# Impurity transport in the Pfirsch–Schlüter regime

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(Received 8 March 1974)

It is shown that the classical inward diffusion of high-$Z$ impurities in toroidal plasmas is enhanced by the Pfirsch–Schlüter effect. Numerical transport coefficients are evaluated. Typically, both density and temperature gradients are found to produce inward impurity diffusion.

In a hydrogen plasma containing high-$Z$ impurity ions, it is well known that the classical cross-field diffusion due to proton-impurity collisions produces inward transport of the impurities, until a steady state is reached in which the impurities are concentrated in the region of maximum proton density.[1] Several recent calculations have shown that in a torus the impurity transport is enhanced in the usual way in the banana[2–4] and plateau[3] regimes of neoclassical theory. Since the low-temperature region near the limiter in present (and possibly future) experiments typically falls into the Pfirsch–Schlüter (or hydrodynamic) regime, it is of interest to complete the theory with a calculation for this regime. Such is the purpose of the present note. The detailed transport equations that we derive will be useful for computational treatments of impurity transport in tokamaks.[5] We adopt a fluid-equation approach; as has been shown in detail for the electron transport problem,[6] the same results could presumably be obtained by appropriate orderings within a kinetic treatment.

We restrict ourselves to the case of a hydrogen plasma containing a single type of impurity ion. We treat the limit where the mass of the impurity ion is very large compared with the mass of the proton, so that the proton-impurity collisions may be treated by a Lorentz model. In this limit, the problem of transport of protons due to proton-impurity collisions is similar to the problem of transport of electrons due to electron-ion collisions; the latter problem has been solved in detail[7–9] for a number of values of the ionic charge $Z$. The similarity of the two problems is apparent when we consider the kinetic equation for protons. There are two collision terms in this equation: one, describing proton–proton collisions, is similar to the electron–electron collision term in the electron transport problem; the other, describing proton-impurity collisions, is similar to the electron-ion collision term in the electron transport problem. In the electron transport problem, the electron–electron collision term is proportional to $n_e$, and the electron–ion collision term is proportional to $n_iZ^2 = n_eZ$. In the problem of proton transport due to collisions with impurities, denoting the protons by $i$ and the impurities by $I$, the proton–proton collision term is proportional to $n_i$, and the proton-impurity collision term is proportional to $n_IZ_I^2$. It follows that the similarity between the two problems becomes exact if we replace $n_e$ by $n_i$, and $Z$ by

$$\alpha = n_IZ_I^2/n_i. \tag{1}$$

Following Braginskii,[9] the force balance equations are then

$$0 = -\nabla p_i + n_ie(\mathbf{E} + \mathbf{v}_i \times \mathbf{B}) + \mathbf{R}, \tag{2}$$

$$0 = -\nabla p_I + n_IZ_Ie(\mathbf{E} + \mathbf{v}_I \times \mathbf{B}) - \mathbf{R}. \tag{3}$$

The force $\mathbf{R}$ is the friction on the protons due to collisions with impurities; to describe the Pfirsch–Schlüter processes, we need keep only the component of $\mathbf{R}$ parallel to the magnetic field, namely,[8,9]

$$R_{||} = -C_1\frac{m_in_i}{\tau_{iI}}u_{||} - C_2n_i\nabla_{||}T_i, \tag{4}$$

where $\mathbf{u} = \mathbf{v}_i - \mathbf{v}_I$, $\tau_{iI} = 3m_i^{1/2}T_i^{3/2}/4(2\pi)^{1/2}n_IZ_I^2e^4\ln\Lambda$, and $C_1$ and $C_2$ are numerical coefficients depending on $\alpha$.

We take the radial components of Eqs. (2) and (3), and eliminate $E_r$ from the two equations; denoting the poloidal and toroidal components of $\mathbf{B}$ and $\mathbf{u}$ by $B_\theta$, $B_T$ and $u_\theta$, $u_T$, respectively, we obtain

$$u_\theta = \frac{B_\theta}{B_T}u_T + \frac{1}{eB_T}\left(\frac{1}{n_i}\frac{\partial p_i}{\partial r} - \frac{1}{n_IZ_I}\frac{\partial p_I}{\partial r}\right). \tag{5}$$

To lowest order in the inverse aspect ratio $\epsilon = r/R$, we have

$$u_{\theta0} = \frac{1}{eB_{T0}}\left(\frac{1}{n_i}\frac{\partial p_i}{\partial r} - \frac{1}{n_IZ_I}\frac{\partial p_I}{\partial r}\right).$$

To first order in $\epsilon$, we have $B_T = B_{T0}/(1 + \epsilon\cos\theta)$ and $B_\theta = B_{\theta0}/(1 + \epsilon\cos\theta)$; we will see that the variation of density and pressure with $\theta$ is much smaller than $\epsilon$, so that in Eq. (5) we may assume that $n_i = n_i(r)$, $p_i = p_i(r)$, $n_I = n_I(r)$, and $p_I = p_I(r)$. For continuity of mass, the flow must be incompressible: $u_\theta = u_{\theta0}/(1 + \epsilon\cos\theta)$. To first order in $\epsilon$, it follows from Eq. (5) that

$$u_T = -\frac{2r\cos\theta}{eRB_{\theta0}}\left(\frac{1}{n_i}\frac{\partial p_i}{\partial r} - \frac{1}{n_IZ_I}\frac{\partial p_I}{\partial r}\right). \tag{6}$$

The parallel components of Eqs. (2) and (3) give

$$\frac{\partial p_i}{r\,\partial\theta} - n_ieE_\theta = -\frac{\partial p_I}{r\,\partial\theta} + n_IZ_IeE_\theta = \frac{B_T}{B_\theta}R_{||}. \tag{7}$$

The $\theta$ component of Eq. (2), in which the term $R_\theta$ is small and may be neglected, gives

$$n_iv_{ir} = -\frac{1}{eB_T}\left(\frac{\partial p_i}{r\,\partial\theta} - n_ieE_\theta\right).$$

The radial proton flux, averaged over the magnetic surface, is

$$\begin{aligned}\Gamma_i &= \langle(1 + \epsilon\cos\theta)n_iv_{ir}\rangle_\theta \\ &= -\frac{1}{eB_{T0}}\left\langle(1 + \epsilon\cos\theta)^2\left(\frac{\partial p_i}{r\,\partial\theta} - n_ieE_\theta\right)\right\rangle_\theta \\ &= -\frac{2r}{eRB_{T0}}\left\langle\cos\theta\left(\frac{\partial p_i}{r\,\partial\theta} - n_ieE_\theta\right)\right\rangle_\theta.\end{aligned} \tag{8}$$

The $\theta$ component of Eq. (3) may be treated in the same way; using Eq. (7), we then have

$$\Gamma_I = -\frac{1}{Z_I}\Gamma_i. \tag{9}$$

Substituting Eq. (4) for $R_{||}$ into Eq. (7), setting $u_{||} \simeq u_T$, and using Eq. (6) for $u_T$, we have

$$\frac{\partial p_i}{r\,\partial\theta} - n_i e E_\theta + C_2 n_i \frac{\partial T_i}{r\,\partial\theta} = C_1 \frac{2m_i r B_T \cos\theta}{\tau_{iI} e R B_\theta^2}\left(\frac{\partial p_i}{\partial r} - \frac{n_i}{n_I Z_I}\frac{\partial p_I}{\partial r}\right). \tag{10}$$

Next, consider the thermal transport. Again following Braginskii,[9] the thermal conduction is

$$\mathbf{q}_{i\perp} = \frac{5}{2}\frac{n_i T_i}{eB^2}\mathbf{B}\times\nabla T_i. \tag{11}$$

The component of Eq. (11) within the magnetic surface gives

$$q_{i\theta} = \frac{B_\theta}{B_T} q_{iT} + \frac{5}{2}\frac{n_i T_i}{eB_T}\frac{\partial T_i}{\partial r}. \tag{12}$$

As before, $n_i$ and $T_i$ cannot depend on $\theta$ to first order in $\epsilon$; the energy equation then demands that $\nabla\cdot\mathbf{q}_i = 0$ to first order in $\epsilon$, i.e., that $q_{i\theta} = q_{i\theta 0}/(1+\epsilon\cos\theta)$. Equation (12) may then be treated in the same way as Eq. (5), to yield

$$q_{iT} = -\frac{5 n_i T_i r\cos\theta}{eRB_{\theta 0}}\frac{\partial T_i}{\partial r}. \tag{13}$$

The total radial energy flux in the proton component, averaged over the magnetic surface, is

$$Q_i = \langle(1+\epsilon\cos\theta)q_{ir}\rangle_\theta + \tfrac{5}{2}T_i\Gamma_i = -\frac{5 r n_i T_i}{eRB_{T0}}\left\langle \cos\theta\frac{\partial T_i}{r\,\partial\theta}\right\rangle_\theta + \tfrac{5}{2}T_i\Gamma_i, \tag{14}$$

where we have used the radial component of Eq. (11) for $q_{ir}$. The parallel thermal conduction is given by[8,9]

$$q_{i\|} = C_2 n_i T_i u_\| - C_3\frac{n_i T_i \tau_{iI}}{m_i}\nabla_\| T_i. \tag{15}$$

Setting $q_{i\|}\simeq q_{iT}$ and $u_\|\simeq u_T$, using Eq. (6) for $u_T$, and Eq. (13) for $q_{iT}$, we have

$$C_3 n_i\frac{\partial T_i}{r\,\partial\theta} = -\frac{2m_i r B_T\cos\theta}{\tau_{iI} e R B_\theta^2}\left[C_2\left(\frac{\partial p_i}{\partial r} - \frac{n_i}{n_I Z_I}\frac{\partial p_I}{\partial r}\right) - \tfrac{5}{2}n_i\frac{\partial T_i}{\partial r}\right]. \tag{16}$$

Substitution of Eq. (16) into Eq. (10) yields a solution for $\partial p_i/r\partial\theta - n_i e E_\theta$; Eq. (8) then gives the proton flux

$$\Gamma_i = -\frac{q^2\rho_i^2}{\tau_{iI}T_i}\left[\left(C_1 + \frac{C_2^2}{C_3}\right)\left(\frac{\partial p_i}{\partial r} - \frac{n_i}{n_I Z_I}\frac{\partial p_I}{\partial r}\right) - \frac{5}{2}\frac{C_2}{C_3}n_i\frac{\partial T_i}{\partial r}\right], \tag{17}$$

where $q = rB_T/RB_\theta$ and $\rho_i = (2m_iT_i)^{1/2}/eB_T$. From Eqs. (14) and (16) we have the proton energy flux

$$Q_i = -\frac{q^2\rho_i^2}{\tau_{iI}}\left[-\frac{5}{2}\frac{C_2}{C_3}\left(\frac{\partial p_i}{\partial r} - \frac{n_i}{n_I Z_I}\frac{\partial p_I}{\partial r}\right) + \frac{25}{4C_3}n_i\frac{\partial T_i}{\partial r}\right] + \tfrac{5}{2}T_i\Gamma_i. \tag{18}$$

The energy flux in the impurity component may be calculated in the same way. Equations (11) and (15) are replaced by

$$\mathbf{q}_{I\perp} = \frac{5}{2}\frac{n_I T_I}{Z_I e B^2}\mathbf{B}\times\nabla T_I, \qquad q_{I\|} = -3.9\frac{n_I T_I \tau_{II}}{m_I}\nabla_\| T_I,$$

where $\tau_{II} = 3m_I^{1/2}T_I^{3/2}/4\pi^{1/2}n_I Z_I^4 e^4\ln\Lambda$. Following the same procedure as above, we have

$$3.9 n_I\frac{\partial T_I}{r\,\partial\theta} = \frac{5 m_I n_I r B_T\cos\theta}{\tau_{II}Z_I e R B_\theta^2}\frac{\partial T_I}{\partial r}, \tag{19}$$

giving

$$Q_I = -1.6\frac{q^2\rho_I^2 n_I}{\tau_{II}}\frac{\partial T_I}{\partial r} + \tfrac{5}{2}T_I\Gamma_I, \tag{20}$$

where $\rho_I = (2m_IT_I)^{1/2}/Z_I e B_T$.

Comparing Eqs. (16) and (19), we see that the variation of $T_I$ on the magnetic surface is roughly $(m_I/m_i)^{1/2}Z_I$ times larger than the variation of $T_i$. From Eqs. (18) and (20), however, we see that the heat conduction in the impurity component is smaller than that in the proton component by a factor $n_I m_I^{1/2}/n_i m_i^{1/2}$.

The numerical coefficients $C_1$, $C_2$, and $C_3$ depend on the quantity $\alpha$ [defined in Eq. (1)] in the same way as the analogous coefficients depend on $Z$ in the electron transport problem. The results are given in Table I for a selection of values of $\alpha$: the values for $\alpha \geq 1$ are taken immediately from Ref. 8; the values for $\alpha\to 0$ have been computed following Braginskii's method of a two-term expansion in Sonine polynomials.[8] Using the values in Table I, the coefficients appearing in Eqs. (17) and (18) are very well fitted by

$$C_1 + \frac{C_2^2}{C_3} = 0.47 + \frac{0.35}{0.66+\alpha},$$
$$\frac{5}{2}\frac{C_2}{C_3} = 0.30 + \frac{0.41}{0.58+\alpha}, \tag{21}$$
$$\frac{25}{4C_3} = \frac{1.13}{\alpha} + 0.50 + \frac{0.56}{0.56+\alpha}.$$

These simple formulas could, for example, be useful in computational treatments.[5]

In cases where there is strong temperature equilibration between protons and electrons, the proton temperature variation on the magnetic surface will be smaller than that given by Eq. (16) because of the high electron parallel thermal conduction. Setting $T_e \equiv T_e(r) = T_{i0}(r)$, we may include this effect by calculating the variation of $T_i$ on the magnetic surface (denoted $\tilde{T}_i$) from $0 = -\nabla\cdot\mathbf{q}_i + Q_{ie}$ to first order in $\epsilon$; here, $Q_{ie} = -3m_e n_e\tilde{T}_i/m_i\tau_e$ where $\tau_e = 3m_e^{1/2}T_e^{3/2}/4(2\pi)^{1/2}n_i e^4\ln\Lambda$. Modifying our previous analysis in this manner, we find that the expression given in Eq. (16) for the variation of $T_i$ on the surface must be reduced by the factor

$$1 + \frac{3m_e n_e}{C_3 m_i n_i}\frac{1}{\tau_e\tau_{iI}}\frac{m_i r^2 B_T^2}{T_i B_\theta^2}.$$

TABLE I. Values of the coefficients $C_1$, $C_2$, and $C_3$ for selected values of $\alpha$.

| $\alpha$ | $C_1$ | $C_2$ | $C_3$ |
|---|---|---|---|
| $\to 0$ | 1.00 | $2.21\alpha$ | $5.52\alpha$ |
| 1 | 0.51 | 0.71 | 3.16 |
| 2 | 0.44 | 0.91 | 4.89 |
| 3 | 0.40 | 1.02 | 6.06 |
| 4 | 0.38 | 1.09 | 6.92 |
| $\infty$ | 0.29 | 1.50 | 12.47 |

This correction factor can be important at very short mean free paths. However, at the Pfirsch–Schlütter/plateau transition for the protons, it is clearly of order $1 + (m_e/m_i)^{1/2}$, and so can be neglected in most cases of practical interest.

There is, however, strong temperature equilibration between protons and impurities in many cases of practical interest, and in these cases we would set $T_{i0}(r) = T_{I0}(r)$ in the above formulas. Owing to the relatively large proton parallel thermal conduction, temperature equilibration will reduce the impurity temperature variation on the magnetic surface from that given by Eq. (19). We may include this effect by adding the term $Q_{Ii} = -3m_i n_i \tilde{T}_I/m_I \tau_{iI}$ to our analysis. With this modification, we find that the expression given in Eq. (19) for the variation of $T_I$ on the surface must be reduced by the factor

$$1 + \frac{3m_i n_i}{3.9 m_I n_I} \frac{1}{\tau_{iI}\tau_{II}} \frac{m_I r^2 B_T^2}{T_I B_\theta^2}\,.$$

This correction factor is important: At the Pfirsch–Schlüter/plateau transition for the protons $(m_i r^2 B_T^2/\tau_{iI}^2 T_i B_\theta^2 \sim 1)$ it is of order $1 + m_i^{1/2} Z_I^2 n_i / m_I^{1/2} n_I$, which is typically large. The heat conduction term in the impurity energy flux, Eq. (20), is reduced by this factor.

A few general remarks may be made. We have found that the classical inward (i.e., in the direction of $\nabla n_i$) diffusion of impurities is enhanced, in a toroidal geometry, by a Pfirsch–Schlüter factor of order $q^2$. The theory is complicated, however, by the appearance of temperature gradient terms. It is interesting to observe that, in neoclassical banana-regime theory,[2–4] the temperature gradient terms are of a sign to produce outward (i.e., in the direction of $-\nabla T_i$) diffusion of impurities. Our present calculations show, however, that this is typically not so in the Pfirsch–Schlüter regime: for, separating density and temperature gradient terms in Eq. (17), the latter (in the limit of large $Z_I$) have the coefficient $5C_2/2C_3 - C_1 - C_2^2/C_3$ which is always negative, corresponding to inward impurity diffusion. (There is one untypical exception, namely the case of an extremely low-impurity density where $\alpha \to 0$, in which case this coefficient vanishes and we are left with a small positive coefficient of order $1/Z_I$, corresponding to outward impurity diffusion.)

In summary, we conclude that the inward impurity diffusion arising from a proton density gradient is enhanced in the usual way by the Pfirsch–Schlüter effect, and that for typical cases a temperature gradient makes a further contribution to the inward diffusion.

This work was supported by U.S. Atomic Energy Commission Contract AT(11-1)-3073.


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