeta}. \tag{48}$$

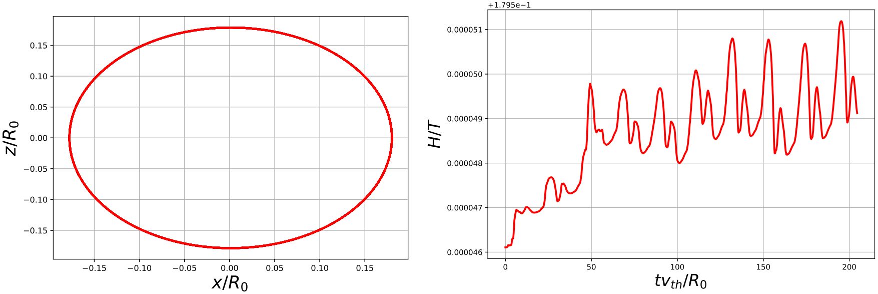
Therefore, $\dot{P}_\zeta = 0$ for the axisymmetrical electromagnetic fields.

Remarks

1. The conservation of P_ζ implies that the deviation of the guiding-center trajectory from the flux surface depends on the sign of U.
2. The broken of axisymmetry corresponds to the guiding center moves to different flux surfaces, i.e., transport phenomena.

Guiding-center Motion

Particle Orbits

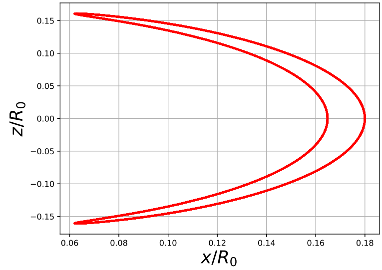


(a) orbit in poloidal plane (b) Hamiltonian

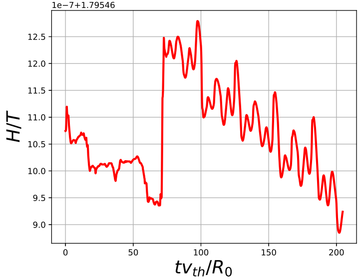
Figure 3: Guiding-center orbit of a circulating particle.

Guiding-center Motion

Particle Orbits



(a) orbit in poloidal plane



(b) Hamiltonian

Figure 4: Guiding-center orbit of a trapped particle.