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Citation: Physics of Fluids B: Plasma Physics **2**, 2563 (1990); doi: 10.1063/1.859380
View online: https://doi.org/10.1063/1.859380
View Table of Contents: http://aip.scitation.org/toc/pfb/2/11
Published by the American Institute of Physics

Canonical coordinates for guiding center particles

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(Received 21 September 1989; accepted 29 June 1990)

Canonical coordinates are obtained for the guiding center Hamiltonian up to a quadrature. The configuration variables are two angles, and the canonical momenta are angular momenta. These coordinates are valid for arbitrary magnetic fields including nonaxisymmetric ones whose field lines may be chaotic. Examples of application of canonical coordinates to tokamak geometry are given.

I. INTRODUCTION

Littlejohn[1] has shown that a Hamiltonian treatment of guiding center motion requires the inclusion of additional terms in the guiding center equations. In the equations of motion these terms are higher order in the gyroradius, and in a strict ordering would be neglected; however, their retention has several advantages: (1) When the fields are time independent energy is explicitly conserved; (2) there is a conserved phase space volume in a Hamiltonian flow; and (3) a Hamiltonian formulation gives an efficient method for finding conserved quantities.

Littlejohn's technique is to expand the single particle Lagrangian in coordinates appropriate for guiding center motion; this is more straightforward than a direct Hamiltonian approach since the variational treatment is covariant: any convenient set of coordinates can be used. When the subsequent transformation to Hamiltonian form is obtained, the most natural variables are not canonical. In this paper we obtain a set of coordinates that are explicitly canonical.

There are advantages to canonical variables. First, the conserved phase space volume element is trivial, $d^2p\, d^2q$; in a noncanonical frame a Jacobian is involved. Second, the equations of motion follow easily from the Hamiltonian in canonical variables; for the noncanonical case an often complicated Poisson bracket must be used. Finally, perturbation theory is especially easy in canonical variables since only one function, H, must be computed in the new variables.

It will be seen that the crux of finding canonical coordinates in phase space lies in proper choice of spatial coordinates, which must be suited to the geometry of the magnetic field. We use the term "canonical field coordinates" to describe the three spatial coordinates that are compatible with canonical phase-space coordinates. Finding canonical field coordinates is equivalent to finding a coordinate system in which one covariant component of both $\mathbf{B}$ and $\mathbf{A}$ vanishes, say $B_r = A_r = 0$. This condition is not, in general, satisfied by an arbitrary set of flux or "straight-field-line" coordinates. We show in Sec. II that such a coordinate system can always be constructed when the field is nonzero. Canonical field coordinates are essentially unique, and conceivably useful in other, non-Hamiltonian contexts.

Some of the issues to be treated in Sec. II have been considered in previous literature. In particular, the guiding center reduction of phase space, from six to four dimensions, is discussed by both Littlejohn[1] and Boozer.[2] However, to our knowledge, canonical field coordinates have not previously been obtained in their general form; they have been obtained previously for special cases. Notably, White, Boozer, and Hay[3] assume a special representation for $\mathbf{B}$ that permits chaotic field lines, but that is not completely general. Boozer[2] obtains canonical coordinates by assuming that B_r can be neglected, a situation that applies only in special circumstances. On the other hand, Boozer has argued that his equations can give trajectories that remain close (within a gyroradius) to the exact guiding center trajectories, even if the radial field is of order unity. We discuss this further below.

It should be emphasized that our argument applies to any toroidal magnetic field: we not only allow for arbitrary aspect ratio, but for arbitrary asymmetry. Most importantly, to permit chaotic field applications we do not assume that the field has well-behaved flux surfaces.

II. CANONICAL COORDINATES

The guiding center equations can be derived from the single particle Lagrangian,

$$L(\mathbf{q},\dot{\mathbf{q}},t) = \tfrac{1}{2}m\dot{\mathbf{q}}^2 + (e/c)\dot{\mathbf{q}}\cdot\mathbf{A}(\mathbf{q},t) - e\Phi(\mathbf{q},t), \qquad (1)$$

by expanding in the ratio of the gyroradius to a typical scale length for field variations. To lowest order the particle gyrates in the magnetic field, $\mathbf{B} = \nabla\times\mathbf{A}$, preserving its magnetic moment, $\mu \equiv \frac{1}{2}mv_\perp^2/B$ (where $\mathbf{v}_\perp = \mathbf{v} - \mathbf{b}\cdot\mathbf{v}$ and $\mathbf{b} = \mathbf{B}/B$). After averaging over the gyration, the approximate Lagrangian can be written as[1]

$$L(\mathbf{x},\dot{\mathbf{x}},t) = \tfrac{1}{2}mu(\mathbf{x},\dot{\mathbf{x}},t)^2 + (e/c)\dot{\mathbf{x}}\cdot A(\mathbf{x},t) - e\Phi(\mathbf{x},t) - \mu B(\mathbf{x},t), \qquad (2)$$

where $\mathbf{x}$ is the position of the guiding center, $\dot{\mathbf{x}}$ is its velocity, and u denotes the parallel velocity $\mathbf{b}\cdot\dot{\mathbf{x}}$. This Lagrangian describes motion of the guiding center along the magnetic field, including the mirror force, and yields all the conventional drifts perpendicular to the field. Equation (2) is also similar to the Lagrangian suggested, but not derived, by Taylor.[4]

As in any Lagrangian system, the variational principle using (2) is completely covariant, and any convenient set of variables can be used to describe the motion.

The Euler–Lagrange equations obtained from (2) are degenerate since the Lagrangian depends only linearly on the components of the guiding center velocity perpendicular

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 0899-8221/90/112563-05$02.00

to $\mathbf{B}$. This implies that the equations of motion in these perpendicular directions are first order in time, and serve merely to define the perpendicular drift velocity in terms of the guiding center position.

When we attempt to construct a Hamiltonian description, the same degeneracy is seen through the definition of the momenta:

$$\mathbf{p} = \frac{\partial L}{\partial \dot{\mathbf{x}}} = mu\mathbf{b} + (e/c)\mathbf{A}. \tag{3}$$

This implies that the two components of $\mathbf{p}$ perpendicular to $\mathbf{b}$ are simply functions of $\mathbf{x}$, through the vector potential. Thus, although the Hamiltonian,

$$H(\mathbf{p},\mathbf{x},t) = p_i \frac{dx^i}{dt} - L(\mathbf{x},\dot{\mathbf{x}},t), \tag{4}$$

appears to depend on six phase-space coordinates, only four are independent. The four-dimensional phase space of independent coordinates is called the "reduced" phase space.

We wish to obtain a set canonical coordinates in the reduced phase space: two configuration coordinates and two canonically conjugate momenta. As Littlejohn has pointed out, Darboux's theorem implies that such canonical variables always exist locally. We will show a physically reasonable set of canonical variables, valid globally (in most cases) can be obtained. The main burden of this construction is to find a coordinate system in which both the vectors $\mathbf{B}$ and $\mathbf{A}$ have only two covariant components.

We first show why such a coordinate system is useful. Recall that in a general coordinate system (x^1,x^2,x^3), any vector can be expanded in the covariant basis

$$\mathbf{B} = B_i\, \nabla x^i,$$

where B_i are the covariant components and summation over repeated indices is implied. Suppose that a coordinate system (r,θ,ζ) exists such that $B_r = A_r = 0$:

$$\begin{aligned}\mathbf{A} &= A_\theta\, \nabla\theta + A_\zeta\, \nabla\zeta,\\ \mathbf{B} &= B_\theta\, \nabla\theta + B_\zeta\, \nabla\zeta.\end{aligned} \tag{5}$$

Then (3) implies that $p_r \equiv 0$ and

$$\begin{aligned}p_\theta &= mub_\theta + (e/c)A_\theta,\\ p_\zeta &= mub_\zeta + (e/c)A_\zeta.\end{aligned} \tag{6}$$

These two equations for $p(r,\theta,\zeta,u,t)$ can be inverted (at least formally) to obtain

$$\begin{aligned}r &= r(p_\theta,p_\zeta,\theta,\zeta,t),\\ u &= u(p_\theta,p_\zeta,\theta,\zeta,t),\end{aligned} \tag{7}$$

providing the Jacobian of the transformation does not vanish. Upon substituting these equations into (4), the remaining terms involving $\dot{\mathbf{x}}$ cancel because of the degeneracy of L, yielding

$$H(p_\theta,p_\zeta,\theta,\zeta,t) = \tfrac{1}{2}mu^2 + e\Phi + \mu B, \tag{8}$$

where u and r are implicitly functions of the canonical variables through (7). Of course, the form of Eq. (8), the particle energy, is not surprising; what is important is that it has been expressed in terms of four canonical coordinates, as shown by (7).

As our notation indicates, we will show that the configuration variables (θ,ζ) can be taken to be angle variables, and $r(p_\theta,p_\zeta,\theta,\zeta,t)$ is a radial variable. Thus the coordinate system is toroidal, and p_θ and p_ζ are canonical angular momenta.

Next we give constructive proof that the required coordinate system exists.

Lemma 1: Let $\mathbf{B}$ be a non-null vector field that is differentiable (actually only Lipschitz is required) and $(\bar{r},\bar{\theta},\bar{\zeta})$ a set of toroidal coordinates. There exists a new set of toroidal coordinates $[r = \bar{r},\ \theta(\bar{r},\bar{\theta},\bar{\zeta}),\ \zeta = \bar{\zeta}]$ such that $B_r = 0$.

Proof: In the new coordinate system (r,θ,ζ) the radial component is given by

$$B_r = B_{\bar{r}} \frac{\partial \bar{r}}{\partial r} + B_{\bar{\theta}} \frac{\partial \bar{\theta}}{\partial r} + B_{\bar{\zeta}} \frac{\partial \bar{\zeta}}{\partial r}. \tag{9}$$

The object is to find (r,θ,ζ) so that $B_r = 0$. Alternatively, to solve this we must find a vector $\mathbf{V}$ such that $\mathbf{V}\cdot\mathbf{B} = 0$, and for which

$$\frac{\partial \bar{x}^i}{\partial r} = V^i(\bar{r},\bar{\theta},\bar{\zeta}). \tag{10}$$

Here we can think of r as parametrizing the field lines of $\mathbf{V}$. The solution of this set of coupled differential equations is the required coordinate transformation $\bar{x}^i \to x^i$.

It is perhaps surprising that Eqs. (10) can be integrated, since most sets of three coupled differential equations cannot; in particular, the equations for the field lines,

$$\frac{dx^i}{ds} = B^i,$$

are not typically integrable: magnetic field lines can be chaotic. However, there is considerable freedom in the choice of $\mathbf{V}$, which can be exploited to make (10) integrable.

Suppose first that $B_{\bar{\theta}}$ is never zero and choose

$$V^{\bar{r}} = 1, \quad V^{\bar{\theta}} = -B_{\bar{r}}/B_{\bar{\theta}}, \quad V^{\bar{\zeta}} = 0, \tag{11}$$

which obviously satisfies $\mathbf{V}\cdot\mathbf{B} = 0$. In this case the solution of (10) for $\bar{r}$ and $\bar{\zeta}$ can be taken to be the trivial solutions

$$\begin{aligned}\bar{\zeta}(r,\theta,\zeta) &= \zeta,\\ \bar{r}(r,\theta,\zeta) &= r.\end{aligned} \tag{12}$$

The remaining equation becomes

$$B_{\bar{\theta}}(r,\bar{\theta},\zeta) \frac{d\bar{\theta}}{dr} + B_{\bar{r}}(r,\bar{\theta},\zeta) = 0, \tag{13}$$

where ζ can be treated as a parameter. In general, (13) cannot be explicitly integrated, but the solution could easily be obtained numerically.

To solve for $\bar{\theta}$, choose the initial condition $\bar{\theta}(r_0,\theta,\zeta) = \theta$ at some arbitrary point r_0. Since the field is Lipschitz, the solution is unique and for each value of r there is a one-to-one relationship between $\bar{\theta}$ and θ. Furthermore, since the fields are periodic functions of θ and ζ, the solution for θ can be written in the form

$$\theta = \bar{\theta} + P(\bar{r},\bar{\theta},\bar{\zeta}), \tag{14}$$

where P is periodic in the angles, and $|\partial P/\partial\bar{\theta}| < 1$. Thus $(\bar{r},\bar{\theta},\bar{\zeta}) \to (r,\theta,\zeta)$ is a good coordinate transformation, which preserves toroidal topology.

Finally, we must consider what happens if $B_{\bar{\theta}}$ has zeros. In this case we first find a new set of coordinates, for example, $(\hat{r},\hat{\theta},\hat{\zeta}) = (\bar{r},\bar{\theta},\bar{\theta} + \omega\bar{\zeta})$, such that the covariant component $B_{\bar\theta}$ does not vanish. This can always be done if the field has no nulls. Once this is accomplished we can proceed as above. □

Note that the transformation (14) that yields the coordinate system in which $B_r = 0$ implies that the required poloidal angle $\bar\theta$ deviates from θ by a quantity of order $B_{\bar r}/B$, which is bounded and may be small in applications. In particular, Boozer[2] uses a similar argument to imply that the neglect of $B_{\bar r}$ will give only nonsecular errors in the guiding center orbits. Here we are concerned more with constructing canonical coordinates in principle, than whether the coordinates can be made canonical by the neglect of unimportant terms.

Lemma 2: Let **A** be the vector potential for **B**. Then in the same coordinate system (r,θ,ζ) as Lemma 1, we can choose a gauge such that $A_r = 0$.

Proof: In general, **A** has three covariant components,

$$\mathbf{A} = A_r \nabla r + A_\theta \nabla\theta + A_\zeta \nabla\zeta. \tag{15}$$

Define

$$\eta = \int^r dr' A_r(r',\theta,\zeta), \quad \psi = A_\theta - \frac{\partial\eta}{\partial\theta}, \quad \chi = -A_\zeta + \frac{\partial\eta}{\partial\zeta}. \tag{16}$$

Then the vector potential is

$$\mathbf{A} = \psi \nabla\theta - \chi\nabla\zeta + \nabla\eta. \tag{17}$$

The last term can be eliminated by a gauge transformation. □

In this coordinate system

$$\mathbf{B} = \nabla\psi\times\nabla\theta - \nabla\chi\times\nabla\zeta, \tag{18}$$

$$= B_\theta \nabla\theta + B_\zeta \nabla\zeta. \tag{19}$$

We call "canonical magnetic coordinates" any system where the covariant radial component of both B and A vanishes:

$$B_r = A_r = 0.$$

The two lemmas show that canonical coordinates exist for any non-null magnetic field. In canonical magnetic coordinates, the canonical momenta are

$$p_\theta = mub_\theta + (e/c)\psi, \quad p_\zeta = mub_\zeta - (e/c)\chi. \tag{20}$$

In general, canonical coordinates are not flux (straight-field-line) coordinates, because in the latter χ must be a function only of ψ. The point is that flux coordinates, like flux surfaces, do not exist in the general case. However, in the canonical coordinate system both ψ and χ are permitted to be arbitrary functions of (r,θ,ζ) as well as time. On the other hand, when flux surfaces exist it is clear that canonical coordinates are a special case of flux coordinates.

III. EXAMPLES

A. Large aspect ratio tokamak geometry

The equilibrium configuration of a large aspect ratio tokamak, with circular limiter and relatively small plasma pressure, was studied by Shafranov.[5] This axisymmetric geometry has well-defined flux surfaces, each with a circular (poloidal) cross section; but the self-force from plasma current displaces the centers of the circles in major radius, so they are not concentric. The conventional coordinates for this geometry, "Shafranov coordinates," will be denoted by (r_S,θ_S,ζ_S); they are conveniently defined in terms of ordinary cylindrical coordinates, (R,ϕ,Z), centered on the toroidal symmetry axis (Z axis). We have

$$R = R_0 - \Delta(r_S) + r_S \cos\theta_S, \quad \phi = -\zeta_S, \quad Z = r_S \sin\theta_S. \tag{21}$$

Here R_0 is the major radius of the magnetic axis, while Δ is the "Shafranov shift," giving the displacement from concentricity of the flux surface labeled by r_S. Tokamak equilibrium theory provides an ordinary differential equation for Δ; here we assume it is a known function.

Shafranov coordinates are not flux coordinates, i.e., they do not admit a representation of the form (18), essentially because the ratio $\mathbf{B}\cdot\nabla\zeta_S/\mathbf{B}\cdot\nabla\theta_S$ is not constant on flux surfaces. However, the variable r_S is an approximate flux surface label, in the sense that

$$\mathbf{B}\cdot\nabla r_S = O(\epsilon^2), \tag{22}$$

where $\epsilon \ll 1$ is the inverse aspect ratio.

Here we derive canonical coordinates for the Shafranov geometry, as an example of the procedure given in Sec. II. It is convenient to depart slightly from the order of that section: we first transform from Shafranov coordinates to flux coordinates, (r_f,θ_f,ζ_f), and then derive canonical coordinates, (r,θ,ζ), in terms of the flux coordinates. In other words, we satisfy Eq. (18) before dealing with Eq. (19).

Since we do not seek exact canonical coordinates, one technical point must be mentioned first. Flux coordinates satisfy

$$B_{r_f} = O(\epsilon^2) = \mathbf{B}\cdot\nabla r_f.$$

Thus, if terms of second order are omitted, flux coordinates are already equivalent to canonical coordinates. It follows that, in order to exemplify the general construction given in Sec. II, we must selectively retain second- and even third-order terms. In particular, distinct canonical coordinates are determined from requiring

$$B_r = O(\epsilon^3) \ll B_{r_f}. \tag{23}$$

While we do not attempt to retain all second- and third-order terms, Eq. (23) and its consequences will be seen to be consistent, in the sense of correctly reproducing the lowest-order corrections to (r_f,θ_f,ζ_f).

Next we construct the (r_f,θ_f,ζ_f) system. We choose $r_f = r_S = r$ (and thus drop the subscripts on r when they are not essential) and $\zeta_f = \zeta_S$, and construct θ_f. Recall[6] that flux coordinates must satisfy

$$\frac{\partial}{\partial\theta_f}\frac{R^2}{J_f} = 0, \tag{24}$$

where J_f is the Jacobian determinant for flux coordinates,

$$J_f \equiv |\nabla r_f\cdot\nabla\theta_f\times\nabla\zeta_f|^{-1}.$$

Since $r_f = r_S$, $\zeta_f = \zeta_S$,

$$J_S = J_f\frac{\partial\theta_f}{\partial\theta_S}.$$

Here $J_S = r_S R(1 - \Delta' \cos\theta_S)$ is the Shafranov Jacobian and the θ dependence of J_f is known from (24), and its r dependence is chosen so that θ_f increases by 2π as θ_S does. Straightforward manipulation yields

$$\frac{\partial\theta_f}{\partial\theta_S} = 1 - (\Delta' + r/R_0)\cos\theta_S,$$

the prime indicating a derivative with respect to $r = r_f = r_S$. We have

$$\theta_f = \theta_S - 2\eta \sin\theta_S, \qquad (25)$$
$$\eta(r) \equiv \tfrac{1}{2}(\Delta' + r/R_0).$$

Noting that $\eta = O(\epsilon)$, we find

$$\cos\theta_S = \cos\theta_f + \eta(\cos 2\theta_f - 1),$$

whence (21) becomes

$$R = R_0 - \Delta + r\cos\theta_f + \eta r(\cos 2\theta_f - 1),$$
$$\phi = -\zeta_f, \quad Z = r\sin\theta_f + \eta r \sin 2\theta_f. \qquad (26)$$

The transition to canonical coordinates begins with the observation that, for an integrable magnetic field, canonical coordinates are a special case of flux coordinates. That is, they satisfy the "strong" version of (18), in which ψ is a function of χ. It follows[6] that the family of axisymmetric flux coordinates is effectively covered by transformations of the form

$$\theta = \theta_f + s(r,\theta_f), \quad \zeta = \zeta_f + qs(r,\theta_f),$$

where

$$q(r) \equiv \frac{d\psi}{d\chi}$$

is the safety factor and s is an arbitrary *periodic* function of θ_f. Then the map $(\theta_f,\zeta_f) \to (\theta,\zeta)$ preserves toroidal topology as well as Eq. (18).

We substitute these forms into Eq. (9) to obtain

$$B_{r_f} - B_{\theta_f}\frac{\partial s}{\partial r} - B_{\zeta_f}\frac{\partial(qs)}{\partial r} = 0 \qquad (27)$$

or, in lowest order,

$$\frac{\partial(qs)}{\partial r} = \frac{B_{r_f}}{B_{\zeta_f}}.$$

Here both factors on the right-hand side are easily computed (see the following), with the result

$$s(r,\theta_f) = 2\sigma(r)\sin\theta_f, \qquad (28)$$

$$\frac{\partial(q\sigma)}{\partial r} = \frac{r}{2qR_0^2}\left(\frac{r}{R_0} + (r\Delta')'\right). \qquad (29)$$

Thus $\sigma(r)$ is straightforwardly computed for any equilibrium with known safety factor and displacement profiles.

After substitution we can express canonical coordinates for a large aspect ratio tokamak as

$$R = R_0 - \Delta + r\cos\theta + \alpha r(\cos 2\theta - 1),$$
$$\phi = -\zeta + 2\sigma q \sin\theta, \qquad (30)$$
$$Z = r\sin\theta + \alpha r \sin 2\theta,$$

with

$$\alpha(r) \equiv \eta - \sigma.$$

Here we noted, from the similarity of Eqs. (25) and (28), that σ enters θ only through the combination $\eta - \sigma$.

As anticipated, $\sigma = O(\epsilon^3)$ and is therefore smaller than the terms neglected in the construction of Shafranov coordinates. That is, $O(\epsilon^3)$ corrections to the flux-coordinate angles make them canonical. Despite the smallness of the geometrical terms involved, they have physical interest, for two reasons. First, they are correctly computed from low-order equilibrium theory: Eq. (29) contains all the information from Eq. (27) that is needed for consistency. Second, the appearance of the σ term in the toroidal angle allows it to enter certain contexts with surprising strength—comparable to other terms of well-known significance. This circumstance is exemplified in the following calculation.

It is instructive to compute the metric coefficients, $g_{\mu\nu}$, defined by

$$ds^2 = dR^2 + R^2 d\phi^2 + dZ^2 = g_{\mu\nu}\, dx^\mu\, dx^\nu.$$

For flux coordinates $(x^1 = r_f, x^2 = \theta_f, x^3 = \zeta_f)$, (26) gives

$$g_{11} = 1 - 2\Delta' \cos\theta,$$
$$g_{12} = r\sin\theta\,[r/R_0 + (r\Delta')'], \qquad (31)$$
$$g_{22} = r^2[1 + 2(r/R_0 + \Delta')\cos\theta],$$
$$g_{33} = R^2;$$

all the other $g_{\mu\nu}$ vanish. The determinant is

$$J_f = rR_0[1 + 2(r/R_0)/\cos\theta].$$

Here and below we neglect $O(\epsilon^2)$ terms. We also conveniently suppress the f subscript in these formulas, which will be shown to pertain to canonical coordinates as well.

As a simple application we compute B_{r_f}. From (18) we have

$$B^{\theta_f} = \chi'/J_f, \quad B^{\zeta_f} = qB^{\theta_f},$$

so that (31) allows us to compute

$$B_{r_f} = g^{12}B^{\theta_f} = (\chi'/R_0)\sin\theta_f[r/R_0 + (r\Delta')'].$$

This result was used in (29).

Turning our attention to the canonical coordinates of Eq. (30), we quickly find Eqs. (31) to be unchanged. However, two additional metric coefficients appear:

$$g_{13} = -2R_0^2(q\sigma)'\sin\theta,$$
$$g_{23} = -2R_0^2 q\sigma\cos\theta. \qquad (32)$$

Here the R_0^2 factors makes these quantities non-negligible, despite the smallness of σ. For example,

$$g_{13} \sim g_{12}.$$

Finally, we verify the canonical nature of our coordinates by computing

$$B_r = g_{12}B^\theta + g_{13}B^\zeta$$
$$= B^\theta(g_{12} + qg_{13}) = B^\theta r\sin\theta\left[\left(\frac{r}{R_0} + (r\Delta')'\right) - \frac{2qR_0^2}{r}\frac{\partial(q\sigma)}{\partial r}\right].$$

This vanishes by virtue of (29).

The result (32) can be useful beyond the domain of Hamiltonian orbit theory. Fluid stability studies, for example, often devolve upon calculations of field line curvature in large aspect ratio geometry. Since B_r terms often complicate such calculations, they might be simplified by use of canonical coordinates. Of course, the results would be correct only if Eq. (32) were taken into account.

B. Canonical description of tokamak orbits

Our second example assumes that canonical coordinates have been found; its intention is to display their application to guiding center orbit theory. The main burden is to obtain explicit expressions for (7), giving the radius and parallel speed in terms of canonical variables; of course, this requires some geometrical simplification.

We consider a tokamak with exact flux surfaces, so that Eq. (18) takes the form

$$\mathbf{B} = q\,\nabla\chi\times\nabla\theta + \nabla\zeta\times\nabla\chi.$$

Our main simplification is to neglect magnetic shear, taking the safety factor q to be a spatial constant. Despite the critical role played by magnetic shear in tokamak confinement, its neglect is harmless whenever radial orbit widths are small compared to the shear length—a circumstance that pertains in all but exceptional cases.

The constancy of q implies

$$\psi = q[\chi - (c/e)p_0], \tag{33}$$

where the integration constant p_0 is needed to account for the toroidal electric field ("Ohmic heating" field):

$$p_0(t) \equiv eE_\zeta t. \tag{34}$$

Here t is the time and E_ζ is the covariant ζ component of the field, which is presumed constant.

In addition to neglecting shear, and with the same goal of explicitness, we assume a large aspect ratio. Then the results of Sec. III A are applicable; in particular,

$$b_\theta + qb_\zeta = 1/b^\theta \cong qR.$$

It is then a simple matter to obtain, from Eqs. (20), (33), and (34),

$$mu = (1/R)(p_\theta/q + p_\zeta + p_0), \tag{35}$$

$$\psi = (c/e)[p_\theta - (b_\theta/R)(p_\zeta + p_0)]. \tag{36}$$

We will use a radial variable based on the toroidal flux ψ so that functional dependence on r is converted to (p_θ, p_ζ) dependence by means of (36). In this regard it is important to notice that the second term in (36) is relatively small: $b_\theta/R = O(\epsilon^2)$. In most contexts this term can be ignored, using

$$\psi = (c/e)p_\theta; \tag{37}$$

in the few cases where the correction term matters it can be treated iteratively, evaluating b_θ/R as a function of p_θ alone. Also note that the R in (35) depends on p_θ in the same way; thus, for example,

$$m\frac{\partial u}{\partial p_\theta} = \frac{1}{qR} - \frac{c}{eR^2}\left(\frac{1}{q}p_\theta + p_\zeta + p_0\right)\frac{\partial R}{\partial\psi}$$
$$= \frac{1}{qR} - \frac{c}{e}\frac{mu}{R}\frac{\partial R}{\partial\psi}. \tag{38}$$

Furthermore the radial variable r may be chosen such that

$$\frac{\partial}{\partial\psi} = \frac{1}{\psi'}\frac{\partial}{\partial r} = \frac{1}{B_0 r}\frac{\partial}{\partial r}.$$

Here B_0 is some (constant) measure of the toroidal field.

Next we verify that our canonical description reproduces the expected tokamak guiding center orbits. Thus we return to Eq. (8) to compute

$$\dot\theta = \frac{\partial H}{\partial p_\theta} = mu\frac{\partial u}{\partial p_\theta} + \frac{c}{eB_0 r}\frac{\partial}{\partial r}(e\Phi + \mu B), \tag{39}$$

$$\dot\zeta = \frac{\partial H}{\partial p_\zeta} = mu\frac{\partial u}{\partial p_\zeta} - \frac{c}{e}\frac{b_\theta}{RB_0 r}\frac{\partial}{\partial r}(e\Phi + \mu B), \tag{40}$$

and

$$\dot r = \frac{c}{e}\frac{1}{B_0 r}\dot p_\theta = -\frac{c}{e}\frac{1}{B_0 r}\frac{\partial H}{\partial\theta}$$
$$= -\frac{c}{e}\frac{1}{B_0 r}\left(-\frac{mu^2}{R}\frac{\partial R}{\partial\theta} + e\frac{\partial\Phi}{\partial\theta} + \mu\frac{\partial B}{\partial\theta}\right). \tag{41}$$

Of course, axisymmetry implies that

$$\dot p_\zeta = 0. \tag{42}$$

These forms are indeed familiar. Note in particular that the curvature drift enters (39) through the second term of (38).

The Hamiltonian simply reveals other properties of the orbits as well. For example, the motion of trapped particles is conveniently examined by restricting attention to the phase-space surface on which $u = 0$ (the location of the banana tips), or

$$p_\theta = -q(p_\zeta + p_0).$$

In view of Eq. (42), the banana tips move according to

$$\dot p_\theta = -q\dot p_0 = -eqE_\zeta.$$

This result expresses the inward motion of trapped particles—the collisionless root of the Ware–Galeev pinch effect.[7,8] The toroidal drift of trapped particles, which affects trapped particle instability, can be found in a similarly straightforward way.

While these applications of the canonical formalism are very simple, the generalization to more interesting and novel issues should be clear. For example, one can easily allow for symmetry-breaking terms in the magnetic field or electrostatic potential. Indeed, the present formalism has proven useful in the numerical investigation of tokamak orbits with asymmetrical perturbations.[9]

ACKNOWLEDGMENTS

We would like to thank Philip Morrison for many helpful discussions. This work was supported by U.S. Department of Energy Contract No. DE-FG05-80ET-53088.

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