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Citation: Physics of Fluids **21**, 87 (1978); doi: 10.1063/1.862083
View online: http://dx.doi.org/10.1063/1.862083
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Pseudo-three-dimensional turbulence in magnetized nonuniform plasma

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(Received 21 June 1977)

A simple nonlinear equation is derived to describe the pseudo-three-dimensional dynamics of a nonuniform magnetized plasma with $T_e \gg T_i$ by taking into account the three-dimensional electron, but two-dimensional ion dynamics in the direction perpendicular to B_0. The equation bears a close resemblance to the two-dimensional Navier–Stokes equation. A stationary spectrum in the frequency range of drift waves is obtained using this equation by assuming a coexisting large amplitude long wavelength mode. The ω-integrated k spectrum is given by $k^{1.8}(1+k^2)^{-2.2}$, while the width of the frequency spectrum is proportional to $k^3(1+k^2)^{-1}$, where k is normalized by c_s/ω_{ci}. The result compares well with the recently observed spectrum in the ATC tokamak.

I. INTRODUCTION

In a low beta, nonuniform plasma embedded in a magnetic field, many instabilities of waves with frequency ω, much below the ion cyclotron frequency, ω_{ci}, and wave vector, $\mathbf{k}$, in the direction almost perpendicular to the magnetic field have been predicted.[1] Experimental observations[2-4] of such pseudo-three-dimensional perturbations indicate the significance of the universality of plasma turbulence of such a nature.

In this paper we first derive a simple model equation which can describe plasma turbulence of this type, namely, that with the temporal time scale being much larger than the ion cyclotron period, and the wave vector in the direction almost perpendicular to the magnetic field. We make two assumptions; (1) the electron temperature is much larger than the ion temperature and (2) the turbulence level is high enough so that the wave-particle interaction can be neglected. The first assumption allows us to treat ions as a cold fluid which drastically simplifies the equations, yet we can retain an effective ion gyroradius by allowing a high electron temperature T_e through $(T_e/m_i)^{1/2}/\omega_{ci}$, where m_i is the ion mass. The second assumption is a natural consequence of strong turbulence.

Particular emphasis is placed on the pseudo-three-dimensionality of the problem. Attempts have been made in the past[5] to describe a magnetized plasma in exactly two-dimensional form (in the direction perpendicular to the ambient magnetic field) for various purposes. However, for a realistic plasma even with small inhomogeneities, the small electron inertia allows a rapid motion of electrons in the direction of the ambient magnetic field and leads to a breakdown of the ideal two-dimensional situation. Quantitatively, if the parallel phase velocity is smaller than the electron thermal speed, the parallel motion of electrons becomes important. In this paper we treat the electrons as a massless fluid.

The model equation derived in this way is shown to have a close resemblance to the two-dimensional Navier–Stokes equation for incompressible fluids. They are identical for $k_\perp^2 \gg [(T_e/m_i)/\omega_{ci}^2]^{-1}$ and deviate from each other for a smaller $k_\perp$. The deviation is due to the effective compressibility produced by the parallel electron motion. After deriving the model equation in Sec. II and discussing its properties and similarities to the Navier–Stokes equation in Sec. III, we make an attempt to explain the recently observed spectrum.[2,3] of density fluctuation in the ATC tokamak using the model equation.

In these experiments density fluctuations with interesting, and somewhat unexpected spectra were observed. The frequency spectrum in the range of a few hundred kHz for a fixed value of a vector wavenumber $\mathbf{k}$ was broad with little or no identifiable peak other than at $\omega = 0$. While the wavenumber ($\mathbf{k}$) spectrum for a fixed ω had a relatively broad peak at $|k| \sim \rho_s^{-1}$, where $\rho_s (= c_s/\omega_{ci})$ is an effective ion gyroradius and c_s is the ion sound speed $[=(T_e/m_i)^{1/2}]$. Dependence of the observed $\mathbf{k}$ spectrum on the direction of $\mathbf{k}$ was weak; the spectral density for the radial wavenumber had a structure almost identical to that for the azimuthal wavenumber.

The broad ω spectrum seems to rule out the possibility that the fluctuation can be explained by a simple weak turbulence theory in which a small deviation from linear eigenmodes is assumed.[6,7] The weak dependency of the $\mathbf{k}$ spectrum on the direction of the wave vector presents further evidence that the observed spectrum cannot be attributed to a simple drift wave turbulence.

The integrated density fluctuation n is found to be approximately three percent of the background density[3]; $|n/n_0| = |e\phi/T_e| \simeq 3\times 10^{-2}$, where ϕ is the total fluctuating potential and T_e is the electron temperature. If we use this value, it can easily be seen that, owing to the mode coupling through the $\mathbf{E}\times\mathbf{B}$ drift,[8] the effective nonlinear frequency shift, $\omega_{ci}|n/n_0|k^4\rho_s^4$, becomes larger than the observed frequency range, $\omega \simeq 10^{-2}\omega_{ci}$.

To obtain the spectrum we assume the coexistence of a large amplitude long wavelength perturbation ($k \ll \rho_s^{-1}$). The turbulence in the short wavelength region ($k \sim \rho_s^{-1}$) is maintained by the scattering due to the long wavelength mode. We derive the width of the ω spectrum as a function of $|k|$ as well as the ω integrated $|k|$ spectrum for

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the wavenumber $\mathbf{k}$ in the direction perpendicular to the ambient magnetic field.

II. MODEL EQUATION

Let us first derive the model nonlinear equation. For illustrative purposes, we assume that the electron temperature is reasonably larger than the ion temperature and we use a cold ion approximation. The cold ion approximation puts a maximum wavenumber approximately $(T_e/T_i)^{1/2}\rho_s^{-1}$ to which the present result is applicable. However, this weakness is overshadowed by the simplicity of the equation. The model equation we use is the equation of continuity for ions in which the parallel ion inertia is neglected. This neglection is commonly used in the type of turbulence discussed here.[5] The negligibly small dependency of the observed spectral density $k_\parallel$ was also confirmed experimentally.[2,3] We have

$$\partial n/\partial t + \nabla_\perp \cdot [n_0(\mathbf{v}_E + \mathbf{v}_p)] = 0\,, \tag{1}$$

where $\nabla_\perp \cdot$ is the divergence operator in the direction perpendicular to the magnetic field, $\mathbf{B}_0$, $\mathbf{v}_E$, and $\mathbf{v}_p$ are the $\mathbf{E}\times\mathbf{B}$ and polarization drifts given respectively by

$$\mathbf{v}_E = -\nabla_\perp\phi \times \mathbf{B}_0/B_0^2\,, \tag{2}$$

$$\mathbf{v}_p = \frac{1}{\omega_{ci}B_0}\left[-\frac{\partial}{\partial t}\nabla_\perp\phi - (\mathbf{v}_E\cdot\nabla_\perp)\nabla_\perp\phi\right], \tag{3}$$

where n and n_0 are the perturbed and unperturbed (but nonuniform) densities, and ω_{ci} is the ion cyclotron frequency. As will be shown later the term which originates from the product of the perturbed density n and $(\mathbf{v}_E+\mathbf{v}_p)$ makes no contribution.

Many authors have assumed an ideal two-dimensional situation,[5,9] and obtained the density perturbation by using Poisson's equation together with the corresponding two-dimensional electron equation. We believe, for example in the presence of a weak shear in the magnetic field, such an assumption is invalid. A slow variation of ϕ in the parallel direction allows the electrons to obey the Boltzmann distribution. The quasi-neutrality condition then gives,

$$n/n_0 = e\phi/T_e\,. \tag{4}$$

If we use this density perturbation in the ion continuity equation, it can easily be shown that

$$\nabla\cdot(n\mathbf{v}_E) = 0\,.$$

Hence, the nonlinear mode coupling in our case originates only from the convective derivation in the polarization drift of ions, the second term in Eq. (3). This makes our approach fundamentally different from the previous two dimensional calculation that uses $v_E n$ nonlinearity.[5,9]

If we expand $\phi(x,t)$ in a spatial Fourier series; $\phi(\mathbf{x},t) = \frac{1}{2}\sum_k(\phi_k(t)e^{i\mathbf{k}\cdot\mathbf{x}} + \text{c.c.})$, where $\mathbf{k}$ is $\mathbf{k}_\perp$, Eqs. (1) to (4) are reduced to

$$\frac{\partial\phi_{\mathbf{k}}(t)}{\partial t} + i\omega_{\mathbf{k}}^*\phi_{\mathbf{k}}(t) = \frac{1}{2}\sum_{\mathbf{k}=\mathbf{k}'+\mathbf{k}''}\Lambda_{\mathbf{k}',\mathbf{k}''}\phi_{\mathbf{k}'}(t)\phi_{\mathbf{k}''}(t)\,. \tag{5}$$

Here, the matrix element $\Lambda_{\mathbf{k}',\mathbf{k}''}$ is given by

$$\Lambda_{\mathbf{k}',\mathbf{k}''} = \frac{1}{1+k^2}(\mathbf{k}'\times\mathbf{k}'')\cdot\hat{z}[(k'')^2-(k')^2]\,, \tag{6}$$

time and space coordinates are normalized by ω_{ci}^{-1} and $\rho_s(=c_s/\omega_{ci})$, (thus k is normalized by ρ_s^{-1}), $\omega_{\mathbf{k}}^*$ is the normalized (by ω_{ci}) drift wave frequency given by

$$\omega_{\mathbf{k}}^* = \frac{-k_yT_e\partial(\ln n_0)/\partial x}{eB_0(1+k^2)\omega_{ci}}\,. \tag{7}$$

Here, the z axis is taken in the direction of B_0 and x in that of the nonuniformity.

Equation (5) is the basic equation which we believe to be appropriate to describe a general class of pseudo-three-dimensional ($k_\parallel \approx 0$) low frequency turbulence in a nonuniform plasma. We note that, because $\omega_{\mathbf{k}}^* \approx 10^{-2}$, even with an amplitude $\phi_k \sim 10^{-2}$, the nonlinear term can dominate near $k=1$ and the equation becomes a Navier–Stokes type; a notion of strong turbulence. There exists, however, an important difference between Eq. (5) and the Navier–Stokes equation; that is, the matrix element $\Lambda_{\mathbf{k}',\mathbf{k}''}$ has a denominator $1+k^2$ which indicates its qualitative change at $k=1$. This is a unique feature of a magnetized plasma. While at $k\ll 1$, the linear term dominates and the weak turbulence signature appears. For a very large value of k, $\omega_{\mathbf{k}}^*$ becomes small and it should be replaced either by the viscous or by the ion Landau damping rate, which contributes to the sink of energy. We also note that mode coupling tends to rotate the k spectrum in the plane perpendicular to the magnetic field, hence, it will isotropize the spectrum in this plane.

III. CONSERVATION LAWS AND RELATION TO NAVIER–STOKES EQUATION

In this section, we derive the conservation laws and discuss the relation of the basic equation, Eq. (5) to the Navier–Stokes equation. For these purposes, we consider a uniform plasma. If we note the identity in the two-dimensional case,

$$\nabla\cdot[(\nabla\phi\times\hat{z})\cdot\nabla]\nabla\phi = [(\nabla\phi\times\hat{z})\cdot\nabla]\nabla^2\phi\,, \tag{8}$$

where $\hat{z}$ is the unit vector in the z (ambient magnetic field) direction, Eq. (5) reduces, in real space, to

$$\frac{\partial}{\partial t}(\nabla^2\phi - \phi) - [(\nabla\phi\times\hat{z})\cdot\nabla]\nabla^2\phi = 0\,. \tag{9}$$

Let us first note the similarity of this expression to the Navier–Stokes equation for the incompressible fluid

$$\left(\frac{\partial}{\partial t} + \mathbf{v}\cdot\nabla\right)\mathbf{v} = -\nabla p\,, \tag{10}$$

$$\nabla\cdot\mathbf{v} = 0\,. \tag{11}$$

In the two-dimensional case, the velocity field $\mathbf{v}$ can be expressed in terms of the z component of its vector potential ψ,

$$\mathbf{v} = \nabla\times\psi\hat{z} = \nabla\psi\times\hat{z}\,. \tag{12}$$

Thus, if we take the curl of Eq. (10) and use Eq. (12), we have

$$(\partial/\partial t)\nabla^2\psi - (\nabla\psi\times\hat{z}\cdot\nabla)\nabla^2\psi = 0\,. \tag{13}$$

If Eq. (9) is compared with the two-dimensional Navier–

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Stokes equation (13), an interesting similarity is seen. They are identical except for the $\partial\phi/\partial t$ term in Eq. (9). This similarity is not surprising because the $\mathbf{E}\times\mathbf{B}$ drift, $-\nabla\phi\times\hat{z}/B_0$, is impossible. In fact, in the previous treatments of the two-dimensional turbulence using $\mathbf{E}\times\mathbf{B}$ coupling,[5] the equation for the electric potential is found to be identical to Eq. (14). The difference in the present case comes from the compressible perturbation due to the parallel electron motion. This is a consequence of the unique property of a plasma which consists of two components of species, electrons and ions with a small mass ratio m_e/m_i. This small mass ratio makes ideal two-dimensional perturbation meaningless for most cases since a small inhomogeneity (or a shear) in the ambient magnetic field on the order of or larger than m_e/m_i allows electrons to shield the potential variation along the field line. This effect gives rise to compressibility for a small perpendicular wavenumber. The basic equation we have derived here indicates that (i) a pseudo-three-dimensional plasma behaves like an incompressible fluid for a short perpendicular wavelength ($k_\perp\gg\rho_s^{-1}$), but for a long perpendicular wavelength the compressibility correction becomes important; and (ii) the nonlinear term of our equation is identical to that of the Navier–Stokes equation if the electrostatic potential is replaced by the z component of the vector potential for the velocity field.

Let us now derive the conserved quantities in our equation. It was shown by Kraichnan,[10] that the two-dimensional Navier–Stokes equation has two inviscid constants of motion, kinetic energy $\int v^2 dV$ and mean-square vorticity $\int(\nabla\times\mathbf{v})^2 dV$, where dV is the volume element. We also find that there are two constants of motion in our equation,

$$W\equiv\int[\phi^2+(\nabla\phi)^2]/2dV \text{ and } U\equiv\int[(\nabla\phi)^2+(\nabla^2\phi)^2]/2dV.$$

If we multiply Eq. (9) by ϕ and integrate over the entire volume, the nonlinear term becomes,

$$\int\phi\nabla\phi\times\hat{z}\cdot\nabla\nabla^2\phi\,dV$$
$$=\int\phi\nabla\cdot[(\nabla\phi\times\hat{z})\nabla^2\phi]dV=\int\nabla\cdot(\phi\nabla^2\phi\nabla\phi\times\hat{z})\,dV.$$

This expression can be written in terms of a surface integral $\int\phi\nabla^2\phi\nabla\phi\times\hat{z}\cdot d\mathbf{S}$. Thus,

$$\frac{1}{2}\frac{\partial}{\partial t}\int[\phi^2+(\nabla\phi)^2]dV\equiv\frac{\partial W}{\partial t}=-\int\mathbf{J}_1\cdot d\mathbf{S}, \tag{14}$$

where $\mathbf{J}_1=-\phi(\partial/\partial t)\nabla\phi+\phi\nabla^2\phi\nabla\phi\times\hat{z}$. This expression is interpreted as the conservation of the total energy density, $(n^2T/n_0+m_in_0v_E^2)/2$, since $n=e\phi/T_e$ and $v_E^2=(\nabla\phi\times\hat{z}/B_0)^2=(\nabla\phi)^2/B_0^2$ in unnormalized form.

Similarly, if we multiply Eq. (9) by $\nabla^2\phi$, we have

$$\frac{1}{2}\frac{\partial}{\partial t}\int[(\nabla\phi)^2+(\nabla^2\phi)^2]dV\equiv\frac{\partial U}{\partial t}=-\int\mathbf{J}_2\cdot d\mathbf{S}. \tag{15}$$

where $\mathbf{J}_2=-\nabla\phi(\partial\phi/\partial t)-\frac{1}{2}(\nabla^2\phi)^2\nabla\phi\times\hat{z}$. Equation (15) shows that the quantity U, which is the sum of the kinetic energy and the squared vorticity, $\Omega^2\equiv(\nabla\times\mathbf{v}_E)^2=(\nabla^2\phi)^2T^2/e^2B_0^2$, is conserved. Now, if we add Eqs. (14) and (15), we obtain

$$\frac{1}{2}\frac{\partial}{\partial t}\int[\phi-(\nabla^2\phi)]^2dV$$
$$+\int[\phi\nabla^2\phi\nabla\phi\times\hat{z}-\tfrac{1}{2}(\nabla^2\phi)^2\nabla\phi\times\hat{z}]\cdot dS=0.$$

This shows that a quantity

$$[n/n_0-(\nabla\times\mathbf{v}_E)^2e^2B_0^2/T^2]^2,$$

is an alternative constant. As is shown in the Appendix, this result may be interpreted as the conservation of squared vorticity enstrophy in a compressible two-dimensional fluid, in which Ω^2/n^2 is conserved in the coordinate moving with the fluid. Two constants of motion indicate that there exist two types of inertia range in which the spectral density for the quantities W and V has different cascading properties.[10] In fact the Gibbs distribution here is given by $\exp(-\alpha W-\beta U)$, and the corresponding spectral density becomes $\langle|\phi_k|^2\rangle=(1+k^2)^{-1}\times(\alpha+\beta k^2)^{-1}$.

IV. STATIONARY SPECTRUM

In this section we obtain the spectral density for $|\phi_{\mathbf{k}}|^2$ using our equation to explain the observed spectrum. The observed level of density fluctuation[3] indicates that $(\sum_{\mathbf{k}}|\phi_{\mathbf{k}}|^2)^{1/2}$ is larger than $\omega_{\mathbf{k}}^*$ in Eq. (5) and that the mode coupling term (the right-hand side) dominates over the linear dispersion term, $\omega_{\mathbf{k}}^*$. Hence, to study the spectrum in this range of k, one cannot use conventional weak turbulence theory[6]; we should consider the plasma to be in the strongly turbulent state.

To date, there exists no well-established theory of strong turbulence. Thus to solve Eq. (5) directly is a difficult task.

There exists neither a sink nor a source term in Eq. (5). This is because we have retained only the dominant terms. In reality, there exist terms representing linear growth and damping, which are important for a range of k where the mode coupling term can be regarded as small. This means that the linear drift wave instability can still exist for a small k (so long as it is larger than the small limit given by collisional damping) which can keep pumping field energy. By retaining only the dominant term as in Eq. (5), we can ask what will be the structure of a stationary spectrum which is maintained by the large mode coupling term.

If the mode coupling term dominates over the linear term, we must include all the possible modes in the system, for example those not directly related to the short wavelength mode. This means we must include the magnetohydrodynamic mode and/or the convective cell mode, if their amplitudes are large. In fact, these modes can have large amplitudes in ϕ because of their intrinsic incompressible nature. For example, the resistive kink mode which seems to exist under most circumstances in a tokamak plasma is found to have a fluctuating fluid velocity of $0.04C_s$.[11] The potential fluctuation associated with this level of velocity fluctuation becomes of order unity. The convective cell mode which is found to be excited directly by the drift wave turbulence in a computer experiment[12] also has a large level of potential fluctuation. In addition, because of similarity to the two-dimensional Navier–Stokes turbulence,

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inverse cascading may occur leading to spectrum condensation at long wavelengths.

In the presence of a large amplitude long wavelength mode $\phi_{\mathbf{k}_0}$, with $|\phi_{\mathbf{k}_0}| \gg |\phi_k|$, $k_0 \ll k \approx 1$, Eq. (5) may be solved by linearizing with respect to the amplitudes of the short wavelength modes. We believe that this approach, even if it seems to have limited applications, is valid for reasonably general cases, because an incompressible mode, $\phi_{\mathbf{k}_0}$, can have a large amplitude fluctuation in electric potential. For example, if we write the fluid velocity perturbation $v_{\mathbf{k}_0}$ in terms of the unnormalized potential fluctuation $\phi_{\mathbf{k}_0}$, $E_{\mathbf{k}} + \mathbf{v}_{\mathbf{k}} \times \mathbf{B}_0 = 0$

$$e\phi_{\mathbf{k}_0}/T_e = (k_0\rho_s)^{-1}(v_{\mathbf{k}}/c_s);$$

hence, even if $v_{\mathbf{k}}/c_s \ll 1$, $e\phi_{\mathbf{k}_0}/T_e$ can be order unity if $k_0 \ll \rho_s^{-1}$.

With these notions, let us construct the wave kinetic equation from Eq. (5) using the renormalization technique described by Kadomtsev.[13] We first integrate Eq. (5) to obtain

$$\phi_{\mathbf{k}}(t) = \frac{1}{2}\sum_{\mathbf{k}=\mathbf{k}'+\mathbf{k}''} \Lambda_{\mathbf{k}',\mathbf{k}''} \times \exp[-i\omega_{\mathbf{k}}^*(t-t')]\phi_{\mathbf{k}'}(t')\phi_{\mathbf{k}''}(t')\,dt'. \tag{16}$$

Now, if we multiply Eq. (5) by $\phi_{\mathbf{k}}^*(t)$ and add the complex conjugate of the product, we have

$$\frac{\partial}{\partial t}|\phi_{\mathbf{k}(t)}|^2 = \frac{1}{2}\sum_{\mathbf{k}=\mathbf{k}_1+\mathbf{k}_2} \Lambda_{\mathbf{k}_1,\mathbf{k}_2}[\phi_{\mathbf{k}_1}(t)\phi_{\mathbf{k}_2}(t)\phi_{\mathbf{k}}^*(t) + \text{c.c.}]. \tag{17}$$

The wave kinetic equation is constructed by substituting Eq. (16) into the right-hand side of Eq. (17) and by taking the ensemble average. We use the random phase approximation. However, because of the presence of the large nonlinear term, we retain the decay rate, $\gamma_{\mathbf{k}}$, of the two-time correlation function compared with the characteristic frequency $\omega_{\mathbf{k}}$. We postulate a Lorentzian shape for the two-time correlation function;

$$\langle\phi_{\mathbf{k}}(t)\phi_{\mathbf{k}'}^*(t')\rangle = \delta_{\mathbf{k},\mathbf{k}'}|\phi_{\mathbf{k}}(t)|^2\exp[-(i\omega_{\mathbf{k}} + \gamma_{\mathbf{k}})(t-t')], \tag{18}$$

where $\gamma_{\mathbf{k}}$ will be obtained later by the renormalization technique. If we now substitute Eq. (16) into the right-hand side of Eq. (17) and use relation (18), we have

$$\frac{\partial|\phi_{\mathbf{k}}(t)|^2}{\partial t} = \frac{1}{2}\,\mathrm{Re}\sum_{\mathbf{k}'}\Lambda_{\mathbf{k}',\mathbf{k}-\mathbf{k}'}\left(\frac{\Lambda_{\mathbf{k}',\mathbf{k}-\mathbf{k}'}|\phi_{\mathbf{k}'}|^2|\phi_{\mathbf{k}-\mathbf{k}'}|^2}{\gamma_{\mathbf{k}'} + \gamma_{\mathbf{k}-\mathbf{k}'} + i(-\omega_{\mathbf{k}}^* + \omega_{\mathbf{k}'} + \omega_{\mathbf{k}-\mathbf{k}'})} + \frac{\Lambda_{\mathbf{k},-\mathbf{k}'}|\phi_{\mathbf{k}}|^2|\phi_{\mathbf{k}'}|^2}{\gamma_{\mathbf{k}} + \gamma_{\mathbf{k}'} + i(\omega_{\mathbf{k}-\mathbf{k}'}^* + \omega_{\mathbf{k}'} - \omega_{\mathbf{k}})} + \frac{\Lambda_{\mathbf{k},\mathbf{k}'-\mathbf{k}}|\phi_{\mathbf{k}}|^2|\phi_{\mathbf{k}-\mathbf{k}'}|^2}{\gamma_{\mathbf{k}} + \gamma_{\mathbf{k}'-\mathbf{k}} + i(\omega_{\mathbf{k}'}^* - \omega_{\mathbf{k}} + \omega_{\mathbf{k}-\mathbf{k}'})}\right) \tag{19}$$

In standard weak turbulent theory,[6] one ignores the $\gamma_{\mathbf{k}}$'s with respect to the $\omega_{\mathbf{k}}$'s in the denominator and represents $(\omega_{\mathbf{k}} - \omega_{\mathbf{k}'} - \omega_{\mathbf{k}''})^{-1}$ by the principal value plus $\pi\delta(\omega_{\mathbf{k}} - \omega_{\mathbf{k}'} - \omega_{\mathbf{k}''})$. Here, because of the large mode coupling term we ignore the frequency mismatch and retain the self-damping term $\gamma_{\mathbf{k}}$. For the small k range where the mode coupling term becomes small one should take the weak turbulence limit. Hence, the present result is applicable only near $k \sim 1$.

We now use the assumption that there exists a large amplitude mode in a long wavelength region. Writing the potential perturbation of the mode as $\phi_{\mathbf{k}_0}$, where $|k_0| \ll 1$, and assuming that $|\phi_{\mathbf{k}_0}|^2 \gg |\phi_{\mathbf{k}}|^2$ and $\gamma_{\mathbf{k}_0} \ll \gamma_{\mathbf{k}}$ for $k \sim \rho_s^{-1}$, we can linearize Eq. (19) with respect to $|\phi_{\mathbf{k}}|^2$.

Let us first obtain $\gamma_{\mathbf{k}}$, which should give the width of the frequency spectrum. For this purpose, we multiply Eq. (5) by $\phi_{\mathbf{k}}^*(t')$ and use Eq. (16) to obtain

$$\frac{\partial}{\partial t}\langle\phi_{\mathbf{k}}(t)\phi_{\mathbf{k}}^*(t')\rangle = -i\omega_{\mathbf{k}}^*\langle\phi_{\mathbf{k}}(t)\phi_{\mathbf{k}}^*(t')\rangle + \frac{1}{4}\Lambda_{\mathbf{k}_0,\mathbf{k}-\mathbf{k}_0}\Lambda_{\mathbf{k},-\mathbf{k}_0}\frac{|\phi_{\mathbf{k}_0}|^2}{\gamma_{\mathbf{k}}}\langle\phi_{\mathbf{k}}(t)\phi_{\mathbf{k}}^*(t')\rangle. \tag{20}$$

Thus, from Eq. (18), we have

$$2\gamma_{\mathbf{k}}^2 = \frac{1}{2}\frac{(\mathbf{k}\times\mathbf{k}_0)^2}{1+k^2}\frac{(k^2 - 2\mathbf{k}_0\cdot\mathbf{k})(k^2 - k_0^2)}{1 + (\mathbf{k}-\mathbf{k}_0)^2}|\phi_{\mathbf{k}_0}|^2. \tag{21}$$

If we expand the right-hand side in the power of $|k_0/k|^2$ and take a simple average over the angle between $\mathbf{k}$ and $\mathbf{k}_0$, we have

$$\gamma_k = \frac{1}{2\sqrt{2}}\frac{k^3 k_0}{1+k^2}|\phi_{\mathbf{k}_0}|\left(1 - \frac{k_0^2}{2k^2}\frac{2k^4 + 4k^2 + 1}{1+k^2}\right). \tag{22}$$

The γ_k so obtained, which represents the width of the frequency spread (around $\omega = 0$), has a different structure from that derived by Dupree and Weinstock[9] due to the finite ion inertia term used here. Namely, our result is k^2 times the Dupree–Weinstock result. This is because the $(\mathbf{E}\times\mathbf{B}_0)n$ term as retained by those authors does not contribute here. For a small value of k, $\gamma_k \propto k^3$; while $\omega_{\mathbf{k}}^* \propto k$, hence γ_k can become smaller than $\omega_{\mathbf{k}}^*$ where the present result becomes inapplicable. If we take the frequency spread at $k=1$ of Ref. 3, $\gamma_k \sim 10^{-2}$. This gives the potential amplitudes of the k_0 mode, $|\phi_{k_0}|$, as approximately 0.2 for $k_0 = 0.1$.

Finally, we obtain the stationary $\mathbf{k}$ spectrum density, $|\phi_{\mathbf{k}}|^2$. For this purpose, we set the left-hand side of Eq. (19) to zero (stationary condition) and equate the damping of $|\phi_{\mathbf{k}}|^2$, [Eq. (22)] to the excitation of $|\phi_{\mathbf{k}}|^2$ by the mode coupling between $\phi_{\mathbf{k}-\mathbf{k}_0}$ and $\phi_{\mathbf{k}_0}$; we ignore the nonlinear term $|\phi_{\mathbf{k}-\mathbf{k}}|^2|\phi_{\mathbf{k}}|^2$

$$\frac{(\mathbf{k}_0\times\mathbf{k})^2(k^2 - 2\mathbf{k}\cdot\mathbf{k}_0)^2}{(1+k^2)^2\gamma_{k-k_0}}|\phi_{\mathbf{k}_0}|^2|\phi_{\mathbf{k}-\mathbf{k}_0}|^2 - 4\gamma_k|\phi_{\mathbf{k}}|^2 = 0. \tag{23}$$

Now, the self-damping rate γ_{k-k_0} of the $k - k_0$ mode can be obtained in a manner similar to γ_k by substituting $\mathbf{k}-\mathbf{k}_0$ for $\mathbf{k}$ in Eq. (22), while $|\phi_{\mathbf{k}-\mathbf{k}_0}|^2$ may be expressed in terms of $|\phi_{\mathbf{k}}|^2$ by the Taylor expansion,

$$|\phi_{\mathbf{k}-\mathbf{k}_0}|^2 = |\phi_{\mathbf{k}}|^2 - \left(\mathbf{k}_0\cdot\frac{\partial}{\partial\mathbf{k}}\right)|\phi_{\mathbf{k}}|^2 + \frac{1}{2}\left(\mathbf{k}_0\cdot\frac{\partial}{\partial\mathbf{k}}\right)^2|\phi_{\mathbf{k}}|^2. \tag{24}$$

If we substitute Eq. (24), γ_k and γ_{k-k_0} from Eq. (22) into Eq. (23), and take the simple average over the angle between $\mathbf{k}$ and $\mathbf{k}_0$, we find that the leading term cancels. The cancelation between the self-damping term and the mode-coupling term has also been noted by Galeev.[14] If we then balance the coefficients of k_0^2, we have the following differential equation for $|\phi_k|^2$,

$$\frac{d^2|\phi_k|^2}{dk^2} + 2\,\frac{1+3k^2}{k(1+k^2)}\frac{d|\phi_k|^2}{dk} + \frac{15k^4 + 18k^2 - 5}{k^2(1+k^2)}|\phi_k|^2 = 0. \tag{25}$$

In the derivation of this equation, the assumption that

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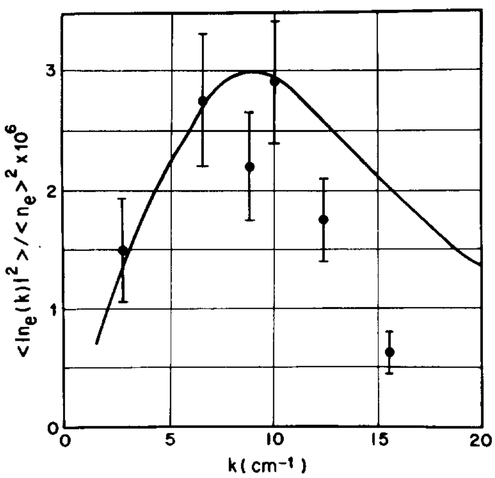

FIG. 1. Comparison of the ω integrated k spectral density between the theory [Eq. (26)], shown by the solid curve, and the experiment by Mazzucato,[2] shown by dots and straight error bars. It is fitted at $k=10^{-1}$ using $\rho_s=10^{-1}$ cm. The discrepancy in the short wavelength is due to the finite ion gyroradius effect and the classic viscous or ion Landau damping which are not included in the theory.

$|\phi_{\mathbf{k}}|^2$ does not depend on the direction of **k** is used again.

Equation (25) is found to have two independent solutions, one having the form approximately equal to $k^{-2.8}$, which spuriously represents the long wavelength spectrum, and the other having a broad peak near $k=1$, which represents the short wavelength spectrum that we are looking for. An approximate analytic solution for the latter has the form

$$|\phi_k|^2 \simeq \frac{k^{1.8}}{(1+k^2)^{2.2}} \quad \text{for } k \lesssim \min\ 10, \left[\frac{T_e}{T_i}\right]^{1/2} . \tag{26}$$

The solution for $k \gtrsim 10$ becomes oscillatory, hence is nonphysical.

If we plot Eq. (24) on the top of the experiment data obtained by Mazzucato,[2] it shows fairly good agreement as shown in Fig. 1. It fits at k (un-normalized) $=10$ cm^{-1} assuming $\rho_s=10^{-1}$ cm. The poor agreement on the short wavelength side is expected because an additional damping and reduced mode coupling coefficient will appear in $k>1$ due to the finite ion gyroradius effect.

V. CONCLUSION

We have derived a model equation which is appropriate to describe the dynamics in the low frequency and short pendicular wavelength of a magnetized nonuniform plasma. The equation has only one nonlinear term which originates from the nonlinear polarization drift. It has a close resemblance to the two-dimensional Navier–Stokes equation for an incompressible fluid. Using the model equation, we obtained the ω-integrated k spectral density as well as the width of the ω spectrum assuming the coexistence of a large amplitude long wavelength potential fluctuation. Since the result does not depend on any particular mode of the system, the spectral density obtained is considered to be universal to a magnetized, nonuniform collisionless plasma.

ACKNOWLEDGMENTS

Discussions with Dr. H. Okuda, Dr. A. Rechester, Academ. R. Z. Sagdeev, Dr. R. E. Slusher, and Dr. C. M. Surko are greatly appreciated.

APPENDIX

Here, we prove that in a two-dimensional compressible plasma fluid,

$$(d/dt)(\Omega^2/n^2)=0 , \tag{A1}$$

where Ω^2 is the generalized entropy as will be defined later and n is the total number density of the plasma. We start with the equation of motion for the ion fluid,

$$m_i\left[\frac{\partial \mathbf{v}}{\partial t}+(\mathbf{v}\cdot\nabla)\mathbf{v}\right]=e(\mathbf{E}+\mathbf{v}\quad \mathbf{B})-\frac{1}{n}\nabla p , \tag{A2}$$

where **E** is the electric field intensity, **B** is the magnetic flux density, and p is the plasma pressure. For general purposes, we consider an electromagnetic perturbation. If we take curl of this equation, and define

$$\mathbf{\Omega}=\nabla\times\mathbf{v}+(e/m_i)\mathbf{B} \tag{A3}$$

while using

$$\nabla\times\mathbf{E}=-\partial\mathbf{B}/\partial t , \tag{A4}$$

$$(\mathbf{v}\cdot\nabla)\mathbf{v}=\tfrac{1}{2}\nabla v^2-\mathbf{v}\times\nabla\times\mathbf{v} , \tag{A5}$$

we have

$$\partial\mathbf{\Omega}/\partial t-\nabla\times(\mathbf{v}\times\mathbf{\Omega})=(1/n^2)\nabla n\times\nabla p . \tag{A6}$$

Now, since $\nabla\cdot\mathbf{\Omega}=0$,

$$\nabla\times(\mathbf{v}\times\Omega)=-\mathbf{\Omega}\nabla\cdot\mathbf{v}+(\mathbf{\Omega}\cdot\nabla)\mathbf{v}-(\mathbf{v}\cdot\nabla)\mathbf{\Omega} . \tag{A7}$$

Furthermore, since we are considering a pseudo-three-dimensional perturbation in which a variation of **v** in the direction of $\mathbf{\Omega}$ is assumed to be zero, (A6) and (A7) give

$$d\mathbf{\Omega}/dt+\mathbf{\Omega}\nabla\cdot\mathbf{v}=(1/m_i n^2)\nabla n\times\nabla p . \tag{A8}$$

Equation (A8) shows that the baroclinic vector $\nabla n\times\nabla p$ becomes a source of B as well as that of the fluid vorticity.

For a cold ion fluid as considered in the main text, $p=0$. If we construct a scalar product of Eq. (A8) with $\mathbf{\Omega}$ and use the continuity equation,

$$\nabla\cdot\mathbf{v}=-d(\ln n)/dt , \tag{A9}$$

we have

$$(d/dt)\ln(\Omega/n)^2=0 , \tag{A10}$$

or

$$(d/dt)(\Omega/n)^2=0 . \tag{A10'}$$

This expression represents a generalized conservation law of enstrophy in a compressible fluid, and can be reduced to Eq. (9) if Boltzmann distributions is used for n.

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