LETTERS

Absolute dissipative drift-wave instabilities in tokamaks

To cite this article: L. Chen *et al* 1980 *Nucl. Fusion* **20** 901

View the article online for updates and enhancements.

ABSOLUTE DISSIPATIVE DRIFT-WAVE INSTABILITIES IN TOKAMAKS*

L. CHEN, M.S. CHANCE, C.Z. CHENG (Plasma Physics Laboratory, Princeton University, Princeton, New Jersey, United States of America)

ABSTRACT. Contrary to previous theoretical predictions, it is shown that the dissipative drift-wave instabilities are absolute in tokamak plasmas. The existence of unstable eigenmodes is shown to be associated with a new eigenmode branch induced by the finite toroidal coupling.

The stability of the drift-wave eigenmode in sheared magnetic fields has been investigated intensively because of its potential importance to transport processes in magnetically confined plasmas such as those in tokamaks. Most of the theories are, however, limited to the slab model. It is now well established that in *slab* geometries, both the collisionless [1–5] and the collisional [6] (dissipative) electrostatic drift-wave eigenmodes are stable in the absence of ion temperature gradients.

The applicability of the slab approximation to tokamak plasmas is, however, not at all clear. In particular, Taylor [7] has suggested that the toroidal couplings may significantly affect the shear-damping mechanism and, thereby, the stability properties. On the other hand, stability studies in toroidal geometries [6, 8] using Taylor's strong-coupling approximation [7] indicate that the eigenmodes are, again, stable for typical values of the shear in tokamaks, i.e. $rq'/q > 1/2$. Here, $q = rB_t/RB_\theta$ is the usual safety factor. In this work, we adopt the ballooning-mode formalism [9] and investigate the stability properties of dissipative drift-wave eigenmodes in toroidal plasmas without using the strong-coupling approximation. We find, both analytically and numerically, that, in contrast to previous theoretical results, unstable eigenmodes do exist. We observe that these unstable eigenmodes are directly related to the appearance of a new *toroidicity-induced* branch [10] which experiences negligible shear damping.

Let us consider electrostatic drift waves in an axisymmetric tokamak with concentric, circular magnetic surfaces. Adopting here the usual (r, θ, ξ) co-ordinates corresponding, respectively, to the (minor) radial, poloidal and toroidal directions, we express the perturbed potential, ϕ, as

$$\phi\left(r,\theta,\xi,t\right) = \sum_j \hat{\phi}_j(s)\exp\left[i\left(m_o\theta + j\theta - n\xi - \omega t\right)\right] \tag{1}$$

where $|j| \ll |m_0|$, $s = (r-r_0)/\Delta r_s$, r_0 is the reference mode-rational surface $m_0 = nq(r_0)$, $\Delta r_s = 1/k_\theta \hat{s}$, $k_\theta = m_0/r_0$ and $\hat{s} = rq'/q$ at $r = r_0$. For simplicity, we ignore temperature gradients and consider only the resistive effects. The two-dimensional eigenmode equation can be derived in a straightforward way by using fluid descriptions for both the electrons and ions and is given by [6, 8]

$$\left[L(j,s) - f_t(j,s)\, T(j,s)\right] \hat{\phi}_j(s) = 0 \tag{2}$$

where

$$L = \left(1 - i\,\frac{\omega\,\nu_{ei}}{k_{\|}^2 v_e^2}\right) b_\theta\left(\hat{s}^2\,\frac{\partial^2}{\partial s^2} - 1\right) - 1 + \frac{\omega_{*e}}{\omega} + \frac{k_{\|}^2 c_s^2}{\omega^2} \tag{3}$$

$$f_t = \left(1 - i\omega\,\nu_{ei}/k_{\|}^2 v_e^2\right)\left(\varepsilon_n\,\omega_{*e}/\omega\right) \tag{4}$$

$$T\,\hat{\phi}_j(s) = \hat{\phi}_{j+1}(s) + \hat{\phi}_{j-1}(s) + \hat{s}\,\frac{\partial}{\partial s}\left[\hat{\phi}_{j+1}(s) - \hat{\phi}_{j-1}(s)\right] \tag{5}$$

$$k_{\|} = (s-j)/qR, \quad b_\theta = k_\theta^2\,\rho_s^2$$

$$\rho_s = c_s/\omega_{ci}, \quad c_s^2 = T_e/M_i, \quad \varepsilon_n = r_n/R$$

$r_n^{-1} = |d\ln N(r)/dr|$, and the rest of notations is standard.

* This work was supported by the U.S. Department of Energy Contract No.EY-76-C-02-3073.

In deriving Eq.(2), we also assume $\tau = T_e/T_i \gg 1$ and $|(\rho_s^2/\tau)(d^2/dr^2 - k_\theta^2)\,\phi| \ll |\phi|$. Note that T in Eq.(5) is the toroidal-coupling operator due to ion ∇B and curvature drifts.

Since, typically, $|m_0| \sim |n| \sim |r_n/\rho_s| \sim O(10^2 - 10^3)$, the large-n ordering, i.e. the ballooning-mode formalism [9] is appropriate here. In the zeroth order, we have, with $z = s - j$, $\hat{\phi}_j(s) = \Phi(z)$ and $\hat{\phi}_{j\pm1}(s) = \Phi(z \mp 1)$, i.e. the eigenmodes are composed of identical structures centred at each mode-rational surface. Equation (2) then reduces to a *one-dimensional* differential-difference equation, i.e.

$$\left[L(z) - f_t(z)\,T(z)\right]\Phi(z) = 0 \tag{6}$$

and

$$T(z)\,\Phi(z) = \Phi(z+1) + \Phi(z-1) + \hat{s}\,(d/dz)\,[\Phi(z-1) - \Phi(z+1)] \tag{7}$$

Fourier-transforming Eq.(6), we obtain the following eigenmode equation describing dissipative drift waves in toroidal plasmas;

$$\left\{\left(d^2/d\hat{\theta}^2\right)\left[d^2/d\hat{\theta}^2 + Q_1(\hat{\theta})\right] + i\alpha\, Q_2(\hat{\theta})\right\}\hat{\phi}(\hat{\theta}) = 0$$

where $\hat{\phi}$ is the Fourier transform of Φ,

$$Q_1(\hat{\theta}) = n_s^2\,\Omega^2\,P(\hat{\theta}) \tag{8}$$

$$P(\hat{\theta}) = 1 - 1/\Omega + Q_2(\hat{\theta}) \tag{9}$$

$$Q_2(\hat{\theta}) = b_\theta\left(1 + \hat{s}^2\,\hat{\theta}^2\right) + \left(2\,\epsilon_n/\Omega\right)\left(\cos\hat{\theta} + \hat{s}\,\hat{\theta}\,\sin\hat{\theta}\right) \tag{10}$$

$$\Omega = \omega/\omega_{*e},\quad n_s^2 = b_\theta\,q^2/\epsilon_n^2$$

$$\alpha = \bar{\nu}\,\Omega^3\left(q^2\,b_\theta/\epsilon_n^2\right)^2$$

and

$\bar{\nu} = \nu_{ei}\,m_e/\omega_{*e}\,m_i$. The boundary condition imposed on Eq.(7) is that the unstable (Im $\Omega > 0$) eigenmodes decay asymptotically as $|\hat{\theta}| \to \infty$.

We first consider the $\alpha \propto \nu_{ei} = 0$ limit [10]. Here, the electron response is adiabatic and the relevant eigenmode equation is given by

$$\left[d^2/d\hat{\theta}^2 + Q_1(\hat{\theta})\right]\hat{\phi}(\hat{\theta}) = 0 \tag{11}$$

The boundary condition is then $\hat{\phi}(\hat{\theta}) \to \exp(i\Omega\eta_s b_\theta^{1/2}\hat{s}\,\hat{\theta}^2/2)$ as $|\hat{\theta}| \to \infty$, i.e. the wave energy is outward-propagating. Furthermore, an examination of $Q_1(\hat{\theta})$, as defined by Eq.(8), indicates that the potential structure consists of a parabolic anti-well plus modulations due to toroidal coupling. Equation (11) has been analysed by using both the interactive WKB [11] and numerical shooting codes. In Fig.1, we plot the eigenmode frequencies $\Omega = \Omega_r + i\Omega_i$ versus the toroidicity ϵ_n for $b_\theta = 0.1$, $\hat{s} = 1$, and $q = 1$ for the lowest eigenstate. The results clearly show the existence of two eigenmode branches. One is a slab-like branch and the other is a new branch induced by the finite toroidicity. The slab-like eigenmodes, similar to the Pearlstein-Berk modes [12] found in the slab limit, correspond to unbounded eigenstates with anti-well potential structures and, hence, experience finite shear damping due to (free) outward energy convection. In fact, as is shown in Fig.1, inclusion of toroidal coupling further

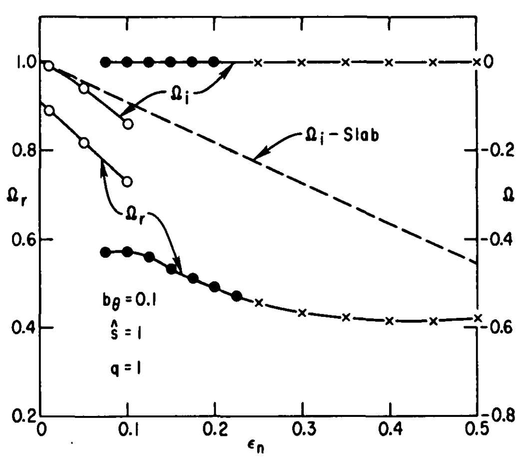

FIG.1. Eigenmode frequencies Ω versus ϵ_n in the $\nu_{ei} = 0$ limit. ○, ●, and X correspond, respectively, to the slab-like, weak, and strong, toroidicity-induced eigenmodes. Ω_i in the 'slab' limit is also shown, where the toroidal coupling term, $(2\epsilon_n/\Omega)(\cos\hat{\theta} + \hat{s}\hat{\theta}\sin\hat{\theta})$, in Eq.(11) is suppressed.

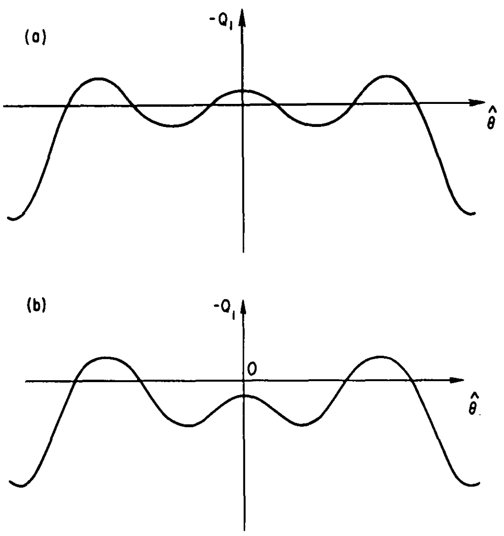

FIG.2. Typical potential structures, $-Q_1$, for the (a) weak and (b) strong toroidicity-induced eigenmodes in the $\nu_{ei}=0$ limit.

enhances the shear damping rates. The toroidicity-induced (T–I) eigenmodes, however, experience negligible shear damping; typically, we find $-\Omega_i \sim O(10^{-3}-10^{-4})$. Typical potential structures corresponding to the weak (smaller ϵ_n) and strong (large ϵ_n) T–I eigenmodes are shown in Fig.2. It is clear from Fig.2 that the T–I eigenmodes correspond to eigenstates quasi-bounded by local potential wells induced by the finite toroidal coupling. The shear damping is negligible here because the convection of wave energies occurs only through the tunnelling leakage. In this respect, the eigenmodes are *quasi-marginally stable.* It is interesting to note that for a certain parameter regime both eigenmode branches can exist simultaneously. Furthermore, we note that the slab-like eigenmodes, having turning points $\pm\theta_t$ close to $\hat{\theta}=0$ (i.e. $|\theta_t| \ll 1$), can be understood by using Taylor's strong-coupling approximation, and the eigenmodes remain damped for $\nu_{ei} \neq 0$ [6]. In this letter, we, therefore, concentrate on the T–I eigenmodes.

We now consider the effects of finite electron resistive dissipation ($\nu_{ei} \neq 0$) on the T–I eigenmodes. For the purpose of this letter, we assume ν_{ei} to be small and perform a perturbative analysis on the weak T–I eigenmodes. Since the tunnelling effects are small, they may be ignored in the present perturbation theory. The corresponding potential structure in the $\nu_{ei}=0$ limit (Fig.2a) then suggests that the eigenmodes can be assumed to be localized at $\hat{\theta}=\theta_0$ where $\theta_0 \neq 0$ and $Q_1'(\theta_0)=0$, i.e.

$$\theta_o b_\theta \hat{s}^2 + (\epsilon_n/\Omega)\left[(\hat{s}-1)\sin\theta_o + \hat{s}\,\theta_o \cos\theta_o\right] = 0 \tag{12}$$

Let $\eta=\hat{\theta}-\theta_0$ and expand Q_1 and Q_2 about $\hat{\theta}=\theta_0$ to $O(\eta^2)$; Eq.(7) becomes

$$\left[(d^2/d\eta^2)(d^2/d\eta^2 + Q_{10} + Q_{10}''\,\eta^2/2) + i\alpha(Q_{20} + Q_{20}''\,\eta^2/2)\right]\hat{\phi}(\eta) = 0 \tag{13}$$

where

$$(Q_{1,2})_0 \equiv Q_{1,2}(\theta_o)$$

and $\mathrm{Re}\, Q_{10}'' < 0$. Setting

$$\phi(t) = \int_{-\infty}^{\infty} d\eta\, \hat{\phi}(\eta)\, \exp(i\eta t)$$

in Eq.(13), we obtain

$$\left[(Q_{10}''/2)(t^2 - it_k^2)\, d^2/dt^2 + t^4 - Q_{10}t^2 + i\alpha\, Q_{20}\right]\phi(t) = 0 \tag{14}$$

Here

$$t_k^2 = \alpha\, Q_{20}''/Q_{10}'' = \alpha/\eta_s^2\, \Omega^2.$$

Eq.(14) can be written as

$$\left[d^2/dy^2 + \lambda - y^2 - i\Lambda/(y^2 - iy_k^2)\right]\phi(y) = 0 \tag{15}$$

where

$$y = t/\beta, \quad \beta = \left(-Q_{10}''/2\right)^{1/4}$$

$$\mathrm{Re}\ \beta > 0, \quad y_k^2 = t_k^2/\beta^2$$

$$\lambda = \left(Q_{10} - it_k^2\right)/\beta^2$$

and

$$\Lambda = \left[\alpha\left(1/\Omega - 1\right) + it_k^4\right]/\beta^4$$

Noting that $|\Lambda| \propto |\alpha| \propto \nu_{ei}$, a perturbative treatment of Eq.(15) can be readily done and we find for the lowest eigenstate, $\lambda = 1 + \lambda_1$, where $\lambda_1 = -\sqrt{\pi}\,(\Lambda/y_k)\exp(-i\pi/4)$. The dispersion relation is then

$$P\left(\theta_0\right) \simeq \Gamma\left(1 + \lambda_1 - \delta\right) \tag{16}$$

Here, $\Gamma = (\beta/\eta_s\Omega)^2$ and δ is included to represent the tunnelling effects.

To further analyse Eq.(16), we need to solve θ_0 from Eq.(12). For this purpose, we note that the T–I eigenmode branch generally exists for $|\epsilon_n/\Omega| > |b_\theta \hat{s}|$ and $|\theta_0| > 1$ for $\hat{s} \sim 1$, so that $\theta_0 \cong \pi/2$. Thus,

$$P\left(\theta_0\right) \simeq 1 + b_\theta\left(1 + \hat{s}^2\pi^2/4\right) - 1/\Omega + \epsilon_n \pi \hat{s}/\Omega \tag{17}$$

$$\Gamma \simeq \left(\epsilon_n \hat{s}/q\Omega\right)\left(\epsilon_n \pi/2\ \Omega b_\theta \hat{s} - 1\right)^{1/2} \tag{18}$$

and

$$\lambda_1 = -\left(1/\Omega - 1\right)\left(\pi\bar{\nu}\,\Omega/\Gamma^3\right)^{1/2}\exp\left(-i\pi/4\right) \tag{19}$$

For parameters of interest here, we note that $|\Omega\Gamma| < 1$. Thus, we have, with $\Omega = \Omega_r + i\Omega_i$ and $|\Omega_i/\Omega_r| < 1$,

$$\Omega_r \simeq \left(1 - \pi\epsilon_n\hat{s}\right)/\left[1 + b_\theta\left(1 + \hat{s}^2\pi^2/4\right)\right] \tag{20}$$

and

$$\Omega_i = \gamma - \gamma_t \tag{21}$$

where

$$\gamma = \left(1 - \Omega_r\right)\left(\pi\bar{\nu}/2\Omega_r\Gamma\right)^{1/2}/\left[1 + b_\theta\left(1 + \hat{s}^2\pi^2/4\right)\right] \tag{22}$$

and γ_t is the small shear damping rate due to the tunnelling, which can be estimated by examining the $\nu_{ei} = 0$ limit [10]. Note that we have calculated Ω_r only to lowest order so that it is independent of the collision $\bar{\nu}$. This is acceptable since we are concerned mainly with the growth rate in this paper.

Equation (21) shows that electron resistive dissipation can destabilize the T–I eigenmodes if $\bar{\nu} > \bar{\nu}_c$, where

$$\bar{\nu}_c = \left(\frac{\nu_{ei}\, m_e}{\omega_{*e}\, m_i}\right)_c = \frac{2\gamma_t^2}{\pi}\left[1 + b_\theta\left(1 + \frac{\hat{s}^2\pi^2}{4}\right)\right]^2\left(\frac{\Gamma}{1-\Omega_r}\right) \tag{24}$$

For $\bar{\nu} \gg \bar{\nu}_c$, we have $\gamma \propto \bar{\nu}^{1/2} \propto \nu_{ei}^{1/2}$, i.e. the growth rates of unstable eigenmodes scale as $\nu_{ei}^{1/2}$. We note that the above perturbative analysis is valid for $|\lambda_1| < 1$, i.e. $\bar{\nu} < \bar{\nu}_p = (\Omega_r\Gamma)^3(1 - \Omega_r)^{-2}/\pi$.

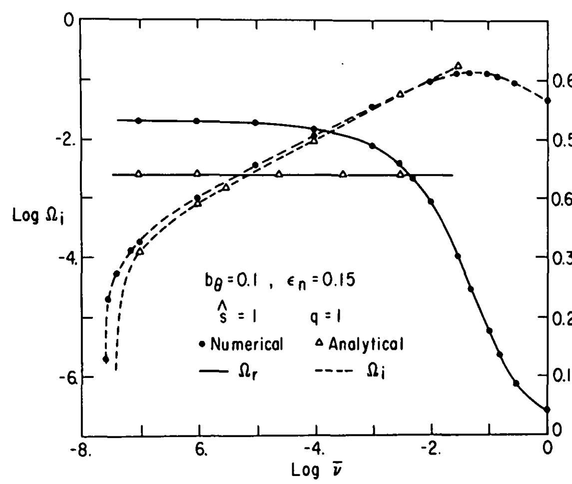

FIG.3. A plot of $\Omega = \Omega_r + i\Omega_i$, $\Omega_i > 0$, versus $\bar{\nu} = \nu_{ei} m_e/\omega_{*e} m_i$.

Finally, we have also solved Eq.(7) numerically in order to verify as well as extend the above analytical results. Figure 3 plots Ω versus the resistivity parameter $\bar{\nu} = \nu_{ei} m_e/\omega_{*e} m_i$ for the case $b_\theta = 0.1$, $\epsilon_n = 0.15$, $q = \hat{s} = 1$. The numerical results clearly demonstrate the properties predicted analytically, i.e. (i) the eigenmode is destabilized for $\bar{\nu} > \bar{\nu}_c \cong 2.5 \times 10^{-8}$, and (ii) $\gamma \propto \bar{\nu}^{1/2}$ for $10^{-2} \gg \bar{\nu} \gg \bar{\nu}_c$. For the present case, Eqs (20) and (21) predict that $\Omega_r \cong 0.4$ and $\gamma \cong \bar{\nu}^{1/2} - \gamma_t$ with $\gamma_t \cong 1.9 \times 10^{-4}$. To obtain a quantitative comparison, we have also plotted the analytical results in Fig.3 up to the perturbation limit $\bar{\nu}_p \cong 0.03$. The agreement is reasonably good. The apparent differences between the agreements of the computed real and imaginary parts with the analysis disappear if the real parts are also plotted on a logarithmic scale and also if the $\bar{\nu}$ dependence is included in Ω_r. For $\bar{\nu} \gtrsim \bar{\nu}_p$, numerical results show that Ω_i starts decreasing with $\bar{\nu}$. We note here that, since the fully collisional equation, Eq.(7), is of fourth order, there exists another solution but it can be shown to be unphysical by examining the $\bar{\nu} \to 0^+$ limit and re-scaling the variables. The details of this analysis as well as the analytical theory for the large-$\bar{\nu}$ limit and a more complete presentation of the numerical results will be published elsewhere.

ACKNOWLEDGEMENT

The authors appreciate useful discussions with Dr. E.J. Valeo.

REFERENCES

[1] ROSS, D.W., MAJAJAN, S.M., Phys. Rev. Lett. **40** (1978) 324; see also TSANG, K.T., CATTO, P.J., WHITSON, J.C., SMITH, J., Phys. Rev. Lett. **40** (1978) 327.

[2] CHEN, Liu, GUZDAR, P.N., WHITE, R.B., KAW, P.K., OBERMAN, C., Phys. Rev. Lett. **41** (1978) 649.

[3] ANTONSEN, T.M., Jr., Phys. Rev. Lett. **41** (1978) 33.

[4] LEE, Y.C., CHEN, Liu, Phys. Rev. Lett. **42** (1979) 708.

[5] LEE, Y.C., CHEN, Liu, NEVINS, W.M., Nucl. Fusion **20** (1980) 482.

[6] CHEN, Liu, GUZDAR, P.N., HSU, J., KAW, P.K., OBERMAN, C., WHITE, R., Nucl. Fusion **19** (1979) 373; see also GUZDAR, P.N., CHEN, Liu, KAW, P.K., OBERMAN, C., Phys. Rev. Lett. **40** (1978) 1566.

[7] TAYLOR, J.B., in Plasma Physics and Controlled Nuclear Fusion Research (Proc. 6th Int. Conf. Berchtesgaden, 1976) Vol.2, IAEA, Vienna (1977) 323.

[8] TANG, W.M., Nucl. Fusion **18** (1978) 1089; see also HORTON, W., ESTES, R.D., KWAK, H., CHOI, D.I., Phys. Fluids **21** (1978) 1366.

[9] CONNOR, J.W., HASTIE, R.J., TAYLOR, J.B., Proc. R. Soc. **A 365** (1979) 1; also GLASSER, A.H., in Finite Beta Theory (Proc. Workshop Varenna, 1977) (COPPI, B., SADOWSKI, W., Eds); LEE, Y.C., Van DAM, J.W., ibid.

[10] Detailed discussions on the eigenmodes in this limit are given by CHEN, Liu, CHENG, C.Z., Princeton Plasma Physics Laboratory Report PPPL-1562 (1979); similar results are also obtained independently by HASTIE, R.J., HESKETH, K.W., TAYLOR, J.B., Nucl. Fusion **19** (1979) 1223, as well as by CHOI, D.J., HORTON, W., Phys. Fluids **23** (1980) 356.

[11] WHITE, R.B., J. Comput. Phys. **31** 3 (1979) 409.

[12] PEARLSTEIN, L.D., BERK, H.L., Phys. Rev. Lett. **23** (1969) 220.

(Manuscript received 19 December 1979
Final manuscript received 14 May 1980)