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Spectrum Cascade by Mode Coupling in Drift-Wave Turbulence

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The spectrum cascade by mode coupling in drift-wave turbulence occurs to larger and smaller values of $|\vec{k}|$ rather than toward lower frequencies. This leads to the dual cascade process; energy cascades to smaller k while enstrophy (square of the vorticity) cascades to larger k, analogous to two-dimensional hydrodynamic turbulence. However, the speed of energy condensation to $k=0$ is much slower than in the hydrodynamic case.

In this Letter, we show that the spectrum cascade by mode coupling in drift-wave turbulence occurs to longer and shorter wavelengths, and that it leads to the dual cascade process where the energy cascades to smaller k, while the entropy cascades to larger k, analogous to two-dimensional hydrodynamic turbulence.[1] This result originates from an intrinsic property of the drift wave in which the linear frequency and the amplitude can be small parameters of the same magnitude.

For wave-wave interactions in drift-wave turbulence the largest coupling occurs through the $\vec{E}\times\vec{B}$ nonlinearity. This allows us to ignore the parallel ion inertia. The best way of deriving the nonlinear equation in such a case is to use the vortex equation for the ion dynamics.[2] If we make the assumption of cold ions, the equation for the vorticity $\Omega=(\nabla_\perp\times\vec{v}_\perp)\cdot\hat{z}$ can be derived by taking the curl of the ion equation of motion and by using $\nabla_\perp\cdot\vec{v}_\perp=-d\ln n/dt$:

$$\frac{d}{dt}\ln\left(\frac{\omega_{ci}+\Omega}{n}\right)=0, \tag{1}$$

where ω_{ci} is the ion cyclotron frequency, n is the number density, and $\vec{v}_\perp$ is the ion velocity in the direction perpendicular to the ambient magnetic field $B_0\hat{z}$. The total derivative includes the convective derivative,

$$\frac{d}{dt}=\frac{\partial}{\partial t}+(\vec{v}_\perp\cdot\nabla), \tag{2}$$

where the leading term in $\vec{v}_\perp$ is given by the $\vec{E}\times\vec{B}$ drift,

$$\vec{v}_\perp=-\nabla\varphi\times\hat{z}/B_0. \tag{3}$$

Here, φ is the electrostatic potential of the drift wave. The number density is given by the electron density as a result of the quasineutrality condition:

$$n=n_0(\vec{x})(1+e\varphi/T_e), \tag{4}$$

where T_e is the electron temperature and the contribution from the Landau pole due to the electron parallel motion is ignored by assuming a saturated state. The vorticity Ω is obtained from Eq. (3),

$$\Omega=(\nabla^2\varphi)/B_0. \tag{5}$$

If we substitute Eqs. (4) and (5) into (1), we obtain the following exact simple nonlinear equation for the drift-wave turbulence[3]:

$$\partial(\nabla^2\varphi-\varphi)/\partial t+[(\nabla\varphi\times\hat{z})\cdot\nabla](\ln n_0-\nabla^2\varphi)=0, \tag{6}$$

where the time and space coordinates are normalized by ω_{ci}^{-1} and $\rho_s=(T_e/m_i)^{1/2}/\omega_{ci}$ and the potential φ by T_e/e.

To find the direction of the spectrum cascade in Eq. (6), we consider three waves with the wave numbers $\vec{k}_1$, $\vec{k}_2$, and $\vec{k}_3$ such that $\vec{k}_1+\vec{k}_2+\vec{k}_3=0$.[4] If we write

$$\varphi(\vec{x},t)=\varphi_{\vec{k}}(t)\exp(i\vec{k}\cdot\vec{x})+\text{c.c.},$$

the coupled equations obtained from Eq. (6) are

$$d\varphi_1/dt + i\omega_1\varphi_1 = \Lambda_{2,3}{}^1\, \varphi_2{}^*\varphi_3{}^*, \tag{7}$$

$$d\varphi_2/dt + i\omega_2\varphi_2 = \Lambda_{3,1}{}^2\, \varphi_3{}^*\varphi_1{}^*, \tag{8}$$

$$d\varphi_3/dt + i\omega_3\varphi_3 = \Lambda_{1,2}{}^3\, \varphi_1{}^*\varphi_2{}^*, \tag{9}$$

where the superscript asterisk indicates the complex conjugates;

$$\varphi_j(t) = \varphi_{\vec{k}_j}(t), \quad j = 1, 2, 3; \tag{10}$$

$$\omega_j = -\frac{\vec{k}_j \times \hat{z}\cdot\nabla \ln n_0}{1 + k_j{}^2}, \quad j = 1, 2, 3, \tag{11}$$

is the normalized drift wave frequency, $\omega_{\vec{k}_j}{}^*/\omega_{ci}$; and the matrix element $\Lambda_{q,r}{}^p$ is given by

$$\Lambda_{q,r}{}^p = (1 + k_p{}^2)^{-1}(\vec{k}_q \times \vec{k}_r)\cdot\hat{z}(k_r{}^2 - k_q{}^2). \tag{12}$$

Note here that typically $\omega_j \simeq 10^{-2}$; hence if $\varphi_j \gtrsim 10^{-2}$, we cannot use the standard method of weak turbulence.[5,6]

The direction of the spectrum cascade may be found by studying the stability of a situation in which one of the modes, 1, 2, or 3, is more highly populated than the others. For this purpose we first assume without loss of generality that $k_j \equiv |\vec{k}_j|$ such that

$$k_1 \leqslant k_2 \leqslant k_3. \tag{13}$$

We first consider a case in which the mode $\vec{k}_2$ is highly populated so that $|\varphi_2| \gg |\varphi_1|, |\varphi_3|$. We can then linearize Eqs. (7) to (9), from which we have

$$\varphi_2 = A_2 \exp(-i\omega_2 t), \quad A_2 = \text{const}, \tag{8$'$}$$

and

$$dA_1/dt = \Lambda_{2,3}{}^1 A_2{}^* A_3{}^* e^{i\theta t}, \tag{7$'$}$$

$$dA_3/dt = \Lambda_{1,2}{}^3 A_2{}^* A_1{}^* e^{i\theta t}, \tag{9$'$}$$

with

$$\varphi_j \equiv A_j(t)\exp(-i\omega_j t), \quad j = 1, 3, \tag{14}$$

and

$$\theta = \omega_1 + \omega_2 + \omega_3 \tag{15}$$

is the frequency mismatch.

From Eqs. (7′) and (9′), we find easily that the instability (exponential growth of A_1 and A_3) occurs when

$$\theta^2 - 4\Lambda_{2,3}{}^1\Lambda_{1,2}{}^3 |A_2|^2 < 0. \tag{16}$$

Inequality (16) shows that the stability is decided by the sign of the product $\Lambda_{2,3}{}^1\Lambda_{1,2}{}^3$.

Now, in view of the assumed relation (13), both of the quantities $k_2{}^2 - k_3{}^2$ and $k_1{}^2 - k_2{}^2$ are negative (or zero) in Eq. (12), while from Fig. 1 we see that $(\vec{k}_2 \times \vec{k}_3)\cdot\hat{z}$ and $(\vec{k}_1 \times \vec{k}_2)\cdot\hat{z}$ have the same sign (if not zero). Hence $\Lambda_{2,3}{}^1\Lambda_{1,2}{}^3 \geqslant 0$ and this situation can be unstable.

On the other hand, since $\Lambda_{3,1}{}^2\Lambda_{1,2}{}^3$ and $\Lambda_{3,1}{}^2\Lambda_{2,3}{}^1$ are always negative (or zero), if modes 1 or 3 are highly populated, then the system is stable. Hence we conclude that the spectrum cascades by simultaneously exciting both shorter- and longer-wavelength modes, and the direction is independent of the magnitude of the linear frequencies ω_1, ω_2, and ω_3 of the three waves.

This interesting result originates from the fact that the matrix elements Λ do not involve ω, while in many weak-turbulence cases, Λ of the highest-frequency mode has a sign which is different from the other two lower-frequency modes.[5,6] Instead of the highest-frequency mode, in our case the mode with the magnitude of wave number *in between* those of the two other wave numbers has the sign of Λ different from the others.

We note here, however, that this situation *does not* allow an excitation of a mode with a frequency higher than ω_2, because even if ω_1 or ω_3 were larger than ω_2, it can be shown that the nonlinearly shifted frequency, $\omega_j - \theta/2$, $j = 1, 3$ [which one can obtain from Eqs. (7′) and (9′)], at the threshold amplitude $|\varphi_2|$ can be shown to become always smaller than or equal to ω_2.

It is possible to obtain the values of $\vec{k}_1$ and $\vec{k}_3$ that produce the maximum growth rate. The maximum growth occurs when the product $\Lambda_{2,3}{}^1\Lambda_{1,2}{}^3$ is maximized. From the cross-product part of Eq. (12), we see first that the angle between $\vec{k}_1$ and $\vec{k}_2$ is $\pi/2$ as shown in Fig. 1. The magnitude of $\vec{k}_1$ and $\vec{k}_3$ that maximizes the product has a complex expression, but if $k_2 \ll 1$, we have

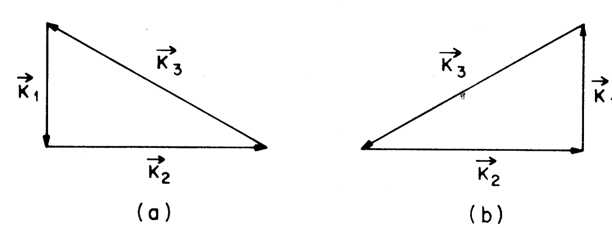

FIG. 1. Two types of available wave-vector–matching condition for $k_1 < k_2 < k_3$. Both cases give a positive product $\Lambda_{2,3}{}^1\Lambda_{1,2}{}^3$, leading to instability if $|\varphi_2| \gg |\varphi_1|$, $|\varphi_3|$. The spectrum cascade occurs simultaneously in the direction of smaller and larger values of wave vector.

the following simple relation:

$$k_{1,\max}^2 \simeq \tfrac{2}{3}k_2^2, \quad k_{3,\max}^2 \simeq \tfrac{5}{3}k_2^2. \qquad (17)$$

If we use $k_{1,\max}$ and $k_{3,\max}$, we can obtain the ratio of energy which is cascaded into wave numbers smaller than k_2 to that into wave numbers larger than k_2. To study this, we first derive the conservation formulas of the equivalent quantum of the three modes, N_1, N_2, and N_3, defined by

$$N_p=(1+k_p^2)|\varphi_p|^2/|k_q^2-k_r^2|, \quad k_q^2 \neq k_r^2. \qquad (18)$$

From Eqs. (7) to (9), we find

$$N_3-N_1=\text{const}, \quad N_2+N_3=\text{const}, \qquad (19)$$

and

$$N_1+N_2=\text{const}.$$

These relations show that a loss of one quantum in N_2 appears as a gain of one quantum in N_1 and N_3, respectively.

Now the energy W of the $\vec{k}$ mode is given by[2] $W=|\varphi_{\vec{k}}|^2(1+k^2)$. Hence, from Eqs. (18) and (19), we see that the fractional energy gains by modes 1 and 3, ΔW_1 and ΔW_3, corresponding to a loss of energy of mode 2, $\Delta W_2=1$, are given by

$$\Delta W_1=\frac{k_3^2-k_2^2}{k_3^2-k_1^2}, \quad \Delta W_3=\frac{k_2^2-k_1^2}{k_3^2-k_1^2}. \qquad (20)$$

At the maximum growth, $\Delta W_1=\frac{1}{3}$ and $\Delta W_3=\frac{2}{3}$. The energy cascades according to the binomial distribution, $\binom{n}{r}(k_1^2/k_2^2)^r(1-k_1^2/k_2^2)^{n-r}$ as shown in Fig. 2. By summing up all the energy cascaded into $k^2<k_2^2$ $(k^2>k_2^2)$, we can obtain the energy which is down (up) shifted in wave number. In this case the ratio of the down- to up-shifted energy approaches approximately 80% to 20% at $n=12$.

We can apply a similar argument to the generalized enstrophy,[2] $U=k^2(1+k^2)|\varphi_{\vec{k}}|^2$. We find that the cascase in U is reversed. The ratio of down-shifted U to up-shifted U approaches approximately 20% to 80% at $n=12$.

These ratios change when the initial value of k_2 is larger because $k_{1,\max}^2$ given in Eq. (17) breaks down and we must consider a nonlocal transfer problem.[6] By repeating the cascade process at large n and by setting an appropriate inertia range, one can obtain the stationary energy spectrum. The unidirectional energy spectrum thus obtained has a power law of k^{-4}. The details will be published elsewhere.[7]

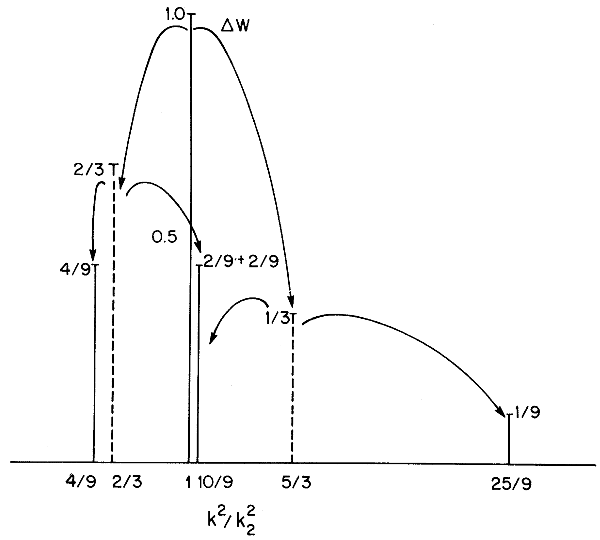

FIG. 2. Cascade of energy from the mode k_2. The amount of energy at each step is shown by the size of the vertical bar and the corresponding value of k^2/k_2^2 is shown at the foot of the bar. Only three steps are shown for illustrative purposes. To find the final ratio of down- to up-shifted energy, one should go on to several more steps. The cascaded energy obeys the binomial distribution.

The spectrum cascade into longer and shorter wavelengths is analogous to the case of two-dimensional hydrodynamic turbulence for an incompressible fluid.[1] In fact the two-dimensional Euler's equation for an incompressible fluid can be described by the stream function ψ satisfying

$$\partial\nabla^2\psi/\partial t-[(\nabla\psi\times\hat{z})\cdot\nabla]\nabla^2\psi=0. \qquad (21)$$

The matrix element is identical to our situation if $1+k_p^2$ is replaced by k_p^2 in Eq. (12). Hence our method can also be applied to Eq. (21). We find that the maximum growth occurs at $k_1^2=(\sqrt{2}-1)k_2^2$, and at $n=12$, 90% of energy is cascaded down in k. Hence the inverse cascade rate of the drift-wave turbulence is smaller than that of the two-dimensional Navier-Stokes turbulence.

Our conclusion that the spectrum cascade in the drift-wave turbulence is of the hydrodynamic type rather than the weak-turbulence type has important implications. For example, the weak-turbulence theory based on nonlinear Landau damping predicts that k_θ $(k\perp\nabla n_0)$ should always decrease (to lower the frequency) and transfer in k_r should be immaterial.[8,9] However, the observation of the production of large-scale vortices in computer simulations[10] of drift-wave turbulence contradicts such a notion. It can be explained in terms of the present theory that predicts isotropic cascading into smaller wave numbers. The condensation of energy at $k \to 0$ indicates a formation of large-scale vortices. This supports the convection process rather than diffusion process as the basic transport mechanism of a magnetized plasma.

Finally, we briefly discuss the effect of a magnetic shear. In the presence of a magnetic shear, Eq. (6) is valid only in a limited region near the mode rational surface. Within a Debye length from the mode rational surface $\vec{k} \cdot \vec{B}_0 \simeq 0$; hence the electrons do not obey the Boltzmann distribution as assumed here. Away from this region, the equation is valid until $k_\parallel$ increases to the point where $k_\parallel v_{\text{th}\,i}/\omega_{ci} \simeq \gamma_N$ ($v_{\text{th}\,i}$ is the ion thermal speed, $k_\parallel$ is the parallel wave number, and $\gamma_N \simeq k^4 |\varphi|$ is the decay rate). Here the parallel ion inertia becomes important and the two-dimensionality assumption breaks down. This occurs typically at a distance of $5\rho_s$ from the mode rational surface. Hence in the presence of a magnetic shear the inverse cascade occurs until $k_\perp \sim (5\rho_s)^{-1}$, at which point the energy may be dissipated to the parallel motion of the ions.

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After the submission of the manuscript, one of the authors (A.H.) learned that the evidence of inverse cascade of energy was also found by the numerical solution of Eq. (6) without the density gradient term.[11]

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Continuous Creation and Annihilation of Coreless Vortices in ^{3}He-A in the Presence of a Heat Flow

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The apparent incompatibility between a "spinning" texture in the bulk and the boundary condition at the surface is shown to be unreal. Topological arguments are used to show that this "incompatibility" can always be eliminated in both open and closed containers by appropriate textural arrangements. The "spinning" process in the bulk corresponds to a continuous nucleation of vortex rings. In an open geometry, these rings will flow downstream along the heat current. In a closed container, they will be devoured entirely by *stationary* singular loops at the surface.

Since the observation of the persistent oscillation in the intensity of ultrasound transmission in ^{3}He-A [1] which indicates that there are periodic motions of the texture $\hat{l}$, three mechanisms have been proposed to explain this phenomenon. All of them involve motions of textures driven by a heat flow—which can be considered as related to a chemical-potential gradient. These mechanisms are as follows: (i) formation of vortex textures and their continuous motions across the chemical-

