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NVS EXTRA HOMEWORK

Before i begin statments and the solution of the propositions, i want to address that the proof i'm about to propose for these Propositions is "My own proof" and includes the possibility of a mistake.

The main goal of this document is to improve my proof writing skills as well as to reflect my current understanding of this subject to my teacher.

PROPOSITION 1:

Consider the N.V.S. $E := \left(\mathcal{C}^0([0,1],\mathbb{R}), \|.\|_\infty\right)$ *and consider the vector subspace :*

$$S := \left\{ f \in \mathcal{C}^0([0,1]) : \quad f(x) = c \quad \forall x \in [0,1], \quad c \in \mathbb{R} \right\}$$

then the quotient space :

$$(E_{\backslash S}, \|.\|_{E_{\backslash S}}) \text{ is } \underline{\text{Banach}}$$

PROPOSED SOLUTION :

We will prove that every convergent series of $E_{\backslash S}$ is convergent in $E_{\backslash S}$. For instance consider the sequence $(x_n)_{n\in\mathbb{N}} \subset E_{\backslash S}$[1], where $x_n = cl(f_n)$ (we did not lose the generality since every element of $E_{\backslash S}$ is class of some function, just without abusing the notations we chose to denote it like that), and :

$$\sum_{n\in\mathbb{N}} \|x_n\|_{E_{\backslash S}} < +\infty$$

where $\|.\|_{E_{\backslash S}}$ denotes the natural norm defined on the quotient space by :

$$\begin{array}{rccc} \|.\|_{E_{\backslash S}} : & E_{\backslash S} & \longrightarrow & \mathbb{R} \\ & cl(f) & \longmapsto & \inf_{x\in cl(f)} \|x\|_\infty \end{array}$$

we will show that :

$$\sum_{n\in\mathbb{N}} x_n \quad \in E_{\backslash S}$$

now for $n \in \mathbb{N}$, we have :

$$\begin{aligned} \|x_n\|_{E_{\backslash S}} &:= \inf_{g\in cl(f_n)} \|g\|_\infty \\ &= \inf_{h\in\mathbb{R}} \|f_n + h\|_\infty \\ &= d(f_n, S) \end{aligned}$$

By the characterization of the infimum, we have for $\varepsilon = \frac{1}{n^2}$ where $n \in \mathbb{N}$ the existence of some function say g_n in x_n such that :

$$\begin{aligned} &\iff \|x_n\|_{E_{\backslash S}} + \frac{1}{n^{1000}} > \|g_n\|_\infty \\ &\iff \|x_n\|_{E_{\backslash S}} > \|g_n\|_\infty - \frac{1}{n^{1000}} \\ &\iff \sum_{n\in\mathbb{N}} \|x_n\|_{E_{\backslash S}} > \sum_{n\in\mathbb{N}} \|g_n\|_\infty - \frac{1}{n^{1000}} \end{aligned}$$

[1] *"i like how the concept of sequences is becoming more and more ambigious sophisticated"*

Now since $(\mathcal{C}^0([0,1],\mathbb{R}), \|.\|_\infty)$ is Banach (an exercise from the worksheet), and we have that the series :

$$\sum_{n\in\mathbb{N}} \|g_n\|_\infty < \sum_{n\in\mathbb{N}} \|x_n\|_{E_{\setminus S}} < +\infty$$

hence the series $\sum_{n\in\mathbb{N}} g_n$ is convergent and in E, now consider the map :

$$\begin{array}{rccc} \Psi : & E & \longrightarrow & E_{\setminus S} \\ & y & \longmapsto & y+S \end{array}$$

this map is continuous, since the norm on $E_{\setminus S}$ was put such that this map is continuous, and it's linear, since for any $z, y \in E$ we have :

$$\Psi(z+y) = \Psi(z) + \Psi(y) = z+S+y+S = (z+y)+S$$

therefore :

$$\begin{aligned} \lim_{n\to\infty} \Psi(\sum_{k=1}^{n} g_k) &= \lim_{n\to\infty} \sum_{k=1}^{n} \Psi(g_k) \\ &= \sum_{k=1}^{\infty} \Psi(g_k) \\ &= \sum_{k=1}^{\infty} x_k \qquad (*) \end{aligned}$$

$\boxed{\text{we have that } \Psi(g_k) = x_k \text{ because } g_k \in x_k}$

and from the other side, we have by the continuity of Ψ :

$$\begin{aligned} \lim_{n\to\infty} \Psi(\sum_{k=1}^{n} g_k) &= \Psi(\lim_{n\to\infty} \sum_{k=1}^{n} g_k) \\ &= \Psi(\sum_{k=1}^{\infty} g_k) \qquad (**) \end{aligned}$$

combining $(*)$ and $(**)$ we get :

$$\sum_{k=1}^{\infty} x_k = \Psi(\sum_{k=1}^{\infty} g_k)$$

and since $\sum_{k=1}^{\infty} g_k$ is convergent and is E, then the LHS, is well defined and is in $E_{\setminus S}$ hence we get that $E_{\setminus S}$ is Banach.

☞ This completes the proof ⤫

PROPOSITION 2:

Let E be an N.V.S. if V is a schauder basis and is countable, then E is separable, i.e. there exists a countable dense subset of E .

PROPOSED SOLUTION :

Let E be an N.V.S. over $\mathbb{R}$ (this idea can be generalized to $\mathbb{C}$) and suppose that V is a set of schauder basis that is countable, then we can find an enumeration $\{q_n\}_{n\in\mathbb{N}}$ of V, i.e. :

$$V = \bigcup_{n\in\mathbb{N}} \{q_n\}$$

i'm going to defin a map :

$$\begin{array}{rccc} g_n : & \overbrace{\mathbb{Q}\times\mathbb{Q}\times\ldots\times\mathbb{Q}}^{n \text{ times}} & \longrightarrow & E \\ & (\lambda_1,\ldots,\lambda_n) & \longmapsto & \lambda_1 q_1 + \ldots + \lambda_n q_n \end{array}$$

we will prove that the set :

$$H := \bigcup_{n\in\mathbb{N}} g_n(\prod_{i=1}^{n}\mathbb{Q}) = \bigcup_{n\in\mathbb{N}} Im(g_n)$$

is dense, trivially H is countable, since its a countable union of countable sets.

Let $x \in E$ and consider the open ball of radius $r \in]0,\infty]$ around x, we will show that there exists an element $z \in H$, such that :

$$\|x - z\| < r$$

since V is schauder basis then :

$$x = \sum_{n=1}^{\infty} \lambda_n q_n \qquad (\lambda_1, \lambda_2, \ldots \in \mathbb{R})$$

and by the characterization of the limit, we have for $\varepsilon = \frac{r}{2}$ we have the existence of $n_0 \in \mathbb{N}$ such that $n > n_0$:

$$\begin{aligned}
&\iff \left| \|x\| - \|\sum_{k=1}^{n_0} \lambda_k q_k\| \right| < \frac{r}{2} \\
&\iff \left| \|\sum_{k>n_0} \lambda_k q_k\| \right| < \left| \|x\| - \|\sum_{k=1}^{n_0} \lambda_k q_k\| \right| < \frac{r}{2} \\
&\iff \|\sum_{k>n_0} \lambda_k q_k\| < \frac{r}{2}
\end{aligned}$$

and put :

$$M := \|\sum_{k=1}^{n_0} \lambda_k q_k\|$$

then we can construct a sequence $(h_n)_{n\in\mathbb{N}} \subset \mathbb{Q}$ such that :

$$|h_n - \lambda_n| < \frac{r}{2M} \qquad (\forall n \leq n_0)$$

we are allowed to choose h_n sufficiently closed to λ_n due to the density of $\mathbb{Q}$ in $\mathbb{R}$, put $v = \sum_{k=1}^{n_0} h_k q_k$ clearly $v \in Im(g_{n_0})$ and we have :

$$\begin{aligned}
\|x - v\| &= \|\sum_{k=1}^{\infty} \lambda_k q_k - \sum_{k=1}^{n_0} \lambda_k q_k\| \\
&\leq \|\sum_{k=1}^{n_0} (\lambda_k - h_k) q_k\| + \|\sum_{k>n_0} \lambda_k q_k\| \\
&< \left|\frac{r}{2M}\right| \|\sum_{k=1}^{n_0} q_k\| + \|\sum_{k>n_0} \lambda_k q_k\| \\
&\leq \frac{r}{2M} \cdot M + \frac{r}{2} \leq \frac{r}{2}
\end{aligned}$$

hence we deduce :

$$\|x - v\| < r \implies v \in B_E(x, r)$$

therefore $v \in B_E(x, r)$. Hence H is dense in E.

☞ This completes the proof ⤬

I may have missed some details in the proofs, & I apologize for sending this late, due to lack of time i couldn't write this in latex, as i wanted it to be well written, i will appreciate any feedback on the proofs. Thanks for your time sir.