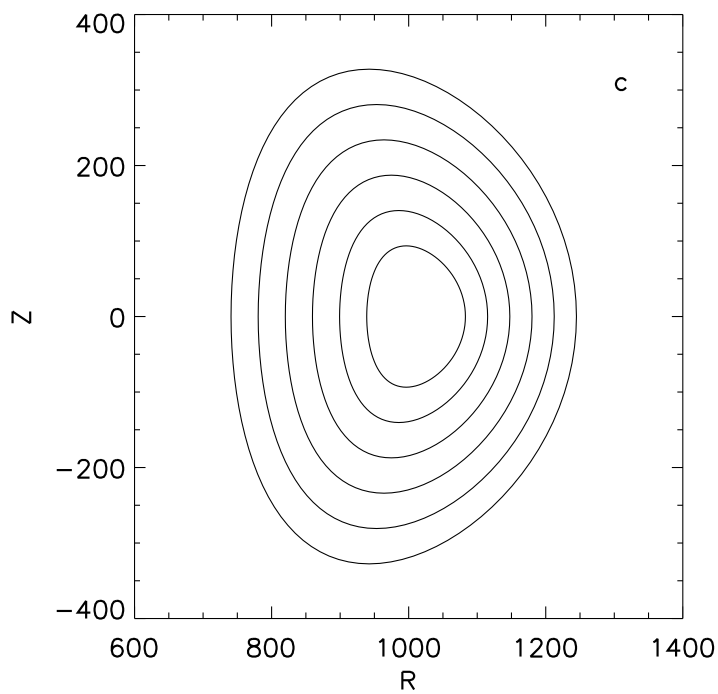
1 Field-Line-Following Coordinates



The toroidal coordinates (r, θ, ζ)

r is a flux surfacee label. In GEM it is defined to be $(R_+ - R_-)/2$, R_- and R_+ are the two points the surface intersects the mid-plane.

θ is an arbitrary "poloidal angle", $-\pi \le \theta \le \pi$. In the interface with EQDSK file, it is the geometrical angle with

$$\tan\theta = (Z - Z_0)/(R - R_0)$$

(r, θ, ζ) is right-handed.

A general axisymmetric equilibrium magnetic field is given by

$$
\begin{aligned}
\mathbf{B} &= \frac{f(\psi)}{R}\hat{\zeta} + \nabla\zeta \times \nabla\psi \\
&= q(\psi)\nabla\psi \times \nabla\theta_f + \nabla\zeta \times \nabla\psi,
\end{aligned}
\tag{1}
$$

The equilibrium magnetic field lines form nested flux-surfaces, $2\pi\Psi(R, Z)$ is the poloidal magnetic flux inside a flux-surface.

The poloidal plasma current is proportional to $f(\Psi)$, a *flux label*. It is usually given as 1D array for RB_T, with other 1D arrays such as q, density and temperature profiles.

θ_f is the straight field line flux coordinate.

The safety factor is $q(\psi) = d\chi/d\psi = \mathbf{B}\cdot\nabla\zeta/\mathbf{B}\cdot\nabla\theta_f$, $2\pi\chi$ is the toroidal magnetic fluxes. It is the poloidal average of the local field line tilt,

$$
\hat{q}(r,\theta) = \frac{\mathbf{B}\cdot\nabla\zeta}{\mathbf{B}\cdot\nabla\theta}
\tag{2}
$$

$$
q(r) = \frac{1}{2\pi}\oint \hat{q}\,d\theta
\tag{3}
$$

Eq. 3 can be written as

$$
2\pi = \oint \frac{f}{Rq\psi'(r)}\frac{1}{\hat{\zeta}\cdot\nabla r \times \nabla\theta}d\theta
\tag{4}
$$

This equation is used to determine $\psi'(r)$ once the flux-surface shape and $f(\psi)$ is given. The flux-surface shape is specified as

$$
\begin{cases}
R(r,\theta) \\
Z(r,\theta)
\end{cases}
\tag{5}
$$

which can be given in analytic form (Miller model) or numerical tables constructed from the eqdsk file with the `gcrz` program.

Since (r, θ, ζ) is right-handed, $\hat{\zeta}\cdot\nabla r \times \nabla\theta > 0$, Eq. 4 implies

$$
\frac{f}{q\psi'} > 0
\tag{6}
$$

The directions of the toroidal field $\mathbf{B}_T$, the poloidal field $\mathbf{B}_p$ and the toroidal plasma current I_p are determined by the signs of f and q, as shown in Table I. This sign relationship is consistent with the often used large aspect ratio expression, $q \approx rB_T/RB_\theta$.

$f > 0,\ q > 0$	$\mathbf{B}_T \parallel \hat{\zeta}$	$I_p \parallel \hat{\zeta}$	$\mathbf{B}_p \parallel \hat{\theta}$
$f > 0,\ q < 0$	$\mathbf{B}_T \parallel \hat{\zeta}$	$-I_p \parallel \hat{\zeta}$	$-\mathbf{B}_p \parallel \hat{\theta}$
$f < 0,\ q > 0$	$-\mathbf{B}_T \parallel \hat{\zeta}$	$-I_p \parallel \hat{\zeta}$	$-\mathbf{B}_p \parallel \hat{\theta}$
$f < 0,\ q < 0$	$-\mathbf{B}_T \parallel \hat{\zeta}$	$I_p \parallel \hat{\zeta}$	$\mathbf{B}_p \parallel \hat{\theta}$

Table 1: Direction of magnetic field and plasma current

The straight-field-line poloidal angle is

$$\theta_f = \int_0^\theta \frac{f}{Rq}\frac{1}{\hat{\zeta}\cdot\nabla\psi\times\nabla\theta}d\theta \tag{7}$$

The field-line-following coordinates

(x, y) are field-line label, meaning

$$\mathbf{b}\cdot\nabla x = \mathbf{b}\cdot\nabla y = 0$$

$\mathbf{b}$ is the unit vector along the equilibrium magnetic field.
x is chosen to label the flux-surface,

$$x = r - r_0$$

y is chosen to be proportional to $q\theta_f - \zeta$, as

$$\begin{aligned}
&\mathbf{B}\cdot\nabla(q\theta_f - \zeta)\\
&= (q(\psi)\nabla\psi\times\nabla\theta_f + \nabla\zeta\times\nabla\psi)\cdot(q\nabla\theta_f - \nabla\zeta)\\
&= 0
\end{aligned} \tag{8}$$

Field-line-following coordinates

$$\begin{cases} x = r - r_0 + L_x/2\\ y = \mathrm{mod}\left(\frac{r_0}{q_0}(q\theta_f - \zeta), L_y\right)\\ z = q_0R_0\left(\theta + \pi q_0/\mid q_0\mid\right)\end{cases} \tag{9}$$

The coefficients for y and z are chosen such that they have the dimension of length.

$$\zeta = \mathrm{mod}\left(\int_0^\theta \hat{q}(r,\theta')d\theta' - \frac{q_0}{r_0}y, 2\pi/\mathsf{lymult}\right) \tag{10}$$

will be used in the $n = 1$ field solver, where the field is Fourier transformed in (θ_f, ζ).

Physical quantities are periodic in (θ, ζ), which mean periodicity in y, but not periodic in z. There is a shift in y as the toroidal coordinate is changed from $(\theta = -\pi, \zeta)$ to $(\theta = \pi, \zeta)$,

$$y(r, \theta = -\pi, \zeta) = \mathrm{mod}\left[\frac{r_0}{q_0}\left(\int_0^{-\pi} \hat{q}(r,\theta')d\theta' - \zeta\right), L_y\right] \tag{11}$$

$$\begin{aligned} y(r, \theta = \pi, \zeta) &= \mathrm{mod}\left[\frac{r_0}{q_0}\left(\int_0^{\pi} \hat{q}(r,\theta')d\theta' - \zeta\right), L_y\right] \\ &= \mathrm{mod}\left[\frac{r_0}{q_0}\left(\int_0^{-\pi} \hat{q}(r,\theta')d\theta' - \zeta + 2\pi q(r)\right), L_y\right] \\ &= \mathrm{mod}\left[y(r, \theta = -\pi, \zeta) + 2\pi\frac{r_0}{q_0}q(r), L_y\right] \end{aligned} \tag{12}$$

Particles moving outside of the simulation box in z will be relocated at the other end, with its y-coordinate shifted by $2\pi r_0 q(r)/q_0$.

A y-grid at $z = 0$ will map to a y value at $z = Lz$ that in general will not coincide with a grid. Linear interpolation is used to obtain the value of a field variable from neighboring grid points. The weights used for linear interpolation are calculated in subroutine weight.

For particle load ballance, z is transformed to $z' = z'(z)$ so that the number of particles in each z'-domain is about constant. The array jfn $= dz/dz'$ then appears in all metric functions.

2 Flux-tube option

- A flux-tube model is made by removing the radial dependence of all equilibrium quantitties

$$
\begin{aligned}
f(r) &= f(r_0) \\
\Psi'(r) &= \Psi'(r_0) \\
T_i(r) &= T_i(r_0) \\
\kappa_{\mathrm{Ti}}(r) &= \kappa_{\mathrm{Ti}}(r_0) \\
R(r,\theta) &= R(r_0,\theta) \\
\left.\frac{\partial B}{\partial \theta}\right|_{r,\theta} &= \left.\frac{\partial B}{\partial \theta}\right|_{r_0,\theta} \\
(\nabla r \cdot \nabla \theta)|_{r,\theta} &= (\nabla r \cdot \nabla \theta)|_{r_0,\theta}
\end{aligned}
$$

- Then use a linear profile for $q(r)$, which is used only for the boundary condition
$$q(r) = q_0 + q'(r - r_0)$$
- Assume perturbations to be periodic in x

Connection to the ballooning representation

Consider linear simulation of a single mode k_y

$$k_y = n\frac{q_0}{r_0}$$

Shift in y across the z-boundary

$$\triangle y(r) = \frac{r_0}{q_0}2\pi[q_0 + (x - L_x/2)q'] = 2\pi r_0 - \pi\hat{s}L_x + 2\pi\hat{s}x$$

Assume at $\theta = -\pi$ the electric potentential is

$$\phi(\theta = -\pi) = e^{\iota k_x x}e^{\iota k_y y}$$

$$k_x = i\frac{2\pi}{L_x}, \qquad i = 0, 1, 2, \ldots$$

Applying the boundary condition, ϕ becomes at $\theta = \pi$,

$$\phi(\theta = \pi) = e^{\iota(k_x - 2\pi\hat{s}k_y)x} e^{\iota k_y y} e^{-\iota k_y(2\pi r_0 - \pi\hat{s}Lx)}$$

Periodidcity in x at $\theta = \pi$ requires that

$$2\pi\hat{s}k_y L_x = 2\pi j$$

choose the smallest box-size in x compatible with periodicity

$$L_x = \frac{1}{nq'}$$

At $\theta = \pi$ the index of radial mode number is reduced by 1. If we write the Fourier expansion of $\phi(x, k_y)$ at $\theta = -\pi$ as

$$\phi(x, k_y, \theta = -\pi) = \sum_i \phi_i e^{\iota 2\pi i x / L_x}$$

then

$$\phi(x, k_y, \theta = \pi) = \sum_i C\phi_{i+1} e^{\iota 2\pi i x / L_x}$$

$$C = e^{-\iota k_y(2\pi r_0 - \pi\hat{s}L_x)} = -e^{-\iota 2\pi n q_0}$$

and the following sequence is continuous

$$\begin{array}{c}
\dots \\
\dots \\
C^{-2}\phi_{-2}(z_0), \quad C^{-2}\phi_{-2}(z_1), \quad C^{-2}\phi_{-2}(z_2), \quad \dots, \quad C^{-2}\phi_{-2}(z_{N-1}), \\
C^{-1}\phi_{-1}(z_0), \quad C^{-1}\phi_{-1}(z_1), \quad C^{-1}\phi_{-1}(z_2), \quad \dots, \quad C^{-1}\phi_{-1}(z_{N-1}), \\
C^{0}\phi_{0}(z_0), \quad C^{0}\phi_{0}(z_1), \quad C^{0}\phi_{0}(z_2), \quad \dots, \quad C^{0}\phi_{0}(z_{N-1}), \\
C^{1}\phi_{1}(z_0), \quad C^{1}\phi_{1}(z_1), \quad C^{1}\phi_{1}(z_2), \quad \dots, \quad C^{1}\phi_{1}(z_{N-1}), \\
C^{2}\phi_{2}(z_0), \quad C^{2}\phi_{2}(z_1), \quad C^{2}\phi_{2}(z_2), \quad \dots, \quad C^{2}\phi_{2}(z_{N-1}), \\
\dots \\
\dots
\end{array} \tag{13}$$

this is the ballooning mode structure $\phi_{k_y}(\theta)$ on the extended poloidal angle space $-\infty < \theta < \infty$.