Quasilinear Theory of Anomalous Transport in Axisymmetric Tokamaks

Stefan Tirkas

University of Colorado, Boulder

July 21, 2020

1 Introduction

2 Quasilinear Theory

3 Tokamak Geometry

For simplicity, we consider an axisymmetric, large aspect-ratio, circular tokamak. This gives the following definition for the equilibrium magnetic field,

$$\boldsymbol{B} = B_\theta \hat{\boldsymbol{e}}_\theta + B_\zeta \hat{\boldsymbol{e}}_\zeta = B_\theta \hat{\boldsymbol{e}}_\theta + B_0(1 - \epsilon \cos\theta)\hat{\boldsymbol{e}}_\zeta = B_0[\frac{\epsilon}{q}\hat{\boldsymbol{e}}_\theta + (1 - \epsilon\cos\theta)\hat{\boldsymbol{e}}_\zeta], \tag{1}$$

where $\epsilon = \frac{r}{R_0} \ll 1$ is the inverse aspect ratio, with $R = R_0 + r\cos\theta$, for r the minor radius, and R_0 the major radius, and q $\simeq \frac{rB_\zeta}{R_0 B_\theta} \sim 1$ is the safety factor[1] - the number of toroidal turns required for one poloidal turn of magnetic field lines. The term $\epsilon\cos\theta$ in R takes into account the change in toroidal radius along the tokamak midplane. Working to $\mathcal{O}(\epsilon)$, the magnetic field magnitude, magnetic field unit vector, and toroidal gradient terms can be written as,

$$B = \sqrt{\boldsymbol{B}\cdot\boldsymbol{B}} = \sqrt{B_0^2[(1-\epsilon\cos\theta)^2 + (\frac{\epsilon}{q})^2]} = B_0\sqrt{1 - 2\epsilon\cos\theta} \simeq B_0(1-\epsilon\cos\theta), \tag{2}$$

$$\hat{\boldsymbol{b}} = \frac{\boldsymbol{B}}{B} = \frac{\frac{\epsilon}{q}\hat{\boldsymbol{e}}_\theta + (1-\epsilon\cos\theta)\hat{\boldsymbol{e}}_\zeta}{1-\epsilon\cos\theta} \simeq \frac{\epsilon}{q}(1+\epsilon\cos\theta)\hat{\boldsymbol{e}}_\theta + \hat{\boldsymbol{e}}_\zeta \simeq \frac{\epsilon}{q}\hat{\boldsymbol{e}}_\theta + \hat{\boldsymbol{e}}_\zeta, \tag{3}$$

$$\nabla = \partial_r \hat{\boldsymbol{e}}_r + \frac{1}{r}\partial_\theta \hat{\boldsymbol{e}}_\theta + \frac{1}{R}\partial_\zeta \hat{\boldsymbol{e}}_\zeta = \partial_r \hat{\boldsymbol{e}}_r + \frac{1}{r}\partial_\theta \hat{\boldsymbol{e}}_\theta + \frac{1}{R_0 + r\cos\theta}\partial_\zeta \hat{\boldsymbol{e}}_\zeta\ . \tag{4}$$

4 Gyrokinetics

Talk about gyrophase-averaging and guiding center coordinates.

4.1 Vlasov Equation

The perturbed, gyrokinetic distribution function is given as a combination of adiabatic and non-adiabatic terms,[2]

$$\delta F = \frac{q}{m}\delta F_a + \delta G, \tag{5}$$

where,

$$\delta F_a = [\delta\Phi\frac{\partial}{\partial\epsilon^*} + (\delta\Phi - \frac{v_\parallel \delta A_\parallel}{c})\frac{\partial}{B\partial\mu}]F_0, \tag{6}$$

$$\delta G_0 = -\frac{q}{m}\langle\delta L\rangle_\alpha \frac{\partial}{B\partial\mu} + \delta H_0, \tag{7}$$

$$\langle\ldots\rangle_\alpha = \frac{1}{2\pi}\int_0^{2\pi}(\ldots)d\alpha, \tag{8}$$

with α as the gyro-phase angle, $\delta L = \delta\Phi - \frac{\boldsymbol{v}\cdot\delta\boldsymbol{A}}{c}$, $\epsilon^* = \frac{v^2}{2} + \frac{q\Phi_0}{m}$, and $\mu = \frac{v_\perp^2}{2B}$. Higher order terms in δG, the perturbed, non-adiabatic distribution function, are dropped. We can simplify things further by choosing for F_0 a Maxwellian equilibrium distribution function, f_M, so that it only depends on ϵ^* and not μ. This gives us a final distribution function,

$$\delta F = \frac{q}{m}\delta\Phi\frac{\partial}{\partial\epsilon^*}f_M + \delta H_0 \ . \tag{9}$$

This distribution function can be plugged into the Vlasov equation and gyrophase-averaged to give the standard gyrokinetic Vlasov equation for a species j,[2]

$$\begin{aligned} &\partial_t\delta H_0 + v_\parallel\nabla_{X_\parallel}\delta H_0 + (\boldsymbol{v}_d + \frac{c\hat{\boldsymbol{b}}\times\nabla_X\langle\delta\Phi\rangle_\alpha}{B})\cdot\nabla_X\delta H_0 \\ &= -\frac{e_j}{m_j}[\partial_t\langle\delta\Phi\rangle_\alpha\partial_{\epsilon^*}f_M - \frac{1}{\omega_{cj}}(\nabla_X\langle\delta\Phi\rangle_\alpha\times\hat{\boldsymbol{b}})\cdot\nabla_X f_M], \end{aligned} \tag{10}$$

where $\boldsymbol{v}_d$, the sum of magnetic curvature and gradient drift terms, is defined as,

$$\boldsymbol{v}_d = \frac{v_\parallel^2 + \frac{1}{2}v_\perp^2}{\omega_{cj}}\frac{\boldsymbol{B}\times\nabla B}{B^2}, \tag{11}$$

with, simplifying to lowest order in ϵ,

$$\begin{aligned} \frac{\boldsymbol{B}\times\nabla B}{B^2} &= \frac{B_0[(1-\epsilon\cos\theta)\hat{\boldsymbol{e}}_\zeta + \frac{\epsilon}{q}\hat{\boldsymbol{e}}_\theta]\times(\partial_r\hat{\boldsymbol{e}}_r + \frac{1}{r}\partial_\theta\hat{\boldsymbol{e}}_\theta + \frac{1}{R}\partial_\zeta\hat{\boldsymbol{e}}_\zeta)B_0(1-\epsilon\cos\theta)}{B_0^2(1-\epsilon\cos\theta)^2} \\ &= \frac{[1-\epsilon\cos\theta)\hat{\boldsymbol{e}}_\zeta + \frac{\epsilon}{q}\hat{\boldsymbol{e}}_\theta]\times[-\frac{1}{R_0}\cos\theta\hat{\boldsymbol{e}}_r + \frac{r}{rR_0}\sin\theta\hat{\boldsymbol{e}}_\theta]}{(1-\epsilon\cos\theta)^2} \\ &= \frac{1}{(1-\epsilon\cos\theta)^2}[-\frac{(1-\epsilon\cos\theta)\cos\theta}{R_0}(\hat{\boldsymbol{e}}_\zeta\times\hat{\boldsymbol{e}}_r) - \frac{(1-\epsilon\cos\theta)\cos\theta}{R_0}(\hat{\boldsymbol{e}}_\zeta\times\hat{\boldsymbol{e}}_\theta) \\ &\qquad - \frac{\epsilon}{qR_0}\cos\theta(\hat{\boldsymbol{e}}_\theta\times\hat{\boldsymbol{e}}_r)] \\ &\simeq (1+2\epsilon\cos\theta)[-\frac{\cos\theta}{R_0}\hat{\boldsymbol{e}}_\theta - \frac{\sin\theta}{R_0}\hat{\boldsymbol{e}}_r] \simeq -\frac{1}{R_0}(\sin\theta\hat{\boldsymbol{e}}_r + \cos\theta\hat{\boldsymbol{e}}_\theta) \end{aligned} \tag{12}$$

The second and third terms on the left-hand side of (10) can be simplified to lowest order in ϵ using (1)-(4) and (11)-(12) as,

$$
\begin{aligned}
v_\parallel \nabla_{X_\parallel} \simeq v_\parallel \nabla_\parallel = v_\parallel(\hat{\boldsymbol{b}} \cdot \nabla) &= v_\parallel(\frac{\epsilon}{q}\hat{\boldsymbol{e}}_\theta + \hat{\boldsymbol{e}}_\zeta) \cdot (\partial_r \hat{\boldsymbol{e}}_r + \frac{1}{r}\partial_\theta \hat{\boldsymbol{e}}_\theta + \frac{1}{R_0 + r\cos\theta}\partial_\zeta \hat{\boldsymbol{e}}_\zeta) \\
&= v_\parallel(\frac{\epsilon}{qr}\partial_\theta + \frac{1}{R}\partial_\zeta) = v_\parallel(\frac{1}{qR_0}\partial_\theta + \frac{1}{R}\partial_\zeta) = \frac{v_\parallel}{qR}(\frac{R}{R_0}\partial_\theta + q\partial_\zeta) \\
&= \frac{v_\parallel}{qR}((1+\epsilon\cos\theta)\partial_\theta + q\partial_\zeta) \simeq \frac{v_\parallel}{qR}(\partial_\theta + q\partial_\zeta) = v_\parallel \frac{\partial}{\partial l},
\end{aligned} \tag{13}
$$

$$
\begin{aligned}
\boldsymbol{v}_d \cdot \nabla_X \simeq \boldsymbol{v}_d \cdot \nabla &= -\frac{v_\parallel^2 + \frac{1}{2}v_\perp^2}{\omega_{cj}}(\sin\theta \hat{\boldsymbol{e}}_r + \cos\theta \hat{\boldsymbol{e}}_\theta) \cdot (\partial_r \hat{\boldsymbol{e}}_r + \frac{1}{r}\partial_\theta \hat{\boldsymbol{e}}_\theta) \\
&= -\frac{v_\parallel^2 + \frac{1}{2}v_\perp^2}{\omega_{cj} R_0}(\sin\theta \partial_r + \frac{\cos\theta}{r}\partial_\theta),
\end{aligned} \tag{14}
$$

with l being the infinitesimal length along the field lines. Note that we have dropped the non-linear $\boldsymbol{E} \times \boldsymbol{B}$ drift term on the left-hand side of (10) because we are interested in linearizing this equation.

4.2 Non-Adiabatic Distribution Function

Using a WKB ansatz the gyrophase-averaged terms can be simplified as,

$$
\langle A(\boldsymbol{x}) \rangle_\alpha = J_0(\sqrt{m_j}\frac{k_\perp v_{\perp j}}{\omega_{cj}})A(\boldsymbol{X}) = J_0(z_j)A(\boldsymbol{X}) \, . \tag{15}
$$

Then, simplifying (10) using (13)-(15) and taking the Fourier transform, gives,

$$
\begin{aligned}
&-i\omega\delta\widetilde{H} + v_\parallel \frac{\partial}{\partial l}\delta\widetilde{H} + i\boldsymbol{v}_d \cdot \boldsymbol{k}_X \delta\widetilde{H} = -\frac{e_j}{m_j}[-i\omega J_0(z_j)\delta\widetilde{\Phi}\partial_{\epsilon^*} f_M \\
&-\frac{i}{\omega_{cj}}J_0(z_j)\delta\widetilde{\Phi}(\boldsymbol{k} \times \hat{\boldsymbol{b}}) \cdot \frac{d}{dr} f_M \hat{\boldsymbol{e}}_r] = i\frac{e_j}{m_j}J_0(z_j)\delta\widetilde{\Phi}[\omega\partial_{\epsilon^*} + \frac{k_\theta}{\omega_{cj}}\frac{d}{dr}]f_M,
\end{aligned} \tag{16}
$$

$$
\Rightarrow \ (v_\parallel \partial_l - i(\omega - \bar{\omega}_{dj}))\delta\widetilde{H} = i\frac{e_j}{m_j}J_0(z_j)\delta\widetilde{\Phi}[\omega\partial_{\epsilon^*} + \frac{k_\theta}{\omega_{cj}}\frac{d}{dr}]f_M, \tag{17}
$$

with the following definitions,

$$
\begin{aligned}
\bar{\omega}_{dj} = -\boldsymbol{v}_d \cdot \boldsymbol{k} &= \frac{k_\theta(v_\parallel^2 + \frac{1}{2}v_\perp^2)}{\omega_{cj} R_0}(\cos\theta + \frac{k_r}{k_\theta}\sin\theta) \\
&= \frac{\omega_{dj}}{2}(\left(\frac{v_\parallel}{v_{Tj}}\right)^2 + \frac{1}{2}\left(\frac{v_\perp}{v_{Tj}}\right)^2)(\cos\theta + \frac{k_r}{k_\theta}\sin\theta),
\end{aligned} \tag{18}
$$

$$
\boldsymbol{k} \times \hat{\boldsymbol{b}} = (k_r \hat{\boldsymbol{e}}_r + k_\theta \hat{\boldsymbol{e}}_\theta + k_\zeta \hat{\boldsymbol{e}}_\zeta) \times (\frac{\epsilon}{q}\hat{\boldsymbol{e}}_\theta + \hat{\boldsymbol{e}}_\zeta) \simeq -k_r \hat{\boldsymbol{e}}_\theta + k_\theta \hat{\boldsymbol{e}}_r, \tag{19}
$$

where ω_{dj} represents the magnetic curvature drift frequency,

$$
\omega_{dj} = 2\frac{n}{dn/dr}\frac{\omega_{*j}}{R_0}, \tag{20}
$$

and ω_{*j} the diamagnetic drift frequency,

$$
\omega_{*j} = \frac{k_\theta T_j}{q_j B}\frac{1}{n}\frac{dn}{dr} \, . \tag{21}
$$

Note that f_M depends only on r due to it being a function of $n_j(r)$ and $T_j(r)$ which are only changing across the circular flux-surfaces - therefore only functions of radius - and that $\nabla_X \simeq \nabla$

to lowest order for the perturbed distribution functions and potential. Next, we can simplify further by plugging in the definition of the Maxwellian, with $k_B T_j \Rightarrow T_j$,

$$f_M(r) = n_j(r)\left(\frac{m_j}{2\pi T_j(r)}\right)^{3/2} e^{-\frac{m_j\epsilon^*}{T_j(r)}}, \tag{22}$$

giving the following values for derivatives,

$$\partial_{\epsilon^*} f_M = -\frac{m_j}{T_j} n_j \left(\frac{m_j}{2\pi T_j}\right)^{3/2} e^{-\frac{m_j\epsilon^*}{T_j(r)}} = -\frac{m_j}{T_j} f_M, \tag{23}$$

$$\begin{aligned}
\frac{d}{dr} f_M &= \frac{dn}{dr}\frac{d}{dn} f_M + \frac{dT}{dr}\frac{d}{dT} f_M = \frac{dn}{dr} e^{-\frac{m_j\epsilon^*}{T_j(r)}} + \frac{dT}{dr}\frac{du}{dT}\frac{d}{du}[n_j \left(\frac{m_j u(T(r))}{2\pi}\right)^{3/2} e^{-m_j\epsilon^* u(T_j(r))}] \\
&= \frac{1}{n}\frac{dn}{dr} f_M + \frac{dT}{dr}\frac{du}{dT} n_j [\frac{3}{2}\left(\frac{m_j u}{2\pi}\right)^{1/2}\left(\frac{m_j}{2\pi}\right) - m_j\epsilon^* \left(\frac{m_j u}{2\pi}\right)^{3/2}] e^{-m_j\epsilon^* u(T_j(r))} \\
&= \frac{1}{n}\frac{dn}{dr} f_M + \frac{dT}{dr}\frac{du}{dT}[\frac{3}{2}u^{-1} - m_j\epsilon^*] f_M = [\frac{1}{n}\frac{dn}{dr} + \frac{dT}{dr}(-\frac{1}{T^2})(\frac{3}{2}u^{-1} - m_j\epsilon^*)] f_M \\
&= [\frac{1}{n}\frac{dn}{dr} - \frac{1}{T}\frac{dT}{dr}(\frac{3}{2} - \frac{m_j v^2}{2T_j})] f_M = [\frac{1}{n}\frac{dn}{dr} - \frac{1}{T}\frac{dT}{dr}(\frac{3}{2} - \frac{v^2}{2v_{Tj}^2})] f_M \\
&= \frac{1}{n}\frac{dn}{dr}[1 + (\frac{1}{2}\left(\frac{v}{v_{Tj}}\right)^2 - \frac{3}{2})\eta_j] f_M,
\end{aligned} \tag{24}$$

given that $u = T^{-1}$, $du = -T^{-2} dT$, $v_{Tj} = \sqrt{\frac{T_j}{m_j}}$, and $\eta_j = \frac{n}{T}\frac{dT}{dn}$. Putting these derivatives into (17) then gives,

$$\begin{aligned}
&(v_\parallel \partial_l - i(\omega - \bar{\omega}_{dj}))\delta\widetilde{H} = i\frac{e_j}{m_j} J_0(z_j)\delta\widetilde{\Phi}[-\frac{m_j}{T_j}\omega + \frac{k_\theta}{\omega_{cj}}\frac{1}{n}\frac{dn}{dr}[1 + (\frac{1}{2}\left(\frac{v}{v_{Tj}}\right)^2 - \frac{3}{2})\eta_j]] f_M \\
\Rightarrow\ &(v_\parallel \partial_l - i(\omega - \bar{\omega}_{dj}))\delta\widetilde{H} = -i\frac{e_j}{T_j} J_0(z_j)\delta\widetilde{\Phi}[\omega - \frac{k_\theta T_j m_j}{q_j B m_j}\frac{1}{n}\frac{dn}{dr}[1 + (\frac{1}{2}\left(\frac{v}{v_{Tj}}\right)^2 - \frac{3}{2})\eta_j]] f_M \\
\Rightarrow\ &(-v_\parallel \partial_l + i(\omega - \bar{\omega}_{dj}))\delta\widetilde{H} = i\frac{e_j}{T_j} J_0(z_j)\delta\widetilde{\Phi}[\omega - \omega_{*j}[1 + (\frac{1}{2}\left(\frac{v}{v_{Tj}}\right)^2 - \frac{3}{2})\eta_j]] f_M
\end{aligned} \tag{25}$$

$$\Rightarrow\ (-v_\parallel \partial_l + i(\omega - \bar{\omega}_{dj}))\delta\widetilde{H} = i\frac{e_j}{T_j} J_0(z_j)\delta\widetilde{\Phi}[\omega - \omega_{*j}^T] f_M, \tag{26}$$

with the following definition,

$$\omega_{*j}^T = \omega_{*j}[1 + (\frac{1}{2}\left(\frac{v}{v_{Tj}}\right)^2 - \frac{3}{2})\eta_j]\ . \tag{27}$$

Then (26) can be simplified further and solved for $\delta\widetilde{H}$ if a few physical assumptions are made for electron drift and ITG modes. First off, trapped particle effects are neglected, i.e.,

$$\omega \gg \omega_{bj}, \tag{28}$$

for ω_{bj} as the trapped particle frequency. We also assume electrons move rapidly in response to the electrostatic potential,

$$k_\parallel v_{Te} \gg \omega \gg k_\parallel v_{Ti}, \tag{29}$$

noting that $v_{Tj} \simeq v_{\parallel j}$. Finally we assume that the frequency associated with magnetic drifts of ions is much smaller than that of the perturbed modes,

$$\omega \gg \omega_{di} \ . \tag{30}$$

Now, (29) and (30) allow the left-hand side of (26) to be rewritten for ions as,

$$i(\omega - \bar{\omega}_{di})(1 + i\frac{v_\parallel}{\omega - \omega_{di}}\frac{\partial}{\partial l}) \simeq i(\omega - \bar{\omega}_{di})(1 + \frac{k_\parallel v_{\parallel i}}{\omega}) \simeq i(\omega - \bar{\omega}_{di}) \ . \tag{31}$$

Finally we can solve for $\delta\widetilde{H}$ explicitly by replacing the left-hand side of (26) with (31), giving the perturbed, non-adiabatic distribution function for ions,

$$\delta\widetilde{H} = \frac{e_j}{T_j} J_0(z_j)\delta\widetilde{\Phi} f_M \frac{\omega - \omega_{*i}^T}{\omega - \bar{\omega}_{di}} \ . \tag{32}$$

References

[1] Wesson 2004

[2] Frieman, Chen 1982