A Pedestrian's Guide to the Ballooning Representation

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1 General form for a linear eignenmode in a tokamak

In these notes, I try to explain the ballooning representation (or ballooning transformation) in the simplest possible way. It is often called the the ballooning transformation, because it is a variable transformation and has a form of a Fourier transform as discussed in Section 4. I often find that young scientists ask me questions about the ballooning transformation and though they may understand the mathematical form, they do not have physical intuition regarding why it is used and what the point of the "extended poloidal angle" is. They might say, "Yeah, I get the math, but what does it really mean?" or "What is the point of it?" Here, I try to make it clear what exactly the eigenfunction along the field line is. Part of the problem is that newcomers are not aware of early work that today appears somewhat less mathematically elegant, but really contains the essence of the physics. To this point, I will follow the formulation in L. Chen, M. Chance and C.Z. Cheng, *Nuclear Fusion* **20** 901 (1980) because their more primitive approach turns out to give a much more physical description.

We will assume the usual large aspect ratio, circular, unshifted flux surface magnetic equilibrium for clarity. E.g. $k_\theta = m_0/r$ and θ is the conventional poloidal angle. We start with the following fairly general representation for a linear eigenmode in tokamak

$$\phi(r, \theta, \zeta, t) = \sum_{m=0}^{\infty} f_m(r) \exp\left[-i\omega_{n_0} t - i n_0 \zeta + i m\theta\right] + \text{c.c.} \tag{1}$$

Each linear eigenmode has one discrete $n = n_0$ and ω_{n_0}. The zero subscript for n is not needed, but I write it just to make very clear we are talking about one particular n. In fact, you will not see the zero subscript in the literature for this reason. We will not write the complex conjugate explicitly any more, but remember it is always there so that ϕ is real. For high-n eigenmodes, $n \gg 1$ there is a better, but nearly equivalent representation where a different indexing of m is used

$$\phi(r, \theta, \zeta, t) = \sum_{j=-N_\theta}^{+N_\theta} f_j(r) \exp\left[-i\omega_{n_0} t - i n_0 \zeta + i(m_0 + j)\theta\right], \tag{2}$$

where N_θ does not need to be that big. Even $N_\theta = 1$ might work in some situations. For a toroidal eigenmode, the poloidal harmonics are coupled. There is only one toroidal mode number with $n = n_0$. If we look at a mode peaking at a particular mode rational surface, the dominant poloidal mode will be $m = m_0 = n_0 q(r_0)$. The next most important modes near r_0 will be $m_0 \pm 1$, then $m_0 \pm 2$, etc. These are what are called "sidebands," e.g. $m = m_0 \pm 1$ are the dominant sidebands, jargon from electrical engineering. Technically N_θ cannot be larger than m_0 and the sum in Eq. (2)

is not infinite so, Eqs. (1) and (2) are not exactly equivalent, but since N_θ order a few, and $m_0 \gg 1$ this does not cause any problems that I am aware of.

We have one particular n_0 and one particular ω_{n_0} for a particular toroidal eigenmode, so we will not write out the ζ and t dependence explicitly any more and simply write

$$\phi_{n_0}(r,\theta) = \sum_{j=-N_\theta}^{+N_\theta} f_j(r) \exp\left[i(m_0+j)\theta\right], \tag{3}$$

where the general form of a linear eigenmode is of course $\phi(r,\theta,\zeta,t) = \phi_{n_0}(r,\theta)\exp\left[-i\omega_{n_0}t - in_0\zeta\right]$

2 Rational surface spacing with constant magnetic shear

In linear toroidal theory, there are a lot of parameters floating around in the equations, so it is easy to get confused. If we assume constant magnetic shear, mode rational surface spacing turns out to be constant and it is very simple understand the details. There is a reference mode rational surface, say where the eigenmode peaks at $m_0 = n_0 q(r_0)$. Since the magnetic field, and hence, the magnetic shear varies slowly (typically slower than the density and temperature gradients), we can assume q' is constant. We can then very easily determine the spacing between rational surfaces, or locations where $m_0 \pm 1 = n_0 q(r)$, e.g. for $m_0 + 1$

$$\begin{aligned} m_0 + 1 = n_0 q(r) &= n_0\left(q(r_0) + (r-r_0)q'\right), \\ 1 &= \Delta r n_0 q', \\ \Delta r &= \frac{1}{n_0 q'} = \frac{r_0}{m_0}\frac{q(r_0)}{r_0 q'} = \frac{1}{k_\theta \hat{s}}, \end{aligned} \tag{4}$$

where $k_\theta = \frac{m_0}{r_0}$, $\hat{s} = \frac{r_0 q'}{q}$. The $m_0 \pm 1$ rational surfaces are located at $r_0 \pm \Delta r$ and the rational surface spacing is constant (for all adjacent j) under the approximation of constant magnetic shear.

3 Translational invariance

Because n is large, the distance between rational surfaces is small and the radial eigenfunctions for each poloidal harmonic $f_j(r)$ are assumed to be the same function, but just shifted. This turns out to be a very useful and good approximation. In cylindrical geometry there is no coupling between poloidal modes and the radial functions are localized to the resonant rational surface within a few ion gyroradaii. The reason for this is that $k_\parallel$ is small only near the rational surface due to to the magnetic shear. In toroidal geometry, the radial modes have a similar shape, but now the poloidal harmonics are coupled. The leap comes by assuming what is sometimes call "translational invariance"

$$f_j(r) = g\left(\frac{r}{\Delta r} - j\right), \tag{5}$$

where g is the *same* function for all poloidal harmonics. You can think of $g(r)$ as being a typical sheared slab radial eigenfunction. It is localized to the rational surface within a few ρ_i. Figure 1 shows a drawing of $f_j(r)$ for the $m_0 \pm 1$ sidebands ($j = \pm 1$) and m_0 ($j = 0$).

With the translational invariance assumption, we can write our toroidal eigenfunction as

$$\phi_{n_0}(r,\theta) = \sum_{j=-N_\theta}^{+N_\theta} g\left(\frac{r}{\Delta r} - j\right) \exp\left[i(m_0 + j)\theta\right]. \tag{6}$$

I would call this one form of the ballooning representation. The more conventional form involves a Fourier transform of g as we will see in the following section. This form (with or without the Fourier transform of g) is widely used in linear theory, but leads to ballooning modes (or streamers) with too large a radial extent. In reality there is a slowly varying radial envelope $A_{n_0}(r)$. Often, a two-scale approximation for r where the fast variation from g and the slow variation from A are treated independently. The more general form that gives physical streamers would have the following form

$$\phi_{n_0}(r,\theta) = A_{n_0}(r) \sum_{j=-N_\theta}^{+N_\theta} g\left(\frac{r}{\Delta r} - j\right) \exp\left[i(m_0 + j)\theta\right]. \tag{7}$$

This form gives the toroidal eigenmodes, or streamers, we typically see from gyrokinetic simulation of ITG modes. Fig. 2 is a sketch of such eigenmodes. We have used this form with scientific visualization to plot "fake" ITG modes. In fact, you can generalize to Miller coordinates and make very nice plots that look just like linear (or nonlinear with multiple n) gyrokinetic simulation results!

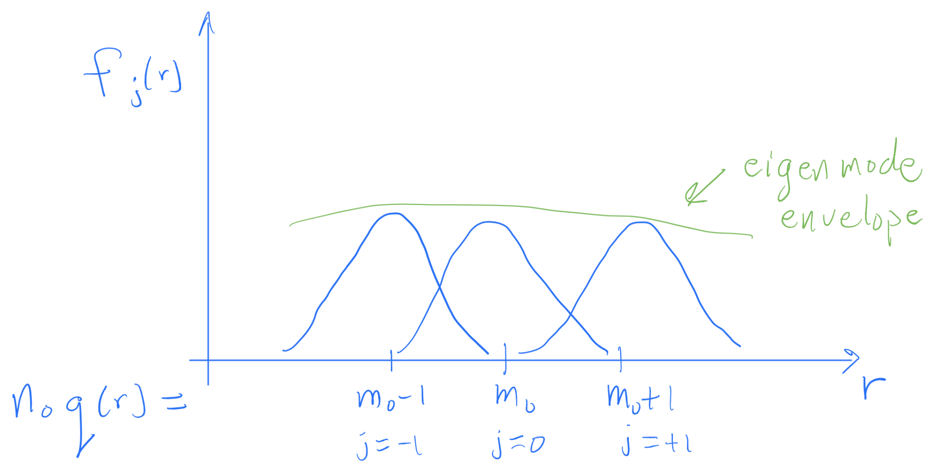

Figure 1: Cartoon showing the $f_j(r)$ functions corresponding to the nearby poloidal harmonics (the sidebands). The poloidal harmonics are coupled leading to the toroidal eigenmode. Assuming translational invariance, the f_j's are all the same function, just shifted. You can thing of them as Gaussian functions with a width of a few ρ_i.

4 Ballooning transformation

The next step towards obtaining the more conventional ballooning transformation is to apply the following Fourier transformation to Eq. (6)

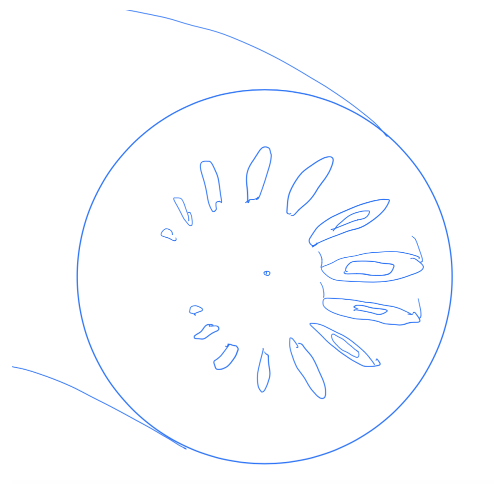

Figure 2: Cartoon showing a typical toroidal eigenmode fairly well represented by Eq. (7).

$$g(u) = \int_{-\infty}^{+\infty} \tilde{g}(\eta) \exp(i\eta u) d\eta. \tag{8}$$

Substituting Eq. (8) for g in Eq. (6), we obtain

$$\begin{aligned} \phi_{n_0}(r,\theta) &= \sum_{j=-N_\theta}^{+N_\theta} \int_{-\infty}^{+\infty} \tilde{g}(\eta) \exp\left(i\eta \left(\frac{r}{\Delta r} - j\right)\right) d\eta \exp\left[i(m_0 + j)\theta\right] \\ &= \sum_m \int_{-\infty}^{+\infty} \tilde{g}(\eta) e^{im\theta} e^{-i(m-m_0)\eta} e^{i\eta\frac{r}{\Delta r}} d\eta \\ &= \sum_m e^{im\theta} \int_{-\infty}^{+\infty} \tilde{g}(\eta) \exp\left[-i\left(m - n_0 q(r_0) - \frac{r}{\Delta r}\right)\right] d\eta \end{aligned} \tag{9}$$

Finally, we use the fact that $\dfrac{r}{\Delta r} = n_0 q' r$, so that $n_0 + \dfrac{r}{\Delta r} = n_0 q(r)$ and obtain

$$\phi_{n_0}(r,\theta) = \sum_m e^{im\theta} \int_{-\infty}^{+\infty} \tilde{g}(\eta) \exp\left[-i\left(m - n_0 q(r)\right)\eta\right] d\eta. \tag{10}$$

Eq. (10) is the celebrated ballooning transformation! This is the form you would plug into the gyrokinetic equations to derive an eigenmode equation (ODE) for $\tilde{g}(\eta)$. Hopefully, from this math, you can see that Eq. (10) and Eq. (6) are equivalent. $\tilde{g}(\eta)$ is the Fourier transformation of $g\left(\frac{r}{\Delta r}\right)$. η is called the extended poloidal angle with $\eta \in (-\infty, +\infty)$. $\tilde{g}(\eta)$ can be thought of as the shape of the eigenfunction along the field line. There is important physics here, where the Fourier transform the local radial eigenmode (with appropriate scaling of the radial variable) is the eigenfunction along the field line using the extended poloidal angle η.

Since $k_\parallel \approx \frac{1}{q(r_0)R_0}(m - n_0 q(r))$, we can see that $k_\parallel$ and r are conjugate variables (Fourier transforms of each other). We can write

$$\exp(ik_\parallel z) = \exp\left[i(m - n_0 q(r))\left(\frac{z}{q(r_0)R_0}\right)\right], \tag{11}$$

where z is the coordinate along the magnetic field line. Looking at Eq. (10), we see that the inverse Fourier transform of η to r is identical (with appropriate normalization) to an inverse Fourier transform of z to r. z and η are essentially the same variable where

$$z = q(r_0)R_0\eta. \tag{12}$$

Due to the large aspect ratio assumption, we can use $R \approx R_0$ and $q = q_0$ in Eq. (12).