Complex Analysis Lecture Notes

Hand written summary from lectures

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`http://farhi.bakir.free.fr/home/index-fr.html`

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- Typos, transcription mistakes, or missing content.
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Chapter 1

Power Series

Lecture 1 08:06 AM Mon, Sep 29 2025

Definition 1.0.1 (Power Series) : A power series is a formal series of the form $\sum_{n=0}^{\infty} a_n z^n$, where $a_n \in \mathbb{C}$ for all $n \in \mathbb{N}_0$.
More generally, given $z_0 \in \mathbb{C}$, a power series centered at z_0 is a formal series of the form:

$$\sum_{n=0}^{\infty} a_n (z - z_0)^n,$$

where $a_n \in \mathbb{C} \quad (\forall n \in \mathbb{N}_0)$

Remark ✎

The set of all complex power series (centered at 0) is denoted by $\mathbb{C}[[z]]$. More generally, given $z_0 \in \mathbb{C}$, the set of all complex power series centered at z_0 is denoted by $\mathbb{C}[[z - z_0]]$.

☞ Operations on Formal Power Series:

Given $z_0 \in \mathbb{C}$, we equip $\mathbb{C}[[z - z_0]]$. with the following operations:

① **Additions:** For all $(a_n)_{n\in\mathbb{N}_0}, (b_n)_{n\in\mathbb{N}_0} \subset \mathbb{C}$:

$$\sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=0}^{\infty} b_n (z - z_0)^n = \sum_{n=0}^{\infty} (a_n + b_n)(z - z_0)^n.$$

② **Multiplication**

$$\sum_{n=0}^{\infty} a_n (z - z_0)^n \times \sum_{n=0}^{\infty} b_n (z - z_0)^n = \sum_{n=0}^{\infty} c_n (z - z_0)^n,$$

where, $c_n := \sum_{k=0}^{n} a_k b_{n-k}$ for all $n \in \mathbb{N}_0$. Also $(c_n)_{n\in\mathbb{N}}$ is called the covolution of the two sequences $(a_n)_{n\in\mathbb{N}_0}$ and $(b_n)_{n\in\mathbb{N}_0}$.

③ **Scalar Multiplication:** For all $\lambda \in \mathbb{C}$, and all $(a_n)_{n\in\mathbb{N}_0} \subset \mathbb{C}$:

$$\lambda \sum_{n=0}^{\infty} a_n (z - z_0)^n = \sum_{n=0}^{\infty} (\lambda a_n)(z - z_0)^n.$$

It's straightforward to verify that $\mathbb{C}\,[[z - z_0]]$ equipped with these operations forms a commutative algebra over $\mathbb{C}$. The Multiplicative identity is the constant power series:

$$1 = 1 + 0 \cdot (z - z_0) + 0 \cdot (z - z_0)^2 + \dots$$

Definition 1.0.2 (Domain of Convergence) : The domain of convergence of a power series $\sum_{n=0}^{\infty} a_n (z-z_0)^n$ is the set of all points $z \in \mathbb{C}$ for which the series converge. The structure of this domain is very specific. Its a disk (possibly with some points in its boundary) centered at z_0.

Proposition 1.0.1 (Abel's Lemma) : Let $\sum_{n=0}^{\infty} a_n (z - z_0)^n$ be a power series and let $z_1 \in \mathbb{C}\setminus\{z_0\}$. Suppose that the sequence $\{a_n(z_1 - z_0)^n\}_{n\in\mathbb{N}_0}$ is bounded. Then, the power series in question converges absolutely (so converges) for every $z \in \mathbb{C}$, such that:

$$|z - z_0| < |z_1 - z_0|$$

Proof. By hypothesis, $\exists M > 0$ such that $\forall n \in \mathbb{N}_0$:

$$|a_n (z_1 - z_0)^n| \leq M$$

Then, for all $z \in \mathbb{C}$ such that $|z - z_0| < |z_1 - z_0|$ we have:

$$\begin{aligned} |a_n(z - z_0)^n| &= \underbrace{|a_n(z_1 - z_0)^n|}_{\leq M} \cdot \underbrace{\left|\frac{z - z_0}{z_1 - z_0}\right|^n}_{<1} \\ &\leq M \underbrace{\left|\frac{z - z_0}{z_1 - z_0}\right|^n}_{<1}. \end{aligned}$$

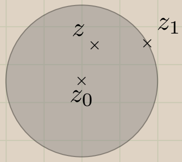
Since $\left|\frac{z-z_0}{z_1-z_0}\right| < 1$ then the geometric series

$$\sum_{n=0}^{\infty} M \left|\frac{z - z_0}{z_1 - z_0}\right|^n \quad \text{Converges .}$$

Thus, the series $\sum_{n=0}^{\infty} |a_n(z-z_0)^n|$ also converges, that is $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ is absolutely convergent. □

Corollary 1.0.2 : Let $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ be a power series which converges at some $z = z_1 \in \mathbb{C}\backslash\{z_0\}$. Then the power series in question converges absolutely (so converges), for every $z \in \mathbb{C}$ such that:

$$|z-z_0| < |z-z_1|$$



Proof. $\sum_{n=0}^{\infty} a_n(z_1-z_0)^n$ converges implies that $a_n(z-z_0)^n \to 0$ as $n \to +\infty$, which implies that the sequence $\{a_n(z_1-z_0)^n\}_{n\geq 0}$ is bounded. *Proposition 1.0.1* permits us to conclude the required result. □

Theorem 1.0.3 (Radius of Convergence) : Let $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ be a power series. Then there exists a unique $R \in [0,\infty]$, called the radius of convergence with the following properties:

① The power series converges absolutely for every $z \in \mathbb{C}$ satisfying $|z-z_0| < R$.

② The power series diverges for every $z \in \mathbb{C}$ satisfying $|z-z_0| > R$. The disk $D(z_0,R) = \{z \in \mathbb{C} : \quad |z-z_0| < R\}$ is called the disk of convergence.

Proof. Define the set $A \subset \mathbb{R}_{\geq 0}$ of nonnegative real numbers for which the sequence $\{|a_n|\, r^n\}_{n\in\mathbb{N}_0}$ is bounded.

$$A := \left\{ r \geq 0 : \quad \sup_{n\in\mathbb{N}_0} |a_n|\, r^n < \infty \right\}$$

we have $A \neq \emptyset$ because $0 \in A$. Define $R := \sup A \in [0,\infty]$, we now show that R has the stated properties.

➭① Let $z \in D(z_0,R)$. By definition of the supremum, there exists $r \in A$, (i.e., $|a_n|\, r^n$ is bounded) such that $|z-z_0| < r \leq R$. Since $|z-z_0| < r$ and $\{|a_n|\, r^n\}_{n\geq 0}$ is bounded, then by Abel's lemma, we deduce that the series $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ converges absolutely.

➭② Let $z \in \mathbb{C}$ such that $|z-z_0| > R$, suppose for contradictions that the power series converges at z. Then by the *Corollary 1.0.2*, it would converge absolutely for any ω with $|\omega - z_0| <$

$|z - z_0|$. In particular, for any r such that:

$$R < r < |z - z_0|$$

the series would converge at points on the circle $C(z_0, r)$, implying $r \in A$. This contradicts the fact that $R = \sup A$. Therefore, the power series diverges.

➼ The Uniqueness of R:

If another $R' \in [0, \infty]$ satisfies the same properties, a point z such that $|z - z_0|$ lies between R and R' would lead to a contradiction regarding the convergence or divergence of the power series. □

1.1 Formulas for Calculating the Radius of Convergence

Proposition 1.1.1 (Hadamard's Formula) : Let $\sum_{n=0}^{\infty} a_n (z - z_0)^n$ be a power series centered at $z_0 \in \mathbb{C}$. Denote by R its radius of convergence. Then:

$$\frac{1}{R} = \lim_{n\to\infty} \sup \sqrt[n]{|a_n|}$$

with the convention $\frac{1}{0} = \infty$ and $\frac{1}{\infty} = 0$

Proof. Let $L := \lim_{n\to\infty} \sup |a_n|^{\frac{1}{n}} \in [0, \infty]$. We must show that $R = \frac{1}{L}$. Let $z \in \mathbb{C} \setminus \{z_0\}$, we distinguish three cases:

➼① If $L = 0$. In this case, we have:

$$0 \leq \lim_{n\to\infty} \inf |a_n|^{\frac{1}{n}} \leq \lim_{n\to\infty} \sup |a_n|^{\frac{1}{n}} = 0$$

Thus, $\lim_{n\to\infty} \inf |a_n|^{\frac{1}{n}} = \lim_{n\to\infty} \sup |a_n|^{\frac{1}{n}} = 0$. This implies that $\lim_{n\to\infty} |a_n|^{\frac{1}{n}}$ exists and equals to 0, so for all n sufficiently large, we have:

$$|a_n|^{\frac{1}{n}} < \frac{1}{2|z - z_0|};$$

That is,

$$|a_n (z - z_0)^n| < \frac{1}{2^n}.$$

Since the geometric series $\sum_{n=1}^{\infty} \frac{1}{2^n}$ converges then the series $\sum_{n=0}^{\infty} |a_n (z - z_0)^n|$ converges $\forall z \in \mathbb{C}$, thus $R = +\infty = \frac{1}{L}$

- ② If $L = +\infty$, we have $L = \lim_{n\to\infty} \sup |a_n|^{\frac{1}{n}} = +\infty$ is equivallent to the fact that the sequence $\left\{|a_n|^{\frac{1}{n}}\right\}_{n\in\mathbb{N}}$ is bounded. Therefore, the sequence:

$$|a_n(z-z_0)^n|^{\frac{1}{n}} = |a_n|^{\frac{1}{n}}\,|z-z_0|$$

is also unbounded. This implies that $|a_n(z-z_0)^n|$ is unbounded, thus $|a_n(z-z_0)^n|$ does not converge to 0 as $n \to \infty$. Hence $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ diverges. Hence $R = 0$.

- ③ If$L \in (0, \infty)$. Let $z \in \mathbb{C}$. We consider two subcases:

❶ If $|z - z_0| < \frac{1}{L}$. Choose r such that $|z-z_0| < r < \frac{1}{L}$, thus $L < \frac{1}{r}$. By defintion of a $\lim_{n\to\infty}\sup$, for all n sufficiently large we have:

$$|a_n|^{\frac{1}{n}} < \frac{1}{r},$$

which implies that:

$$|a_n(z-z_0)^n| < \underbrace{\left(\frac{|z-z_0|}{r}\right)^n}_{<1}.$$

Since $\left|\frac{z-z_0}{r}\right| < 1$, the geometric series $\sum_{n=0}^{\infty}\left|\frac{z-z_0}{r}\right|^n$ converges. By comparison, the power series $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ converges absolutely.

❷ If $(|z-z_0| > \frac{1}{L})$. In this case, we have:

$$\begin{aligned}\lim_{n\to\infty}\sup|a_n(z-z_0)^n|^{\frac{1}{n}} &= \lim_{n\to\infty}\sup\left(|a_n|^{\frac{1}{n}}\,|z-z_0|\right)\\ &= L\,|z-z_0| > 1\end{aligned}$$

Thus, $\{a_n(z-z_0)^n\}_{n\in\mathbb{N}}$ is unbounded, hence $|a_n(z-z_0)^n|$ does not converge to zero as $n \to \infty$, implying that $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ diverges. Therefore:

$$R = \frac{1}{L}.$$

□

Lecture 2 `08:00 AM Mon, Oct 06 2025`

Proposition 1.1.2 (Ratio Test Formula) : Let $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ be a power series. Suppose that the limit

$$\alpha = \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right|$$

exists (i.e., $\in [0, \infty]$). Then the radius of convergence R of the power series in question is $R = \alpha$.

Proof. We use the d'Allembert rule for the series

$$\sum_{n=0}^{\infty} a_n (z - z_0)^n \qquad (z \in \mathbb{C} \setminus \{z_0\}).$$

Let $z \in \mathbb{C} \setminus \{z_0\}$. we have:

$$\begin{aligned}
\lim_{n\to\infty} \left| \frac{a_{n+1}(z - z_0)^{n+1}}{a_n (z - z_0)^n} \right| &= \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| \cdot |z - z_0| \\
&= |z - z_0| \cdot \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| \\
&= \frac{|z - z_0|}{\alpha}
\end{aligned}$$

By the d'Allembert rule, we have:

- The series $\sum_{n=0}^{\infty} a_n (z - z_0)^n$ converges if

$$\frac{|z - z_0|}{\alpha} < 1 \quad \text{i.e.} \quad |z - z_0| < \alpha.$$

- The series $\sum_{n=0}^{\infty} a_n (z - z_0)^n$ diverges if

$$\frac{|z - z_0|}{\alpha} > 1 \quad \text{i.e.} \quad |z - z_0| > \alpha.$$

Hence $R = \alpha$.

□

Example: Determine the radius of convergence of the power series $\sum_{n=0}^{\infty} \frac{z_n}{n!}$ where $z_0 = 0$.

1 st Method: (By Hadamard formula)

We must compute $\lim_{n\to\infty} \sup \left(\frac{1}{n!}\right)^{\frac{1}{n}}$. By the stirling formula, we have that:

$$n! \sim_{+\infty} n^n e^{-n} \sqrt{2\pi n}.$$

Thus we get:

$$(n!)^{\frac{1}{n}} \sim_{+\infty} n e^{-1} (2\pi n)^{\frac{1}{2n}}.$$

Thus

$$\left(\frac{1}{n!}\right)^{\frac{1}{n}} \sim_{+\infty} \frac{e}{n} (2\pi n)^{-\frac{1}{2n}} \to 0 \text{ as } n \to +\infty .$$

Thus $R = \frac{1}{0} = +\infty$.

This means that the power series $\sum_{n=0}^{\infty} \frac{z^n}{n!}$ converges for all $z \in \mathbb{C}$.

2 nd METHOD:

We use *Proposition* 2 . we have:

$$\lim_{n\to\infty} \left| \frac{\frac{1}{n!}}{\frac{1}{(n+1)!}} \right| = \lim_{n\to\infty} \frac{(n+1)!}{n!}$$
$$= \lim_{n\to\infty} (n+1) = +\infty.$$

Thus $R = +\infty$

1.2 Analytic Functions

Definition 1.2.1 : Let Ω be a non empty open subset of $\mathbb{C}$ and let $z_0 \in \Omega$.

Let $f : \Omega \longrightarrow \mathbb{C}$ be a map. then:

1. f is said to be analytic at z_0 if there exists $r > 0$ and a complex sequence $(a_n)_{n\in\mathbb{N}_0}$ such that $D(z_0, r) \subset \Omega$ and:

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n \quad (\forall z \in D(z_0, r)) \,.$$

2. f is said to be analytic on Ω if its analytic at every point of Ω.

Example:

1. Every complex polynomial is analytic on $\mathbb{C}$. Indeed, let $P \in \mathbb{C}[\mathbb{Z}]$, and $z_0 \in \mathbb{C}$. since $P(z + z_0) \in \mathbb{C}[\mathbb{Z}]$, we can write:

$$P(z + z_0) = \sum_{n=0}^{d} a_n z^n \quad (d \in \mathbb{N}_0).$$

Substituting z by $(z - z_0)$, we get:

$$P(z) = \sum_{n=0}^{d} a_n (z - z_0)^n,$$

which is a power series centered at z_0 with infinite randius of convergence. Thus, P is analytic at z_0. Since z_0 was arbitrary, P is analytic on $\mathbb{C}$.

2. The function $z \longrightarrow \frac{1}{z}$ is analytic on $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$. Indeed, let $z_0 \in \mathbb{C}^*$ arbitrary. For $z \in D(z_0, |z_0|)$, we have:
$$\left| \frac{z - z_0}{z_0} \right| < 1.$$
We can write
$$\begin{aligned} \frac{1}{z} &= \frac{1}{z_0 + (z - z_0)} \\ &= \frac{1}{z_0} \cdot \frac{1}{1 + \frac{z - z_0}{z_0}} \\ &= \frac{1}{z_0} \cdot \sum_{n=0}^{\infty} (-1)^n \left(\frac{z - z_0}{z_0} \right)^n \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{z_0^{n+1}} (z - z_0)^n, \end{aligned}$$
which is a power series centered at z_0, valid on $D(z_0, |z_0|)$. Hence $z \longrightarrow \frac{1}{z}$ is analytic at z_0. Since $z_0 \in \mathbb{C}^*$ was arbitrary, then $z \longrightarrow \frac{1}{z}$ is analytic on $\mathbb{C}^*$.

1.2.1 Properties of Analytic Functions

Proposition 1.2.1 : Let Ω be a non empty open subset of $\mathbb{C}$ and let $z_0 \in \Omega$. If $f, g : \Omega \longrightarrow \mathbb{C}$ are analytic at z_0, then the same is for $(f+g)$ and $(f \cdot g)$. Moreover, if f and g are represented by power series with radii of convergence R_f and R_g respectively then $(f+g)$ and $(f \cdot g)$ are represented by power series with radii of convergence $\geq \min(R_f, R_g)$.

Proof. Exercise. □

Corollary 1.2.2 : Let Ω be a non empty open subset of $\mathbb{C}$ and let $f, g : \Omega \longrightarrow \mathbb{C}$. If f and g are both analytic on Ω, then the same is for $(f+g)$ and $(f \cdot g)$.

Proposition 1.2.3 (Analyticity $\implies$ Continuity) : Let Ω be a non empty open subset of $\mathbb{C}$ and let $z_0 \in \Omega$. Let also $f : \Omega \longrightarrow \mathbb{C}$ be a map. If f is analytic at z_0 then f is continuous at z_0

Proof. Suppose that f is analytic at z_0 then there exists $R > 0$ and a complex sequence $(a_n)_{n \in \mathbb{N}_0}$

such that $D(z_0, R) \subset \Omega$ and:

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n \quad (\forall z \in D(z_0, R))$$

In particular, $f(z_0) = a_0$. Thus for all $z \in D(z_0, R)$ we have:

$$\begin{aligned} f(z) - f(z_0) &= \sum_{n=1}^{\infty} a_n (z - z_0)^n \\ &= (z - z_0) \sum_{n=1}^{\infty} a_n (z - z_0)^{n-1} \\ &= (z - z_0) \sum_{n=0}^{\infty} a_{n+1} (z - z_0)^n \qquad (1) \end{aligned}$$

By the Hadamard formula, we see that the power series $\sum_{n=0}^{\infty} a_{n+1}(z - z_0)^n$ has the same radius of convergence as the original power series $\sum_{n=0}^{\infty} a_n (z - z_0)^n$. Consequently, the power series $\sum_{n=0}^{\infty} a_{n+1}(z - z_0)^n$ converges absolutely for $|z - z_0| < R$. Let $r \in \mathbb{R}$ such that $0 < r < R$. Then for all $z \in D(z_0, r)$, we have from (1) the estimate:

$$\begin{aligned} |f(z) - f(z_0)| &= |z - z_0| \cdot \left| \sum_{n=0}^{\infty} a_{n+1} (z - z_0)^n \right| \\ &\leq |z - z_0| \sum_{n=0}^{\infty} |a_{n+1}| \, |z - z_0|^n \\ &\leq |z - z_0| \underbrace{\sum_{n=0}^{\infty} |a_{n+1}| \cdot r^n}_{<+\infty \text{ since } r<R}. \end{aligned}$$

Taking the limit as $z \to z_0$, we conclude that $\lim_{z \to z_0} f(z) = f(z_0)$, so f is continuous at z_0. ☐

Corollary 1.2.4 (Immediate) : Let Ω be a non empty open subset of $\mathbb{C}$ and $f : \Omega \longrightarrow \mathbb{C}$. If f is analytic on Ω, then f is continuous on Ω.

Proposition 1.2.5 (Composition of Analytic functions) : Let Ω_1 and Ω_2 be two nonempty open subsets of $\mathbb{C}$ and let $f : \Omega_1 \longrightarrow \Omega_2$ and $g : \Omega_2 \longrightarrow \mathbb{C}$ be two maps. Let also $z_0 \in \Omega_1$. If f is analytic at z_0 and g is analytic at $f(z_0)$, then $(g \circ f)$ is analytic at z_0.

Proof. Exercise ☐

Corollary 1.2.6 (Immediate) : Let Ω_1 and Ω_2 be two nonempty open subsets of $\mathbb{C}$ and let $f : \Omega_1 \longrightarrow \Omega_2$ and $g : \Omega_2 \longrightarrow \mathbb{C}$ be two maps. If f is analytic on Ω_1 and g is analytic on Ω_2 then $(g \circ f)$ is analytic on Ω_1.

Proposition 1.2.7 (Quotient of Analytic Functions) : Let Ω be a nonempty open subsets of $\mathbb{C}$ and let $z_0 \in \Omega$. Let also $f, g : \Omega \longrightarrow \mathbb{C}$ be two functions which are both analytic at z_0 and such that $g(z_0) \neq 0$. Then the function $\frac{f}{g}$ is analytic at z_0.

Proof. Since $g(z_0) \neq 0$ then the function $h : w \longrightarrow \frac{1}{w}$ is analytic at $g(z_0)$ (as seen in previous examples). Therefore, by *Proposition* 1.2.5, the function $\frac{1}{g} = h \circ g$ is anayltic at z_0.
It then follows from *Proposition* 1.2.1 that the product $f \cdot \left(\frac{1}{g}\right)$ is analytic at z_0. □

Corollary 1.2.8 (Immediate) : Let Ω be a non empty open subset of $\mathbb{C}$ and let $f, g : \Omega \longrightarrow \mathbb{C}$ be two analytic functions on Ω such that $g(z) \neq 0$ for every $z \in \Omega$. Then the function $\frac{f}{g}$ is analytic on Ω .

Example: Every rational function is analytic on its domain of definition. This is because a rational function is a quotient of two polynomials, and polynomials are analytic on $\mathbb{C}$.

- -

1.3 Power series define Analytic functions

Theorem 1.3.1 : A power series with a positive radius of converges defines an analytic function on its disk of convergence.

Proof. Let $\sum_{n=0}^{\infty} a_n(z - z_0)^n$ be a power series $(z_0 \in \mathbb{C}, (a_n)_{n\in\mathbb{N}}) \subset \mathbb{C})$ with radius of convergence $R > 0$. Define the function f on the disk $D(z_0, R)$ by:

$$f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n.$$

We must show that f is analytic on $D(z_0, R)$. Let $z_1 \in D(z_0, R)$ arbitrary. We will show that f is analytic at z_1. For $z \in D(z_1, R - |z_1 - z_0|)$, we have

$$|z - z_0| \overset{T.I}{\leq} \underbrace{|z - z_1|}_{<R-|z_1-z_0|} + |z_1 - z_0| < R$$

Thus $D(z_1, R - |z_1 - z_0|) \subset D(z_0, R)$, so the power series $\sum_{n=0}^{\infty} a_n(z - z_0)^n$ converges absolutely. so:

$$\begin{aligned}
f(z) &= \sum_{n=0}^{\infty} a_n(z - z_0)^n \\
&= \sum_{n=0}^{\infty} a_n((z - z_1) + (z_1 - z_0))^n \\
&= \sum_{n=0}^{\infty} a_n \sum_{k=0}^{n} \binom{n}{k} (z - z_1)^k (z_1 - z_0)^{n-k} \\
&= \sum_{k=0}^{\infty} \left(\sum_{n=k}^{\infty} a_k \binom{n}{k} (z_1 - z_0)^{n-k} \right) (z - z_1)^k
\end{aligned}$$

The interchange of summation is justified by the absolute convergence of the double series for $z \in D(z_1, R - |z_1 - z_0|)$. This express $f(z)$ as a power series in $(z - z_1)$ in the disk $D(z_1, R - |z_1 - z_0|)$, proving that f is analytic at z_1. Since z_1 was arbitrary in $D(z_0, R)$, then f is analytic on $D(z_0, R)$. $\square$

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Example: The power series $\sum_{n=0}^{\infty} \frac{z^n}{n!}$ has radius of convergence $R = +\infty$. Therefore (by the previous Theorem), it defines an analytic function on the whole complex plane $\mathbb{C}$.

Definition 1.3.1 : The analytic function on $\mathbb{C}$ defined by:

$$\exp(z) = e^z := \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

is called the exponential function.

Definition 1.3.2 (Entire function) : A complex function $f : \mathbb{C} \longrightarrow \mathbb{C}$ which is analytic on the whole complex plane $\mathbb{C}$ is called an entire function.

Example:

① Every complex polynomial is an entire function.

② The exponential function $\exp(z)$ is an entire function.

1.3.1 Properties of the exponential function

Proposition 1.3.2 : The exponential function defines the following properties:

① $\forall z_1, z_2 \in \mathbb{C}$, we have:

$$e^{z_1+z_2} = e^{z_1} \cdot e^{z_2} \text{ and } e^{z_1-z_2} = \frac{e^{z_1}}{e^{z_2}}.$$

② for all $z \in \mathbb{C}$, we have $e^z \neq 0$.

③ (Euler's formula): $\forall \theta \in \mathbb{R}$, we have:

$$e^{i\theta} = \cos\theta + i\sin\theta.$$

④ $\forall z \in \mathbb{C}$, we have:

$$e^z = 1 \iff z \in 2\pi i\mathbb{Z}.$$

More generally, for all $z, z' \in \mathbb{C}$, we have:

$$e^z = e^{z'} \iff z - z' \in 2\pi\mathbb{Z}.$$

So, the exponential function is periodic with period $2\pi i$.

Proof.

➼ ① $\forall z_1, z_2 \in \mathbb{C}$, we have

$$\begin{aligned}
e^{z_1} \cdot e^{z_2} &= \sum_{k=0}^{+\infty} \frac{z_1^k}{k!} \cdot \sum_{\ell=0}^{+\infty} \frac{z_2^\ell}{\ell!} \\
&= \sum_{k,\ell \in \mathbb{N}_0} \frac{z_1^k z_2^\ell}{k!\ell!} \\
&= \sum_{n=0}^{+\infty} \left(\sum_{k,\ell \in \mathbb{N}_0, k+\ell=n} \frac{z_1^k z_2^\ell}{k!\ell!} \right) \\
&= \sum_{n=0}^{+\infty} \left(\sum_{k=0}^{n} \frac{z_1^k z_2^{n-k}}{k!(n-k)!} \right) \\
&= \sum_{n=0}^{+\infty} \frac{1}{n!} \underbrace{\left(\sum_{k=0}^{n} \frac{n!}{k!(n-k)!} z_1^k z_2^{n-k} \right)}_{=\sum_{k=0}^{n} \binom{n}{k} z_1^k z_2^{n-k} = (z_1+z_2)^n} \\
&= \sum_{n=0}^{+\infty} \frac{1}{n!} (z_1 + z_2)^n = e^{z_1+z_2},
\end{aligned}$$

next, we have:

$$e^{z_1 - z_2} \cdot e^{z_2} \overset{\text{by the first formula}}{=} e^{z_1 - z_2 + z_2} = e^{z_1}.$$

Hence $e^{z_1 - z_2} = \frac{e^{z_1}}{e^{z_2}}$, as required.

➼ ② For all $z \in \mathbb{C}$, we have:

$$e^z \cdot e^{-z} \overset{(1)}{=} e^{z-z} = e^0 = 1.$$

Thus $e^z \neq 0$.

➼ ③ (Euler's Formula).

For all $\theta \in \mathbb{R}$, we have:

$$
\begin{aligned}
e^{i\theta} &= \sum_{n=0}^{+\infty} \frac{(i\theta)^n}{n!} \\
&= \sum_{n=0}^{+\infty} i^n \frac{\theta^n}{n!} \\
&= \sum_{n \in \mathbb{N}_0,\ n \text{ is even}} i^n \frac{\theta^n}{n!} + \sum_{n \in \mathbb{N}_0,\ n \text{ is odd}} i^n \frac{\theta^n}{n!} \\
&= \sum_{k=0}^{+\infty} i^{2k} \frac{\theta^{2k}}{(2k)!} + \sum_{k=0}^{+\infty} i^{2k+1} \frac{\theta^{2k+1}}{(2k+1)!} \\
&= \underbrace{\sum_{k=0}^{+\infty} (-1)^k \frac{\theta^{2k}}{(2k)!}}_{\cos\theta} + i \underbrace{\sum_{k=0}^{+\infty} (-1)^k \frac{\theta^{2k+1}}{(2k+1)!}}_{\sin\theta} \\
&= \cos\theta + i\sin\theta,
\end{aligned}
$$

as required.

➻ ④ Let $z \in \mathbb{C}$ and write

$$z = x + iy \qquad (x, y \in \mathbb{R}).$$

we have

$$
\begin{aligned}
e^z &= e^{x+iy} \\
&\overset{(1)}{=} e^x \cdot e^{iy} \\
&\overset{(3)}{=} e^x(\cos y + i\sin y) \\
&= e^x \cos y + ie^x \sin y.
\end{aligned}
$$

Thus

$$
\begin{aligned}
e^z = 1 &\iff \begin{cases} e^x \cos y = 1 \\ e^x \sin y = 0 \end{cases} \iff \begin{cases} \cos y = e^{-x} > 0 \\ \sin y = 0 \end{cases} \\
&\iff \begin{cases} \exists k \in \mathbb{Z}: \quad y = 2\pi k \\ e^{-x} = \cos 2\pi k = 1 \end{cases} \iff \begin{cases} \exists k \in \mathbb{Z}: \quad y = 2\pi k \\ x = 0 \end{cases} \\
&\iff z = 2\pi k i \qquad (k \in \mathbb{Z}) \\
&\iff z \in 2\pi\mathbb{Z},
\end{aligned}
$$

as required. □

1.3.2 Trigonometric and hyperbolic functions

Definition 1.3.3 (Complex Trigonometric functions) : We define the trigonometric functions cosine and sine by:

$$\cos z := \sum_{n=0}^{+\infty} (-1)^n \frac{z^{2n}}{(2n)!},$$
$$\sin z := \sum_{n=0}^{+\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!} \qquad (\forall z \in \mathbb{C}).$$

Clearly, these functions extend the real functions cos and sin. The power series defining cos and sin have infinite radius of convergence, thus (By a previous theorem) cos and sin are analytic on $\mathbb{C}$; that is, cos and sin are entire functions.

Remark ✎

We easily verify the extended Euler's formula:

$$e^{iz} = \cos z + i \sin z \qquad (\forall z \in \mathbb{C}).$$

From this formula, we derive:

$$\cos z = \frac{e^{iz} + e^{-iz}}{2},$$
$$\sin z = \frac{e^{iz} - e^{-iz}}{2i} \qquad (\forall z \in \mathbb{C}).$$

These functions are not bounded in $\mathbb{C}$, when you replace $x \leftarrow ix$, you get $\cos ix = \cosh x$.

Exercise

Using property ④ of *Proposition 1.3.2* and Euler's formula, show the following properties:

① The functions cos and sin are both 2π-periodic.

② The set of zeros of $z \mapsto \cos z$ is $(\frac{\pi}{2} + \pi\mathbb{Z})$, while the set of zeros of $z \mapsto \sin z$ is $\pi\mathbb{Z}$.

③ For all $z \in \mathbb{C}$, we have

$$\cos^2 z + \sin^2 z = 1.$$

☞ For Example, for ③: By the Euler formula, we have for all $z \in \mathbb{C}$:

$$\begin{aligned}\cos^2 z + \sin^2 z &= \left(\frac{e^{iz}+e^{-iz}}{2}\right)^2 + \left(\frac{e^{iz}-e^{-iz}}{2i}\right)^2 \\ &= \frac{4}{4} = 1\end{aligned}$$

Definition 1.3.4 (Complex hyperbolic functions) : We define the hyperbolic functions cosh and sinh by:

$$\begin{aligned}\cosh z &:= \sum_{n=0}^{+\infty} \frac{z^{2n}}{(2n)!} = \frac{e^z+e^{-z}}{2} = \cos(iz), \\ \sinh z &:= \sum_{n=0}^{+\infty} \frac{z^{2n+1}}{(2n+1)!} = \frac{e^z-e^{-z}}{2} = -i\sin(iz) \qquad (\forall z \in \mathbb{C}).\end{aligned}$$

Clearly, these definitions extend the real functions cosh and sinh. Like the trigonometric functions cos and sin, the hyperbolic functions cosh and sinh are also entire functions.

Exercise

Using the expressions of cosh and sinh in terms of cos and sin, verify the following properties:

① The functions cosh and sinh are both $2\pi-$periodic.

② The set of zeros of cosh is $(\frac{\pi}{2}i + \pi i\mathbb{Z})$, while the set of zeros of sinh is $\pi i\mathbb{Z}$.

③ For all $z \in \mathbb{C}$, we have

$$\cosh^2 z - \sinh^2 z = 1.$$

Definition 1.3.5 (Further trigonometric and hyperbolic functions) : We define the following functions:

$$\begin{aligned}\tan z &:= \frac{\sin z}{\cos z} \qquad \left(\forall z \in \mathbb{C}\backslash\left(\frac{\pi}{2}+\pi\mathbb{Z}\right)\right), \\ \cot z &:= \frac{\cos z}{\sin z} \qquad (\forall z \in \mathbb{C}\backslash\pi\mathbb{Z}), \\ \tanh z &:= \frac{\sinh z}{\cosh z} \qquad \left(\forall z \in \mathbb{C}\backslash\left(\frac{\pi}{2}i+\pi i\mathbb{Z}\right)\right), \\ \coth z &:= \frac{\cosh z}{\sinh z} \qquad (\forall z \in \mathbb{C}\backslash\pi i\mathbb{Z}).\end{aligned}$$

This clearly extends the well-known real functions tan, cot, tanh, and coth. Note that each of these four functions is analytic in its domain of definition (according to the previous results on analytic functions).

1.4 Holomorph functions

Definition 1.4.1 : Let Ω be a nonempty open subset of $\mathbb{C}$ and z_0 be a point in Ω. Let also $f : \Omega \longrightarrow \mathbb{C}$ be a map.

- We say that f is holomorphic at z_0 if the limit

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists and belong to $\mathbb{C}$. In this case, the limit is called the derivative of f at the point z_0 and denoted by $f'(z_0)$.

- We say that f is holomorphic on Ω if it is holomorphic at every point in Ω. In this case, the function

$$\begin{array}{rccc} f': & \Omega & \longrightarrow & \mathbb{C} \\ & z & \longmapsto & f'(z) \end{array}$$

is called the derivative of f.

Proposition 1.4.1 (Holomorphy of power series) : Let $z_0 \in \mathbb{C}$, $(a_n)_{n \in \mathbb{N}_0} \subset \mathbb{C}$, and S be the power series

$$S(z) = \sum_{n=0}^{+\infty} a_n (z - z_0)^n.$$

Suppose that S has a positive radius of convergence R. Then S' is holomorphic on $D(z_0, R)$ and we have for all $z \in D(z_0, R)$:

$$\begin{aligned} S'(z) &= \sum_{n=0}^{+\infty} n a_n (z - z_0)^{n-1} \\ &= \sum_{n=0}^{+\infty} (n+1) a_{n+1} (z - z_0)^n. \end{aligned}$$

Proof. For simplicity, suppose without loss of generality that $z_0 = 0$. First, remark that by using the Hadamard formula, the power series

$$\sum_{n=1}^{+\infty} n a_n z^{n-1} = \sum_{n=0}^{+\infty} (n+1) a_{n+1} z^n$$

has the same radius of convergence R as S. It follows that $\sum_{n=1}^{+\infty} n a_n z^{n-1}$ is absolutely convergent on $D(0, R)$; That is, for all $0 < r < R$, the series $\sum_{n=1}^{+\infty} n \, |a_n| \, r^{n-1}$ converges. Now, let $z_1 \in D(0, R)$

be arbitrary and show that S is holomorphic at z_1. Choose $r \in \mathbb{R}$ such that $|z_1| < r < R$. For all $z \in D(0,r) \setminus \{z_1\}$, we have

$$\begin{aligned}
\frac{S(z) - S(z_1)}{z - z_1} &= \frac{\sum_{n=0}^{+\infty} a_n z^n - \sum_{n=0}^{+\infty} a_n z_1^n}{z - z_1} \\
&= \sum_{n=0}^{+\infty} a_n \frac{z^n - z_1^n}{z - z_1} \\
&= \sum_{n=1}^{+\infty} a_n \sum_{k=0}^{n-1} z^k z_1^{n-1-k} \\
&= \sum_{n=1}^{+\infty} a_n \sum_{k=0}^{n-1} z^k z_1^{n-1-k} \qquad (*).
\end{aligned}$$

Next, we show that this last series of functions converges normally on $D(0,r) \setminus \{z_1\}$. For $z \in D(0,r) \setminus \{z_1\}$, we have:

$$\begin{aligned}
\left| a_n \sum_{k=0}^{n-1} z^k z_1^{n-1-k} \right| &\le |a_n| \sum_{k=0}^{n-1} \underbrace{|z|^k}_{<r} \underbrace{|z_1|^{n-1-k}}_{<r} \\
&\le |a_n| \sum_{k=0}^{n-1} r^{n-1} \\
&= n\,|a_n|\, r^{n-1} \qquad \text{(independent on } z\text{)}.
\end{aligned}$$

Since the series $\sum_{n=1}^{+\infty} n\,|a_n|\,r^{n-1}$ converges (as explained at the beginning of this of this proof) then the series of function $\sum_{n=1}^{+\infty} a_n \sum_{k=0}^{n-1} z^k z_1^{n-1-k}$ converges normally (no uniformally) on $D(0,r) \setminus \{z_1\}$. Therefore, we can interchange the limit as $z \to z_1$ and the summation for computing $\lim_{z \to z_1} \sum_{n=1}^{+\infty} \sum_{k=0}^{n-1} z^k z_1^{n-1-k}$. Doing so, we get according to $(*)$;

$$\begin{aligned}
\lim_{z \to z_1} \frac{S(z) - S(z_1)}{z - z_1} &= \sum_{n=1}^{+\infty} \lim_{z \to z_1} a_n \sum_{k=0}^{n-1} z^k z_1^{n-1-k} \\
&= \sum_{n=1}^{+\infty} a_n \sum_{k=0}^{n-1} z_1^k z_1^{n-1-k} \\
&= \sum_{n=1}^{+\infty} n a_n z_1^{n-1} \in \mathbb{C}.
\end{aligned}$$

Hence S is holomorphic at z_1 and we have

$$\begin{aligned}
S'(z_1) &= \sum_{n=1}^{+\infty} n a_n z_1^{n-1} \\
&= \sum_{n=0}^{+\infty} (n+1) a_{n+1} z_1^n.
\end{aligned}$$

Since z_1 is arbitrary in $D(0, R)$ then S is holomorphic on $D(0, R)$ and we have for all $z \in D(0, R)$:

$$S'(z) = \sum_{n \geq 1} n a_n z^{n-1} = \sum_{n=0}^{+\infty} (n+1) a_{n+1} z^n.$$

□

Lecture 4

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Corollary 1.4.2 (Infinite differentiability of power series) : Let $z_0 \in \mathbb{C}$, $(a_n)_{n \in \mathbb{N}_0} \subset \mathbb{C}$, and S be the power series

$$S(z) := \sum_{n=0}^{+\infty} a_n (z - z_0)^n.$$

Suppose that S has a positive radius of convergence R. Then S is infinitely $\mathbb{C}-$differentiable on $D(z_0, R)$ and we have for all $k \in \mathbb{N}_0$ and all $z \in D(z_0, R)$:

$$\begin{aligned} S^{(k)}(z) &= \sum_{n=k}^{+\infty} n(n-1) \dots \ (n-k+1) a_n (z - z_0)^{n-k} \\ &= \sum_{n=0}^{+\infty} (n+k)(n+k-1) \dots \ (n+1) a_{n+k} (z - z_0)^n \\ &= \sum_{n=0}^{+\infty} \frac{(n+k)!}{n!} a_{n+k} (z - z_0)^n. \end{aligned}$$

In particular, we have for all $k \in \mathbb{N}_0$:

$$S^{(k)}(z_0) = k! a_k.$$

Corollary 1.4.3 (Analytic functions are $\mathbb{C}$-infintely differentiable) : Let Ω be a nonempty open subset of $\mathbb{C}$ and $z_0 \in \Omega$. Let also $f : \Omega \longrightarrow \mathbb{C}$ be a map.

① If f is analytic at z_0 then f is infinitely $\mathbb{C}$-differentiable (no holomorphic) on some neighborhood of z_0 and we have in that neighborhood:

$$\boxed{f(z) = \sum_{n=0}^{+\infty} \frac{f^{(n)}(z_0)}{n!} (z - z_0)^n}$$

Taylor's formula

② If f is analytic on Ω then f is infinitely $\mathbb{C}$-differentiable (so holomorphic) on Ω.

Proof. Represent f by a power series in S in a neighborhood of z_0 and apply Corollary 3. □

Remark ✎

$$\text{Analytic} \implies \text{holomorphic}$$

② Cauchy *(1825)*:

f_n holomorphic + f' is continuous $\implies$ f is analytic.

③ Goursat *(1900)*:

f is holomorphic $\implies$ f is analytic.

Definition 1.4.2 : Let Ω be a nonempty open subset of $\mathbb{C}$ and $f : \Omega \longrightarrow \mathbb{C}$ be a map. An antiderivative of f is a holomorphic function $F : \Omega \longrightarrow \mathbb{C}$ such that $F' = f$.

Proposition 1.4.4 (Existence of Local antiderivatives) : Let Ω be a nonempty open subset of $\mathbb{C}$ and $z_0 \in \Omega$. Let also $f : \Omega \longrightarrow \mathbb{C}$ be a map. If f is analytic at z_0 then f admits an antiderivative in a neighborhood of z_0. Precisely, $\exists r > 0$ and $F : D(z_0, r) \longrightarrow \mathbb{C}$ analytic such that $F'(z) = f(z)$ for all $z \in D(z_0, r)$.

Proof. Suppose that f is analytic at z_0. then $\exists r > 0, \exists (a_n)_{n\in\mathbb{N}_0} \subset \mathbb{C}$ such that for all $z \in D(z_0, r)$:

$$f(z) = \sum_{n=0}^{+\infty} a_n (z - z_0)^n.$$

Define $F : D(z_0, r) \longrightarrow \mathbb{C}$ by

$$F(z) = \sum_{n=0}^{+\infty} \frac{a_n}{n+1} (z - z_0)^{n+1} = \sum_{n=1}^{+\infty} \frac{a_{n-1}}{n} (z - z_0)^n \qquad (\forall z \in D(z_0, r)).$$

The Hadamard formula shows that this last power series has the name radius of convergence as the original power series $\sum_{n=0}^{+\infty} a_n (z - z_0)^n$ representing f (which is $\geq r$). Consequently, F is well-defined on $D(z_0, r)$, and by the previous results, F is even analytic on $D(z_0, r)$ so holomorphic on $D(z_0, r)$ and for all $z \in D(z_0, r)$:

$$
\begin{aligned}
F'(z) &= \sum_{n=1}^{+\infty} \frac{a_{n-1}}{n} n(z-z_0)^{n-1} \\
&= \sum_{n=1}^{+\infty} a_{n-1}(z-z_0)^{n-1} \\
&= \sum_{n=0}^{+\infty} a_n(z-z_0)^n = f(z).
\end{aligned}
$$

Thus, F is an antiderivative of f on $D(z_0, r)$, completing the proof. □

Remark ✎

The rules of differentiation for analytic/holomorphic functions are the same as those of real-valued functions. For example:

$$
\begin{aligned}
(fg)' &= f'g + fg' \\
(f \circ g) &= g' \cdot (f' \circ g).
\end{aligned}
$$

On the other hand, the derivatives of known elementary functions, such that $z \to e^z$, $z \to \cos z$,$z \to \sin z$, etc are the same as in the real case. For example:

$$
\begin{aligned}
(e^z)' &= e^z \qquad (\forall z \in \mathbb{C}) \\
(\sin z)' &= \cos z \qquad (\forall z \in \mathbb{C})
\end{aligned}
$$

Proof.

$$
e^z = \sum_{n=0}^{+\infty} \frac{z^n}{n!} \qquad R = +\infty.
$$

$$
\begin{aligned}
(e^z)' &= \sum_{n=1}^{+\infty} \frac{n}{n!} z^{n-1} \\
&= \sum_{n=1}^{+\infty} \frac{z^{n-1}}{(n-1)!} \\
&= \sum_{n=0}^{+\infty} \frac{z^n}{n!} = e^z.
\end{aligned}
$$

□

1.5 The Cauchy-Riemann equations

Theorem 1.5.1 (Cauchy-Riemann equations) : Let Ω be a nonempty open subset of $\mathbb{C}$, $z_0 = x_0 + iy_0$ with $(x_0, y_0 \in \mathbb{C})$ a point in Ω, and $f : \Omega \longrightarrow \mathbb{C}$ be a map. Let $P : \mathrm{Re} f : \Omega \longrightarrow \mathbb{R}$ and $Q : \mathrm{Im} f : \Omega \longrightarrow \mathbb{R}$ so that

$$f(z) = P(x, y) + iQ(x, y).$$

for all $z = x + iy \in \Omega$, with $x, y \in \mathbb{R}$ then f is holomorphic at z_0 if and only if P and Q are differentiable at (x_0, y_0) and satisfy the following Cauchy-Riemann equations at (x_0, y_0) :

$$\begin{cases} \frac{\partial P}{\partial x}(x_0, y_0) = \frac{\partial Q}{\partial y}(x_0, y_0) \\ \frac{\partial P}{\partial y}(x_0, y_0) = -\frac{\partial Q}{\partial x}(x_0, y_0) \end{cases}$$

Proof.

$$(\Longrightarrow)$$

Suppose that f in holomorphic at z_0. Then for $h = u + iv \quad (u, v \in \mathbb{R})$, sufficiently small, we have:

$$f(z_0 + h) = f(z_0) + \cosh + o(h),$$

with $c = c_1 + ic_2 \in \mathbb{C} \quad (c_1, c_2 \in \mathbb{R})$. expanding this, we find:

$$P(x_0 + u, y_0 + v) + iQ(x_0 + u, y_0 + v) = P(x_0, y_0) + iQ(x_0, y_0) + (c_1 + ic_2)(u + iv) + o(u, v).$$

Identifying real and imaginary parts gives:

$$P(x_0 + u, y_0 + v) = P(x_0, y_0) + c_1 u - c_2 v + o(u, v),$$

$$Q(x_0 + u, y_0 + r) = Q(x_0, y_0) + c_2 u + c_1 v + o(u, v).$$

$$\frac{\partial P}{\partial x}(x_0, y_0) = c_1, \quad \frac{\partial P}{\partial y}(x_0, y_0) = -c_2, \quad \frac{\partial Q}{\partial x}(x_0, y_0) = c_2, \quad \frac{\partial Q}{\partial y}(x_0, y_0) = c_1.$$

Thus, P and Q indeed satisfying the the Cauchy-Riemann condition at (x_0, y_0).

$$(\Longleftarrow)$$

Conversly, suppose that P and Q are differentiable at (x_0, y_0) and satisfy the Cauchy-Riemann conditions at this point. Set

$$c_1 := \frac{\partial P}{\partial x}(x_0, y_0) = \frac{\partial Q}{\partial y}(x_0, y_0) \in \mathbb{R}$$

$$c_2 := \frac{\partial Q}{\partial x}(x_0, y_0) = -\frac{\partial f}{\partial y}(x_0, y_0) \in \mathbb{R}$$

By hypothesis, for $(u, v) \in \mathbb{R}^2$ sufficiently small, we have:

$$P(x_0 + u, y_0 + v) = P(x_0 + y_0) + c_1 u - c_2 v + o(u, v)$$
$$Q(x_0 + u, y_0 + v) = Q(x_0, y_0) + c_2 u + c_1 v + o(u, v).$$

Then, setting $h = u + iv$:

$$\begin{aligned} f(z_0 + h) &= P(x_0 + u, y_0 + v) + iQ(x_0 + u, y_0 + v) \\ &= P(x_0, y_0) + iQ(x_0, y_0) + \underbrace{(c_1 + ic_2)}_{c}(u + iv) + o(u, v) \\ &= f(z_0) + ch + o(h), \end{aligned}$$

with $c = c_1 + ic_2$. This shows that f is holomorphic at z_0. The theorem is proved. □

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Corollary 1.5.2 (Cauchy-Riemann equations on an open set) : Let Ω be an open subset of $\mathbb{C}$ and $f : \Omega \longrightarrow \mathbb{C}$ be a map. Let $P := \mathrm{Re} f : \Omega \longrightarrow \mathbb{R}$ and $Q := \mathrm{Im} f : \Omega \longrightarrow \mathbb{R}$, so that

$$f(z) = P(x, y) + iQ(x, y)$$

for all $z = x + iy \in \Omega$, with $x, y \in \mathbb{R}$. Then f is holomorphic on Ω if and only if P and Q are differentiable on Ω and satisfy the following Cauchy-Riemann equations:

$$\frac{\partial P}{\partial x} = \frac{\partial Q}{\partial y},$$
$$\frac{\partial P}{\partial y} = -\frac{\partial Q}{\partial x},$$

on Ω.

1.5.1 The isolated zeros thereom

☞ Some toplogical remainders:

Definition 1.5.1 (Limit points) : Let E be a toplogical space, $A \subset E$, $x \in E$. we say that x is a limit point of A if every neighborhood of x intersect A in a point different from

x; That is,

$$\forall V \in \mathcal{V}(x): \quad V \cap (A \setminus \{x\}) \neq \emptyset.$$

Note that this is equivalent to $x \in \overline{A \setminus \{x\}}$. The set of all limit points of A is denoted by A' and is called the derived set of A.

⤐ In metric spaces, we have the following equivalent definition:

Definition 1.5.2 : Let E be a metric space, $A \subset E$, and $x \in E$. We say that x is a limit point of A if there exists a sequence $(x_n)_{n\in\mathbb{N}}$ in $A \setminus \{x\}$ that converges to x .

- -

Definition 1.5.3 : Let E be a toplogical space and $A \subset E$.

① We say that an element $a \in A$ is isolated if its not a limit point of A.

② We say that A is a discrete set if all its points are isolated.

Example: The set $\mathbb{N}$ and $\mathbb{Z}$ are discete in $\mathbb{R}$, whereas the set $\mathbb{Q}$ is not (even though its countable).

Proposition 1.5.3 : In $\mathbb{R}^n (n \in \mathbb{N})$, every discrete set is at most countable.

Proof. ⤐ Exercise ! □

Theorem 1.5.4 (The isolated zeros theorem) : Let Ω be a nonempty connected open set in $\mathbb{C}$ and let f be a an analytic function on Ω that is not identically zero. Then the zero of f in Ω are all isolated; In other words, the set of zeros of f in Ω is discrete.

Proof. We proceed by contradiction. Suppose that there exists a zero $z_0 \in \Omega$ of f that is not isolated; i.e., z_0 is a limit point of the set of all zeros of f in Ω. Therefore, there exists a sequence $(z_k)_{k\geq 1}$ of zeros of f in Ω, with all terms distinct from z_0, that converges to z_0. Since f is analytic at z_0, $\exists r > 0$ and a power series representation:

$$f(z) = \sum_{n=0}^{+\infty} a_n (z - z_0)^n \qquad (\forall z \in D(z_0, r))$$

with $a_n \in \mathbb{C}$ for all $n \in \mathbb{N}_0$.

① Let us first show that the coefficients $a_n (n \in \mathbb{N})$ must necessarily all be zero. We proceed by contradictiom, assuming the contrary, and consider

$$P := \min \{n \in \mathbb{N}_0 : a_n \neq 0\}.$$

We then have for all $z \in D(z_0, r)$:

$$\begin{aligned} f(z) &= \sum_{n=p}^{+\infty} a_n (z - z_0)^n \\ &= (z - z_0)^p \left[a_p + a_{p+1}(z - z_0) + a_{p+2}(z - z_0)^2\right] \\ &= (z - z_0)^p \sum_{n=0}^{+\infty} a_{n+p}(z - z_0)^n. \end{aligned}$$

By specializing to $z = z_k$ (for k sufficiently large so that $z_k \in D(z_0, r)$) we find that (for $k \geq 1$ sufficiently large):

$$\underbrace{f(z_k)}_{=0} = \underbrace{(z_k - z_0)^p}_{\neq 0} \sum_{n=0}^{+\infty} a_{n+p}(z_k - z_0)^n.$$

Hence

$$\sum_{n=0}^{+\infty} a_{n+p}(z_k - z_0)^n = 0,$$

taking the limit as $k \to +\infty$ and noting the normal convergence of the series on the left, we obtain: $a_p = 0$. This contradicts the definition of p and shows that $a_n = 0$ for all $n \in \mathbb{N}_0$. In follows from this that:

$$f(z) = 0 \qquad (\forall z \in D(z_0, r)).$$

② Now, we will show that f is identically zero on all Ω, which will yield the desired contradiction. Let $\omega \in \Omega$ be arbitrary and show that $f(\omega) = 0$. Since Ω is connected (no path-connected, as its an open subset of $\mathbb{C}$), there exists a continuous path $\gamma : [0, 1] \longrightarrow \Omega$ such that $\gamma(0) = z_0$ and $\gamma(1) = \omega$. Consider

$$t_0 := \sup \{t \in [0, 1] : (f \circ \gamma)(t) = 0\}$$

The supremum exists because the set is non empty, as it contains 0, and it is bounded from above by 1. Since f and γ are continuous, the function $f \circ \gamma$ are continuous on $[0, 1]$. Consequently, the set

$$\{t \in [0, 1] : (f \circ \gamma)(t) = 0\} = (f \circ \gamma)^{-1}(\{0\})$$

which is closed in $[0, 1]$. Therefore t_0 belongs to this set; in other words, we have

$$(f \circ \gamma)(t_0) = 0 \qquad (1)$$

Let us show that $t_0 = 1$. Suppose for contradiction, that $t_0 < 1$. By the reasoning from the previous part of this proof (replacing z_0 by $\gamma(t_0)$, which is a limit point of the zero of f), there exists $r' > 0$ such that

$$f(z) = 0 \qquad (\forall z \in D(\gamma(t_0), r')).$$

FOr $\varepsilon > 0$, sufficiently small, we have:

$$\gamma(t_0 + \varepsilon) \in D(\gamma(t_0), r'),$$

(by the continuity of γ). Therefore:

$$g(\gamma(t_0 + \varepsilon)) = 0,$$

i.e. $(f \circ \gamma)(t_0 + \varepsilon) = 0$. This contradicts the very definition of t_0 as the supremum. Hence, necessarily $t_0 = 1$. This gives, from (1), $f(\omega) = 0$. Since ω was arbitrary in Ω, we have $f \equiv 0$ on Ω. Contradiction. This final contradiction ensures that the zeros of f in Ω are all isolated. The theorem is proved. `there is a tiny error in this proof, will be fixed next time.`

$\square$

Corollary 1.5.5 (Principle of analytic continuation) : Let f and g be two analytic functions on a nonempty connected open subset Ω of $\mathbb{C}$ that coincide on a subset $A \subset \Omega$ pocessing a limit point in Ω. Then f and g are identical on Ω.

Proof. Let $\varphi := f - g$. Then φ is analytic on Ω and vanishes on the set $A \subset \Omega$, which has a limit point $a \in \Omega$. So $a \in \overline{A \setminus \{a\}} \subset \overline{A}$. Since φ vanishes on A and is continuous on Ω, then it vanishes on $\overline{A} \cap \Omega$. In particular, φ vanishes at a. Therefore, a is a non-isolated zero of φ. By the isolated zero theorem, this implies that $\varphi \equiv 0$ on Ω; That is, $f \equiv g$ on Ω. $\square$

Example: Let us show that (without using the extended Euler formulas) that for all $z \in \mathbb{C}$, we have:

$$\cos^2 z + \sin^2 z = 1.$$

consider $f(z) := \cos^2 z + \sin^2 z$ and $g(z) := 1$. f and g are analytic on $\mathbb{C}$ (which is an connected open subset of $\mathbb{C}$) and coincide on $\mathbb{R}$, which pocesses a limit point in $\mathbb{C}$. Thus, by the principle of analytic continuation $f \equiv g$ on $\mathbb{C}$; i.e.

$$\cos^2 z + \sin^2 z = 1 \qquad (\forall z \in \mathbb{C}).$$

1.5.2 Multiplicity of a zero of an analytic function

Theorem 1.5.6 : Let Ω be a nonempty connected open subset of $\mathbb{C}$, and let f be an analytic function on Ω, not identically zero. Let $z_0 \in \Omega$ be a zero of f. Then there exists a unique positive integer p and a unique analytic function g on Ω does not vanish at z_0, such that

$$f(z) = (z - z_0)^p g(z) \qquad (\forall z \in \Omega)$$

Proof. ➬ **Existence of p and g :**

Since f is analytic at z_0, $\exists r > 0$ and a complex sequence $(a_n)_{n\in\mathbb{N}_0}$ such that $D(z_0, r) \subset \Omega$ and

$$f(z) = \sum_{n=0}^{+\infty} a_n (z - z_0)^n \qquad (\forall z \in D(z_0, r)).$$

Since f is not identically zero on Ω, it is certainly not identically zero on $D(z_0, r)$ (By the isolated zeros theorem). Thus the coefficients a_n are not all zero. We can therefore define

$$p := \min \{n \in \mathbb{N}_0 : a_n \neq 0\}.$$

since $a_0 = f(z_0)$, we have $p \geq 1$. Then, for all $z \in D(z_0, r)$:

$$\begin{aligned} f(z) &= \sum_{n=p}^{+\infty} a_n (z - z_0)^n \\ &= (z - z_0)^p \sum_{n=0}^{+\infty} a_{n+p} (z - z_0)^n \end{aligned}$$

Now, define $g : \Omega \longrightarrow \mathbb{C}$ by:

$$g(z) := \begin{cases} \frac{f(z)}{(z-z_0)^p} & \text{if } z \neq z_0, \\ a_p & \text{if } z = z_0. \end{cases}$$

We observe that:

- g is analytic on $\Omega \backslash \{z_0\}$ (as a quotitent of two analytic functions on $\Omega \backslash \{z_0\}$).

- For $z \in D(z_0, r)$, $g(z) = \sum_{n=0}^{+\infty} a_{n+p}(z - z_0)^n$ which shows that g is analytic at z_0. Hence, g is analytic on Ω. Moreover, we have

  $$f(z) = (z - z_0)^p g(z) \qquad (\forall z \in \Omega)$$

  and $g(z_0) = a_p \neq 0$.

Uniqueness of p and g :

Suppose there exists $p_1, p_2 \in \mathbb{N}$ and analytic functions g_1, g_2 on Ω that do not vanish at z_0, such that

$$f(z) = (z - z_0)^{p_1} g_1(z) = (z - z_0)^{p_2} g_2(z) \qquad (\forall z \in \Omega).$$

Then, for all $z \in \Omega \setminus \{z_0\}$, we have:

$$g_1(z) = (z - z_0)^{p_2 - p_1} g_2(z).$$

If $p_1 < p_2$ then taking the limit as $z \to z_0$ and using the continuity of g_1 and g_2 at z_0, we obtain $g_1(z_0) = 0$, which contradicts the hypothesis $g_1(z_0) \neq 0$. Therefore, we must have $p_1 \geq p_2$. By symmetry, we also have $p_2 \geq p_1$, so $p_1 = p_2$. Then, from above, we get

$$\forall z \in \Omega \setminus \{z_0\} : g_1(z) = (z - z_0)^{p_2 - p_1} g_2(z),$$

since $p_2 - p_1 = 0$, we get

$$g_1(z) = g_2(z).$$

Since g_1 and g_2 are continuous at z_0, taking the limit as $z \to z_0$ gives $g_1(z_0) = g_2(z_0)$. Hence $g_1 \equiv g_2$ on Ω, which completes the proof of uniqueness. $\square$

Definition 1.5.4 : In the context of the above theorem, the positive integer p is called the multiplicity of the zero z_0 of f. if z_1 is not a zero of f, its multiplicity is conventionally taken to be 0.

Chapter 2

The Cauchy integral formula on a circle and applications

Lecture 6 08:09 AM Mon, Oct 27 2025

Theorem 2.0.1 : Let $z_0 \in \mathbb{C}$ and $f : \mathbb{C} \longrightarrow \mathbb{C}$ be a function, analytic at z_0. Let also R be the radius of convergence of the power series representing f in a neighborhood of z_0. Then for every $n \in \mathbb{N}_0$ and every $0 < r < R$, we have:

$$\boxed{f^{(n)}(z_0) = \frac{n!}{2\pi r^n} \int_0^{2\pi} f(z_0 + re^{i\theta}) e^{-ni\theta} d\theta}$$

In particular, for $n = 0$ and $0 < r < R$.

$$\boxed{f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + re^{i\theta}) d\theta}$$

Proof. By hypothesis, $\exists!$ complex sequence $(a_n)_{n\in\mathbb{N}_0}$ such that for all $z \in D(z_0, R)$

$$f(z) = \sum_{n=0}^{+\infty} a_n (z - z_0)^n \qquad (\forall z \in D(z_0, R))$$

Fix $r \in (0, R)$. For every $\theta \in [0, 2\pi]$, taking $z = z_0 + re^{i\theta} \in C(z_0, r) \subset D(z_0, R)$ in the formula above, we obtain

$$f(z_0 + re^{i\theta}) = \sum_{n=0}^{+\infty} a_n r^n e^{in\theta}.$$

Since the trigonometric series on the right converges normally (so uniformally) with respect to $\theta \in [0, 2\pi]$ (Because for all $n \in \mathbb{N}_0$, we have $\left|a_n r^n e^{ni\theta}\right| = |a_n r^n|$, and the series $\sum_{n=0}^{+\infty} |a_n| r^n$ converges by properties of power series) then that is the Fourier series of the function $\theta \mapsto f(z_0 + re^{i\theta})$. Consequently, for every $n \in \mathbb{N}_0$, the Fourier coefficients are given by:

$$a_n r^n = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + re^{i\theta}) e^{-ni\theta} d\theta.$$

Thus

$$a_n = \frac{1}{2\pi r^n} \int_0^{2\pi} f(z_0 + re^{i\theta}) e^{-ni\theta} d\theta.$$

Comparing this with Taylor's formula

$$a_n = \frac{f^{(n)}(z_0)}{n!}$$

, we obtain for every $n \in \mathbb{N}_0$:

$$f^{(n)}(z_0) = \frac{n!}{2\pi r^n} \int_0^{2\pi} f(z_0 + re^{i\theta}) e^{-in\theta} d\theta,$$

as required. □

Remark ✎

In the context of *Theorem* 1, the right-hand side of the second formula (corresponding to $n = 0$):

$$f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + re^{i\theta}) d\theta.$$

is precisely the average value of f on the circle $C(z_0, r)$. Therefore, according to *Theorem 1*, the average value of f on any circle centered at z_0 and contained in $D(z_0, R)$ is equal to the value of f at z_0.

2.1 Analytic continuation of power series

Theorem 2.1.1 : Let Ω be a nonempty open subset of $\mathbb{C}$, and let $f : \Omega \longrightarrow \mathbb{C}$ be an analytic function on Ω. Let $z_0 \in \Omega$, and let $\sum_{n=0}^{+\infty} a_n(z-z_0)^n$ (where $a_n \in \mathbb{C}$, for all $n \in \mathbb{N}_0$) be the unique power series representing f in a small neighborhood $D(z_0, r) \subset \Omega$ of z_0 with (with $r > 0$). Then, the power series $\sum_{n=0}^{+\infty} a_n(z-z_0)^n$ converges on every disk centered at z_0 and contained in Ω, and it sum remains $f(z)$.

Proof. We have

$$f(z) = \sum_{n=0}^{+\infty} a_n(z-z_0)^n \qquad (\forall z \in D(z_0, r)).$$

Let $R > r$ such that $D(z_0, R) \subset \Omega$ and show that the above formula remains valid on $D(z_0, R)$. Given $n \in \mathbb{N}_0$, by Cauchy's integral formula on a circle, we have for all $0 < \sigma < r$:

$$a_n = \frac{1}{2\pi\sigma^n} \int_0^{2\pi} f(z_0 + \sigma e^{i\theta}) e^{-ni\theta} d\theta.$$

hence $\sigma \mapsto \frac{1}{2\pi\sigma^n} \int_0^{2\pi} f(z_0 + \sigma e^{i\theta}) e^{-ni\theta} d\theta$ is constant (with respect to σ) in $(0, r)$. This suggests us to define

$$\begin{array}{rccl} \varphi : & (0, R) & \longrightarrow & \mathbb{C} \\ & \sigma & \longmapsto & \varphi(\sigma) = \frac{1}{2\pi\sigma^n} \int_0^{2\pi} f(z_0 + \sigma e^{i\theta}) e^{-ni\theta} d\theta. \end{array}$$

(φ is constant in $(0, r)$. with value a_n). Since f is analytic (so continuous) on $D(z_0, R) \subset \Omega$ then φ is well-defined and differentiable on $(0, R)$. We now show that φ remains constant on $(0, R)$ on $(0, R)$. By applying results from measure theory and integration (which allow differentiation under the integral sign), we have for all $\sigma \in (0, R)$:

$$\begin{aligned} \varphi'(\sigma) &= -\frac{n\sigma^{-n-1}}{2\pi} \int_0^{2\pi} f(z_0 + \sigma e^{i\theta}) e^{-ni\theta} d\theta + \frac{\sigma^{-n}}{2\pi} \int_0^{2\pi} f'(z_0 + \sigma e^{i\theta}) e^{(-n+1)i\theta} d\theta. \\ &= \frac{\sigma^{-n-1}}{2\pi} \int_0^{2\pi} \underbrace{\left[-nf(z_0 + \sigma e^{i\theta}) e^{-ni\theta} + \sigma f'(z_0 + \sigma e^{i\theta}) e^{(-n+1)i\theta} \right]}_{\frac{d}{d\theta}(-if(z_0+\sigma e^{i\theta})e^{-ni\theta})} d\theta \\ &= \frac{\sigma^{-n-1}}{2\pi} \underbrace{\left[-if(z_0 + \sigma e^{i\theta}) e^{-ni\theta} \right]_0^{2\pi}}_{=0} = 0. \end{aligned}$$

This shows that φ is constant on $(0, R)$, and its value is a_n (since $\varphi \equiv a_n$ on $(0, r)$). Thus, we have shown that:

$$a_n = \frac{1}{2\pi\sigma^n} \int_0^{2\pi} f(z_0 + \sigma e^{i\theta}) e^{-ni\theta} d\theta \qquad (\forall n \in \mathbb{N}_0, \forall \sigma \in (0, R))$$

This formula allows us to estimate $|a_n|$ in terms of n. Giving $n \in \mathbb{N}_0, \sigma \in (0, R)$, and setting $M(f) := \sup_{z \in C(z_0,\sigma)} |f(z)|$, we have:

$$\begin{aligned} a_n &= \left| \frac{1}{2\pi\sigma^n} \int_0^{2\pi} f(z_0 + \sigma e^{i\theta}) e^{-ni\theta} d\theta \right| \\ &\leq \frac{1}{2\pi\sigma^n} \int_0^{2\pi} \underbrace{\left| f(z_0 + \sigma e^{i\theta}) \right|}_{\leq M(f)} d\theta \\ |a_n| &\leq \frac{M(f)}{\sigma^n} \qquad \forall n \in \mathbb{N}_0, \forall \sigma \in (0, R) \end{aligned}$$

So $\forall z \in D(z_0, \sigma)$:

$$|a_n (z - z_0)^n| \leq M(f) \left| \frac{z - z_0}{\sigma} \right|^n$$

implying that the power series $\sum_{n=0}^{+\infty} a_n (z - z_0)^n$ converges absolutely on $D(z_0, \sigma), \forall \sigma \in (0, R)$. Consequently, $\sum_{n=0}^{+\infty} a_n (z - z_0)^n$ converges (absolutely) on $D(z_0, R)$. Further, define

$$g(z) = \sum_{n=0}^{+\infty} a_n (z - z_0)^n \qquad (\forall z \in D(z_0, R)).$$

Since g is analytic on $D(z_0, R)$ and coincides with f on $D(z_0, r)$, by the principle of analytic continuation, g coincides with f on $D(z_0, R)$. Hence

$$f(z) = \sum_{n=0}^{+\infty} a_n (z - z_0)^n \qquad (\forall z \in D(z_0, R)),$$

completing the proof. □

Corollary 2.1.2 : Let Ω be a nonempty open subset of $\mathbb{C}$ and z_0 be a point in Ω. Let also $f : \Omega \longrightarrow \mathbb{C}$ be an analytic function on Ω and $\sum_{n=0}^{+\infty} a_n (z - z_0)^n$ (where $(a_n)_{n \in \mathbb{N}_0} \subset \mathbb{C}$) be a power series representing f in a neighborhood of z_0. Then the radius of convergence R of that power series satisfies;

$$R \geq \operatorname{dist}(z_0, \mathbb{C} \backslash \Omega).$$

Proof. Set $\sigma := \operatorname{dist}(z_0, \mathbb{C} \backslash \Omega)$. By definition of S, we have $D(z_0, \sigma) \subset \Omega$. (Indeed, if $z \in \mathbb{C} \backslash \Omega$ then $d(z, z_0) \geq \inf_{w \in \mathbb{C} \backslash \Omega} d(w, z_0) = d(z_0, \mathbb{C} \backslash \Omega) = \sigma$, thus $z \in \mathbb{C} \backslash D(z_0, \sigma)$). Thus, by *Theorem 2*, the power series $\sum_{n=0}^{+\infty} a_n (z - z_0)^n$ converges on $D(z_0, \sigma)$. Hence $R \geq \sigma$, as required. □

Example: ☞ VERY IMPORTANT.

Consider the function $f(z) = \frac{z}{e^z - 1}$ with the convention $f(0) = 1$. For all z in a small neighborhood of θ, with $z \neq 0$, we have:

$$\begin{aligned} f(z) &= \frac{z}{z + \frac{z^2}{2!} + \frac{z^3}{3!} + \dots} \\ &= \frac{1}{1 + \frac{z}{2!} + \frac{z^2}{3!} + \dots}, \end{aligned}$$

Which also holds for $z = 0$, showing that f is analytic at 0 (as a quotient of two analytic functions at 0, with the denominator nonzero at 0). Moreover, the zero of $z \mapsto e^z - 1$ are the complex numbers $2k\pi i$ with $k \in \mathbb{Z}$, implying that f is analytic on $\mathbb{C} \backslash 2\pi i\mathbb{Z}$ (as a quotient of two analytic functions on this domain, with the denominator nonzero at any point of the domain). In conclusion, f is analytic on the domain

$$\begin{aligned} \Omega &= (\mathbb{C} \backslash 2\pi\mathbb{Z}) \cup \{0\} \\ &= \mathbb{C} \backslash 2\pi i\mathbb{Z}^*. \end{aligned}$$

Consider the power series expansion of f at 0 :

$$\frac{z}{e^z - 1} = \sum_{n=0}^{+\infty} B_n \frac{z^n}{n!}, \qquad (*)$$

where $B_n (n \in \mathbb{N}_0)$ are the famous Bernoulli numbers, which appear in nearly all branches of mathematics (number theory, analysis, combinatorics, algebraic geometry, etc...).:w

$$B_0 = 1, B_1 = -\frac{1}{2}, B_2 = \frac{1}{6}, B_3 = 0, B_4 = \frac{1}{42}, \dots$$

$$(B + 1)^n = B^n \qquad (\forall n \geq \mathbb{N}_2).$$

for $n = 2$:

$$B^2 + 2B + 1 = B_2 \implies B_1 = -\frac{1}{2}$$

for $n = 3$:

$$(B + 1)^3 = B^3 \implies 3B_2 = \frac{1}{2} \implies B_2 = \frac{1}{6}.$$

Theorem 2 shows that fomula $(*)$ is valid on the largest disk centered at 0 and contained in Ω, which is $D(0, 2\pi)$. Hence

$$\frac{z}{e^z - 1} = \sum_{n=0}^{+\infty} B_n \frac{z^n}{n!} \qquad (\forall z \in \mathbb{C}, |z| < 2\pi).$$

Remark ✎

It can be shown that

$$B_{2n+1} = 0 \quad \forall n \geq 1,$$

and

$$B_{2n} \sim_{+\infty} (-1)^{n-1} \cdot \frac{2(2n)!}{(2\pi)^{2n}}.$$

This allows us to directly verify (using Hadamard formula) that the radius of convergence of the power series $\sum_{n=0}^{+\infty} B_n \frac{z^n}{n!}$ is exactly 2π.

Lecture 7

08:08 AM Mon, Nov 03 2025

Theorem 2.1.3 (Cauchy's integral formula on a circle II) : Let f be an analytic function on a nonempty open subset Ω of $\mathbb{C}$ containing a closed disk $\overline{D}(z_0, R)$ (with $z_0 \in \Omega$, and $R > 0$). Then for any $n \in \mathbb{N}_0$, we have

$$\boxed{f^{(n)}(z_0) = \frac{n!}{2\pi R^n} \int_0^{2\pi} f(z_0 + Re^{i\theta}) e^{-ni\theta} d\theta.}$$

In particular, for $n = 0$:

$$\boxed{f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + Re^{i\theta}) d\theta}$$

➼ To prove this theorem, we need the following toplogical lemma (corollary):

Corollary 2.1.4 : In the context of *Theorem 4* , $\exists \alpha > 0$ such that $D(z_0, R + \alpha) \subset \Omega$.

Proof. Since $\overline{D}(z_0, R)$ is compact in $\mathbb{C}$ (because it's closed and bounded in $\mathbb{C}$) and $\mathbb{C} \backslash \Omega$ is closed in $\mathbb{C}$ (because Ω is open in $\mathbb{C}$) and $\overline{D}(z_0, R) \cap (\mathbb{C} \backslash \Omega) = \emptyset$ (since $\overline{D}(z_0, R) \subset \Omega$ by hypothesis) then $d(\overline{D}(z_0, R), \mathbb{C} \backslash \Omega) > 0$. Set $\alpha := d(\overline{D}(z_0, R), \mathbb{C} \backslash \Omega) > 0$ and show that:

$$\mathbb{C} \backslash \Omega \subset \mathbb{C} \backslash D(z_0, R + \alpha) \tag{1}$$

Let us show **(1)**. Let $z \in \mathbb{C} \backslash \Omega$ and set

$$u_z := z_0 + \frac{z - z_0}{|z - z_0|} R$$

(so $u_z \in C(z_0, R) \subset \overline{D}(z_0, R)$) On the one hand, we have:

$$|z - u_z| = d(z, u_z) \geq d(\mathbb{C} \backslash \Omega, \overline{D}(z_0, R)) = \alpha.$$

On the other hand, we have:

$$\begin{aligned} |z - u_z| &= \left| z - z_0 - \frac{z - z_0}{|z - z_0|} R \right| \\ &= \left| (z - z_0) \left(1 - \underbrace{\frac{R}{|z - z_0|}}_{<1} \right) \right| \quad (\text{since } |z - z_0| > R,\ z \in \mathbb{C} \backslash \Omega \subset \overline{D}(z_0, R)) \\ &= \left(1 - \frac{R}{|z - z_0|} \right) |z - z_0| = |z - z_0| - R. \end{aligned}$$

Comparing the two results, we get

$$|z - z_0| - R \geq \alpha,$$

i.e.,

$$|z - z_0| \geq R + \alpha$$

i.e.,

$$z \in \mathbb{C} \backslash D(z_0, R + \alpha),$$

as required. Inclusion **(1)** is then proved. Hence $D(z_0, R + \alpha) \subset \Omega$, completing the proof. □

❄ PROOF OF THEOREM 4

Proof. By *Corollary* 5 , $\exists \alpha > 0$ such that $D(z_0, R + \alpha) \subset \Omega$. Then, by *Theorem* 2, $\exists (a_n)_{n \in \mathbb{N}_0} \in \mathbb{C}$ such that

$$f(z) = \sum_{n=0}^{+\infty} a_n (z - z_0)^n \quad (\forall z \in D(z_0, R + \alpha)).$$

Finally, by *Theorem* 1, we have for all $n \in \mathbb{N}_0$:

$$f^{(n)}(z_0) = \frac{n!}{2\pi R^n} \int_0^{2\pi} f(z_0 + Re^{i\theta}) e^{-ni\theta} d\theta \quad (\text{since } R < R + \alpha).$$

The Theorem is then proved. □

2.2 Cauchy's inequalities

Cauchy's inequalities (also called Cauchy's estimates) are fundamental bounds on the derivatives (of different orders) of an analytic function in terms of the function it self.

Theorem 2.2.1 (Cauchy's inequalities) : Let f be an analytic function on a nonempty open subset Ω of $\mathbb{C}$ containing a closed disk $\overline{D}(z_0, R)$ (with $z_0 \in \Omega, R > 0$). Then for any $n \in \mathbb{N}_0$, the nth derivative of f at z_0 satisfies the inequality

$$\boxed{\left|f^{(n)}(z_0)\right| \leq \frac{n!}{R^n} M_f(R),}$$

where $M_f(R) := \sup_{z \in C(z_0,R)} |f(z)|$ is the maximum modulus of f on the circle of radius R centered at z_0.

If we know the values of f analytic function in a circle R, we are able to know it's values in the interior.

Proof. By *Theorem 4 (Last section)*, we have for all $n \in \mathbb{N}_0$:

$$f^{(n)}(z_0) = \frac{n!}{2\pi R^n} \int_0^{2\pi} f(z_0 + Re^{i\theta}) e^{-ni\theta} d\theta.$$

which gives:

$$\begin{aligned}
\left|f^{(n)}(z_0)\right| &= \frac{n!}{2\pi R^n} \left| \int_0^{2\pi} f(z_0 + Re^{i\theta}) e^{-ni\theta} d\theta \right| \\
&\leq \frac{n!}{2\pi R^n} \int_0^{2\pi} \left|f(z_0 + Re^{i\theta})\right| \underbrace{\left|e^{-ni\theta}\right|}_{=1} d\theta \\
&= \frac{n!}{2\pi R^n} \int_0^{2\pi} \left| \underbrace{f(z_0 + Re^{i\theta})}_{\leq \sup_{0 \leq \theta \leq 2\pi} |f(z_0 + Re^{i\theta})|} \right| d\theta. \\
&\leq \frac{n!}{2\pi R^n} 2\pi \sup_{\theta \in [0,2\pi]} \left|f(z_0 + Re^{i\theta})\right|.
\end{aligned}$$

Since $\theta \in [0, 2\pi] \iff z_0 + Re^{i\theta} \in C(z_0, R)$, then $\sup_{\theta \in [0,2\pi]} \left|f(z_0 + Re^{i\theta})\right| = \sup_{z \in C(z_0,R)} |f(z)| = M_f(R)$. Hence

$$\left|f^{(n)}(z_0)\right| \leq \frac{n!}{R^n} M_f(R),$$

as required. $\square$

2.3 Lionville's theorem

Theorem 2.3.1 (Lionville's theorem) : Every entire function (i.e., analytic on $\mathbb{C}$) that is bounded is necessarily constant.

Proof. Let f be an entire function bounded by some constant $M > 0$ (i.e., $|f(z)| \leq M, \forall z \in \mathbb{C}$). We will show that $f'(z_0) = 0$, $\forall z_0 \in \mathbb{C}$. Since f is analytic on $\mathbb{C}$ then, by applying Cauchy's inequalities (i.e. *Theorem* 1) for $n = 1$, we obtain for all $R > 0$:

$$|f'(z_0)| \leq \frac{1}{R} \sup_{z \in C(z_0,R)} |f(z)| \leq \frac{M}{R}$$

By letting $R \to +\infty$, we obtain $f'(z_0) = 0$, since z_0 was arbitrary in $\mathbb{C}$, then f' is identically zero on $\mathbb{C}$, implying that f is constant, as required. $\square$

2.3.1 Proof of the fundamental theorem of Algebra via Lionville's theorem

We first prove the following theorem, from which we immediately derive the fundamental theorem of Algebra.

Theorem 2.3.2 : Every non constant polynomial in $\mathbb{C}[Z]$ has at least one zero in $\mathbb{C}$.

Proof. **Through Lionville's theorem.** Let $P \in \mathbb{C}[Z]$ be a non constant polynomial and write:

$$P(z) = a_0 z^n + a_1 z^{n-1} + \ldots + a_n,$$

with $n \in \mathbb{N}, a_0, a_1, \ldots, a_n \in \mathbb{C}$, with $a_0 \neq 0$. Assume for contradiction that P has no root in $\mathbb{C}$. Then the function $f : \mathbb{C} \longrightarrow \mathbb{C}$ defined by

$$f(z) = \frac{1}{P(z)} \qquad (\forall z \in \mathbb{C}),$$

is analytic on $\mathbb{C}$ (as a rational function is well defined on the whole complex plane $\mathbb{C}$); i.e., f is an entire function. Let us show in addition that f is bounded. For all $z \in \mathbb{C}^*$, we have:

$$\begin{aligned} |P(z)| &= \left| z^n \left(a_0 + \frac{a_1}{z} + \frac{a_2}{z^2} + \ldots + \frac{a_n}{z^n} \right) \right| \\ &= |z|^n \cdot \left| a_0 + \frac{a_1}{z} + \frac{a_2}{z^2} + \ldots + \frac{a_n}{z^n} \right|. \end{aligned}$$

Since $a_0 \neq 0$ and

$$\lim_{|z| \to \infty} \frac{a_1}{z} = \lim_{|z| \to \infty} \frac{a_2}{z^2} = \ldots \lim_{|z| \to \infty} \frac{a_n}{z^n} = 0.$$

Then $\lim_{|z|\to\infty} |P(z)| = +\infty$. Thus

$$\lim_{|z|\to\infty} |f(z)| = 0.$$

Therefore, there exists $R > 0$ such that

$$|f(z)| \leq 1 \qquad (\forall z \in \mathbb{C}, |z| > R).$$

On the other hand, since $z \mapsto |f(z)|$ is continuous on the compact set $\overline{D}(z_0, R)$, there exists $M > 0$ such that

$$|f(z)| \leq M \qquad \forall z \in \overline{D}(0, R)$$

Hence, for all $z \in \mathbb{C}$:

$$|f(z)| \leq \max(1, M).$$

This shows that f is bounded applying Lionville's theorem, we deduce that f is constant; so $\frac{1}{P}$ is constant, implying that P is constant, Contradiction. This contradiction confirms that P has atleast one zero in $\mathbb{C}$, as required. □

Corollary 2.3.3 (Fundamental Theorem of Algebra) : Every polynomial $P \in \mathbb{C}[Z]$ splits over $\mathbb{C}$.

Proof. Proceed by induction on the degree of P and use *Theorem* 7 □

2.4 The maximum modulus principle

The maximum modulus principle ensures that the modulus of a non-constant analytic function on a nonempty connected open set $\Omega \subset \mathbb{C}$, continuous on $\overline{\Omega}$, can attain its maximum only on the boundary $\partial\Omega := \overline{\Omega}\backslash\Omega$, and never in Ω. We begin by establishing the following key lemma (corollary).

Corollary 2.4.1 : Let f be a non-constant analytic function on a non-empty connected open set $\Omega \subset \mathbb{C}$ containing a closed disk $\overline{D}(z_0, R)$ (with $z_0 \in \Omega$, and $R > 0$). Then, we have:

$$|f(z_0)| < \max_{z\in\overline{D}(z_0,R)} |f(z)|.$$

Proof. We distinguish two cases:

Case 1: (if $f(z_0)=0$). By the isolated zero theorem, f is not identically zero on $\overline{D}(z_0,R)$. Hence

$$\max_{z\in\overline{D}(z_0,R)}|f(z)|>0=|f(z_0)|,$$

as required.

Case 2: (if $f(z_0)\neq 0$). Since the function $z\mapsto f(z)-f(z_0)$ is analytic on Ω and not identically zero on Ω (which is connected) then it's zero z_0 has a finite multiplicity $p\in\mathbb{N}$. Therefore, in a neighborhood of z_0, we have:

$$f(z)-f(z_0)=a(z-z_0)^p+o\left((z-z_0)^p\right), \qquad (✈)$$

with $a\in\mathbb{C}^*$. Next, consider the exponential form of the nonzero complex number $\frac{a}{f(z_0)}$; that is, $\frac{a}{f(z_0)}=\rho e^{i\theta}$, where $\rho>0$, and $\theta\in[0,2\pi)$. and define, for $\varepsilon>0$ sufficiently small:

$$z_\varepsilon := z_0+\varepsilon e^{i\frac{\theta}{\rho}}.$$

From (✈), we have (for ε sufficiently small).

$$\begin{aligned}
f(z_\varepsilon)-f(z_0)&=a(z_\varepsilon-z_0)^p+o\left((z-z_0)^p\right)\\
&=\underbrace{a}_{f(z_0)\rho e^{i\theta}}\left(e^pe^{-i\theta}\right)+o\left((z-z_0)^p\right)\\
&=f(z_0)\rho\varepsilon^p+o\left(\varepsilon^p\right).
\end{aligned}$$

Hence

$$\begin{aligned}
f(z_\varepsilon)&=f(z_0)+f(z_0)\rho\varepsilon^p+o(\varepsilon^p).\\
&=f(z_0)\left(1+\rho\varepsilon^p\right)+o\left(\varepsilon^p\right).
\end{aligned}$$

Hence

$$\left|\frac{f(z_\varepsilon)}{f(z_0)}\right|=1+\rho\varepsilon^p+o\left(\varepsilon^p\right)>1$$

for all ε sufficiently small since $z_\varepsilon\in\overline{D}(z_0,R)$ for ε sufficiently small then

$$|f(z_0)|<\max_{z\in\overline{D}(z_0,R)}|f(z)|,$$

as required. ☐

Lecture 8

`08:02 AM Mon, Nov 10 2025`

Remark ✎

In the context of last theorem, we provide another proof using Cauchy's inequality.

Proof. **Second proof of Corollary 9 (Via the Cauchy formula).** Suppose for contradiction that $|f(z_0)| = \max_{z\in\overline{D}(z_0,R)} |f(z)|$. Then we have for all $r \in [0, R]$ and all $\theta \in [0, 2\pi]$:

$$|f(z_0)| - \left|f(z_0 + re^{i\theta})\right| \geq 0.$$

It follows that for all $r \in [0, R]$:

$$\begin{aligned}
0 &\leq \int_0^{2\pi} \left(|f(z_0)| - \left|f(z_0 + re^{i\theta})\right|\right) d\theta \\
&= 2\pi\,|f(z_0)| - \int_0^{2\pi} \left|f(z_0 + re^{i\theta})\right| d\theta \\
&\leq 2\pi\,|f(z_0)| - \underbrace{\left|\int_0^{2\pi} f(z_0 + re^{i\theta}) d\theta\right|}_{2\pi|f(z_0)|\text{Cauchy's formula}} \\
&= 2\pi\,|f(z_0)| - 2\pi\,|f(z_0)| = 0.
\end{aligned}$$

Hence, for all $r \in [0, R]$:

$$\int_0^{2\pi} \left(|f(z_0)| - \left|f(z_0 + re^{i\theta})\right|\right) d\theta = 0 \tag{1}$$

Given $r \in [0, R]$, since the function $\theta \mapsto |f(z_0)| - \left|f(z_0 + re^{i\theta})\right|$ (where $\theta \in [0, 2\pi]$) is continuous and non-negative on $[0, 2\pi]$, we deduce from (1) (according to the properties of the Riemann integrals) that for all $r \in [0, R]$:

$$|f(z_0)| - \left|f(z_0 + re^{i\theta})\right| = 0, \quad (\forall \theta \in [0, 2\pi]).$$

Consequently, we have:

$$\left|f(z_0 + re^{i\theta})\right| = |f(z_0)|\,, \quad (\forall r \in [0, R]\,, \forall \theta \in [0, 2\pi])\,.$$

In other words, $|f|$ is constant on $\overline{D}(z_0, R)$, so on $D(z_0, R)$. Thus (according to the result of a tutorial exercise) f is also constant on $D(z_0, R)$. Finally, since Ω is connected, we conclude (by the Analytic continuation principle) that f is constant on the whole Ω. Contradiction, this contradiction ensures that

$$|f(z_0)| < \max_{z\in\overline{D}(z_0,R)} |f(z)|\,,$$

as required. □

Theorem 2.4.2 (Maximum Modulus Principle) : Let Ω be a nonempty bounded connected open subset of $\mathbb{C}$, and let $f : \overline{\Omega} \longrightarrow \mathbb{C}$ be a function continuous on $\overline{\Omega}$ and analytic on Ω. Then, we have:

$$\max_{z\in\overline{\Omega}} |f(z)| = \max_{z\in\partial\Omega} |f(z)| .$$

Moreover, if this maximum is attained at some point $z_0 \in \Omega$, then f is necessarly constant on $\overline{\Omega}$.

Proof. Since Ω is bounded, $\overline{\Omega}$ is compact. As f is continuous on $\overline{\Omega}$, so is $|f|$. Therefore, $|f|$ is bounded and attains its supremum on $\overline{\Omega}$. That is, there exists $u \in \overline{\Omega}$ such that:

$$|f(u)| = \max_{z\in\overline{\Omega}} |f(z)| .$$

We now consider two cases:

- **Case 1:** (if $u \in \Omega$). Since Ω is open and $u \in \Omega$, there exists $R > 0$ such that $\overline{D}(u, R) \subset \Omega$, then:

$$|f(u)| \le \max_{z\in\overline{D}(u,R)} |f(z)| \le \max_{z\in\overline{\Omega}} |f(z)| \,(= |f(u)|).$$

  Hence,

$$|f(u)| = \max_{z\in\overline{D}(u,R)} |f(z)| .$$

  By the above corollary, f must be constant on Ω, and by continuity on $\overline{\Omega}$. Thus, the conclusion of the theorem hold in this case.

- **Case 2:** (if $u \notin \Omega$). Then $u \in \partial\Omega$, and we have:

$$\max_{z\in\partial\Omega} |f(z)| = \max_{z\in\overline{\Omega}} |f(z)| = |f(u)| .$$

  Now, if there exists $z_0 \in \Omega$ such that

$$|f(z_0)| = \max_{z\in\partial\Omega} |f(z)| = \max_{z\in\overline{\Omega}} |f(z)| ,$$

  then applying the reasoning of **Case 1** with u replaced by z_0, we conclude that f is constant on $\overline{\Omega}$. This completes the proof.

□

Corollary 2.4.3 (Minimum Modulus Principle) : Let Ω be a nonempty bounded connected open subset of $\mathbb{C}$, and let $f : \overline{\Omega} \longrightarrow \mathbb{C}$ be a function continuous on $\overline{\Omega}$ and analytic on Ω. Suppose that f has no zeros in Ω. Then, we have:

$$\min_{z\in\overline{\Omega}} |f(z)| = \min_{z\in\partial\Omega} |f(z)| .$$

Moreover, if this minimum is attained at some point $z_0 \in \Omega$, then f is necessarily constant on $\overline{\Omega}$.

Proof. The corollary is trivial if f vanishes at some point on $\partial\Omega$. Otherwise (i.e., if f does not vanish at any point in $\overline{\Omega}$), apply Maximum Modulus Principle to the function $\frac{1}{f}$. □

2.4.1 Proof of the fundamental theorem of Algebra via the Minimum Modulus Principle

We now give an alternative proof of the Fundamental Theorem of Algebra. We are lead to reprove Theorem 7 (stating that every non-constant polynomial in $\mathbb{C}[\mathbb{Z}]$ has atleast one zero in $\mathbb{C}$). So, let $p \in \mathbb{C}[\mathbb{Z}]$ be a non-constant polynomial and write

$$p(z) = a_0 z^n + a_1 z^{n-1} + \ldots + a_{n-1} z + a_n,$$

with $n \in \mathbb{N}$, $a_0, a_1, \ldots, a_n \in \mathbb{C}$, and $a_0 \neq 0$. Assume for contradiction that p has no zeros in $\mathbb{C}$. Since

$$\lim_{z\to+\infty} |p(z)| = +\infty$$

(as seen in previous proof of Theorem 7) then there exists $R > 0$ satisfying:

$$\forall z \in \mathbb{C} : |z| = R \implies |p(z)| > |p(0)| .$$

Thus, we have

$$\min_{z\in\overline{D}(z_0,R)} |p(z)| \leq |p(0)| .$$

while

$$\min_{z\in C(0,R)} |p(z)| > |p(0)| .$$

Hence,

$$\min_{z\in\overline{D}(0,R)} |p(z)| \neq \min_{z\in C(0,R)} |p(z)| ,$$

which is in contradiction with Minimum Modulus Principle. This contradiction ensures that p has atleast one zero in $\mathbb{C}$, as required.

2.4.2 Open Mapping Theorem

Theorem 2.4.4 (Open Mapping Theorem) : Let Ω be an nonempty connected open subset of $\mathbb{C}$ and $f : \Omega \longrightarrow \mathbb{C}$ be a non-constant analytic function on Ω. Then $f(\Omega)$ is an open subset of $\mathbb{C}$. In other words, every non-constant analytic function is an open mapping.

Proof. We aim to show that $f(\Omega)$ is a neighborhood of each of its points. Let $z_0 \in \Omega$ be an arbitrary point in Ω, and let us show that $f(\Omega)$ is a neighborhood of $\omega_0 := f(z_0)$. Consider the analytic function $g(z) = f(z) - \omega_0 = f(z) - f(z_0)$, which is not identically zero on Ω and clearly has a zero at z_0. By the Isolated Zeros Theorem, there exists $r > 0$, such that $D(z_0, r) \subset \Omega$ and g doesn't vanish except at z_0 itself. In particular, g does not vanish on the circle $C := C(z_0, \frac{r}{2})$; that is, $\omega_0 \notin f(C)$. Since $f(C)$ is compact set in $\mathbb{C}$ (as the continuous image of a compact set) it follows that $\rho := d(\omega_0, f(C)) > 0$. We now claim that the points of $\mathbb{C} \backslash f(\Omega)$ are closer to the curve $f(C)$ than the point ω_0. Indeed, take an aribtrary point $\omega_1 \in \mathbb{C} \backslash f(\Omega)$.

$$\boxed{d(\omega_1, f(C)) \overset{??}{<} d(\omega_1, \omega_0)}$$

The function $z \mapsto \frac{1}{f(z) - \omega_1}$ is analytic on Ω (since $f(z) - \omega_1 \neq 0$) by the choice of ω_1) and non-constant on Ω. By the maximum modulus principle applied to this function on the closed disk $\overline{D}(z_0, \frac{r}{2})$ (whose boundary in $C(z_0, \frac{r}{2}) = C$), we have:

$$\left| \frac{1}{f(z_0) - \omega_1} \right| < \max_{z \in C} \left| \frac{1}{f(z) - \omega_1} \right| = \frac{1}{\min_{z \in C} |f(z) - \omega_1|} = \frac{1}{d(\omega_1 - f(C))}.$$

That is,

$$d(\omega_1, \omega_0) > d(\omega_1, f(C)).$$

confirming our claim. Since the points in the disk $D(\omega, \frac{\rho}{2})$ are clearly closer to the point ω_0 than the curve $f(C)$. Indeed, for all $\omega \in D(\omega_0, \frac{\rho}{2})$, we have

$$\begin{aligned} \rho := d(\omega_0, f(C)) \leq d(\omega) &\leq d(\omega_0, \omega) + d(\omega, f(C)) \\ &< \frac{\rho}{2} + d(\omega_1, f(C)). \end{aligned}$$

Thus,

$$d(\omega, f(C)) > \frac{\rho}{2} > d(\omega_1, \omega_0).$$

It follows from the previous claim that $D(\omega_0, \frac{\rho}{2}) \subset f(\Omega)$. This shows that $f(\Omega)$ is a neighborhood of ω_0. Since ω_0 is aribtrary in $f(\Omega)$, it follows that $f(\Omega)$ is an open subset of $\mathbb{C}$, as required. $\square$