CHAPTER 0

Contents

CHAPTER 1

Topology of n-dimensional R

1.1. The Algebraic structure of $\mathbb{R}^n$

The space $\mathbb{R}^n$ consists of all $n-$tuples $x = (x_1, x_2, \dots, x_n)$ of real numbers. That is, $\mathbb{R}^n$ is the cartesian product of n copies of $\mathbb{R}$, $\mathbb{R}^n = \mathbb{R} \times \mathbb{R} \times \dots \times \mathbb{R}$ (n factors).
The members x of $\mathbb{R}^n$ are called the points of $\mathbb{R}^n$, They can also be viewed as vectors, and $\mathbb{R}$ as a field of scalars.
We define addition in $\mathbb{R}^n$ as coordinate-wise addition, if $x = (x_1, x_2, ..., x_n)$ and $y = \left(y_1, y_2, \dots, y_n\right)$ then $x + y = \left(x_1 + y_1, x_2 + y_2, \dots, x_n + y_n\right)$
also for $\alpha \in \mathbb{R}$, and $x \in \mathbb{R}$ we defined $\alpha x = (\alpha x_1, \alpha x_2, \dots, \alpha x_n)$
we refer to such multiplication of vector by a scalar as a scalar multiplication
With these notations of addition and scalar multiplication, $\mathbb{R}^n$ possess the algebraic structure of vector space. We shall have need for a notion of dot product of two vectors x and y. This is defined by $x \cdot y = x_1 y_1 + x_2 y_2 + \dots + x_n y_n$. where $y = \left(y_1, y_2, \dots, y_n\right)$ and $x = (x_1, x_2, \dots, x_n)$. The dot product satisfies certain conditions that we summarize in a theorem

Theorem 1.1.1
Let $x, y, z \in \mathbb{R}^n$ and let $\alpha \in \mathbb{R}$, then

1. $x \cdot y = y \cdot x$
2. $x\left(y + z\right) = x \cdot y + x \cdot z$
3. $(\alpha x)\, y = \alpha\left(x \cdot y\right)$

Proof. Easy □

Remark

Observe that $0_{\mathbb{R}^n} x = 0_{\mathbb{R}}$*, but* $x \cdot y$ *can equal* 0 *without either* x *or* y *being* 0*. Two vectors* x *and* y *for which* $x \cdot y = 0$ *are said to be orthogonal vectors.Geometrically, two orthogonal vectors in* $\mathbb{R}^n$ *are perpendicular.*

1.2. The Metric structure of $\mathbb{R}^n$

Definition 1.2.1 Euclidean Norm

Let $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$. We define the euclidean norm by:

$$||x||_2 = \sqrt{\sum_{i=1}^{n} x_i^2}$$

Note that: $||x||_2 \geq 0, \quad \forall\, x \in \mathbb{R}^n$

Definition 1.2.2 Euclidean Distance

For $x, y, z \in \mathbb{R}^n$, The euclidean distance between x and y is defined by:

$$d_2\left(x, y\right) = ||x - y||_2$$

In terms of coordinates, with $x = (x_1, x_2, ..., x_n)$ and $y = \left(y_1, y_2, ..., y_n\right)$, then:

$$||x - y||_2 = \sqrt{\sum_{i=1}^{n} \left(x_i - y_i\right)^2}$$

Also observe that we can express the norm via the dot product: $||x||_2^2 = x \cdot x$ or $||x||_2 = \sqrt{x \cdot x}$

Theorem 1.2.1 Cauchy-Schawrtz Inequality

For $x, y \in \mathbb{R}^n$, we have :

$$\left|x \cdot y\right| \leq ||x||.||y||$$

Proof. Let $x, y \in \mathbb{R}^n$, for all $t \in \mathbb{R}$, we have $P(t) = \left(x - ty\right)\left(x - ty\right) \geq 0$, for all $t \in \mathbb{R}$, then we get:

$$\begin{aligned} P(t) &= x \cdot x - 2tx \cdot y + y \cdot yt^2 \\ &= ||x||^2 - 2txy + t^2||y||^2 \end{aligned}$$

Then The function $P(t)$ is quadratic in t, since it is non-negative for all t, it cannot have two distinct roots, hence it's descriminant cannot be positive, that is

$$4\left(xy\right)^2 - 4||x||^2||y||^2 \leq 0$$

it follows that

$$\left|x \cdot y\right| \leq ||x||.||y||$$

□

Theorem 1.2.2 Norm properties
Let $x, y \in \mathbb{R}^n$, and $\alpha \in \mathbb{R}$, then :

(i) $||x|| \geq 0$, and $||x|| = 0$ if and only if $x = 0$

(ii) $||\alpha x|| = |\alpha|\, ||x||$

(iii) $||x + y|| \leq ||x|| + ||y||$ (triangle inequality)

Proof. • The proof of (i) is immediate.

- $\forall\, \alpha \in \mathbb{R}, x \in \mathbb{R}^n$, we have $||\alpha x|| = \sqrt{\sum_{i=1}^{n} (\alpha x_i)^2} = \sqrt{\alpha^2 \sum_{i=1}^{n} x_i} = |\alpha|\, ||x||$
- To prove (iii) we calculate:

$$\begin{aligned} ||x+y||^2 = \left(x+y\right)\left(x+y\right) &= xx + 2xy + xy \\ &\leq ||x||^2 + 2||x||||y|| + ||y||^2 \\ &= \left(||x|| + ||y||\right)^2 \end{aligned}$$

The inequality follows from the Cuachy-Schawrtz inequality

□

1.3. Elementary Toplogy of n-dimensional R

Let $x_0 \in \mathbb{R}^n$, and let $r > 0$, the set:

$$B\left(x_0, r\right) = \{x \in \mathbb{R}^n : ||x - x_0|| < r\}$$

is called the open ball of radius r centered at x_0, it's also called the r-neighborhood of x_0. For instance, for $n = 2$, and $x_0 = \left(x, y\right) \in \mathbb{R}^2$, the open ball around x_0 with radius $r > 0$ is the open disk,

$$B\left(x_0, r\right) = \left\{\left(x, y\right) \in \mathbb{R}^2 : \quad (x - x_0)^2 + \left(y - y_0\right)^2 \leq r^2\right\}$$

and when $n = 1$, and $x_0 \in \mathbb{R}$, $B(x_0, r) = (x_0 - r, x_0 + r)$.
We denote by $\overline{B}(x_0, r) = \{x \in \mathbb{R}^n : \|x - x_0\| \leq r\}$, the set $\overline{B}(x_0, r)$ is called the closed ball around x_0

Definition 1.3.1 Open set
Let $E \subset \mathbb{R}^n$, The set E is called an open set in $\mathbb{R}^n$ if each of it's points has a sufficiently small open ball around it completly contained in E, that is, for each $x \in E$ there exists $r > 0$ such that $B(x, r) \subset E$

Example *The whole space E and the empty set ∅ are open.*

Example *The open ball is an open set*

Proof. Let $x \in B(x_0, r)$, take $\delta = r - \|x - x_0\| > 0$, we should prove $B(x, \delta) \subset B(x_0, r)$, let $y \in B(x, \delta)$, we have

$$\begin{aligned} \|y - x_0\| = \|y - x + x - x_0\| &\leq \|y - x\| + \|x - x_0\| \\ &< \delta + \|x - x_0\| = r \end{aligned}$$

That is $\|y - x_0\| < r$. Hence, $y \in B(x_0, r)$ □

Definition 1.3.2 The boundary
The boundary of a set $E \subset \mathbb{R}^n$, denoted ∂E, is the set of points $x \in \mathbb{R}^n$ such that every ball around x intersect E and the complement of E, that is for every $\varepsilon > 0$

$$\partial E = \{x \in \mathbb{R}^n : B(x, \varepsilon) \cap E \neq \emptyset, B(x, \varepsilon) \cap E^c \neq \emptyset\}$$

Clearly $\partial E = \partial(E^c)$

Example *The boundary of $B(x_0, r)$ is the sphere $S(x_0, r) = \{x \in \mathbb{R}^n : \|x - x_0\| = r\}$*

Remark *Show that an open set can not contain any of it's boundary points*

Definition 1.3.3
Let $E \subset \mathbb{R}^n$, A point $x_0 \in E$ is an interior point of E if there exist $\delta > 0$ such that $B(x_0, \delta) \subset E$. The set of all interior points of E is called the interior of E, and

denoted by E°.

Note that $E^\circ \subset E$ and E is open if and only if $E^\circ = E$

Example $\mathbb{R}^2$, $E = (0,1] \times [0,2]$

Definition 1.3.4 Closed set

A set $E \subset \mathbb{R}^n$ is said to be closed if it's complement E^c in $\mathbb{R}^n$ is open.

Theorem 1.3.1

A set E is closed in $\mathbb{R}^n$ if and only if $\partial E \subset E$

Proof.

$$(\implies)$$

Suppose that A is closed,then E^c is open, it follows from exercise 1 that $E^c \cap \partial E = \varnothing$ Hence $\partial E \subset E$

$$(\impliedby)$$

Conversly,let $x \in E^c$, since $\partial E \subset E$ it follows that x is no a boundary point of E, by the definition, there exist an $r > 0$ such that $B(x,r) \cap E = \varnothing$, that is $B(x,r) \subset E^c$ and so E^c is open. □

Definition 1.3.5

Let $E \subset \mathbb{R}^n$, we define the closure of E, denoted by $\overline{E}$, to be the set $\overline{E} = \partial E \cup E$

Remark

- $E \subset \overline{E}$
- *E is a closed set* $\iff$ $\overline{E} = E$

Example

- *by taking* $E = \mathbb{R}$, $\overline{\mathbb{Q}} = \mathbb{R}$, *and* $\mathbb{Q} = \varnothing$
- *by taking* $E = \mathbb{Z}$, $\overline{\mathbb{Z}} = \mathbb{Z}$,
- *by taking* $E = \mathbb{R}^2$, $\overline{\mathbb{Q} \times \mathbb{Q}} = \mathbb{R}^2$

Example

1. *Consider the set* $\mathbb{Z} \subset \mathbb{R}$. *we have* $\mathbb{Z}^{\circ} = \emptyset$ *and* $\partial\mathbb{Z} = \overline{\mathbb{Z}} = \mathbb{Z}$.*So* $\mathbb{Z}$ *is closed.*
2. *Consider the set* $\mathbb{Q} \subset \mathbb{R}$, $\mathbb{Q}^{\circ} = \emptyset$ *and* $\overline{\mathbb{Q}} = \partial\mathbb{Q} = R$

Definition 1.3.6
Let $E \subset \mathbb{R}^n$ and $x_0 \in \mathbb{R}^n$, we call x_0 an accumulation point or limit point of E if every open ball around x_0 contains at least one point $x \in E$ distinct from x_0, that is for every $r > 0$ we have $(B(x_0, r) - \{x_0\}) \cap E \neq \emptyset$.
If x_0 is not an accumulation point of E is called an isolated point of E.

The next proposition provides a useful characterisation of closness

Theorem 1.3.2 Closeness Characterisation
A set $E \subset \mathbb{R}^n$ is closed iff E contains all its limit points.

Proof.

$$(\implies)$$

Suppose E is closed and let x_0 be a limit point of E .If $x_0 \notin E$ then $x_0 \in E^c$ which is open. Therefore, there exists $\delta > 0$ such that $B(x_0, \delta) \subset E^c$ then $B(x_0, \delta) \cap E = \emptyset$ which is a contradiction with x_0 being a limit point of E

$$(\impliedby)$$

Suppose that E contains all of its limit points. we need to show that E is a closed set. Let $x \in E^c$, x is not a limit point of E then by the definition there exist $\delta > 0$ such that $B(x, \delta) \cap E = \emptyset$.Hence $B(x, \delta) \subset E^c$ so E^c is open which gives us that E is closed □

1.4. Convergence of sequences of $\mathbb{R}^n$

We reserve the letter n for the dimension and we use letters i, j, k for the index,that is, we denote by $(x_k)_{k\in\mathbb{N}}$ a sequence of vectors in $\mathbb{R}^n$ and the components vector $x_k = (x_{k\cdot 1}, x_{k\cdot 2}, \dots, x_{k\cdot n})$, We now define the limit of a sequence in $\mathbb{R}^n$.

Definition 1.4.1 Convergent Sequence

Let (x_k) be a sequence of vectors in $\mathbb{R}^n$ and $x \in \mathbb{R}^n$. We say that $(x_l)_{l\in\mathbb{N}}$ converges to x if $d(x_k, x) = \|x_k - x\| \to 0$ as $k \to \infty$. We write $x_k \to x$. that is:

$$x_k \to x \iff \forall \varepsilon > 0, \exists N \in \mathbb{N}, \forall k > N \implies \|x_k - x\| < \varepsilon$$

Example

- *Now let* $(x_k)_{k\in\mathbb{N}} = \left(\frac{1}{k}, k^2e^{-k}, k^3\right) \subset \mathbb{R}^3$, *is a sequence on* $\mathbb{R}^3$, *this sequence is not convergent.Indeed, Suppose that there exist* $x = (x_1, x_2, x_3)$ *such that*$x_k \to x$.

$$\|x_k - x\|^2 = \left(\frac{1}{k} - x_1\right)^2 + \left(k^2e^{-k} - x_2\right)^2 + \left(k^3 - x_3\right)^2 \to \infty$$

  we can see that the sequence is divergent.

- *Now consider the sequence* $(x_k)_{k\in\mathbb{N}} = \left(\frac{1}{k}, \frac{k+1}{k}\right)$ *is convergent.Indeed, we have*

$$\|x_k - x\|^2 = \frac{1}{k^2} + \frac{1}{k^2} = \frac{2}{k^2} \to 0 \quad \textit{as } k \to \infty$$

Definition 1.4.2 Bounded Sequence

Just like in $\mathbb{R}$, a sequence $(x_n)_{n\in\mathbb{N}} \subset \mathbb{R}^n$ is said to be bounded if there exist $M > 0$ with $\|x_k\| \le M \quad \forall k \ge 1$, and a sequence which is not bounded is called unbounded.In addition, every convergent sequence is bounded and any unbounded sequence diverges.

Theorem 1.4.1

Let $(x_n)_{n\in\mathbb{N}}$ and $\left(y_n\right)_{n\in\mathbb{N}}$ be two sequences in $\mathbb{R}^n$ with $x_k \to x$ and $y_k \to y$, where $x, y \in \mathbb{R}^n$, then :

1. for all $\alpha, \beta \in \mathbb{R}$, $\quad \alpha x_k + \beta y_k \to \alpha x + \beta y$
2. $\|x_k\| \to \|x\|$
3. $\langle x_k, y_k\rangle \to \langle x, y\rangle$
4. $x_k \to x \quad$ if and only if $\quad x_{k_i} \to x_i \quad \forall i = 1, \dots, n$

Proof. (i)

$$||\alpha x_k + \beta y_k - \alpha x - \beta y|| \leq |\alpha|\, ||x_k - x|| + \left|\beta\right| ||y_k - y||$$

(ii)

$$|||x_k|| - ||x||| \leq ||x_k - x||$$

(iii)

$$\begin{aligned}\left|\langle x_k, y_k\rangle - \langle x, y\rangle\right| &= \left|\langle x_k - x, y_k\rangle + \langle x, y_k - y\rangle\right| \\ &\leq \left|\langle x_k - x, y_k\rangle\right| + \left|\langle x, y_k - y\rangle\right| \\ &\leq \underbrace{||x_k - x||||y_k||}_{\text{Bounded}} + ||y_k - y||||x|| \to 0\end{aligned}$$

(iv)

$$\left|x_{k_i} - x_i\right| \leq ||x_k - x||$$

□

By property (iv) of the above theorem, we see that to study the convergence of a sequence in $\mathbb{R}^n$, it's sufficient to study the convergence of it's components sequences in $\mathbb{R}$.
Next, we would like to restat the definition of Cuachy sequence in $\mathbb{R}^n$.

Definition 1.4.3
Let (X, d) be a metric space

1. A sequence $(x_k) \subset \mathbb{R}^n$ is called a Cauchy sequence if :

$$d\left(x_k, x_j\right) \to 0 \quad \text{as} \quad k, j \to \infty$$

i.e.

$$\forall\, \varepsilon > 0, \quad \exists N \in \mathbb{N}, \quad \forall\, i, j > N : \quad ||x_k - x_j|| < \varepsilon$$

2. X is called a complete metric space if every Cauchy sequence in X converges to a point in X

As in $\mathbb{R}$, every Cauchy sequence is bounded.

Theorem 1.4.2

1. If $x_k \to x$ then every subsequence $x_{k_n} \to x$
2. Every convergent sequence is a Cauchy sequence
3. A Cauchy sequence converges, if it has a convergent subsequence

Proof. $x_k \to x$, is equivalent to $\forall\, \varepsilon > 0, \quad \exists N \in \mathbb{N}, \quad \forall\, k > N: \quad \|x_k - x\| < \varepsilon$
Let $\left(x_{k_n}\right)$ be a subsequence. we have :

$$\forall\, \varepsilon > 0, \quad \exists N_1 = \phi(N), \quad \underbrace{\phi(k)}_{=k_m} > N_1 \implies \|x_{k_m} - x\| < \varepsilon$$

$$\|x_k - x\| \leq \|x_k - x_{k_m}\| + \|x_{k_m} - x\| \leq \varepsilon$$

$$\forall\, \varepsilon > 0, \quad \exists N_1 \in \mathbb{N}, \quad \forall\, n > N_1: \quad \|x_k - x\| \leq \frac{\varepsilon}{2}$$
$$\forall\, \varepsilon > 0, \quad \exists N_2 \in \mathbb{N}, \quad \forall\, n > N_2: \quad \|x_{k_m} - x\| \leq \frac{\varepsilon}{2}$$

□

Theorem 1.4.3
$\mathbb{R}^n$, is complete.

Proof. Since (x_k) is Cauchy, for each $i = 1, \dots, n$ we have:

$$\left|x_{k_i} - x_{j_i}\right| \leq \|x_k - x_j\| \to 0 \quad \text{as} \quad k, j \to \infty$$

Hence each component sequence $\left(x_{k_i}\right)$ is Cauchy in $\mathbb{R}$, since $\mathbb{R}$ is complet, then there exists $x_i \in \mathbb{R}$ such that $x_{k_i} \to x_i$ for each $i = 1, \dots, n$. Taking $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$, then $x_k \to x$.

□

Theorem 1.4.4
Let S be a subset of $\mathbb{R}^n$, and let $x \in \mathbb{R}^n$. Then $x \in \overline{S}$ if and only if there exist a sequence of points in S that converges to x.

Proof. ($\Longleftarrow$)

Suppose that $x \in \overline{S}$, if $x \in S$, we take $x = x_k$, if $x \notin S$, $x \in \partial S$, then for all $k \in \mathbb{N}$, there exist $x_k \in S$, such that $x_k \in B\left(x, e^{-k}\right)$, it follows that $\|x_k - x\| < e^{-k}$.

($\Longrightarrow$)

By applying the definition, Easy! □

Definition 1.4.4

Let $S \subset \mathbb{R}^n$, the diameter $d(S)$ of S is

$$d(S) = \sup\left\{\|x - y\| : \quad x, y \in S\right\}$$

Example

$$d(B(x, r)) = 2r$$

$$d(\mathbb{Z}^n) = +\infty$$

Definition 1.4.5

A set S in $\mathbb{R}^n$ is called bounded if there exist $R > 0$ such that $S \subset B(0, R)$, Equivalently, if its diameter is finite.

Theorem 1.4.5 Bolzano-Weirstrass

1. Every bounded sequence in $\mathbb{R}^n$ has a convergent subsequence.
2. Every bounded infinite set in $\mathbb{R}^n$ has a limit point

CHAPTER 2

Functions forms $\mathbb{R}^n$ to $\mathbb{R}^n$

2.1. Functions from $\mathbb{R}^n$ to $\mathbb{R}^m$

A function f from a subset Ω of $\mathbb{R}^n$, called the domain of f, into $\mathbb{R}^m$ is a rule that assigns to each point $x = (x_1, x_2, \dots, x_n) \in \Omega$, a unique point $y \in \mathbb{R}^m$, we write $f : \Omega \longrightarrow \mathbb{R}^m$, $y = f(x)$, and we call $f(x)$ the image of x under f, the set of all images is called the range of f, and is written as:

$$rang\left(f\right) = \left\{f(x) : x \in \mathbb{R}\right\}$$

Since y is a vector in $\mathbb{R}^m$, then $f(x) = y = \left(y_1, y_2, \dots, y_m\right)$, The components y_j are uniquely defined by x and the function f.

Hence they define the functions $f_j : \Omega \longrightarrow \mathbb{R}$, by the rule $y_j = f_j(x)$, for all $j = 1, \dots, m$.

These functions are called the components functions of f, we then write

$f(x) = (f_1(x), f_2(x), \dots, f_n(x))$

Example

1. *The function* $f\left(x,y\right) = \left(y, 2xy, \ln\left(x^2+y^2+1\right)\right)$ *is defined everywhere in* $\mathbb{R}^2$ *it is a vector-valued function* $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ *with composent functions:*

$$f_1\left(x,y\right) = y, \quad f_2\left(x,y\right) = 2xy, \quad f_3\left(x,y\right) = \ln\left(x^2+y^2+1\right)$$

2. *The function* $f\left(x,y\right) = \ln\left(1-x^2-y^2\right)$ *is real-valued function defined on the set:*

$$\Omega = \left\{\left(x,y\right) \in \mathbb{R}^2 : x^2+y^2 < 1\right\}$$

3. *The function* $f(x) = (\cos x, \sin x, \alpha x)$ *is a vector-valued function in one variable* $x \in \mathbb{R}$ *with values in* $\mathbb{R}^3$.

Associated with any function $f : \Omega \subset \mathbb{R}^n \longrightarrow \mathbb{R}$ are two types of sets which play a special rule in the devlopment of our subject. The graph of f and the level set of f

Definition 2.1.1 The Graph of a function
Let $f : \Omega \subset \mathbb{R}^n \longrightarrow \mathbb{R}$, the graph of f is the subset of $\mathbb{R}^{n+1}$(a hypersurface) defined as follows

$$graph\left(f\right) = \left\{(x, f(x)) : \quad x \in \Omega\right\}$$

In the important case of a function of variables $\left(x, y\right) \in \Omega \subset \mathbb{R}^2$ the graph of f is the surface.

Definition 2.1.2 The Level of a function
Let $f : \Omega \subset \mathbb{R}^n \longrightarrow \mathbb{R}$, and $\alpha \in \mathbb{R}$, the subset S_α of $\mathbb{R}^n$ defined by:

$$S_\alpha = \left\{x \in \Omega : \quad f(x) = \alpha\right\}$$

is called the level set of f of level α.
Note that the graph of f be regarded as the level set S of a function
$F : \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$, defined by $F(x, y) = f(x) - y$, that is:

$$graph\left(f\right) = S = \left\{\left((x_1, \dots, x_n), y\right) : F\left(x_1, \dots, x_n, y_n\right) = 0\right\} = \left\{\left(x, y\right) : F\left(x, y\right) = 0\right\}$$

Example *The function $f\left(x, y\right) = e^{-\left(x^2+y^2\right)}$ its graph is:*

$$graph\left(f\right) = \left\{\left(x, y, e^{-\left(x^2+y^2\right)}\right), (x, y) \in \mathbb{R}^2\right\}$$

And the level of f of level α is:

$$S_\alpha\left(f\right) = \left\{(x, y) \in \mathbb{R}^2 : \quad e^{-\left(x^2+y^2\right)} = \alpha\right\} = \begin{cases} \varnothing & if \;\; \alpha \leq 0 \;\; or \;\; \alpha > 1 \\ x^2 + y^2 = -\ln(\alpha) \end{cases}$$

Definition 2.1.3 The Algebraic Operations
Let $f, g : \Omega \subset \mathbb{R}^n \longrightarrow \mathbb{R}^m$. The sum of f and g is defined by:

$$\left(f + g\right)(x) = f(x) + g(x), \quad \forall x \in \Omega$$

and the difference:

$$\left(f - g\right)(x) = f(x) - g(x), \quad \forall x \in \Omega$$

When $m = 1$ the product of f and g is defined by:

$$fg(x) = f(x)\,g(x)\,, \quad \forall\, x \in \Omega$$

and the quotiont by:

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}, \quad \forall\, x \in \Omega \text{ with } g(x) \neq 0$$

Definition 2.1.4 The Image and The Inverse Image
Given a function $f : \Omega \subset \mathbb{R}^n \longrightarrow \mathbb{R}^m$ and subsets $A \subset \mathbb{R}^n$ and $B \subset \mathbb{R}^m$,we define the image of A under f by:

$$f(A) = \left\{ f(x) : x \in A \right\}$$

and the inverse image of B under f by:

$$f^{-1}(B) = \left\{ x \in \Omega : f(x) \in B \right\}$$

2.2. Limits of functions

To simplify the notation, the euclidean norm in $\mathbb{R}^n$ and $\mathbb{R}^m$ will be both denoted by $||.||$

Definition 2.2.1 Limit of a function
Let $f : \Omega \subset \mathbb{R}^n \longrightarrow \mathbb{R}^m$ and a is an accumulation point of Ω, a point $b \in \mathbb{R}^m$ is said to be the limit of f at a written as $\lim_{x\to a} f(x) = b$ if:

$$\forall\, \varepsilon > 0, \quad \exists \delta > 0, \quad \forall\, x \in \Omega : \quad 0 < ||x - a|| < \delta \implies ||f(x) - b|| < \varepsilon$$

this is,

$$\lim_{x\to a} f(x) = b \iff \lim_{x\to a} ||f(x) - b|| = 0$$

Theorem 2.2.1 Uniqueness of the limit
If $\lim_{x\to a} f(a)$ exists then it is unique.

Proof. Assume $\lim_{x\to a} f(a)$ exists, and suppose it is not unique, so there exist $b, b' \in \mathbb{R}^m$

such that:

$$\begin{cases} \forall\, \varepsilon > 0, \exists \delta_1 > 0, \forall\, x \in \Omega : \quad \|x - a\| < \delta_1 \implies \|f(x) - b\| < \frac{\varepsilon}{2} \\ \forall\, \varepsilon > 0, \exists \delta_2 > 0, \forall\, x \in \Omega : \quad \|x - a\| < \delta_2 \implies \|f(x) - b'\| < \frac{\varepsilon}{2} \end{cases}$$

let $\varepsilon > 0$ and take $\delta = \min(\delta_1, \delta_2)$ then we have:

$$\|b - b'\| \le \|b - f(x)\| + \|f(x)\, b'\| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

since ε is arbitrary, we get $\|b - b'\| = 0$ which gives us $b = b'$ contradiction. □

Example *Show that* $\lim_{(x,y)\to(0,0)} f(x,y) = 0$,

$$f(x,y) = \frac{|x|^{5/2} + |y|^{5/2}}{x^2 + y^2}$$

Let $\varepsilon > 0$, *we need to find* $\delta > 0$ *such that* $\forall\, x \in \Omega$:

$$0 < \|x - a\| < \delta \implies \|f(x,y)\| < \varepsilon$$

We have:

$$\left|f(x,y)\right| = \frac{|x|^{5/2} + |y|^{5/2}}{x^2 + y^2} \le \frac{2\left(x^2 + y^2\right)^{5/4}}{x^2 + y^2} = 2\left(x^2 + y^2\right)^{1/4} = 2\left(\sqrt{x^2 + y^2}\right)^{1/2}$$
$$< 2\sqrt{\delta}$$

Hence, by taking $\delta = \left(\frac{\varepsilon}{2}\right)^2$ *we get* $\left|f\left(x,y\right)\right| < \varepsilon$

$$f(x,y) = \frac{\sin\left(e^{-\frac{1}{x^2+y^2}}\right)}{e^{-\frac{1}{x^2+y^2}}}$$

$\lim_{(x,y)\to(0,0)} f(x,y) = 1$

Theorem 2.2.2

For $f, g : \Omega \subset \mathbb{R}^n \longrightarrow \mathbb{R}^m$ and let a be an accumulation point of Ω, we have:

$$\lim_{x\to a}\left(\alpha f(x) + \beta g(x)\right) = \alpha \lim_{x\to a} f(x) + \beta \lim_{x\to a} g(x)$$

And when $m = 1$,

$$\lim_{x\to a} f(x)g(x) = \lim_{x\to a} f(x) \lim_{x\to a} g(x)$$
$$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$$

where $g(x) \neq 0, \forall\, x \in \Omega$ and $\lim_{x\to a} g(x) \neq 0$

Proof.

$$\lim_{x\to a} f(x) = b \iff \forall\, \varepsilon > 0, \exists \delta_1, \forall\, x \in \omega : \quad \|x - a\| < \delta_1 \implies \|f(x) - b\| < \frac{\varepsilon}{2M}$$

$$\lim_{x\to a} g(x) = b' \iff \forall\, \varepsilon > 0, \exists \delta_2, \forall\, x \in \omega : \quad \|x - a\| < \delta_2 \implies \|g(x) - b'\| < \frac{\varepsilon}{2\|b\|}$$

$$\lim_{x\to a} fg = bb' \iff \forall\, \varepsilon > 0, \exists \delta > 0, \forall\, x \in \omega : \quad 0 < \|x - a\| < \delta \implies \|f(x)g(x) - bb'\|$$
$$= \|f(x)g(x) - bg(x) + bg(x) - bb'\| \quad \leq$$

□

Theorem 2.2.3

Let $f(x) = (f_1, ..., f_n) : \omega \longrightarrow \mathbb{R}^m$ and let b be an accumulation point of Ω then:

$$\lim_{x\to a} f(x) = b \iff \lim_{x\to a} f_j(x) = b_j \quad \forall\, j = 1, \dots, m$$

Proof.

$$(\implies)$$

Suppose that $\lim_{x\to a} f(x) = b$, then $\forall\, j = 1, \dots, m$

$$\left|f_j(x) - b_j\right| \leq \|f(x) - b\|$$

and then :

$$\forall\, \varepsilon > 0, \quad \exists > 0, \forall\, x : x \in \Omega : \quad 0 < \|x - a\| < \delta \implies \left|f_j(x) - b_j\right| < \varepsilon$$

Thus $\lim_{x\to a} f_j(x) = b_j$

$$(\impliedby)$$

Suppose that for all $j = 1, \dots, m$, $\lim_{x\to a} f_j(x) = b_j$.

$$\forall\, \varepsilon > 0, \quad \exists \delta_j > 0, \quad \forall\, x \in \Omega : \quad 0 < \|x - a\| < \delta_j \implies \|f_j(x) - b_j\| < \frac{\varepsilon}{\sqrt{m}}$$

$$\forall\, \varepsilon > 0, \quad \exists \delta = min_{j=1,\dots,m}\delta_j : \quad \forall\, x \in \Omega : \quad 0 < \|x - a\| < \delta \implies \|f(x) - b\|$$

$$\|f(x) - b\| = \sqrt{\sum_{j=1}^{m} \left|f_j(x) - b_n\right|^2} \leq \left(\sum_{j=1}^{m} \frac{\varepsilon^2}{m}\right)^{1/2} = \varepsilon$$

then $\lim_{x\to a} f(x) = b$ □

Example

$$f(x,y) = \left(e^x, \ln\left(1+|y|\right), \frac{x^2}{1+y^2}\right)$$

$$\lim_{(x,y)\to(0,0)} f(x,y) = (1,0,0)$$

Theorem 2.2.4

Let $f : \Omega \subset \mathbb{R}^n \longrightarrow \mathbb{R}^m$, and let a be an accumulation point of Ω, then:

$$\lim_{x\to a} f(x) = b \iff \left(\forall\,(x_k) \subset \Omega : \quad \lim_{x\to\infty} x_k = a \implies \lim_{k\to\infty} f(x_k) = b\right)$$

Proof.

$$(\implies)$$

Suppose that $\lim_{x\to a} f(x) = b$

$$\forall\,\varepsilon > 0, \exists\delta > 0, \quad \forall\, x \in \Omega : \quad 0 < ||x - a|| < \delta \implies ||f(x) - b|| < \varepsilon$$

Let $(x_k) \subset \Omega$ such that $\lim_{k\to\infty} x_k = a$.
For $\delta > 0, \exists N \in \mathbb{N} : \forall\, k > N \quad ||x_k - a|| < \delta \implies ||f(x_k - b)|| < \varepsilon$

$$\forall\,\varepsilon > 0, \quad \exists N \in \mathbb{N} : \quad \forall\, k > N \implies ||f(x_k) - b|| < \varepsilon$$

Then $\lim_{k\to\infty} f(x_k) = b$

$$(\impliedby)$$

Suppose that $\forall\,(x_k) \subset \Omega$, $\lim_{k\to\infty} x_k = a \implies \lim_{k\to\infty} f(x_k) = b$ and suppose that $\lim_{x\to a} f(x) \neq b$ then:

$$\exists\varepsilon > 0, \quad \forall\,\delta > 0, \quad \exists x \in \Omega : \quad ||x - a|| < \delta \text{ and } ||f(x) - b|| > \varepsilon$$

for $\delta = \frac{1}{k}, \quad \exists x_k \in \Omega : ||x_k - a|| < \frac{1}{k}$ and $||f(x_k) - b|| > \varepsilon$

$$\lim_{k\to\infty} x_k = a \quad \lim_{k\to\infty} f(x_k) = b$$

contradiction! then $\lim_{x\to a} f(x) = b$ □

By the uniqueness of the limit of f at point a, b is also the limit as x approcahes a long of any curve passing from a, this fact will be raised to explain why certain functions fail to have a limit at a point

Example

1. *Let* $f(x,y) = \frac{xy+y^2}{x^2+y^2}$
   - Show that the limit of f *at* $(0,0)$ *doesn't exist.*

$$\text{for } x = y \quad \lim_{x\to 0} f(x,x) = 1$$
$$\text{for } x = -y \quad \lim_{x\to 0} f(x,-x) = 0$$

   consequently, the limit doesn't exist !

2. *Let* $f(x,y) = \frac{2xy^2}{x^2+y^4}$
   - Show that the limit of f *at* $(0,0)$ *doesn't exist.*

$$\text{for } x = \alpha y \quad \lim_{x\to 0} f(x,\alpha x) = \infty$$
$$\text{for } x = y^2 \quad \lim_{x\to 0} f(y^2,y) = 1$$

   consequently, the limit doesn't exist !

2.3. Continious functions

Definition 2.3.1 Continious functions
Let $f : \Omega \subset \mathbb{R}^n \longrightarrow \mathbb{R}^m$,where Ω is an open set of $\mathbb{R}^n$ and $a \in \Omega$, we say that f is continious at a if :

$$\forall\, \varepsilon > 0, \exists \delta > 0, \forall\, x \in \Omega : \quad \|x-a\| < \delta \implies \|f(x) - f(a)\| < \varepsilon$$

Observe that with this definition, every function f is continious at each isolated point of a, that is every point of Ω that is not a limit of Ω.
If a is an accumulation point of Ω, then the definition of continiuity is given by:

$$f \text{ is continious at } a \iff \lim_{x\to a} f(x) = f(a)$$

More over, we say that f is continious on Ω if it's continious at each point of Ω.

Example

$$f(x,y) = \begin{cases} \frac{x^2y}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 \quad \text{if } (x,y) = (0,0) \end{cases}$$

$$\lim_{(x,y)\to(0,0)} \left|f(x,y)\right| = \lim_{(x,y)\to(0,0)} \left|\frac{x^2y}{x^2+y^2}\right| \le \lim_{(x,y)\to(0,0)} \left|y\right| = 0$$

since $\lim_{(x,y)\to(0,0)} f(x,y) = 0 = f(0,0)$, *the function* f *is continious at* $(0,0)$,

as immediate consequence of the definition of continiuity and the properties of the limit is the following properties

Theorem 2.3.1

1. $f : \Omega \subset \mathbb{R}^n \longrightarrow \mathbb{R}^m$, then $f = \left(f_1, \dots, f_n\right)$ is continiious at a if and only if f_j is continious at a for all $j = 1, \dots, n$
2. $f : \Omega \subset \mathbb{R}^n \longrightarrow \mathbb{R}^m$ then f is continious at a if for all $(x_k) \subset \Omega$, if $\lim_{k\to\infty} x_k = a \implies \lim_{n\to\infty} f(x_k) = f(a)$
3. The sum, diffirence, product and quotient of two continious functions, whenever defined, is continious

Example *Let* $P_i : \mathbb{R}^n \longrightarrow \mathbb{R}$ *be the projection function on the* i^{th} *coordinate axis, that is:*

$$\begin{array}{rccc} P_i : & \mathbb{R}^n & \longrightarrow & \mathbb{R} \\ & (x_1, \dots, x_n) & \longmapsto & x_i \end{array}$$

Then for $i = 1, \dots, n$, *each* P_i *is continious on* $\mathbb{R}^n$

$$\forall\, x, y \in \mathbb{R}^n \quad |P_i(x) - P_i(0)| = \left|x_i - y_i\right| \le \|x - y\| < \varepsilon$$

$$\begin{array}{rccc} f : & \mathbb{R}^n & \longrightarrow & \mathbb{R} \\ & x & \longmapsto & \|x\| \end{array}$$

Theorem 2.3.2

Let $f : \Omega \subset \mathbb{R}^n \longrightarrow \mathbb{R}^m$. Then f is continuous at $a \in \Omega$ if and only if $f(x_k) \to f(a)$ wherever (x_k) is a sequence of points of Ω such that $x_k \to a$.

Definition 2.3.2

Let $f : \Omega \longrightarrow \mathbb{R}^n$ and $g : U \subset \mathbb{R}^n \longrightarrow \mathbb{R}^p$ be two functions such that $f(\Omega) \subset U$.

then the composition of f and g is defined by

$$g \circ f(x) = g\left(f(x)\right) \quad g \circ f : \Omega \longrightarrow \mathbb{R}^p$$

Theorem 2.3.3
Let $f : \Omega \subset \mathbb{R}^m \longrightarrow \mathbb{R}^n$ and $g : U \subset \mathbb{R}^m \longrightarrow \mathbb{R}^p$, such that f is continious at a and g is continious at $f(a)$, then $g \circ f$ is continious at a

Proof. Suppose that f is continuous at a then for all $(x_k) \subset \Omega$, such that $\lim_{k\to\infty} x_k = a$, we get $\lim_{k\to\infty} f(x_k) = f(a)$, and g is continious at $f(a)$, $\lim_{k\to\infty} g\left(f(x_k)\right) = g \circ f(a)$, then $g \circ f$, is continuous at a □

Example *Let $f : \mathbb{R}^2 \longrightarrow R$ defined by:*

$$f\left(x, y\right) = \begin{cases} e^{-\frac{1}{|x-y|}} & x \neq y \\ 0 & x = y \end{cases}$$

Show that f is continuous on $\mathbb{R}^2$. Since f is a composition of continuous functions, the only points where f may not be continuous are the points pn the line $y = x$, that is, at points (a, a) where $a \in \mathbb{R}$. We investegate the continuity of f at these points. Setting $t = x - y$, the function becomes:

$$f(t) = \begin{cases} e^{-\frac{1}{|t|}} & t \neq 0 \\ 0 & t = 0 \end{cases}$$

Now since:

$$\lim_{t\to 0} f(t) = f(a, a) = \lim_{\left(x,y\right)\to(a,a)} f(x, y) = 0$$

Then f is continuous at all points on $x = y$ then f is contiuous on $\mathbb{R}^2$

2.4. Linear Transformation

Definition 2.4.1
A function $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is called linear transformation, if $\forall\, x, y \in \mathbb{R}^n$, we have

$$T\left(x + \alpha y\right) = T(x) + \alpha T\left(y\right)$$

Theorem 2.4.1

A function $T : \mathbb{R}^n \longrightarrow \mathbb{R}$ is a linear transformation if and only if there exist a unique $w \in \mathbb{R}^n$, such that $T(x) = w \cdot x$ for all $x \in \mathbb{R}^n$.

Proof.

$$[\Longrightarrow]$$

Suppose that there exist $w \in \mathbb{R}^n$ such that $T(x) = w \cdot x$

$$\begin{aligned} \forall x, y \in \mathbb{R}^n, \forall \alpha \in \mathbb{R}, \quad T\left(x + \alpha y\right) &= w\left(x + \alpha y\right) \\ &= w \cdot x + \alpha w \cdot y \\ &= T(x) + \alpha T\left(y\right) \end{aligned}$$

$$(\Longleftarrow)$$

Let $(e_1, \dots, e_n)$ is a basis of $\mathbb{R}^n$, then for all $x \in \mathbb{R}^n$,

$$x = \sum_{i=1}^{n} x_i e_i$$

$$\begin{aligned} T(x) = T\left(\sum_{i=1}^{n} x_i e_i\right) &= \sum_{i=1}^{n} T(x_i e_i) \\ &= \sum_{i=1}^{n} x_i T(e_i) \\ &= x \cdot w \quad w = (T(e_1), \dots, T(e_n)) \end{aligned}$$

For the unicity, suppose that there exist another w and w' where $T(x) = w'x = wx$

$$\begin{aligned} T(x) = wx = w'x &\Longrightarrow wx - w'x = 0 \\ &\Longrightarrow (w - w')\, x = 0 \quad \forall x \in \mathbb{R}^n \\ &\Longrightarrow \|w - w'\|_2^2 = 0 \quad \Longrightarrow w = w' \end{aligned}$$

$\square$

Theorem 2.4.2

Any linear transformation $T : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is continious.

Proof. Let $x, y \in \mathbb{R}^n$, let $(e_1, \dots, e_n)$ be some cannonical bases, we have

$$\begin{aligned}\|T(x) - T(y)\| = \|T\left(x-y\right)\| &= \|T\left(\sum\left(x_i - y_i\right)e_i\right)\| \\ &= \|\sum(x_i - y_i)T(e_i)\| \\ &\leq \sum_{i=1}^{n}\left|x_i - y_i\right| \|T(e_i)\| \\ &\leq \sup_{i=\overline{1,n}}\left|x_i - y_i\right| \sum_{j=1}^{n}\|T\left(e_j\right)\| \leq \|x-y\| \overbrace{\sum_{i=1}^{n}\|T(e_i)\|}^{M}\end{aligned}$$

Continious functions on compact sets. □

2.5. Compact sets

Definition 2.5.1

A set $\Omega \subset \mathbb{R}^n$ is compact if every sequence in Ω has a convergent subsequence whose limit is on Ω.

Theorem 2.5.1

A set $\Omega \subset \mathbb{R}^n$, is compact if and only if Ω is closed and bounded

Proof. Suppose that Ω is closed and bounded, let $(x_k) \subset \Omega$ then by bolzano-weirstrass (x_k) has a subsequence (x_k) convergent to a limit $x_0 \in \overline{\Omega}$, and by Ω is closed then $x_0 \in \Omega$. Suppose that Ω is compact, then Ω is closed, indeed, $(x_k) \subset \Omega$ and $x_k \to x_0$, then $x_0 \in \Omega$, Ω is unbounded.

$$\forall k \in \mathbb{N}, \exists x_k \in \Omega : \quad \|x_k\| \geq k$$

(x_k) does not contain any convergent subsequence, contradiction with Ω is compact. □

Theorem 2.5.2

A set $\Omega \subset \mathbb{R}^n$ is closed and bounded if and only if every infinite subset of Ω has an accumulation point that belongs to Ω.

Theorem 2.5.3

let $f : \Omega \subset \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be a continious, if Ω is compact then $f(\Omega)$, is compact.

Proof. Let $(y_k) \subset f(\Omega)$, then

$$\forall k \in \mathbb{N}, \exists (x_k) \subset \Omega : \quad f(x_k) = y_k$$

since Ω is compact, then there exist $(x_{nk}) \subset \Omega$, such that (x_{nk}) converges to a point x_0 in Ω, since f is continuous then

$$\lim_{n\to\infty} f(x_{nk}) = f(x_0)$$

which is a subsequence of $f(x_n) = y_n$, hence converges in $f(\Omega)$, therefore $f(\Omega)$ is compact ☐

Theorem 2.5.4 Weirstrass
let $f : \Omega \longrightarrow \mathbb{R}$ be a continious function and Ω is compact, then f attains it's maximum and minimum value in Ω.

Proof. The set $f(\Omega)$ is a compact subset of $\mathbb{R}$.Hence it is bounded and closed, Let $[c, d]$ be the smallest closed interval containing $f(\Omega)$. Such an interval exist since $f(\Omega)$ is bounded. Since $f(\Omega)$ is closed $c \in f(\Omega)$ and $d \in f(\Omega)$ then $\max f(\Omega) = d$ and $\min f(\Omega) = c$. ☐

2.6. Uniform continuity

Definition 2.6.1
Let $\Omega \subset \mathbb{R}^n$ and let $f : \Omega \longrightarrow \mathbb{R}^m$, we say f is uniformally continious on Ω if for every $\varepsilon > 0$, there exist $\delta > 0$ such that if $x, y \in \Omega$ and $||x - y|| < \delta$, then $||f(x) - f(y)|| < \varepsilon$

Theorem 2.6.1
Let $f : \Omega \subset \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be a continious functions if Ω is compact then f is uniformally continious.

Proof. suppose that f is continious and Ω is compact with f not uniformally continious, we have that f is not uniformally continious on Ω, then :

$$\exists \varepsilon > 0, \forall n \in \mathbb{N}, \exists (x_n)_{n\in\mathbb{N}}, \left(y_n\right)_{n\in\mathbb{N}} \subset \Omega : \quad ||x_n - y_n|| < \frac{1}{n} \text{ and } ||f(x_n) - f(y_n)|| \geq \varepsilon$$

x_n and $y_n \in \Omega$, and Ω is compact set by definition x_n has a convergent subsequence $(x_{nk})_{k\in\mathbb{N}}$ and the sequence $\left(y_{nk}\right)_{k\in\mathbb{N}} \subset \Omega$ then $\left(y_{kn}\right)$ has a convergent subsequence $\left(y_{k'n}\right)$

converges on Ω, then the subsequences $(x_{k'n})$ and $\left(y_{k'n}\right)$ are convergent to x and y in Ω respectively and by the above

$$\|x_{k'n} - y_{k'n}\| \leq \frac{1}{k'n}$$

we deduce that $x = y$, then

$$\|f(x_{k'n}) - f(y_{k'n})\| \geq \varepsilon$$

$0 \geq \varepsilon > 0$ contradiction then f is uniformally continious. □

2.7. Connectedness

Definition 2.7.1

A subset $\Omega \subset \mathbb{R}^n$ is said to be connected if its not possible to write $\Omega = (A \cap \Omega) \cup (B \cap \Omega)$, with A and B are open sets of $\mathbb{R}^n$ and that $\Omega \cap A$ and $B \cap \Omega$ both non empty and

$$A \cap B \cap \Omega = \emptyset$$

Theorem 2.7.1

let $f : \Omega \subset \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be a continious function and Ω is connected, then $f(\Omega)$ is connected

Proof. assume the contrary, that $f(\Omega)$ is not connected, then $f(\Omega) = A' \cup B'$, are open sets in $f(\Omega)$ such that $A' \cap B' = \emptyset$, let $A = \Omega \cap f^{-1}(A')$, $B = \Omega \cap f^{-1}(B')$, A and B are open sets of Ω, and $A \cap B = \emptyset$ and $\Omega = A \cup B$, contradiction with Ω is connected set hence $f(\Omega)$ is connected □

Theorem 2.7.2

let Ω be a connected set of $\mathbb{R}^n$ and $f : \Omega \longrightarrow \mathbb{R}$ is continious, and let $c \in \mathbb{R}$ such that

$$f(x_1) \leq c \leq f(x_2)$$

then by definition of an interval there exist $x_0 \in \Omega$ such that $f(x_0) = c$

Proof. we have $f(\Omega)$ is connected set on $\mathbb{R}$, then $f(\Omega)$ is an interval, and

$$f(x_1) \leq c \leq f(x_2)$$

then $\exists x_0 \in \Omega$ such that $f(x_0) = c$, where $c \in [f(x_1), f(x_2)]$. □

2.8. Path-connectedness

Definition 2.8.1

let x, y be points of metric space X, a path in X from x to y is a continious function $\gamma : [a, b] \longrightarrow X$, where $\gamma(a) = x$ and $\gamma(b) = y$

Definition 2.8.2

A metric space X is called pathwise connected if any two points in X can be joined by a path in X

Corollary 2.8.1 *Any pathwise connected metric space is connected*

Theorem 2.8.1

Continious images of pathwise connected sets are pairwise connected

Theorem 2.8.2

Let Ω be an open set connected in $\mathbb{R}^n$, then Ω is pathwise connected

Proof. Let x_0 be fixed point of X, and consider the sets A and B defined by :

$$A = \left\{x \in \Omega : x \text{ can be joined with } x_0 \text{ by a path in } \Omega \right\}$$

$$B = \{z \in \Omega : z \notin A\}$$

we have that $A \cup B = \Omega$, now we show that A and B are open sets of Ω, let $x \in A \subset \Omega$ then there exist $r > 0$, such that $B(x, r) \subset \Omega$, let $y \in B(x, r)$ and since $B(x, r)$ is convex then we can join x with y by a segment and x joined by x_0 because $x \in A$, then y is joined with x_0 in Ω then $y \in A$, hence A is an open set of Ω.
Secondly we show that B is an open set, let $z \in B \subset \Omega$, there exist $\delta > 0$ such that $B(z, \delta) \subset \Omega$, suppose that there exist $x \in B(z, \delta)$ and $x \notin B$ then $x \in A$ and we can join z to x (convexity of the open ball), and we can join x to x_0, so $z \in A$ which is a contradiction since we took $z \in B$, therefore $x \in B$ hence B is an open set, we have that Ω is a connected set of X, therefore either $A = \emptyset$ or $B = \emptyset$, since A contains x_0 then $B = \emptyset$, so $A = \Omega$. $\square$

Proposition 2.8.1 *Let Ω be an open set in $\mathbb{R}^n$. Then Ω is connected if and only if for any two points in Ω, they can be joined a polynomial path.*

2.9. Convex Sets

Definition 2.9.1

A non empty set $\mathscr{C}$ in $\mathbb{R}^n$ is said to be convex if the line segment joining any two points x and y of $\mathscr{C}$ is contained in $\mathscr{C}$, that is

$$x, y \in \mathscr{C} \implies \gamma(t) = tx + (1-t)y \in \mathscr{C} \quad \forall\, t \in [0,1]$$

Example *Any open or closed ball of X is a convex set, let $x, y \in B(r,a)$ and $t \in [0,1]$, then*

$$\begin{aligned} \|tx + (1-t)y - a\| = \|t(x-a) + (1-t)(y-a)\| &\leq t\|x-a\| + (1-t)\|y-a\| \\ &\leq tr + (1-t)r \\ &\leq r \end{aligned}$$

then we have

$$tx + (1-t)y \in B(r,a)$$

CHAPTER 3

Differentiable functions from $\mathbb{R}^n$ to $\mathbb{R}^m$

Definition 3.0.1

Let $f : \Omega \subset \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be a function defined on an open set Ω of $\mathbb{R}^n$, we say that f is differentiable at a if there is a linear transformation $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ such that

$$\lim_{x \to a} \frac{\|f(x) - f(a) - T(x-a)\|_m}{\|x-a\|_n} = 0$$

if f is differentiable at every point of Ω, we say that f is differentiable on Ω

Remark *Setting $h = x - a \neq 0$, then $x \to a$ in $\mathbb{R}^n$ is equivalent to $h \to 0$, and so the limit of the above definition is equivalent to*

$$\lim_{h \to 0} \frac{\|f(a+h) - f(a) - T(h)\|_m}{\|h\|_n} = 0$$

Theorem 3.0.1

If $f : \Omega \subset \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is differentiable at a if and only if there exist a function $\varepsilon(x)$ so that for $x \in \Omega$, we have

$$f(x) = f(a) + T(x-a) + \varepsilon(x)\|x-a\|$$

with $\varepsilon(x) \to 0$ as $x \to a$

Proof. set

$$\varepsilon(x) = \frac{f(x) - f(a) - T(x-a)}{\|x-a\|}$$

now if f is differentiable at a, then $\lim_{x \to a} \varepsilon(x) = 0$, conversely if

$$f(x) = f(a) + T(x-a) + \varepsilon(x)\|x-a\|$$

the result also holds, since $x \neq a$, we have

$$\lim_{x\to a} \frac{f(x) - f(a) - T(x-a)}{||x-a||} = \lim_{x\to a} \varepsilon(x) = 0$$

therefore f is differentiable at a □

Proposition 3.0.1 *Let $f : \Omega \subset \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be a function defined on a open subset $\Omega \subset \mathbb{R}^n$ and $a \in \Omega$. If f is differentiable at a, then T is unique.*

Proof. Suppose T, S are linear transformations satisfiying

$$f(a+h) = f(a) + T(h) + \varepsilon_1(h)||h||$$
$$f(a+h) = f(a) + S(h) + \varepsilon_2(h)||h||$$

with

$$\varepsilon_1(h) \to 0$$
$$\varepsilon_2(h) \to 0$$

as $h \to 0$, now by substracting the two equations we get

$$(T-S)(h) = (\varepsilon_1 - \varepsilon_2)||h||$$

setting $L = T - S$ and dividing by $||h||$, it follows that

$$\frac{||L(h)||}{||h||} = ||\varepsilon_1 - \varepsilon_2|| \to 0 \quad \text{as} \quad h \to 0$$

now, let $x \in \mathbb{R}^n$ be any non zero vector and let $t \in \mathbb{R}$, take $h = tx$, then $h \to 0$ is equivalent to $t \to 0$, and since L is linear, we get the following

$$0 = \lim_{h\to 0} \frac{||L(h)||}{||h||} = \lim_{t\to 0} \frac{||L(tx)||}{||tx||} = \lim_{t\to 0} \frac{||L(x)||}{||x||} = \frac{||L(x)||}{||x||}$$

thus $L(x) = 0$, for all $x \in \mathbb{R}^n$ which means $L \equiv 0$. Hence $S = T$ □

Remark *As we show that T is unique we denoted T by $d_f^{(a)}$ or $\mathscr{D}_f^{(a)}$, the differentiable (derivative) of f at the point a.*

Proposition 3.0.2 *If f is differentiable at a, then it is continuous at a*

Proof. Since f is differentiable at a, we have

$$f(x) = f(a) + d_f^{(a)}(x-a) + \varepsilon(x)||x-a||$$

by taking limit on both sides

$$\lim_{x\to a} f(x) = f(a)$$

since $\left(\lim_{x\to a} d_f^{(a)}(x-a) = 0\right)$, due to continuity of d_f. $\square$

Theorem 3.0.2

let f and g be function from an open set Ω in $\mathbb{R}^n$ to $\mathbb{R}^m$ diffirentiable at $a \in \Omega$ and let $c \in \mathbb{R}$, then :

(1) $f+g$ is diffirentiable at a and $d_{f+g}(a) = d_f(a) + d_g(a)$

(2) cf is diffirentiable at a and $d_{cf}(a) = cd_f(a)$

(3) if $m = 1$, then $f \cdot g(x) = f(x) \cdot g(x)$ is diffirentiable at a and $d_{f\cdot g}(a) = f(a) \cdot d_g(a) + d_f(a)g(a)$

(4) if $m = 1$, and $g(a) \neq 0$, then $\left(\frac{f}{g}(a)\right)$ is diffirentiable at a and $d_{\frac{f}{g}} = \frac{g(a)d_f^{(a)} - f(a)d_g^{(a)}}{g(a)^2}$

Proof. we have f and g are diffirentiable at a, i.e.

$$f(a+h) = f(a) + d_f^{(a)}(h) + \varepsilon_1(h)||h|| \quad \varepsilon_1(h) \to 0 \quad \text{as} \quad h \to 0$$
$$g(a+h) = g(a) + d_g^{(a)}(h) + \varepsilon_2(h)\,||h|| \quad \varepsilon_2(h) \to 0 \quad \text{as} \quad h \to 0$$

we get

$$\begin{aligned} fg\,(a+h) - fg(a) &= f(a+h)g(a+h) - f(a)g(a) \\ &= \left(f(a+h) - f(a)\right)g(a+h) + f(a)\left(g(a+h) - g(a)\right) \\ &= \left(d_f^{(a)}(h) + \varepsilon_1(h)||h||\right)g(a+h) + f(a)\left(d_g^{(a)}(h) + \varepsilon_2(h)||h||\right) \\ &= d_f^{(a)}(h)g(a+h) + d_g^{(a)}(h)f(a) + \left(\varepsilon_1(h)g(a+h) + \varepsilon_2(h)f(a)\right)||h|| \end{aligned}$$

on the other side we have

$$\frac{fg(a+h) - fg(a) - g(a)d_f^{(a)} - f(a)d_g^{(a)}(h)f(a)}{||h||}$$
$$= \frac{\left(g(h+a) - g(a)\right)d_f^{(a)}(h) + \left(\varepsilon_1(h)g(a+h) + \varepsilon_2(h)f(a)\right)||h||}{||h||}$$

the RHS tends to zero as $h \to 0$, therefore we get the equality. $\square$

Theorem 3.0.3
Let Ω be an open set in $\mathbb{R}^n$ and $f : \Omega \longrightarrow \mathbb{R}^m$ and $g : U \longrightarrow \mathbb{R}^p$ where U is an open set in $\mathbb{R}^m$ with $f(\Omega) \subset U$, if f is diffirentiable at $a \in \Omega$ and g is diffirentiable at $f(a)$ then $g \circ f$ is diffirentiable at a and $d_{g\circ f}^{(a)} = d_g^{(f(a))} \cdot d_f^{(a)}$

Proof. f and g are diffirentiable at a, then let $y = f(x)$ and $b = f(a)$, we have :

$$f(x) = f(a) + d_f^{(a)}(x-a) + \varepsilon_1(x)\|x-1\|$$
$$g(y) = g(b) + d_g^{(b)}(y-b) + \varepsilon_1(y)\|y-b\|$$

Substituting the first equation into the second we get

$$\begin{aligned} g\left(f(x)\right) &= g(f(a)) + d_g^{(f(a))}\left(d_f^{(a)}(x-a) + \varepsilon_1(x)\|x-a\|\right) + \varepsilon_2\left(f(x)\right)\|d_f^{(a)}(x-a) + \varepsilon_1\|x-a\|\| \\ &= g\left(f(a)\right) + d_g^{(f(a))} d_f^{(a)}(x-a) + \\ &\quad \underbrace{\left[d_g^{(f(a))}\left(\varepsilon_1(x)\|x-a\|\right) + \varepsilon_2\left(f(x)\right)\|d_f^{(a)}(x-a) + \varepsilon_1(x)\|x-a\|\|\right]}_{\|x-a\|\varepsilon(x)} \end{aligned}$$

the proof is complete if we show that $\lim_{x\to a}\varepsilon(x) = 0$ where

$$\varepsilon(x) = \frac{d_g^{(f(a))}\left(\varepsilon_1(x)\|x-a\| + \varepsilon_2\left(f(x)\right)\|d_f^{(a)}(x-a) + \varepsilon_1(x)\|x-a\|\|\right)}{\|x-a\|}$$

we get

$$\varepsilon(x) = d_g^{(f(a))}\left(\varepsilon_1(x)\right) + \frac{1}{\|x-a\|}\left(\varepsilon_2\left(f(x)\right)\|d_f^{(a)}(x-a) + \varepsilon_1(x)\|x-a\|\|\right)$$

clearly $\lim_{x\to a} d_g^{(f(a))}\left(\varepsilon_1(x)\right) = 0$, we must show the rest tends to zero

$$\begin{aligned} \|\frac{1}{\|x-a\|}\left(\varepsilon_2\left(f(x)\right)\|d_f^{(a)}(x-a) + \varepsilon_1(x)\|x-a\|\|\right)\| &\le \|\varepsilon_2\left(f(x)\right)\|c\frac{\|x-a\|}{\|x-a\|} + \|\varepsilon_1(x)\| \\ &= c\|\varepsilon_2(f(x))\| + \varepsilon_1(x) \to 0 \quad \text{as} \quad x \to a \end{aligned}$$

□

Proposition 3.0.3 *Let Ω be an open set in $\mathbb{R}^n$ and $f : \Omega \longrightarrow \mathbb{R}^m$ with component functions $f = \left(f_1, \dots, f_n\right)$, then f is diffirentiable at a if f_j is diffirentiable at a, for all $j = 1, \dots, m$, Moreover*

$$d_f^{(a)} = \left(d_{f_1}^{(a)}, d_{f_2}^{(a)}, \dots, d_{f_n}^{(a)}\right)$$

Proof. let $f = \left(f_1, f_2, \dots, f_n\right)$ and $d_f^{(a)} = (\lambda_1, \dots, \lambda_n)$, we have the vectorial equality from

$$f(x) = f(a) + d_f^{(a)}(x-a) + \varepsilon_1(x)\|x-a\|$$

then

$$f_j(x) = f_j(a) + \lambda_j(x-a) + \varepsilon_j(x)\|x-a\| \quad \forall j = 1, \dots, m$$

then for all $j = 1, \dots, m$:

$$\sup_{j=1,\dots,m} \|\varepsilon_j(x)\| \leq \|\varepsilon(x)\| \leq m \sup_{j=1,\dots,m} \|\varepsilon_j(x)\|$$

suppose that f is diffirentable at a, then $\varepsilon(x) \to 0$ as $x \to a$, to show that $f_j(x)$ is diffirentiable we have to show that $e_j(x)$ goes to 0 as x tends to a, from the above inequality, we get

$$\forall j = 1, \dots, n \quad \lim_{x\to a} e_j(x) = 0$$

now suppose that $f_j(x)$ is diffirentiable at a for all $j = 1, \dots, n$, again from the above inequality, by taking the limit we get the inverse implication. $\square$

One should notice by the above proposition, we see that to study the differentiability of a function $f : \Omega \subset \mathbb{R}^n \longrightarrow \mathbb{R}^m$, it suffices to study the differentiability of it's component functions $f_j : \Omega \subset \mathbb{R}^n \longrightarrow \mathbb{R}$, for $j = \overline{1, m}$. Hence, we turn to real valued functions of several variables.

Example

1. *let*

$$\begin{aligned} f :\ \Omega \subset \mathbb{R}^n &\longrightarrow \mathbb{R}^n \\ x &\longmapsto x_0 \end{aligned}$$

f is diffirentiable at any point $a \in \mathbb{R}^n$ and $d_f^{(a)} = 0$

$$\frac{\|f(x) - f(a) - d_f^{(a)}(x-a)\|_n}{\|x-a\|_m} = 0 \to 0 \text{ as } x \to a$$

2. *let*

$$\begin{aligned} f :\ \mathbb{R}^n &\longrightarrow \mathbb{R}^m \\ x &\longmapsto T(x) \end{aligned}$$

where T is a linear transformation, f is differentiable at each point $a \in \mathbb{R}^n$, and we have $d_f^{(a)}(h) = T(h)$

$$\frac{\|f(x) - f(a) - T(x-a)\|_n}{\|x-a\|} = 0 \to 0 \text{ as } x \to a$$

3. *define the function*

$$\begin{aligned} f :\ \mathbb{R}^n &\longrightarrow \mathbb{R} \\ x &\longmapsto \|x\|^2 \end{aligned}$$

we have that f is diffirentiable at any point of $\mathbb{R}^n$

$$\begin{aligned} f(a+h)-f(a) = \|x+h\|^2 - \|a\|^2 &= \langle a+h, a+h\rangle - \|a\|^2 \\ &= \|a\|^2 + 2\langle a,h\rangle + \|h\|^2 - \|a\|^2 \\ &= 2\langle a,h\rangle + \|h\|^2 \end{aligned}$$

hence

$$f(a+h) - f(a) - 2\langle a,h\rangle = \underbrace{\|h\|}_{\varepsilon(h)}\|h\|$$

4. *Consider the following function*

$$\begin{aligned} f: \quad \mathbb{R}^2 &\longrightarrow \mathbb{R}^2 \\ (x,y) &\longmapsto (x+y, x-y) \end{aligned}$$

we have :

$$d_f^{(a)} = \begin{pmatrix} a_1+a_2 & a_1-a_2 \end{pmatrix}$$

and

$$d_f^{(a)}h = \begin{pmatrix} a_1+a_2 & a_1-a_2 \end{pmatrix}\begin{pmatrix} h_1 \\ h_2 \end{pmatrix}$$

5. *Consider the following*

$$\begin{aligned} f: \quad \mathbb{R}^2 &\longrightarrow \mathbb{R} \\ (x,y) &\longmapsto xy \end{aligned}$$

$$\begin{aligned} f(a_1+h_1, a_2+h_2) - f(a_1,a_2) &= (a_1+h_1)(a_2+h_2) - a_1a_2 \\ &= a_1h_2 + a_2h_1 + \|h\|\underbrace{\frac{h_1h_2}{\|h\|}}_{\varepsilon(h)} \end{aligned}$$

therefore we can see that

$$d_f^{(a_1,a_2)}(h_1,h_2) = a_1h_1 + a_2h_2$$

6. *We will leave this as an exercise for the reader*

$$\begin{aligned} f: \quad \mathbb{R}^n &\longrightarrow \mathbb{R} \\ (x_1,\ldots,x_n) &\longmapsto |x_1| + \ldots + |x_n| \end{aligned}$$

3.1. Partial and directional derivatives

Let $f : \Omega \subset \mathbb{R}^n \longrightarrow \mathbb{R}$ where Ω is an open set of $\mathbb{R}^n$ and $a \in \mathbb{R}^n$, if f is diffirentiable at a then there exist a linear function $\lambda : \mathbb{R}^n \longrightarrow \mathbb{R}$ such that

$$f(x) = f(a) + \lambda^{(a)}(x-a) + \varepsilon(x)\|x-a\|$$

with $\lim_{x\to a} \varepsilon(x) = 0$, we know that $\lambda(x)$ is an inner product product by some fixed vector $w \in \mathbb{R}^n$, that is

$$\lambda(x) = \langle w, x\rangle \quad w \in \mathbb{R}^n$$

this vector which depends on a, is called the gradient of f at a and is denoted by $\nabla f(a)$, we can write

$$f(x) = f(a) + \langle \nabla f(a), x-a\rangle + \varepsilon(x)\|x-a\|$$

we want to get a convenient and explicit form of the gradient

Definition 3.1.1

Let $f : \Omega \longrightarrow \mathbb{R}$, ,and let Ω be an open set of $\mathbb{R}^n$, and let $a \in \Omega$. the directional derivative of f at a in direction of a nonzero vector $u \in \mathbb{R}^n$, denoted by $d_{u,f}^{(a)}$ is defined by

$$d_{u,f}^{(a)} = \lim_{t\to 0} \frac{f\left(a+tu\right) - f(a)}{t}$$

wherever the limit exist

Theorem 3.1.1

If $f : \Omega \subset \mathbb{R}^n \longrightarrow \mathbb{R}$ is diffirentiable at $a \in \Omega$, then for every $u \neq 0$, where $u \in \mathbb{R}^n$, $d_{u,f}^{(a)}$ exist and

$$d_{u,f}^{(a)} = \langle \nabla f(a), u\rangle$$

Proof. Since f is differentiable at a we have

$$f(x) = f(a) + \langle \nabla f(a), x-a\rangle + \varepsilon(x)\|x-a\|$$

by setting $x = a + tu$, this yields

$$f\left(a+tu\right) - f(a) = \langle \nabla f(a), tu\ \rangle + \varepsilon\left(a+tu\right)\|tu\|$$

so we get

$$d_{u,f}^{(a)} = \lim_{t\to 0} \frac{f(a+tu) - f(a)}{t} = \lim_{t\to 0} \left(\langle \nabla f(a), u \rangle + \varepsilon (a+tu) \|u\| \frac{|t|}{t} \right)$$
$$= \langle \nabla f(a), u \rangle$$

□

certain directions are special, namely those of the standard bases elements $(e_i)_{i=\overline{1,n}}$ the direction of the coordinate axes

Corollary 3.1.1 *For nonzero vectors $u, v \in \mathbb{R}^n$ and $c, d \in \mathbb{R}$, such that $cu + dv \neq 0$, we have*

$$d_{cu+dv,f}^{(a)} = cd_{u,f}^{(a)} + dd_{v,f}^{(a)}$$

Corollary 3.1.2 $\left|d_{u,f}^{(a)}\right|$ *is maximized in the direction of $u = \nabla f(a)$*

Proof. By the Cauchy-Schwartz inequality

$$\left|d_{u,f}^{(a)}\right| = \left|\langle \nabla f(a), u \rangle\right| \leq \|\nabla f(a)\| \|u\|$$

with equality only if u is a scalar multople of $\nabla f(a)$ □

Definition 3.1.2

Let $f : \Omega \subset \mathbb{R}^n \longrightarrow \mathbb{R}$, the directional derivative of f at a is the direction of e_i is denoted by

$$\frac{\partial f}{\partial x_i}(a) = d_{e_i,f}^{(a)} = \langle \nabla f(a), e_i \rangle = \lim_{t\to 0} \frac{f(a_1, \dots, a_i + t, \dots, a_n) - f(a_1, \dots, a_n)}{t}$$

is simply the ordinary derivative of f considered as a function of x_i solely, and considering the other components as fixed, since

$$\lambda(x) = \langle w, x \rangle \quad i = \overline{1,n}$$

we have

$$\nabla f(a) = \left(\frac{\partial f}{\partial x_1}(a), \frac{\partial f}{\partial x_2}(a), \dots, \frac{\partial f}{\partial x_n}(a) \right)$$

and so

$$f(x) = f(a) + \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(a)(x_i - a_i) + \varepsilon(x)\|x - a\|$$

Example *Consider this function*

$$f(x,y) = \begin{cases} 0 & x = 0 \text{ or } y = 0 \\ 1 & \text{otherwise} \end{cases}$$

$$\frac{\partial f}{\partial x}(0,0) = \lim_{t\to 0} \frac{f(t,0) - f(0,0)}{t} = 0$$
$$\frac{\partial f}{\partial y}(0,0) = \lim_{t\to 0} \frac{f(t,0) - f(0,0)}{t} = 0$$

but when

$$\begin{aligned} \frac{\partial f}{\partial x}\left(x,y\right) &= \lim_{t\to 0} \frac{f\left(x+t,y\right) - f(x,y)}{t} \\ &= \lim_{t\to 0} \frac{f\left(t,y\right) - f(0,y)}{t} \\ &= \lim_{t\to 0} \frac{1}{t} \to \infty \end{aligned}$$

partial derivatives at $(0,0)$ *exist but the function is not even diffirentiable, but the reverse is true i.e. if a function is diffirentiable then the partial derivatives exist*

Example *Let* $f(x,y,z) = 2x^2 + 3y^2 + z^2$, *find the directional derivative of* f *at* $a = (2,1,3)$ *in the direction* $v = (1,0,-2)$, *we have*

$$\begin{aligned} \nabla f(x,y,z) &= \left(4x, 6y, 2z\right) \\ \nabla f(2,1,3) &= (8,6,6) \\ d_{v,f}^{(a)} &= \langle (8,6,6), (1,0,-2) \rangle = -4 \end{aligned}$$

in the other hand

$$\lim_{t\to 0} \frac{f\left((2,1,3) + (t,0,-2t)\right) - f(2,1,3)}{t} = -4$$

Next we show how matrices arrive in connection with derivations, we shall see that if $f : \Omega \subset \mathbb{R}^n \longrightarrow \mathbb{R}^m$ *is diffirentiable at* $a \in \Omega$, *then the partial derivative of it's component functions* $\frac{\partial f_j}{\partial x_i}(a)$ *exist and determine the linear transformation* $d_f^{(a)}$ *completely*

Theorem 3.1.2
Let Ω be an open set in $\mathbb{R}^n$ and $f : \Omega \longrightarrow \mathbb{R}^m$ be diffirentiable at $a \in \Omega$, then $\frac{\partial f_j}{\partial x_i}(a)$ exist and the standard matrix representation of $d_f^{(a)}$ is the $m \times n$ matrix, given by :

$$d_f^{(a)} = \left(\frac{\partial f_j}{\partial x_i}\right)_{\substack{1 \leq j \leq m \\ 1 \leq i \leq n}}$$

Proof. Let $d_f^{(a)} = \left(d_{f_1}^{(a)}, \dots, d_{f_m}^{(a)}\right) = T$, and let $\{e_1, \dots, e_n\}$ and $\{u_1, \dots, u_m\}$ be the standard bases of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively, by the definition of a linear map the j^{th} entry of the standard of the standard matrix of T, say c_{ji} is given by the j^{th} component of the vector

$$T(e_i) = \sum_{i=1}^{m} c_{ij} u_j$$

since f is diffirentiable at a, then f_j are diffirentiable for all $j = 1, \dots, m$, hence each $\frac{\partial f_j}{\partial x_i}(a)$ exist and

$$\frac{\partial f_j}{\partial x_i}(a) = \left\langle \nabla f_j(a), e_i \right\rangle = d_{f_j}^{(a)}(e_i)$$

so we have

$$T(e_i) = \sum_{j=1}^{m} d_{f_j}^{(a)}(e_i) u_j = \sum_{j=1}^{m} \frac{\partial f_j}{\partial x_i}(a) u_j$$

then $c_{ji} = \frac{\partial f_j}{\partial x_i}(a)$, therefore we can write our matrix :

$$\begin{pmatrix} \frac{\partial f_1}{\partial x_1}(a) & \dots & \frac{\partial f_1}{\partial x_n}(a) \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1}(a) & \dots & \frac{\partial f_m}{\partial x_n}(a) \end{pmatrix} = \begin{pmatrix} d_{f_1}^{(a)} \\ \vdots \\ d_{f_n}^{(a)} \end{pmatrix}$$

□

Definition 3.1.3
The standard matrix $d_f^{(a)}$ is called the Jacobian matrix of f at a.

$$d_f^{(a)} = \begin{pmatrix} \frac{\partial f}{\partial x_1}(a) & \dots & \frac{\partial f_1}{\partial x_n}(a) \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1}(a) & \dots & \frac{\partial f_m}{\partial x_n} \end{pmatrix} = \begin{pmatrix} d_{f_1}^{(a)} \\ \vdots \\ d_{f_n}^{(a)} \end{pmatrix}$$

and reduces the problem of computing the differential of differentiable function f to that of computing the partial derivatives of it's components $f_1, \dots, f_n$

Definition 3.1.4

The matrix $d_f^{(a)}$ is called the Jacobian matrix of f at a, where $m = n$, the determinant of $d_f^{(a)}$ is called the Jacobian of f at a and is denoted $\mathscr{J}_f^{(a)}$, where

$$\mathscr{J}_f^{(a)} = det\left(d_f^{(a)}\right)$$

Example *Let*

$$\begin{array}{cccc} f: & \mathbb{R}^3 & \longrightarrow & \mathbb{R}^2 \\ & (x,y,z) & \longmapsto & \left(\frac{x^2+y^2-z^2}{2}, xy\right) \end{array}$$

what is $d_f^{(3,1,0)}$*?, let's find it out !*

$$d_f^{(x,y,z)} = \begin{pmatrix} x & y & -z \\ y & x & 0 \end{pmatrix} \Longrightarrow d_f^{(3,1,0)} = \begin{pmatrix} 3 & 1 & 0 \\ 1 & 3 & 0 \end{pmatrix}$$

Corollary 3.1.3(Chain Rule) *let* $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ *and let* $g : \mathbb{R}^m \longrightarrow \mathbb{R}^p$

$$d_{g\circ f}^{(a)} = d_g^{(f(a))} \cdot d_f^{(a)} = \underbrace{\begin{pmatrix} \frac{\partial g_1}{\partial y_1}(f(a)) & \dots & \frac{\partial g_1}{\partial y_m}(f(a)) \\ \vdots & \ddots & \vdots \\ \frac{\partial g_p}{\partial y_1}(f(a)) & \dots & \frac{\partial g_p}{\partial y_m}(f(a)) \end{pmatrix}}_{(p\times m)} \underbrace{\begin{pmatrix} \frac{\partial f_1}{\partial x_1}(a) & \dots & \frac{\partial f_1}{\partial x_n}(a) \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1}(a) & \dots & \frac{\partial f_m}{\partial x_n}(a) \end{pmatrix}}_{(m\times n)}$$

Example *Let* $f\left(x,y\right) = (e^{x+y}, e^{x-y})$ *and* $\phi : \mathbb{R} \longrightarrow \mathbb{R}^2$ *a curve in* $\mathbb{R}^2$ *with* $\phi(0) = (0,1)$ *and* $\phi'(0) = (1,1)$*, find the tangent vector to the image of the curve* $\phi(t)$ *under* f *at* $t = 0$

let

$$\begin{array}{cccc} \Psi: & \mathbb{R} & \longrightarrow & \mathbb{R}^2 \\ & x & \longmapsto & f\circ\phi(t) \end{array}$$

we clearly have

$$\Psi'(t) = d_f^{(\phi(t))} \cdot \phi'(t)$$

and

$$\Psi'(0) = d_f^{(\phi(0))} \cdot \phi'(0) = d_f^{(0,1)} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

calculating $d_f^{(0,1)}$ seperately

$$d_f^{(0,1)} = \begin{pmatrix} e & e \\ e^{-1} & -e^{-1} \end{pmatrix}$$

hence we find :

$$\Psi'(0) = \begin{pmatrix} e & e \\ e^{-1} & -e^{-1} \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2e \\ 0 \end{pmatrix}$$

Theorem 3.1.3 Differentiability Criterion
Let $f : \Omega \longrightarrow \mathbb{R}^n$ where Ω is an open set in $\mathbb{R}^n$, if all partial derivative $\frac{\partial f_j}{\partial x_i}$ for $j = \overline{1,m}, i = \overline{1,n}$ exist in a neighborhood of $a \in \Omega$ and are continuous, then f is differentiable at a

Proof. It suffices to prove the result for a real-valued function f, we must show that :

$$\lim_{x \to a} \frac{\left| f(x) - f(a) - \langle \nabla f(a), x-a \rangle \right|}{||x-a||} = 0$$

we let

$$g_i(t) = f\left(x_1, \dots, x_{i-1}, t, a_{i+1}, \dots, a_m\right)$$

and let

$$v_i = \begin{cases} v_0 = a \quad i = 0 \\ (x_1, \dots, x_i, a_{i+1}, \dots, a_m) \quad i = \overline{1,n} \end{cases}$$

if $x \in B\left(a, r\right)$ for all $i = \overline{1,n}$, then we have

$$v_i \in B\left(a, r\right)$$

the function g_i maps the interval between a_i and x_i to $\mathbb{R}$ and

$$g_i'(t) = \frac{\partial f}{\partial x_i}\left(x_1, \dots, x_{i+1}, t, a_{i+1}, \dots, a_m\right)$$

by the one variable Mean Value Theorem, there is ζ_i in the interval between x_i and a_i such that

$$g_i(x_i) - g_i(a_i) = (x_i - a_i)\, g_i'(\zeta_i) = (x_i - a_i)\frac{\partial f}{\partial x_i}\left(x_1, \dots, \zeta_i, \dots, a_m\right)$$
$$g_i(x_i) - g_i(a_i) = f(v_i) - f(v_{i-1})$$

we have

$$\begin{aligned} f(x) - f(a) - \langle \nabla f(a), x-a \rangle &= \sum_{i=1}^{n} \left(f(v_i) - f(v_{i-1}) \right) - \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(a)\,(x_i - a_I) \\ &= \sum_{i=1}^{n} (x_i - a_i) \left(\frac{\partial f}{\partial x_i}(x_1, \dots, \zeta_i, \dots, a_m) - \frac{\partial f}{\partial x_i}(a) \right) \to 0 \end{aligned}$$

and

$$\begin{aligned} \left| f(x) - f(a) - \langle \nabla f(a), x-a \rangle \right| &\leq \sum_{i=1}^{n} |x_i - a_i| \left| \frac{\partial f}{\partial x_i}(\zeta_i) - \frac{\partial f}{\partial x_i}(a) \right| \\ &\leq \|x-a\| \sum_{i=1}^{n} \left| \frac{\partial f}{\partial x_i}(\zeta_i) - \frac{\partial f}{\partial x_i}(a) \right| \end{aligned}$$

then

$$0 \leq \frac{\left| f(x) - f(a) - \langle \nabla f(a), x-a \rangle \right|}{\|x-a\|} \leq \sum_{i=1}^{n} \left| \frac{\partial f}{\partial x_i}(x_1, \dots, x_{i-1}, \zeta_i, a_{i+1}, \dots, a_m) - \frac{\partial f}{\partial x_i}(a) \right|$$

by the continuity of $\frac{\partial f}{\partial x_i}$ for all $i = 1, \dots, n$ at a we get

$$\lim_{x \to a} \sum_{i=1}^{n} \left| \frac{\partial f}{\partial x_i}(x_1, \dots, \zeta_i, \dots, a_m) - \frac{\partial f}{\partial x_i}(a) \right| = 0$$

hence we get the result

$$\lim_{x \to a} \frac{\partial f}{\partial x_i}(x_1, \dots, x_{i-1}, \zeta_i, \dots, a_m) = \frac{\partial f}{\partial x_i}(a)$$

□

Example *Let*

$$f(x,y) = \begin{cases} \left(x^2 + y^2\right) \sin\left(\frac{1}{\sqrt{x^2+y^2}}\right) & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

f is diffirentiable at each point $\left(x, y\right) \in \mathbb{R}^2 \backslash (0,0)$

$$\forall\, x, y \in \mathbb{R}^2 \backslash \{(0,0)\} \qquad \frac{\partial f}{\partial x}\left(x,y\right) = 2y \sin\left(\frac{1}{\sqrt{x^2+y^2}}\right) - \frac{y}{\sqrt{x^2+y^2}} \cos\left(\frac{1}{\sqrt{x^2+y^2}}\right)$$

$$\frac{\partial f}{\partial x}(0,0) = \lim_{t \to 0} \frac{f(t,0) - f(0,0)}{t} = 0$$

$$\frac{\partial f}{\partial y}(0,0) = \lim_{t \to 0} \frac{f(0,t) - f(0,0)}{t} = 0$$

hence we get

$$\frac{\partial f}{\partial x}(x,y) = \begin{cases} 2x\sin\left(\frac{1}{\sqrt{x^2+y^2}}\right) - \frac{x}{\sqrt{x^2+y^2}} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

we have $\frac{\partial f}{\partial x}$ *and* $\frac{\partial f}{\partial y}$ *are not continuous at* $(0,0)$*, however*

$$\begin{aligned} \frac{\left|f(x,y) - f(0,0) - \langle \nabla f(a), (x,y)\rangle\right|}{\|(x,y)\|} &= \frac{(x^2+y^2)\sin\left(\frac{1}{\sqrt{x^2+y^2}}\right)}{\|(x,y)\|} \\ &= \frac{(x^2+y^2)}{\sqrt{x^2+y^2}}\sin\left(\frac{1}{\sqrt{x^2+y^2}}\right) \\ &= \sqrt{x^2+y^2}\sin\left(\frac{1}{\sqrt{x^2+y^2}}\right) \longrightarrow 0 \text{ as } (x,y) \to 0 \end{aligned}$$

then the function f *is differentiable at* $(0,0)$

Theorem 3.1.4 The Mean Value Theorem

Suppose $f : \Omega \subset \mathbb{R}^n \longrightarrow \mathbb{R}$ is a differentiable function on the convex set Ω, let $a, b \in \Omega$ and $\zeta(t) = ta + (1-t)b$, then there exist $c \in \zeta((0;1))$ such that

$$f(b) - f(a) = \langle \nabla f(c), a - b\rangle$$

Proof. Let

$$\phi(t) = f \circ \zeta(t)$$

we have for all t

$$\begin{aligned} \phi'(t) = d_f^{(\zeta(t))}\zeta'(t) &= \langle \nabla f(\zeta(t)), \zeta'(t)\rangle \\ &= \langle \nabla f(\zeta(t)), a - b\rangle \end{aligned}$$

by the one variable mean value, we have

$$\begin{aligned} \phi(1) - \phi(0) &= \phi'(\mu) \quad \mu \in (0;1) \\ \phi(1) - \phi(0) &= f(\zeta(1)) - f(\zeta(0)) \\ &= f(a) - f(b) = \langle \nabla f(\zeta(\mu)), a - b\rangle \end{aligned}$$

take $\zeta(\mu) = c$, we get

$$f(a) - f(b) = \langle \nabla f(c), a - b\rangle$$

□

Corollary 3.1.4 *Let $f : \Omega \subset \mathbb{R}^n \longrightarrow \mathbb{R}$ be differentiable function on a convex subset K of Ω , if*

$$\|\nabla f(x)\| \leq M \quad \forall x \in K$$

then

$$\left|f(x) - f(y)\right| \leq M\|x - y\| \quad \forall x, y \in K$$

Proof. Let $x, y \in K$, and K is convex subset of Ω, then there is $c_{x,y}$ such that

$$f(x) - f(y) = \left\langle \nabla f(c_{x,y}), x - y \right\rangle$$

by Cauchy-Schwartz inequality we get

$$\left|f(x) - f(y)\right| \leq \|\nabla f(c_{x,y})\|\|x - y\| \leq M\|x - y\| \quad \forall x, y \in K$$

□

Corollary 3.1.5 *Let $f : \Omega \subset \mathbb{R}^n \longrightarrow \mathbb{R}$ with Ω is a convex set in $\mathbb{R}^n$, if*

$$\nabla f(x) = 0 \quad \forall x \in \Omega$$

then f is constant

Proof. let $a \in \Omega$, by the Mean Value Theorem for all $x \in \Omega$ there is c_x such that

$$f(x) - f(a) = \langle \nabla f(c_x), x - a \rangle = 0 \implies f(x) = f(a) \quad \forall x \in \Omega$$

□

Corollary 3.1.6 *let $f : \Omega \subset \mathbb{R}^n \longrightarrow \mathbb{R}$, with Ω an open connected set in $\mathbb{R}^n$, if*

$$\nabla f(x) = 0 \quad \forall x \in \Omega$$

then f is constant in Ω

Proof. Let $a \in \Omega$, we define

$$A = \left\{x \in \Omega : f(x) = f(a)\right\}$$

it suffies to show that A is closed an open at the same time (to conclude that $A = \Omega$), it's clear that A is closed since

$$f^{-1}\left(\left\{f(a)\right\}\right) = A$$

and $\{f(a)\}$ is closed, and f is continuous, we show that A is open set, let $x \in A \subset \Omega$, there exist $\delta > 0$ then there exist $B(x,\delta) \subset \Omega$, we let $y \in B(x,\delta)$, since $B(x,\delta)$ is convex, then by the above corollary f is constant in $B(x,\delta)$, then $f(y) = f(x) = f(a)$, then $y \in A$, therefore we get

$$B(x,\delta) \subset A \implies A \text{ is open set}$$

then $A = \Omega$, because $a \in A$ i.e $(A \neq \emptyset)$ □

Corollary 3.1.7 *Let $f : \Omega \subset \mathbb{R}^n \longrightarrow \mathbb{R}^m$ where Ω is open connected set in $\mathbb{R}^n$ if*

$$d_f^{(x)} = 0 \quad \forall x \in \Omega$$

then f is constant in Ω

Corollary 3.1.8 *Let Ω be an open bounded convex set in $\mathbb{R}^n$ and $f : \overline{\Omega} \longrightarrow \mathbb{R}^m$ with $f \in \mathscr{C}^1(\overline{\Omega})$, then there exist a constant $M > 0$ such that for all $x, y \in \overline{\Omega}$*

$$\|f(x) - f(y)\| \leq M\|x - y\|$$

Proof. Let $f = (f_1, \dots, f_n)$, since Ω is convex then for all $x, y \in \Omega$, we apply Mean Value Theorem to each component and by cauchy schwartz inequality we have

$$\begin{aligned}\|f(x) - f(y)\|^2 = \sum_{j=1}^{m} \left|f_j(x) - f_j(y)\right|^2 &= \sum_{j=1}^{m} \left|\left\langle \nabla f_j(c_j), x - y \right\rangle\right| \\ &\leq \sum_{j=1}^{m} \|\nabla f_j(c_j)\|^2 \|x - y\|^2 \\ &\leq \|x - y\|^2 \sum_{j=1}^{m} \|\nabla f_j(c_j)\|^2\end{aligned}$$

for all $x \in \Omega$,

$$\nabla f_j(c_j) \leq M_j$$

$\overline{\Omega}$ is compact set, then for all $x, y \in \Omega$

$$\|f(x) - f(y)\| \leq M\|x - y\| \quad \text{with} \quad M = \sqrt{\sum_{j=1}^{m} M_j}$$

□