Stationary Spectrum of Strong Turbulence in Magnetized Nonuniform Plasma

Akira Hasegawa and Kunioki Mima[(a)]
Bell Laboratories, Murray Hill, New Jersey 07974
(Received 15 November 1976)

A stationary spectrum in the frequency range much below the ion cyclotron frequency is obtained for a strongly turbulent nonuniform plasma. The ω-integrated k spectrum is given by $k^{1.8}(1+k^2)^{-2.2}$, while the width of the frequency spectrum is proportional to $k^3(1+k^2)^{-1}$, where k is normalized by c_s/ω_{ci}. The result compares well with the recently observed spectrum in the ATC tokamak.

In recent experiments of microwave[1] and CO_2 laser[2] scattering from collective modes at a frequency range of a few hundred kHz in the ATC tokamak, density fluctuations with interesting, and somewhat unexpected, spectra were observed. The frequency (ω) spectrum for a fixed value of a vector wave number $\vec{k}$ was broad with no identifiable peak other than at $\omega=0$. While the wave number ($\vec{k}$) spectrum for a fixed ω had a relatively broad peak at $|\vec{k}|\sim\rho_s^{-1}$, where ρ_s ($=c_s/\omega_{ci}$) is an effective ion gyroradius; c_s and ω_{ci} are the ion sound speed and the ion cyclotron frequency. Dependence of the observed $\vec{k}$ spectrum on the direction of $\vec{k}$ was weak; the spectral density for the radial wave number had a structure almost identical to that for the azimuthal wave number.

The broad ω spectrum rules out the possibility that the fluctuation can be explained by a simple weak-turbulence theory in which a small deviation from linear eigenmodes is assumed.[3] The weak dependency of the $\vec{k}$ spectrum on the direction of the wave number rules out turbulence directly excited by a drift-wave instability.[4]

The integrated density fluctuation n is found to be approximately 3% of the background density; $|n/n_0| = |e\varphi/T_e| \simeq 3\times10^{-2}$, where φ is the total fluctuating potential and T_e is the electron temperature. If we use this value, it can be easily seen that, because of the mode coupling through $\vec{E}\times\vec{B}$ drift,[5] the effective nonlinear frequency shift, $\omega_{ci}|n/n_0|k^4\rho_s^4$, becomes comparable to the observed frequency range, $\omega\simeq10^{-2}\omega_{ci}$.

In this Letter an attempt is made to explain qualitatively the observed spectrum, which we believe to be universal to magnetized, collisionless, nonuniform plasma. This Letter has two basic goals. One is the identification of the important nonlinear term and the derivation of a simple nonlinear equation which is appropriate to a general class of quasi-two-dimensional low-frequency turbulence. The other is the solution of the equation using the renormalized-weak-turbulence technique[6] and comparison of the result with the experimental data. To obtain the solution we assume the existence of a large-amplitude long-wavelength perturbation ($k\ll\rho_s^{-1}$). The turbulence in the short-wavelength region ($k\sim\rho_s^{-1}$) is maintained by the scattering due to the long-wavelength mode. We derive the width of the ω spectrum as a function of $|\vec{k}|$ as well as the ω-integrated $|\vec{k}|$ spectrum for the wave number $\vec{k}$ in the direction perpendicular to the ambient magnetic field.

Let us first derive the nonlinear equation. For illustrative purposes, we assume that the electron temperature is reasonably larger than the ion temperature and use a cold-ion approximation.[7] The model equation that we use is the equation of continuity for ions in which the parallel ion inertia is neglected[8]:

$$(\partial n/\partial t) + \nabla_\perp\cdot[n_0(\vec{v}_E+\vec{v}_p)] = 0, \tag{1}$$

where $\nabla_\perp\cdot$ is the divergence operator in the direction perpendicular to the magnetic field $\vec{B}_0$, $\vec{v}_E$, and $\vec{v}_p$ are $\vec{E}\times\vec{B}$ and polarization drifts given, respectively, by

$$\vec{v}_E = (-\nabla_\perp\varphi\times\vec{B}_0)/B_0^2, \tag{2}$$

$$\vec{v}_p = \frac{1}{\omega_{ci}B_0}\left[-\frac{\partial}{\partial t}\nabla_\perp\varphi - (\vec{v}_E\cdot\nabla_\perp)\nabla_\perp\varphi\right], \tag{3}$$

n and n_0 are the perturbed and unperturbed (but nonuniform) density, and ω_{ci} is the ion cyclotron frequency.

Many authors assume an ideal two-dimensional situation here,[9] and obtain n using Poisson's equation together with another two-dimensional electron equation. We believe, however, that in the presence of a weak shear in the magnetic field, such an assumption is invalid. A slow variation of φ in the parallel direction allows the electrons to obey the Boltzmann distribution. The quasineutrality condition gives, then,

$$n/n_0 = e\varphi/T_e. \tag{4}$$

The nonlinear mode coupling in our case originates from the convective derivative in the polarization drift of ions. This makes our approach fundamentally different from previous two-dimensional calculations that use the $(\vec{E}\times\vec{B}_0)n$ nonlinearity[6,9]; in our case this term has no contribution because its divergence is zero.

If we expand $\varphi(\vec{x}, t)$ in spatial Fourier series so that

$$\varphi(\vec{x}, t) = \tfrac{1}{2}\sum_{\vec{k}}[\varphi_{\vec{k}}(t)e^{i\vec{k}\cdot\vec{x}} + \text{c.c.}],$$

where $\vec{k}$ is $\vec{k}_\perp$, Eqs. (1) to (4) are reduced to

$$\frac{\partial\varphi_{\vec{k}}(t)}{\partial t} + i\omega_{\vec{k}}^*\varphi_{\vec{k}}(t) = \frac{1}{2}\sum_{\vec{k}=\vec{k}'+\vec{k}''}\Lambda_{\vec{k}',\vec{k}''}\varphi_{\vec{k}'}(t)\varphi_{\vec{k}''}(t). \tag{5}$$

Here the matrix element $\Lambda_{\vec{k}',\vec{k}''}$ is given by

$$\Lambda_{\vec{k}',\vec{k}''} = \frac{1}{1+k^2}(\vec{k}'\times\vec{k}'')\cdot\vec{e}_z[(k'')^2-(k')^2], \tag{6}$$

time and space are normalized by ω_{ci}^{-1} and $\rho_s(=c_s/\omega_{ci})$ (thus k is normalized by ρ_s^{-1}), $\omega_{\vec{k}}^*$ is the normalized (by ω_{ci}) drift-wave frequency given by

$$\omega_{\vec{k}}^* = \frac{-k_yT_e\partial(\ln n_0)/\partial x}{eB_0(1+k^2)\omega_{ci}}.$$

Here the z axis is taken in the direction of $\vec{B}_0$ and x in that of the nonuniformity.

Equation (5) is the basic equation which we believe to be appropriate to describe a general class of quasi-two-dimensional ($k_\parallel\approx 0$) low-frequency turbulence in a nonuniform plasma. We note that, because $\omega_{\vec{k}}^*\approx 10^{-2}$, even with an amplitude $\varphi_{\vec{k}}\sim 10^{-2}$, the nonlinear term can dominate near $k=1$ and the equation becomes a Navier-Stokes type—a notion of strong turbulence. While at $k\ll 1$, the linear term dominates and the weak-turbulence signature appears. For a very large value of k, $\omega_{\vec{k}}^*$ becomes small and it should be replaced either by viscous or by ion Landau-damping rate, which contributes to the sink of energy. We also note that the mode coupling tends to rotate the $\vec{k}$ spectrum in the plane perpendicular to the magnetic field, hence will isotropize the spectrum in this plane.

Let us now try to solve the equation using the renormalized-weak-turbulence technique[6] and by assuming an existence of large-amplitude perturbation at a long-wavelength regime. We first integrate Eq. (5) to obtain

$$\varphi_{\vec{k}}(t) = \frac{1}{2}\sum_{\vec{k}=\vec{k}'\times\vec{k}''}\Lambda_{\vec{k}',\vec{k}''}\int^t \exp[-i\omega_{\vec{k}}^*(t-t')]\,\varphi_{\vec{k}'}(t')\varphi_{\vec{k}''}(t')\,dt'. \tag{7}$$

Now, if we multiply Eq. (5) by $\varphi_{\vec{k}}^*(t)$ and add the complex conjugate of the product, we have

$$\frac{\partial}{\partial t}|\varphi_{\vec{k}}(t)|^2 = \frac{1}{2}\sum_{\vec{k}=\vec{k}_1+\vec{k}_2}\Lambda_{\vec{k}_1,\vec{k}_2}[\varphi_{\vec{k}_1}(t)\varphi_{\vec{k}_2}(t)\varphi_{\vec{k}}^*(t)+\text{c.c.}]. \tag{8}$$

The wave kinetic equation is constructed by substituting Eq. (7) into the right-hand side of Eq. (8) and by taking ensemble average. We use the random-phase approximation here. However, because of the presence of the large nonlinear term, we retain the decay rate, $\gamma_{\vec{k}}$, of the two-time correlation function compared with the characteristic frequency $\omega_{\vec{k}}$. We postulate here a Lorentzian shape for the two-time correlation function;

$$\langle\varphi_{\vec{k}}(t)\varphi_{\vec{k}'}^*(t')\rangle = \delta_{\vec{k},\vec{k}'}|\varphi_{\vec{k}}(t)|^2\exp[-(i\omega_{\vec{k}}+\gamma_{\vec{k}})(t-t')], \tag{9}$$

where $\gamma_{\vec{k}}$ will be obtained later by the renormalization technique. If we now substitute Eq. (7) into the right-hand side of Eq. (8) and use the relation (9), we have the following wave kinetic equation:

$$\frac{\partial|\varphi_{\vec{k}}(t)|^2}{\partial t} = \frac{1}{2}\sum_{\vec{k}'}\Lambda_{\vec{k}',\vec{k}-\vec{k}'}\left[\frac{\Lambda_{\vec{k}',\vec{k}-\vec{k}'}}{\gamma_{\vec{k}'}+\gamma_{\vec{k}-\vec{k}'}}|\varphi_{\vec{k}'}|^2|\varphi_{\vec{k}-\vec{k}'}|^2 + \frac{\Lambda_{\vec{k},-\vec{k}'}}{\gamma_{\vec{k}}+\gamma_{\vec{k}'}}|\varphi_{\vec{k}}|^2|\varphi_{\vec{k}'}|^2 + \frac{\Lambda_{\vec{k},\vec{k}'-\vec{k}}}{\gamma_{\vec{k}}+\gamma_{\vec{k}'-\vec{k}}}|\varphi_{\vec{k}}|^2|\varphi_{\vec{k}-\vec{k}'}|^2\right]. \tag{10}$$

In the derivation of this expression the frequency mismatch is assumed to be smaller than $\gamma_{\vec{k}}$, which is valid near $k=1$. The second and the third terms in the right-hand side are the self-interaction terms to be used for the renormalization of the propagator, and the first term represents the mode-mode coupling.

We now use the assumption that there exists a large-amplitude mode in a long-wavelength region. We shall discuss later the physical origin of such a long-wavelength perturbation. Writing the potential perturbation of the mode as $\varphi_{\vec{k}_0}$, where $|k_0| \ll 1$, and assuming that $|\varphi_{\vec{k}_0}|^2 \gg |\varphi_{\vec{k}}|^2$ and $\gamma_{\vec{k}_0} \ll \gamma_{\vec{k}}$ for $k \sim \rho_s^{-1}$, we can linearize Eq. (10) with respect to $|\varphi_{\vec{k}}|^2$.

Let us first obtain $\gamma_{\vec{k}}$, which should give the width of the frequency spectrum. For this purpose, we multiply Eq. (5) by $\varphi_{\vec{k}}^*(t')$ and use Eq. (7) to obtain

$$\partial\langle\varphi_{\vec{k}}(t)\varphi_{\vec{k}}^*(t')\rangle/\partial t = -i\omega_{\vec{k}}^*\langle\varphi_{\vec{k}}(t)\varphi_{\vec{k}}^*(t')\rangle + \tfrac{1}{4}\Lambda_{\vec{k}_0,\vec{k}-\vec{k}_0}\Lambda_{\vec{k},-\vec{k}_0}(|\varphi_{\vec{k}_0}|^2/\gamma_{\vec{k}})\langle\varphi_{\vec{k}}(t)\varphi_{\vec{k}}^*(t')\rangle. \tag{11}$$

Thus, from Eq. (9), we have

$$\gamma_{\vec{k}} = \frac{1}{2\sqrt{2}}\frac{k^3k_0}{1+k^2}|\varphi_{\vec{k}_0}|\left[1-\frac{k_0^2}{2k^2}\frac{2k^4+4k^2+1}{1+k^2}\right], \tag{12}$$

where a simple average over the angle between $\vec{k}$ and $\vec{k}_0$ is taken assuming no dependency of $|\varphi_k|^2$ on the angle, and the lowest-order term in $|k_0/k|^2$ is retained. The obtained $\gamma_{\vec{k}}$, which represents the width of the frequency spread (around $\omega = 0$), has a different structure from that derived by Dupree and Weinstock[10] due to the finite ion-inertia term used here. For a small value of k, $\gamma_{\vec{k}} \propto k^3$, while $\omega_{\vec{k}}^* \propto k$, hence γ_k can become smaller than $\omega_{\vec{k}}^*$ where the present result becomes inapplicable. If we take the frequency spread at $k = 1$ of Ref. 2, $\gamma_k \sim 10^{-2}$. This gives the potential amplitudes of k_0 mode, $|\varphi_{k_0}|$, approximately[11] 0.2 for $k_0 = 0.1$.

Finally, we obtain the stationary $\vec{k}$ spectrum density, $|\varphi_{\vec{k}}|^2$. For this purpose, we set the left-hand side of Eq. (10) to zero (stationary condition) and equate the damping of $|\varphi_{\vec{k}}|^2$, [Eq. (12)] to the excitation of $|\varphi_{\vec{k}}|^2$ by the mode coupling between $\varphi_{\vec{k}-\vec{k}_0}$ and $\varphi_{\vec{k}_0}$. We then expand $|\varphi_{\vec{k}-\vec{k}_0}|^2$ around $|\varphi_{\vec{k}}|^2$ in the power of $\vec{k}_0$;

$$|\varphi_{\vec{k}-\vec{k}_0}|^2 = |\varphi_{\vec{k}}|^2 + (\vec{k}_0\cdot\partial/\partial\vec{k})|\varphi_{\vec{k}}|^2 + \tfrac{1}{2}(\vec{k}_0\cdot\partial/\partial\vec{k})^2|\varphi_{\vec{k}}|^2.$$

It then turns out that the leading term of the mode coupling just cancels with that of the damping term.[12] Hence, by balancing the terms in the order k_0^2, we obtain the following differential equation for $|\varphi_k|^2$,

$$\frac{d^2|\varphi_k|^2}{dk^2} + 2\,\frac{1+3k^2}{k(1+k^2)}\frac{d|\varphi_k|^2}{dk} + \frac{15k^4+18k^2-5}{k^2(1+k^2)}|\varphi_k|^2 = 0. \tag{13}$$

In the derivation of this equation, the assumption that $|\varphi_{\vec{k}}|^2$ does not depend on the direction of $\vec{k}$ is used again.

Equation (13) is found to have two independent solutions, one having a form $\sim k^{-2.8}$, which spuriously represents the long-wavelength spectrum, and the other having a broad peak near $k = 1$, which represents the short-wavelength spectrum that we are looking for. An approximate analytic solution for the latter has a form

$$|\varphi_k|^2 \simeq k^{1.8}/(1+k^2)^{2.2} \text{ for } k \lesssim \min[10, (T_e/T_i)^{1/2}]. \tag{14}$$

The solution for $k \gtrsim 10$ becomes oscillatory and is hence nonphysical.

Figure 1 shows the comparison between the theoretical spectral density given by Eq. (14) (solid curve), and the experimental data published by Mazzucato.[2] It is fitted at k (un-normalized) = 10 cm^{-1} assuming $\rho_s = 10^{-1}$ cm. A good agreement is seen in the long-wavelength side. The poor agreement in the short-wavelength side is expected because an additional damping and reduced mode-coupling coefficient will appear in $k > 1$ due to the finite-ion-gyroradius effect.

We now justify the assumption of the existence of a large-amplitude, long-wavelength mode. Such a mode may be excited directly by a drift-wave instability.[13] Because the mode-coupling coefficient is small for a long-wavelength region, the nonlinear damping rate is small. This indicates that a drift wave with a long wavelength, even if the growth rate is small, can build up to a large-amplitude level. The other candidates are a magnetohydrodynamic or tearing mode[11] and/or a long-wavelength convective cell[14] which are pertinent to a nonuniform plasma which is continuously heated (for example by Ohmic current).

In conclusion, we have derived a model equation which is appropriate to describe low-frequency, short-wavelength dynamics of a magnetized nonuniform plasma and obtained the ω-integrated k spectral density as well as the width of the ω spectrum assuming the turbulence is maintained by potential fluctuations in a long-wavelength region. Since the result does not depend on any particular mode of the system, the

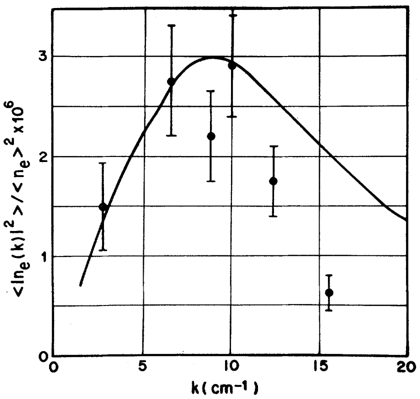

FIG. 1. Comparison of the ω-integrated k spectral density between the theory [Eq. (14)], shown by the solid curve, and the experiment by Mazzucato (Ref. 1), shown by dots and straight error bars. It is fitted at $k = 10\ \mathrm{cm}^{-1}$ using $\rho_s = 10^{-1}$ cm. Discrepancy in the short wavelength is due to the finite–ion-gyroradius effect and the classic viscous or ion Landau damping which are not included.

obtained spectral density is considered to be universal to a magnetized, nonuniform collisionless plasma.

Discussions with Dr. H. Okuda, Dr. R. Z. Sagdeev, Dr. R. E. Slusher, and Dr. C. M. Surko are greatly appreciated.

[(a)]On leave from Faculty of Engineering, Osaka University, Suita, Japan.

[1]E. Mazzucato, Phys. Rev. Lett. 36, 792 (1976).

[2]R. E. Slusher and C. M. Surko, Phys. Rev. Lett. 37, 1747 (1976).

[3]R. Z. Sagdeev and A. A. Galeev, in *Nonlinear Plasma Theory*, edited by T. M. O'Neil and D. L. Book (Benjamin, New York, 1969), p. 103.

[4]C. W. Horton, Phys. Rev. Lett. 37, 1269 (1976).

[5]A. Hasegawa, Phys. Lett. 57A, 143 (1976); A. Hasegawa and L. Chen, Phys. Rev. Lett. 36, 1362 (1976).

[6]T. H. Dupree, Phys. Fluids 9, 1773 (1966); J. Weinstock, Phys. Fluids 12, 1045 (1969); H. Okuda and J. M. Dawson, Phys. Fluids 16, 408 (1973).

[7]This approximation puts a maximum wave number $\sim (T_e/T_i)^{1/2}\rho_s^{-1}$ ($\sim 2\rho_s^{-1}$ for ATC) to which the present result is applicable. This limitation can be waived by a use of Vlasov equation for ions; the result will be published elsewhere.

[8]This assumption is commonly used in the type of turbulence discussed here (cf. Ref. 6). Negligibly small dependency of the observed spectral density on $k_\parallel$ was also confirmed in the experiments in Refs. 1 and 2.

[9]For example, J. B. Taylor and B. McNamara, Phys. Fluids 14, 1492 (1971); D. Montgomery *et al.*, Phys. Fluids 15, 815 (1972); T. H. Dupree, Phys. Fluids 17, 100 (1974).

[10]This $\gamma_{\vec{k}}$ is k^2 times the $\gamma_{\vec{k}}$ of Dupree and Weinstock. The difference originates from the different nonlinear terms taken [the nonlinear polarization vs $n(\vec{E}\times\vec{B}_0)$]. If the Boltzmann distribution for the number density is used as here, $n=\varphi$, and the $n(\vec{E}\times\vec{B}_0)$ term gives no mode coupling since $\nabla_\perp\cdot n(\vec{E}\times\vec{B}_0)=0$. $\vec{E}\times\vec{B}_0$ drift does not produce any dissipation because $\vec{E}\cdot\vec{v}_E=0$ [T. H. Dupree and D. J. Tetrault, Bull. Am. Phys. Soc. 21, 1115 (1976)], while the polarization drift does, $\vec{E}\cdot\vec{v}_p\neq 0$, due to the ion inertia effect.

[11]We show that this number is quite reasonable, if, for example, we consider $\vec{\varphi}_{k_0}$ to be an imcompressible magnetohydrodynamic mode [note that we need only electric field for the mode coupling]. If we take a recent numerical result of the resistive-kink mode [D. B. Waddel, M. N. Rosenbluth, D. A. Monticello, and R. B. White, Nucl. Fusion 16, 528 (1976)], the saturated perturbed fluid velocity $\tilde{v}_{k_0}$ is given by $c_s/25$. Then, because $k_0\varphi_{k_0}$ (unnormalized) $=\tilde{v}_{k_0}B_0$, we see $e\varphi_{k_0}/T_e=(k_0\rho_s)^{-1}/25 \sim 0.4$.

[12]The cancellation in the dominant term between the damping due to the orbit diffusion and the excitation due to the mode-mode coupling was noted first by A. A. Galeev, Zh. Eksp. Teor. Fiz. 57, 1361 (1969) [Sov. Phys. JETP 30, 737 (1970)]. Because of this effect, the damping rate of the total wave spectrum (integrated in k space) is much smaller than $\gamma_{\vec{k}}$. This explains the long-lived turbulence observed in the experiments.

[13]No long-wavelength density fluctuation is observed by the laser scattering. However, the source can be a potential fluctuation of an incompressible mode as shown in Eq. (3). The drift-wave instability excited by a temperature gradient is a suitable candidate for an excitation of incompressible perturbation.

[14]C. Z. Cheng and H. Okuda, Phys. Rev. Lett. 38, 708, 1037(E) (1977).