Electrostatic Gyrokinetic Equation and Radial Particle Flux

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Gyrokinetic theory is a basic tool to describe the low-frequency ($|\omega| \ll |\omega_c|$) phenomena in magnetically confined plasmas. In this manuscript, we summarize the classical nonlinear gyrokinetic equations.

1 Gyrokinetic Ordering

We first list the ordering of spatiotemporal scales and fluctuation strength. The relative fluctuation levels are estimated by the ordering

$$|\frac{\delta F}{F}| \sim |\frac{Ze\delta\Phi}{T}| \sim |\frac{\delta B}{B}| \sim \mathcal{O}(\epsilon). \tag{1}$$

We adopt the following ordering for the scales of microscopic fluctuations

$$|\frac{\partial_t}{\omega_c}| \sim \mathcal{O}(\epsilon), \quad |\rho_c \nabla_\parallel| \sim \mathcal{O}(\epsilon), \quad |\vec{\rho}_c \cdot \nabla_\perp| \sim \mathcal{O}(1). \tag{2}$$

The scales of macroscopic quantities, e.g., B_0 and F_0, are

$$|\frac{\partial_t}{\omega_c}| \sim \mathcal{O}(\epsilon^3), \quad |\vec{\rho}_c \cdot \nabla| \sim \mathcal{O}(\epsilon). \tag{3}$$

2 Toroidal Geometry

For simplicity, we consider an axisymmetric, low-β, large aspect-ratio tokamak with concentric circular magnetic surfaces, with the usual right-handed flux coordinate system (r, θ, ζ), corresponding to the minor radius, poloidal and toroidal angle, respectively. We therefore may use the $s - \alpha$ model with $\alpha = 0$ for the background magnetic field: The equilibrium magnetic field is given by $\vec{B} = B_0[(1-\epsilon\cos\theta)\hat{e}_\zeta + \epsilon/q\hat{e}_\theta]$, with $\epsilon = r/R_0 \ll 1$ and R_0 the major radius. Specifically, one has

$$\begin{aligned} R &= R_0 + r\cos\theta, \\ \zeta_c &= \frac{\pi}{2} - \zeta, \\ z &= r\sin\theta. \end{aligned} \tag{4}$$

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3 Frieman-Chen equation

The fluctuating distribution function can be decomposed into

$$\delta F = \underbrace{\frac{e_j}{m_j}\delta\Phi\partial_\varepsilon F_0}_{\text{particle adiabatic}} + \underbrace{\delta H(\vec{X},\varepsilon,\mu,\sigma,\xi,t)}_{\text{particle non-adiabatic}}, \tag{5}$$

assuming the background distribution to be Maxwellian.

By taking the usual gyrokinetic ordering, one can derive the nonlinear gyrokinetic equation in electrostatic limit as

$$\begin{aligned} & \partial_t\delta H + v_\parallel \nabla_{X_\parallel}\delta H + (\vec{v}_d + \frac{c}{B}\hat{b}\times\nabla_X\langle\delta\Phi\rangle_\xi)\cdot\nabla_X\delta H \\ = & -\frac{e_j}{m_j}[\partial_t\langle\delta\Phi\rangle_\xi\partial_\varepsilon F_0 - \frac{1}{\omega_{cj}}(\nabla_X\langle\delta\Phi\rangle_\xi\times\hat{b})\cdot\nabla_X F_0]. \end{aligned} \tag{6}$$

Here,

$$\vec{v}_d = \hat{b}\times[\frac{v_\parallel^2}{\omega_{cj}}\hat{b}\cdot\nabla\hat{b} + \frac{1}{2}\frac{v_\perp^2}{\omega_{cj}}\nabla(\ln B)], \tag{7}$$

is the magnetic drift velocity. j denotes the particle species, and $\langle A\rangle_\xi$ is the gyrophase average, i.e.,

$$A(\vec{x}) = A(\vec{X}+\vec{\rho}_c) = e^{\vec{\rho}\cdot\nabla_X}A(\vec{X}) = e^{-i\sin\xi\hat{\Lambda}}A(\vec{X}) = \sum_{n=-\infty}^{+\infty} J_n(\hat{\Lambda})A(\vec{X})e^{-in\xi}, \tag{8}$$

with J_n being the n-th order Bessel function, so one obtains

$$\langle A(\vec{x})\rangle = J_0(\hat{\Lambda})A(\vec{X}), \tag{9}$$

with $\hat{\Lambda}^2 = -(2\mu B/\omega_c^2)\nabla_\perp^2$.

In toroidal geometry, the gradient operators can be written as

$$\nabla = \hat{e}_r\frac{\partial}{\partial r} + \frac{\hat{e}_\theta}{r}\frac{\partial}{\partial\theta} + \frac{\hat{e}_\zeta}{R_0 + r\cos\theta}\frac{\partial}{\partial\zeta}, \tag{10}$$

$$v_\parallel\nabla_{X_\parallel} = \frac{v_\parallel}{qR}(\partial_\theta + q\partial_\zeta), \tag{11}$$

and

$$\vec{v}_d\cdot\nabla_X = -\frac{v_\parallel^2 + \frac{1}{2}v_\perp^2}{\omega_{cj}R_0}(\sin\theta\partial_r + \frac{\cos\theta}{r}\partial_\theta). \tag{12}$$

4 Radial Particle Flux

In the electrostatic limit, the particle flux of j-species can be calculated simply by its definition, yielding

$$\begin{aligned} \Gamma_{jr} &= -\frac{c}{B_0}\langle\langle(\frac{\partial_\theta}{r}\delta\phi)\delta F_j\rangle_s\rangle_v \\ &= -\frac{c}{B_0}\langle\langle[(\frac{\partial_\theta}{r}\delta\phi^*)\delta F_j + (\frac{\partial_\theta}{r}\delta\phi)\delta F_j^*]\rangle_s\rangle_v \\ &= \frac{ick_\theta}{B_0}\langle\langle(\delta\phi^*\delta F_j - \delta\phi\delta F_j^*)\rangle_s\rangle_v, \end{aligned} \tag{13}$$

where $\langle A \rangle_s$ denotes the magnetic surface average, $\langle A \rangle_v$ is the velocity integration. Noting the quasineutrality condtion and assuming j denotes ion species,

$$\sum_j Z_j \langle\langle \delta F_j \rangle_s \rangle_v = \langle\langle \delta F_e \rangle_s \rangle_v, \tag{14}$$

Eq.(13) implies that

$$\begin{aligned} \sum_j Z_j \Gamma_{jr} &= \frac{ick_\theta}{B_0} \sum_j Z_j \langle\langle(\delta\phi^* \delta F_j - \delta\phi \delta F_j^*)\rangle_s\rangle_v \\ &= \frac{ick_\theta}{B_0} \langle\langle(\delta\phi^* \delta F_e - \delta\phi \delta F_e^*)\rangle_s\rangle_v, \end{aligned} \tag{15}$$

by using the decomposition Eq.(5), Eq.(15) can be further cast into

$$\sum_j Z_j \Gamma_{jr} = \frac{ick_\theta}{B_0} \langle\langle(\delta\phi^* \delta H_e - \delta\phi \delta H_e^*)\rangle_s\rangle_v, \tag{16}$$

therefore we have $\sum_j Z_j \Gamma_{jr} = 0$ in the adiabatic electron limit. It is also worthwhile noting that, here we have not adopted the quasilinear or any other specific model, the discussion is generally valid.