Quasilinear Theory of Anomalous Transport in Axisymmetric Tokamaks

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1 Introduction

2 Tokamak Geometry

For simplicity, we consider an axisymmetric, large aspect-ratio, circular tokamak. This gives the following definition for the equilibrium magnetic field,

$$\boldsymbol{B} = B_\theta \hat{\boldsymbol{e}}_\theta + B_\zeta \hat{\boldsymbol{e}}_\zeta = B_\theta \hat{\boldsymbol{e}}_\theta + B_0(1-\epsilon\cos\theta)\hat{\boldsymbol{e}}_\zeta = B_0[\frac{\epsilon}{q}\hat{\boldsymbol{e}}_\theta + (1-\epsilon\cos\theta)\hat{\boldsymbol{e}}_\zeta], \tag{1}$$

where $\epsilon = \frac{r}{R_0} \ll 1$ is the inverse aspect ratio, with $R = R_0 + r\cos\theta$, for r the minor radius, and R_0 the major radius, and q $\simeq \frac{rB_\zeta}{R_0 B_\theta} \sim 1$ is the safety factor - the number of toroidal turns required for one poloidal turn of magnetic field lines. The term $\epsilon\cos\theta$ in R takes into account the change in toroidal radius along the tokamak midplane. Working to $\mathcal{O}(\epsilon)$ at the most, the magnetic field magnitude, magnetic field unit vector, and toroidal nablaient terms can be written as,

$$B = \sqrt{\boldsymbol{B}\cdot\boldsymbol{B}} = \sqrt{B_0^2[(1-\epsilon\cos\theta)^2 + (\frac{\epsilon}{q})^2]} = B_0\sqrt{1-2\epsilon\cos\theta} \simeq B_0(1-\epsilon\cos\theta), \tag{2}$$

$$\hat{\boldsymbol{b}} = \frac{\boldsymbol{B}}{B} = \frac{\frac{\epsilon}{q}\hat{\boldsymbol{e}}_\theta + (1-\epsilon\cos\theta)\hat{\boldsymbol{e}}_\zeta}{1-\epsilon\cos\theta} \simeq \frac{\epsilon}{q}(1+\epsilon\cos\theta)\hat{\boldsymbol{e}}_\theta + \hat{\boldsymbol{e}}_\zeta \simeq \frac{\epsilon}{q}\hat{\boldsymbol{e}}_\theta + \hat{\boldsymbol{e}}_\zeta, \tag{3}$$

$$\nabla = \partial_r \hat{\boldsymbol{e}}_r + \frac{1}{r}\partial_\theta \hat{\boldsymbol{e}}_\theta + \frac{1}{R}\partial_\zeta \hat{\boldsymbol{e}}_\zeta = \partial_r \hat{\boldsymbol{e}}_r + \frac{1}{r}\partial_\theta \hat{\boldsymbol{e}}_\theta + \frac{1}{R_0 + r\cos\theta}\partial_\zeta \hat{\boldsymbol{e}}_\zeta\ . \tag{4}$$

3 Gyrokinetics

$\boldsymbol{Talkaboutgyrophase - averagingandguidingcentercoordinates.}$

3.1 Vlasov Equation

The perturbed, gyrokinetic distribution function is given as a combination of adiabatic and non-adiabatic terms,[1]

$$\delta F = \frac{q}{m}\delta F_a + \delta G, \tag{5}$$

where,

$$\delta F_a = [\delta\Phi \frac{\partial}{\partial \epsilon^*} + (\delta\Phi - \frac{v_\parallel \delta A_\parallel}{c})\frac{\partial}{B\partial\mu}]F_0, \tag{6}$$

$$\delta G_0 = -\frac{q}{m}\langle \delta L\rangle_\alpha \frac{\partial}{B\partial\mu} + \delta H_0, \tag{7}$$

$$\langle \ldots \rangle_\alpha = \frac{1}{2\pi}\int_0^{2\pi}(\ldots)d\alpha, \tag{8}$$

with α as the gyro-phase angle, $\delta L = \delta\Phi - \frac{\boldsymbol{v}\cdot\delta\boldsymbol{A}}{c}$, $\epsilon^* = \frac{v^2}{2} + \frac{q\Phi_0}{m}$, and $\mu = \frac{v_\perp^2}{2B}$. Higher order terms in δG, the perturbed, non-adiabatic distribution function, are dropped. We can simplify things further by choosing a Maxwellian equilibrium distribution function, F_0, so that it only depends on ϵ^* and not μ. This gives us a final distribution function,

$$\delta F = \frac{q}{m}\delta\Phi \frac{\partial}{\partial\epsilon^*}F_0 + \delta H_0 \ . \tag{9}$$

This distribution function can be plugged into the Vlasov equation and gyrophase-averaged to give the standard gyrokinetic Vlasov equation for a species j,[1]

$$\begin{aligned}
&\partial_t \delta H_0 + v_\parallel \nabla_\parallel \delta H_0 + (\boldsymbol{v}_d + \frac{c\hat{\boldsymbol{b}}\times\nabla_X\langle\delta\Phi\rangle_\alpha}{B})\cdot\nabla_X \delta H_0 \\
&= -\frac{e_j}{m_j}[\partial_t\langle\delta\Phi\rangle_\alpha \partial_{\epsilon^*}F_0 - \frac{1}{\omega_{cj}}(\nabla_X\langle\delta\Phi\rangle_\alpha \times \hat{\boldsymbol{b}})\cdot\nabla_X F_0],
\end{aligned} \tag{10}$$

where $\boldsymbol{v}_d$, the sum of magnetic curvature and gradient drift terms, is defined as,

$$\boldsymbol{v}_d = \frac{v_\parallel^2 + \frac{1}{2}v_\perp^2}{\omega_{cj}}\frac{\boldsymbol{B}\times\nabla B}{B^2}, \tag{11}$$

with, simplifyingto lowest order in ϵ,

$$\begin{aligned}
\frac{\boldsymbol{B}\times\nabla B}{B^2} &= \frac{B_0[(1-\epsilon\cos\theta)\hat{\boldsymbol{e}}_\zeta + \frac{\epsilon}{q}\hat{\boldsymbol{e}}_\theta]\times(\partial_r\hat{\boldsymbol{e}}_r + \frac{1}{r}\partial_\theta\hat{\boldsymbol{e}}_\theta + \frac{1}{R}\partial_\zeta\hat{\boldsymbol{e}}_\zeta)B_0(1-\epsilon\cos\theta)}{B_0^2(1-\epsilon\cos\theta)^2} \\
&= \frac{[1-\epsilon\cos\theta)\hat{\boldsymbol{e}}_\zeta + \frac{\epsilon}{q}\hat{\boldsymbol{e}}_\theta]\times[-\frac{1}{R_0}\cos\theta\hat{\boldsymbol{e}}_r + \frac{r}{rR_0}\sin\theta\hat{\boldsymbol{e}}_\theta]}{(1-\epsilon\cos\theta)^2} \\
&= \frac{1}{(1-\epsilon\cos\theta)^2}[-\frac{(1-\epsilon\cos\theta)\cos\theta}{R_0}(\hat{\boldsymbol{e}}_\zeta\times\hat{\boldsymbol{e}}_r) - \frac{(1-\epsilon\cos\theta)\cos\theta}{R_0}(\hat{\boldsymbol{e}}_\zeta\times\hat{\boldsymbol{e}}_\theta) \\
&\qquad - \frac{\epsilon}{qR_0}\cos\theta(\hat{\boldsymbol{e}}_\theta\times\hat{\boldsymbol{e}}_r)] \\
&\simeq (1+2\epsilon\cos\theta)[-\frac{\cos\theta}{R_0}\hat{\boldsymbol{e}}_\theta - \frac{\sin\theta}{R_0}\hat{\boldsymbol{e}}_r] \simeq -\frac{1}{R_0}(\sin\theta\hat{\boldsymbol{e}}_r + \cos\theta\hat{\boldsymbol{e}}_\theta)
\end{aligned} \tag{12}$$

The second and third terms on the left-hand side of (10) can be simplified to lowest order in ϵ using (1)-(4) and (11)-(12) as,

$$\begin{aligned}
v_\parallel\nabla_\parallel &= v_\parallel(\hat{\boldsymbol{b}}\cdot\nabla) = v_\parallel(\frac{\epsilon}{q}\hat{\boldsymbol{e}}_\theta + \hat{\boldsymbol{e}}_\zeta)\cdot(\partial_r\hat{\boldsymbol{e}}_r + \frac{1}{r}\partial_\theta\hat{\boldsymbol{e}}_\theta + \frac{1}{R_0 + r\cos\theta}\partial_\zeta\hat{\boldsymbol{e}}_\zeta) \\
&= v_\parallel(\frac{\epsilon}{qr}\partial_\theta + \frac{1}{R}\partial_\zeta) = v_\parallel(\frac{1}{qR_0}\partial_\theta + \frac{1}{R}\partial_\zeta) = \frac{v_\parallel}{qR}(\frac{R}{R_0}\partial_\theta + q\partial_\zeta) \\
&= \frac{v_\parallel}{qR}((1+\epsilon\cos\theta)\partial_\theta + q\partial_\zeta) \simeq \frac{v_\parallel}{qR}(\partial_\theta + q\partial_\zeta),
\end{aligned} \tag{13}$$

$$
\begin{aligned}
\boldsymbol{v}_d \cdot \nabla_X &= -\frac{v_\parallel^2 + \frac{1}{2}v_\perp^2}{\omega_{cj}}(\sin\theta\hat{\boldsymbol{e}}_r + \cos\theta\hat{\boldsymbol{e}}_\theta) \cdot (\partial_r\hat{\boldsymbol{e}}_r + \frac{1}{r}\partial_\theta\hat{\boldsymbol{e}}_\theta) \\
&= -\frac{v_\parallel^2 + \frac{1}{2}v_\perp^2}{\omega_{cj}}(\sin\theta\partial_r + \frac{\cos\theta}{r}\partial_\theta)
\end{aligned}
\tag{14}
$$

Note that we have dropped the non-linear $\boldsymbol{E}\times\boldsymbol{B}$ drift term on the left-hand side of (10) because we are interested in linearizing this equation.

3.2 Non-Adiabatic Distribution Function

Using a WKB ansatz the gyrophase-averaged terms can be simplified as,

$$
\langle A(\boldsymbol{x})\rangle_\alpha = J_0(\frac{k_\perp v_\perp}{\omega_{cj}})A(\boldsymbol{X}) \tag{15}
$$

References

[1] Frieman, Chen 1982