On Fluid ETG model

Haotian Chen

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For illustrative purposes, we consider electrostatic ETG with adiabatic ions

$$\delta n_i = -\frac{en_0}{T_i}\delta\phi, \tag{1}$$

and assume isothermal electrons with $\delta T_e = 0$. We take 3D description, that is $k_\parallel \neq 0$, therefore this model can not handle zonal problems, which have $k_\parallel = 0$.

The continuity and momentum equations of electrons are

$$\frac{\partial n_e}{\partial t} + \nabla \cdot (n_e \vec{v}_e) = 0, \tag{2}$$

$$m_e \frac{d\vec{v}_e}{dt} = -e(-\nabla\phi + \frac{\vec{v}_e \times \vec{B}}{c}) - \frac{\nabla(P_e + \delta P_e)}{n_e}, \tag{3}$$

i.e.,

$$m_e \frac{d\vec{v}_e}{dt} = (1+\tau)e\nabla\phi - e\frac{\vec{v}_e \times \vec{B}}{c} - \frac{\nabla P_e}{n_e}, \tag{4}$$

where we have used Eq.(1) and $\tau = T_e/T_i$.

From the momentum equation we know, the electron parallel velocity can be calculated by

$$m_e \hat{b} \cdot \frac{d\vec{v}_e}{dt} = (1+\tau)e\nabla_\parallel \delta\phi, \tag{5}$$

we then have, to the lowest order,

$$\vec{v}_\parallel = (1+\tau)\frac{e}{m_e}\partial_t^{-1}\nabla_\parallel\delta\phi, \tag{6}$$

i.e.,

$$v_\parallel \sim (1+\tau)\frac{k_\parallel e\delta\phi}{m_e\omega}. \tag{7}$$

On the other hand, the perpendicular component of momentum equation gives,

$$\hat{b} \times \frac{d\vec{v}_e}{dt} = (1+\tau)\frac{e}{m_e}\hat{b} \times \nabla\phi - \omega_{ce}\vec{v}_{e,\perp} - \frac{\hat{b} \times \nabla P_e}{m_e n_e}, \tag{8}$$

which could be solved perturbatively.

To the lowest order, the perpendicular drift is

$$\vec{v}_{e,\perp,0} = (1+\tau)\frac{e}{m_e\omega_{ce}}\hat{b}\times\nabla\phi - \frac{\hat{b}\times\nabla P_e}{m_e\omega_{ce}n_e} \equiv (1+\tau)\vec{v}_E + \vec{v}_D, \tag{9}$$

so the total lowest order drift motion can be written as

$$\vec{v}_{e,0} = \vec{v}_\parallel + (1+\tau)\vec{v}_E + \vec{v}_D. \tag{10}$$

To the next order, the polarization drift is given by

$$\vec{v}_{e,1} = -\frac{1}{\omega_{ce}}\hat{b}\times\frac{d\vec{v}_{e,0}}{dt} = -\frac{1}{\omega_{ce}}(\partial_t + \vec{v}_{e,0}\cdot\nabla)(\hat{b}\times\vec{v}_{e,0}). \tag{11}$$

Since

$$\hat{b}\times\vec{v}_{e,0} = -\frac{e(1+\tau)}{m_e\omega_{ce}}\nabla_\perp\delta\phi + \frac{\nabla_\perp P_e}{m_e\omega_{ce}n_e}, \tag{12}$$

therefore, Eq.(11) can be approximated, in the second order, by

$$\begin{aligned}
\vec{v}_{e,1} &\simeq -\frac{1+\tau}{\omega_{ce}}[\partial_t + ((1+\tau)\vec{v}_E + \vec{v}_D)\cdot\nabla](\hat{b}\times\vec{v}_E) \\
&= \{\partial_t + \frac{1}{m_e\omega_{ce}}[(1+\tau)e\hat{b}\times\nabla_\perp\phi - \frac{\hat{b}\times\nabla P_e}{n_e}]\cdot\nabla_\perp\}\frac{e(1+\tau)}{m_e\omega_{ce}^2}\nabla_\perp\delta\phi \\
&= \frac{e(1+\tau)}{m_e\omega_{ce}^2}\partial_t\nabla_\perp\delta\phi - [\frac{\hat{b}\times\nabla P_e}{n_e}\cdot\nabla_\perp]\frac{e(1+\tau)}{m_e^2\omega_{ce}^3}\nabla_\perp\delta\phi \\
&\quad + \frac{e^2(1+\tau)^2}{m_e^2\omega_{ce}^3}[\hat{b}\times\nabla_\perp\phi\cdot\nabla_\perp]\nabla_\perp\delta\phi.
\end{aligned} \tag{13}$$

Note that incompressibility $\nabla\cdot\vec{v}_{e,\perp,0}$ in slab geometry, Eq.(2) yields

$$\partial_t\delta n_e + n_e\nabla\cdot(\vec{v}_\parallel + \vec{v}_{e,1}) + \nabla\delta n_e\cdot\vec{v}_D + (1+\tau)\nabla n_e\cdot\vec{v}_E = 0, \tag{14}$$

where we have defined the small parameter $\epsilon \ll 1$ by

$$\frac{k_\parallel}{k_\perp} \sim \frac{\rho_e}{r_n} \equiv \mathcal{O}(\epsilon), \tag{15}$$

and noted the orderings

$$\frac{\omega}{\omega_{ce}} \sim \frac{k_\perp\rho_e}{2\tau}\frac{\rho_e}{r_n} \equiv \mathcal{O}(\epsilon^{3/2}), \tag{16}$$

$$\frac{\delta n_e}{n_e} \sim \epsilon, \quad k_\perp^2\rho_e^2 \sim \epsilon, \quad \tau \sim 1, \quad \frac{v_D}{v_E} \sim 1, \quad k_\parallel^2 r_n^2 \sim \epsilon, \tag{17}$$

and then found

$$\frac{n_e\nabla\cdot\vec{v}_\parallel}{\nabla n_e\cdot\vec{v}_E} \sim \frac{4\tau^2k_\parallel^2r_n^2}{k_\perp^2\rho_e^2} \sim \mathcal{O}(1), \quad \frac{n_e\nabla\cdot\vec{v}_{e,1}}{\nabla n_e\cdot\vec{v}_E} \sim k_\perp\rho_e\frac{r_n}{\rho_e}\frac{\omega}{\omega_{ce}} \sim \epsilon, \tag{18}$$

$$\frac{\nabla\delta n_e\cdot\vec{v}_D}{\nabla n_e\cdot\vec{v}_E}\sim k_\perp\rho_e\frac{r_n}{\rho_e}\frac{\delta n_e}{n_e}\sim\sqrt{\epsilon},\quad \frac{\nabla_\parallel\delta n_e\cdot\vec{v}_\parallel}{\nabla n_e\cdot\vec{v}_E}\sim\frac{\delta n_e}{n_e}k_\parallel r_n\frac{k_\parallel}{k_\perp}\frac{\omega_{ce}}{\omega}\sim\epsilon^{3/2},\quad \frac{\nabla n_e\cdot\vec{v}_{e,1}}{\nabla n_e\cdot\vec{v}_E}\sim\epsilon^{3/2}. \tag{19}$$

Substitute $\delta n_e=\delta n_i$ into Eq.(14), we get

$$\begin{aligned}
& -n_e\partial_t\frac{e\delta\phi}{T_i}+n_e(1+\tau)\frac{e}{m_e}\partial_t^{-1}\nabla_\parallel^2\delta\phi \\
& +\frac{en_e(1+\tau)}{m_e\omega_{ce}^2}\partial_t\nabla_\perp^2\delta\phi-\frac{e(1+\tau)}{m_e^2\omega_{ce}^3}[\hat{b}\times\nabla P_e\cdot\nabla_\perp]\nabla_\perp^2\delta\phi \\
& +\frac{e^2n_e(1+\tau)^2}{m_e^2\omega_{ce}^3}[\hat{b}\times\nabla_\perp\delta\phi\cdot\nabla_\perp]\nabla_\perp^2\delta\phi \\
& +\nabla_\perp\frac{e\delta\phi}{T_i}\cdot\frac{\hat{b}\times\nabla P_e}{m_e\omega_{ce}}+(1+\tau)\frac{e}{m_e\omega_{ce}}\nabla n_e\cdot\hat{b}\times\nabla\phi \\
= \; & 0,
\end{aligned} \tag{20}$$

which is valid to the ϵ order .

With the electron dynamic normalization

$$\frac{e\delta\phi}{T_i}=\Phi,\quad \frac{\partial_x n_e}{n_e}=-\frac{1}{r_n},\quad \frac{\partial_x T_e}{T_e}=-\frac{1}{r_t},\quad ,\frac{r_n}{r_t}=\eta_e,\quad \rho_e=\sqrt{\frac{\tau m_e}{m_i}}\rho_i,\quad \frac{\vec{x}}{\rho_e}\to\vec{x},\quad \frac{\rho_e}{r_n}\omega_{ce}t\to t, \tag{21}$$

we have

$$\begin{aligned}
& -(1-\frac{1+\tau}{2\tau}\nabla_\perp^2)\partial_t\Phi+\frac{1+\tau}{2\tau}\frac{r_n^2}{\rho_e^2}\partial_t^{-1}\nabla_\parallel^2\Phi+\frac{(1+\tau)(1+\eta_e)}{4\tau}\partial_y\nabla_\perp^2\Phi \\
& +\frac{1+\eta_e}{2\tau}\partial_y\Phi+\frac{(1+\tau)^2}{\tau^2}\frac{r_n}{4\rho_e}[\hat{b}\times\nabla_\perp\Phi\cdot\nabla_\perp]\nabla_\perp^2\Phi \\
= \; & 0.
\end{aligned} \tag{22}$$

Alternatively, if one adopts the ion dynamic normalization

$$\frac{e\delta\phi}{T_i}=\Phi,\quad \frac{\partial_x n_e}{n_e}=-\frac{1}{r_n},\quad \rho_e=\sqrt{\frac{\tau m_e}{m_i}}\rho_i,\quad \frac{\vec{x}}{\rho_i}\to\vec{x},\quad \frac{\rho_i}{r_n}\omega_{ci}t\to t, \tag{23}$$

Eq.(22) becomes,

$$\begin{aligned}
& -(1-\frac{1+\tau}{2}\frac{m_e}{m_i}\nabla_\perp^2)\partial_t\Phi+\frac{1+\tau}{2}\frac{r_n^2}{\rho_i^2}\frac{m_i}{m_e}\partial_t^{-1}\nabla_\parallel^2\Phi+\frac{\tau(1+\tau)(1+\eta_e)m_e}{4m_i}\partial_y\nabla_\perp^2\Phi \\
& +\frac{1+\eta_e}{2}\partial_y\Phi+\frac{m_e(1+\tau)^2}{m_i}\frac{r_n}{4\rho_i}[\hat{b}\times\nabla_\perp\Phi\cdot\nabla_\perp]\nabla_\perp^2\Phi \\
= \; & 0.
\end{aligned} \tag{24}$$