Chapter 03 : Stirling Numbers

Section 1 : Stirling numbers of the second kind

Lemma 0.1 : Recursive relation

The stirling numbers of second kind satisfy the following recurrence

$$\left\{ n \atop k \right\} = \left\{ n-1 \atop k-1 \right\} + k \left\{ n-1 \atop k \right\} \tag{1}$$

Proof is left as an exercise

Theorem 0.1 :

Let x be a variable :

$$x^n = \sum_{i=1}^{n} \left\{ n \atop i \right\} (x)_i \tag{2}$$

Proof is left as an exercise

Section 2 : Stirling numbers of the third kind

Definition 1.1 :

The Stirling Numbers of the third kind (Lah numbers) , The Lah numbers count the number of ways to partition a set of n elements into k non empty lists (ordered subsets) , they are denoted $L(n, k)$ or $\left\lfloor n \atop k \right\rfloor$

Lemma 1.1 :

Recurrence relation of Lah Numbers

$$\left\lfloor n \atop k \right\rfloor = \left\lfloor n-1 \atop k-1 \right\rfloor + (n+k-1) \left\lfloor n-1 \atop k \right\rfloor \tag{1.1}$$

Theorem 1.1 :

Let x be a variable :

$$x^{(n)} = \sum_{k=1}^{n} \left\lfloor n \atop k \right\rfloor (x)_k \tag{1.2}$$

Examples:

$$x(x+1) = x(x-1) + 2x \tag{1.3}$$

$$x(x+1)(x+2) = x(x-1)(x-2) + 6x(x-1) + 6x \tag{1.4}$$

Lemma 1.2 :

$$\left\lfloor n \atop k \right\rfloor = \sum_{j=k}^{n} \left[n \atop j \right] \left\{ j \atop k \right\} \tag{1.5}$$

$$\begin{aligned} x^{(n)} &= \sum_{j=0}^{n} \left[n \atop j \right] x^j_1 \\ &= \sum_{j=0}^{n} \left[n \atop j \right] \sum_{k=0}^{j} \left\{ j \atop k \right\} (x)_k \\ x^{(n)} &= \sum_{k=0}^{n} \left[\sum_{j=k}^{n} \left[n \atop j \right] \left\{ j \atop k \right\} \right] (x)_k \end{aligned}$$

Section 2 : Stirling numbers of the third kind

Some Notations:

$$\begin{bmatrix} n \\ k \end{bmatrix} = |\Delta(n,k)| = (-1)^{n-k}\Delta(n,k)$$

$$\left\lfloor \begin{matrix} n \\ k \end{matrix} \right\rfloor = |L(n,k)| = (-1)^{n-k}L(n,k)$$

Exercises: Prove the Following

- $x^n = \sum_{k=0}^{n} (-1)^{n-k} \begin{Bmatrix} n \\ k \end{Bmatrix} x^{(n)}$
- $(x)_n = \sum_{k=0}^{n} (-1)^{n-k} \begin{bmatrix} n \\ k \end{bmatrix} x^k$
- $(x)_n = \sum_{k=0}^{n} (-1)^{n-k} \left\lfloor \begin{matrix} n \\ k \end{matrix} \right\rfloor x^{(n)}$

Theorem 1.2 :

- $\sum_{j=k}^{n} \Delta(n,j)S(j,k) = \delta_{n,k}$
- $\sum_{j=k}^{n} S(n,j)\Delta(j,k) = \delta_{n,k}$

Let $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ be two given sequences , then we have the following equivalence :

$$a_n = \sum_{k=0}^{n} (-1)^{n-k} \begin{bmatrix} n \\ k \end{bmatrix} b_k \iff b_n = \sum_{k=0}^{n} \begin{Bmatrix} n \\ k \end{Bmatrix} a_k \tag{1.6}$$