Appendix A

Log-Chi-Square Distribution

This appendix provides the mathematical proofs for the log-transformed version of chi-squared random variables.

Chi-squared random variables $\chi \sim \chi^2(k)$ follows the pdf:

$$pdf(\chi;\, k) = \frac{\chi^{(k/2)-1}e^{-\chi/2}}{2^{k/2}\Gamma\left(\frac{k}{2}\right)} \tag{A.1}$$

Setting L=k/2 into Eqn. A.1

$$pdf(\chi) = \frac{\chi^{L-1}e^{-\chi/2}}{2^{L}\Gamma(L)} \tag{A.2}$$

Applying the variable change theorem, which states that: if $y = \phi(x)$ with $\phi(c) = a$ and $\phi(d) = b$, then:

$$\int_a^b f(y)\, dy = \int_c^d f[\phi(x)]\frac{d\phi}{dx}dx \tag{A.3}$$

into the log-transformation, which changes the random variables $\Lambda = ln(\chi)$, we have:

$$\begin{aligned} d\chi &= e^{\Lambda}d\Lambda \\ \frac{\chi^{L-1}e^{-\chi/2}}{2^{L}\Gamma(L)}d\chi &= \frac{(e^{\Lambda})^{L-1}e^{-e^{\Lambda}/2}}{2^{L}\Gamma(L)}e^{\Lambda}d\Lambda \end{aligned}$$

In other words, we have:

$$pdf(\Lambda; L) = \frac{e^{L\Lambda - e^{\Lambda}/2}}{2^L \Gamma(L)} \tag{A.4}$$

From the PDF given in Eqn. A.4, a characteristic function can be computed. By definition, the characteristic function (CF) $\varphi_X(t)$ for a random variable X is computed as:

$$\begin{aligned}\varphi_X(t) = \mathrm{E}\left[e^{itX}\right] &= \int_{-\infty}^{\infty} e^{itx}\, dF_X(x) \\ &= \int_{-\infty}^{\infty} e^{itx} f_X(x)\, dx\end{aligned}$$

with $\varphi_x(t)$ is the characteristic function, $F_X(x)$ is the CDF function of X and $f_X(x)$ is the PDF function of X. Thus the characteristic function for the log-chi-squared distribution is defined as:

$$\varphi_\Lambda(t) = \int_0^{\infty} e^{itx} \frac{e^{Lx - e^x/2}}{2^L \Gamma(L)}\, dx \tag{A.5}$$

The Gamma function is defined over the complex domain as: $\Gamma(z) = \int_0^\infty e^{-x} x^{z-1} dx$. Thus $\Gamma(L+it) = \int_0^\infty e^{-x} x^{L+it-1} dx$. Set $x = e^z/2$ then $dx = e^z/2dz$, we have $\Gamma(L+it) = \int_0^\infty e^{itz} \frac{e^{Lz - e^z/2}}{2^{L+it}} dz$

That is:

$$\varphi_\Lambda(t) = 2^{it} \frac{\Gamma(L+it)}{\Gamma(L)} \tag{A.6}$$

Consequently, the first and second derivative of log-chi-squared distribution can be computed. The first derivative is given as:

$$\frac{\partial \varphi_\Lambda(t)}{\partial t} = \frac{i 2^{it} \Gamma(L+it)}{\Gamma(L)} \left[\ln 2 + \psi^0(L+it)\right] \tag{A.7}$$

due to

$$\begin{aligned}
\frac{\partial \Gamma(x)}{\partial x} &= \Gamma(x)\psi^0(x), \\
\frac{\partial \Gamma(L+it)}{\partial t} &= i\Gamma(L+it)\psi^0(L+it), \\
\frac{\partial 2^{it}}{\partial t} &= i2^{it}\ln(2), \\
\partial(u\cdot v)/\partial t &= u\cdot\partial v/\partial t + v\cdot\partial u/\partial t,
\end{aligned}$$

where $\psi^0()$ denotes the di-gamma function.

Meanwhile, the second derivative can be written as:

$$\frac{\partial^2 \varphi_\Lambda(t)}{\partial t^2} = \frac{i^2 2^{it}\Gamma(L+it)}{\Gamma(L)}\Big(\big[\ln 2 + \psi^0(L+it)\big]^2 + \psi^1(L+it)\Big) \tag{A.8}$$

due to:

$$\begin{aligned}
\frac{d2^{it}\Gamma(L+it)}{dt} &= i2^{it}\Gamma(L+it)\left[\ln 2 + \psi^0(L+it)\right], \\
\frac{d\psi^0(t)}{dt} &= \psi^1(t), \\
\frac{d\psi^0(L+it)}{dt} &= i\psi^1(L+it), \\
\partial(u\cdot v)/\partial t &= u\cdot\partial v/\partial t + v\cdot\partial u/\partial t,
\end{aligned}$$

with $\psi^1()$ denotes the tri-gamma function.

The n^{th} moments of random variable X can be computed from the derivatives of its characteristic function as:

$$\mathrm{E}\left(\Lambda^n\right) = i^{-n}\,\varphi_\Lambda^{(n)}(0) = i^{-n}\,\left[\frac{d^n}{dt^n}\varphi_\Lambda(t)\right]_{t=0} \tag{A.9}$$

Thus

$$\begin{aligned}
\mathrm{E}\left(\Lambda\right) &= i^{-1}\left[\frac{d\varphi_\Lambda(t)}{dt}\right]_{t=0} \\
&= i^{-1}\left[\frac{i2^{it}\Gamma(L+it)}{\Gamma(L)}\left[\ln 2 + \psi^0(L+it)\right]\right]_{t=0}
\end{aligned}$$

That is

$$avg(\Lambda) = \psi^0(L) + ln(2) \tag{A.10}$$

Similarly,

$$\begin{aligned} \mathrm{E}\left(\Lambda^2\right) &= i^{-2}\left[\frac{d^2\varphi_\Lambda(t)}{dt^2}\right]_{t=0} \\ &= \left[\frac{2^{it}\Gamma(L+it)}{\Gamma(L)}\left(\left[\ln 2 + \psi^0(L+it)\right]^2 + \psi^1(L+it)\right)\right]_{t=0} \end{aligned}$$

That is:

$$E(\Lambda^2) = \left[\psi^0(L) + \ln(2)\right]^2 + \psi^1(L) \tag{A.11}$$

Thus

$$var(\Lambda) = E(\Lambda^2) - E^2(\Lambda) = \psi^1(L) \tag{A.12}$$

A.1 Averages and Variances of POLSAR Covariance Matrix Determinant and Log-Determinant

In this section, the expected value and variance value of these mixture of random variables is derived

$$\chi_L^d \sim \prod_{i=0}^{d-1} \chi(2L-2i) \tag{A.13}$$

$$\Lambda_L^d \sim \sum_{i=0}^{d-1} \Lambda(2L-2i) \tag{A.14}$$

given the averages and variances of individual components.

$$avg\left[\chi(2L)\right] = 2L \tag{A.15}$$

$$var\left[\chi(2L)\right] = 4L \tag{A.16}$$

$$avg\left[\Lambda(2L)\right] = \psi^0(L) + \ln 2 \tag{A.17}$$

$$var\left[\Lambda(2L)\right] = \psi^1(L) \tag{A.18}$$

Making use of the mutual indepent property of each component X_i, the variance and expectation of the sumation and product of random variables can be written as:

$$\begin{aligned}
avg\left(\sum_{i=1}^{n} X_i\right) &= \sum_{i=1}^{n} avg(X_i), \\
var\left(\sum_{i=1}^{n} X_i\right) &= \sum_{i=1}^{n} var(X_i), \\
avg\left(\prod_{i=1}^{n} X_i\right) &= \prod_{i=1}^{n} avg(X_i), \\
var\left(\prod_{i=1}^{n} X_i\right) &= \prod_{i=1}^{n} \left[avg^2(X_i) + var(X_i)\right] - \prod_{i=1}^{n} avg^2(X_i).
\end{aligned}$$

Thus they can be computed as:

$$\begin{aligned}
avg\left[\chi_L^d\right] &= 2^d \cdot \prod_{i=0}^{d-1}(L-i), \\
var\left[\chi_L^d\right] &= \prod_{i=0}^{d-1} 4(L-i)(L-i+1) - \prod_{i=0}^{d-1} 4(L-i)^2, \\
avg\left[\Lambda_L^d\right] &= d \cdot \ln 2 + \sum_{i=0}^{d-1} \psi^0(L-i), \\
var\left[\Lambda_L^d\right] &= \sum_{i=0}^{d-1} \psi^1(L-i)
\end{aligned}$$

A.2 Deriving the Characteristic Functions for the Consistent Measures of Distance

Given that the characteristic function (CF) of the elementary log-chi square distributions can be written as:

$$CF_{\Lambda(2L)}(t) = 2^{it}\Gamma(L+it)/\Gamma(L)$$

the CF for the following random variables, which are combinations of the above elementary random variables, can be derived

$$
\begin{aligned}
\Lambda_L^d &\sim \sum_{i=0}^{d-1} \Lambda(2L-2i) \\
\mathbb{L} &\sim \Lambda_L^d - d \cdot \ln(2L) \\
\mathbb{D} &\sim \mathbb{L} - d \cdot \ln L + \sum_{i=0}^{d-1} \psi^0(L-i) \\
\mathbb{C} &\sim \sum_{i=0}^{d-1} [\Lambda(2L-2i) - \Lambda(2L-2i)]
\end{aligned}
$$

Since

$$
\begin{aligned}
CF_{\sum X_i}(t) &= \prod CF_{X_i}(t) \\
CF_{x+k}(t) &= e^{itk} CF_x(t)
\end{aligned}
$$

we have:

$$
CF_{\chi_L^d}(t) = \frac{2^{idt}}{\Gamma(L)^d} \prod_{j=0}^{d-1} \Gamma(L-j+it) \tag{A.19}
$$

$$
CF_{\mathbb{L}} = \frac{1}{L^{idt}\Gamma(L)^d} \prod_{j=0}^{d-1} \Gamma(L-j+it) \tag{A.20}
$$

$$
CF_{\mathbb{D}} = \frac{1}{\Gamma(L)^d} \prod_{j=0}^{d-1} e^{idt\psi^0(L-j)} \Gamma(L-j+it) \tag{A.21}
$$

Also due to

$$
\begin{aligned}
CF_{-\Lambda(2L)}(t) &= 2^{-it} \frac{\Gamma(L-it)}{\Gamma(L)} \\
\Delta(2L) &\sim \Lambda(2L) - \Lambda(2L) \\
\Gamma(L-it)\Gamma(L+it) &= \Gamma(2L)B(L-it, L+it) \\
CF_{\Delta(2L)}(t) &= \frac{\Gamma(2L)B(L-it, L+it)}{\Gamma^2(L)}
\end{aligned}
$$

we arrive at:

$$CF_{\mathbb{C}} \quad = \quad \prod_{j=0}^{d-1} \frac{\Gamma(2L-2j)B(L-j-it, L-j+it)}{\Gamma^2(L-j)} \tag{A.22}$$

with $\Gamma()$ and $B()$ denotes Gamma and Beta functions respectively.

A.3 SAR intensity as a special case of POLSAR covariance matrix determinant

In this appendix, the following results for SAR intensity I are shown to be special cases of the results given in this thesis for the determinant of POLSAR's covariance matrix $det|C_v|$. Specifically, not only the following results from chapter 3, i.e. $d = L = 1$, is reviewed:

$$I \sim \bar{I} \cdot pdf\left[e^{-R}\right] \tag{A.23}$$

$$\log_2 I \sim \log_2 \bar{I} + pdf\left[2^{D-2^D}\right] \tag{A.24}$$

$$\frac{I}{\bar{I}} = \mathbb{R} \sim pdf\left[e^{-R}\right] \tag{A.25}$$

$$\log_2 I - \log_2 \bar{I} = \mathbb{D} \sim pdf\left[2^D e^{-2^D} \ln 2\right] \tag{A.26}$$

$$\log_2 I_1 - \log_2 I_2 = \mathbb{C} \sim pdf\left[\frac{2^c}{(1+2^c)^2} \ln 2\right] \tag{A.27}$$

$$avg(\mathbb{D}) = -\gamma / \ln 2 \tag{A.28}$$

$$var(\mathbb{D}) = \frac{\pi^2}{6} \frac{1}{\ln^2 2} \tag{A.29}$$

$$mse(\mathbb{D}) = \frac{1}{\ln^2 2}(\gamma^2 + \pi^2/6) = 4.1161 \tag{A.30}$$

but also the following well-known results for multi-look SAR, i.e. $d = 1, L > 1$ is also considered:

$$I \sim pdf\left[\frac{L^L I^{L-1} e^{-LI/\bar{I}}}{\Gamma(L)\bar{I}^L}\right] \tag{A.31}$$

$$N = \ln I \sim pdf\left[\frac{L^L}{\Gamma(L)} e^{L(N-\bar{N}) - Le^{N-\bar{N}}}\right] \tag{A.32}$$

It will be shown that all of these results are special cases of the result derived previously and rewritten below:

$$|C_v| \sim \frac{|\Sigma_v|}{(2L)^d} \prod_{i=0}^{d-1} \chi^2(2L-2i) \tag{A.33}$$

$$\ln|C_v| \sim \ln|\Sigma_v| + \sum_{i=0}^{d-1} \Lambda(2L-2i) - d \cdot \ln 2L \tag{A.34}$$

$$\frac{|C_v|}{|\Sigma_v|} = \mathbb{R} \sim \frac{1}{(2L)^d} \prod_{i=0}^{d-1} \chi^2(2L-2i) \tag{A.35}$$

$$\ln|C_v| - \ln|\Sigma_v| = \mathbb{D} \sim \sum_{i=0}^{d-1} \Lambda(2L-2i) - d \cdot \ln 2L \tag{A.36}$$

$$\ln|C_{1v}| - \ln|C_{2v}| = \mathbb{C} \sim \sum_{i=0}^{d-1} \Delta(2L-2i) \tag{A.37}$$

$$avg(\mathbb{D}) = \sum_{i=0}^{d-1} \psi^0(L-i) - d \cdot \ln L \tag{A.38}$$

$$var(\mathbb{D}) = \sum_{i=0}^{d-1} \psi^1(L-i) \tag{A.39}$$

$$mse(\mathbb{D}) = \left[\sum_{i=0}^{d-1} \psi^0(L-i) - d \cdot \ln L\right]^2 + \sum_{i=0}^{d-1} \psi^1(L-i) \tag{A.40}$$

This appendix also derives new results for multi-look SAR data, which can be thought of either as extensions of the corresponding single-look SAR results or as simple cases of the POLSAR results presented above. They are:

$$\frac{I}{\bar{I}} = \mathbb{R} \sim \frac{1}{2L} \chi^2(2L) \tag{A.41}$$

$$\ln I - \ln \bar{I} = \mathbb{D} \sim \Lambda(2L) - \ln 2L \tag{A.42}$$

$$\ln I_1 - \ln I_2 = \mathbb{C} \sim \Delta(2L) \tag{A.43}$$

$$avg(\mathbb{D}) = \psi^0(L) - \ln L \tag{A.44}$$

$$var(\mathbb{D}) = \psi^1(L) \tag{A.45}$$

$$mse(\mathbb{D}) = \left[\psi^0(L) - \ln L\right]^2 + \psi^1(L) \tag{A.46}$$

The derivation process detailed below consists of two-phases. The first phase collapses the generic multi-dimensional POLSAR results into the classical one-dimensional SAR domain. Mathematically this means setting the dimensional number in POLSAR to $d = 1$ and collapsing the POLSAR covariance matrix into the variance measure in SAR, which also equals the SAR intensity i.e. $|C_v| = I, |\Sigma_v| = \bar{I}$.

The output of the first phase, in the general case, is applicable to multi-look SAR data, where $d = 1$ but $L > 1$. The second phase simplifies the multi-look results into single-look results. Mathematically, that means setting $L = 1$ in the multi-look result and converting from natural logarithmic domain to the base-2 logarithm used in 3.

A.4 Original Domain: SAR Intensity and its ratio

Setting $d = 1$, $|C_v| = I$ and $|\Sigma_v| = \bar{I}$ into Eqns. A.33 and A.35 we have:

$$\begin{aligned} I &\sim \frac{\bar{I}}{2L}\chi^2(2L) \\ \frac{I}{\bar{I}} = \mathbb{R} &\sim \frac{1}{2L}\chi^2(2L) \end{aligned}$$

Or in PDF forms, and applying the variable change theorem we have:

$$\begin{aligned} \frac{2LI}{\bar{I}} &\sim pdf\left[\frac{x^{L-1}e^{-x/2}}{2^L\Gamma(L)}\right] \\ \frac{I}{\bar{I}} &\sim pdf\left[\frac{x^{L-1}e^{-x/2}}{2^L\Gamma(L)} \cdot dx/dt\right]_{x=2L\cdot t} \\ &\sim pdf\left[\frac{L^L t^{L-1}e^{-Lt}}{\Gamma(L)}\right] \\ I &\sim pdf\left[\frac{L^L t^{L-1}e^{-Lt}}{\Gamma(L)} \cdot dt/dx\right]_{t=x/\bar{I}} \\ &\sim pdf\left[\frac{L^L x^{L-1}e^{-Lx/\bar{I}}}{\bar{I}^L\Gamma(L)}\right] \end{aligned}$$

Thus we have the following results for multi-look SAR

$$I \sim pdf\left[\frac{L^L x^{L-1} e^{-Lx/\bar{x}}}{\bar{I}^L \Gamma(L)}\right] \tag{A.47}$$

$$\frac{I}{\bar{I}} = \mathbb{R} \sim pdf\left[\frac{L^L x^{L-1} e^{-Lx}}{\Gamma(L)}\right] \tag{A.48}$$

Now setting $L = 1$, these results become:

$$I \sim pdf\left[\frac{e^{x/\bar{I}}}{\bar{I}}\right] \tag{A.49}$$

$$\frac{I}{\bar{I}} = \mathbb{R} \sim pdf\left[e^{-x}\right] \tag{A.50}$$

which is the same as in chapter 3.

A.5 Log-transformed domain: SAR log-intensity and the log-distance

The result for multi-look SAR data written in log-transformed domain can be derived from two different approaches. The first is to follow the simplification method, where the results for log-transformed POLSAR data are simplified into a log-transformed multi-look SAR result.

The second approach is to apply log-transformation into the results derived in the previous section. In this section, it is shown that both approaches would result in identical results.

Setting $d = 1$, $|C_v| = I$ and $|\Sigma_v| = \bar{I}$ into Eqns. A.34 and A.36 we have

$$\begin{aligned} \ln I &\sim \ln \bar{I} + \Lambda(2L) - \ln 2L \\ \ln I - \ln \bar{I} = \mathbb{L} &\sim \Lambda(2L) - \ln 2L \end{aligned}$$

Or in PDF form, and applying variable change theorem we have:

$$\begin{aligned}
\ln I - \ln \bar{I} + \ln 2L &\sim pdf\left[\frac{e^{Lx-e^x/2}}{2^L\Gamma(L)}\right] \\
\ln I - \ln \bar{I} &\sim pdf\left[\frac{e^{Lx-e^x/2}}{2^L\Gamma(L)}\cdot dx/dt\right]_{x=t+\ln 2L} \\
&\sim pdf\left[\frac{L^L e^{Lt-Le^t}}{\Gamma(L)}\right] \\
\ln I &\sim pdf\left[\frac{L^L e^{Lt-Le^t}}{\Gamma(L)}\cdot dt/dx\right]_{t=x-\ln \bar{I}} \\
&\sim pdf\left[\frac{L^L e^{L(x-\bar{N})-Le^{x-\bar{N}}}}{\Gamma(L)}\right]
\end{aligned}$$

with $\bar{N} = \ln \bar{I}$.

Thus the first approach arrives at

$$\ln I = \mathbb{N} \sim pdf\left[\frac{L^L e^{L(x-\bar{N})-Le^{x-\bar{N}}}}{\Gamma(L)}\right] \quad \text{(A.51)}$$

$$\ln I - \ln \bar{I} = \mathbb{L} \sim pdf\left[\frac{L^L e^{Lt-Le^t}}{\Gamma(L)}\right] \quad \text{(A.52)}$$

In the second approach, log-transformation is applied on previous result for multi-look SAR intensity and its ratio in the original domain (Eqns. A.48 and A.47). This also arrives at the same results as above, the detailed working however is omitted here for brevity.

To compute summary statistics for the multi-look SAR dispersion, set $d = 1$ into the Eqns. A.40, A.38 and A.39 we have:

$$\begin{aligned}
avg(\mathbb{L}) &= \psi^0(L) - \ln L \\
var(\mathbb{L}) &= \psi^1(L) \\
mse(\mathbb{L}) &= \left[\psi^0(L) - \ln L\right]^2 + \psi^1(L)
\end{aligned}$$

This completes the first phase of the derivation process. The second phase of simplification involves setting $L = 1$ into the above results for multi-look SAR data, and converting natural logarithms into base-2 logarithms. First, setting $L = 1$ makes the above results become

$$\begin{aligned}
\ln I = \mathbb{N} &\sim pdf\left[e^{(x-\bar{N})-e^{x-\bar{N}}}\right] \\
\ln I - \ln \bar{I} = \mathbb{L} &\sim pdf\left[e^{x-e^{x}}\right] \\
avg(\mathbb{L}) &= \psi^{0}(1) = -\gamma \\
var(\mathbb{L}) &= \psi^{1}(1) = \pi^{2}/6 \\
mse(\mathbb{L}) &= \left[\psi^{0}(1)\right]^{2} + \psi^{1}(1) = \gamma^{2} + \pi^{2}/6
\end{aligned}$$

with γ denotes the Euler-Mascharoni constant. Then to convert to base-2 logarithm from natural logarithmic transformation, the variable change theorem is invoked. That is:

$$\begin{aligned}
\log_2 I = \mathbb{N}_2 &\sim pdf\left[e^{(x-\bar{N})-e^{x-\bar{N}}} \cdot dx/dt\right]_{x=t\cdot\ln 2} \\
\mathbb{N}/\ln 2 = \mathbb{N}_2 &\sim pdf\left[e^{(t\cdot\ln 2-\bar{N})-e^{t\cdot\ln 2-\bar{N}}} \ln 2\right]_{\bar{N}_2=\bar{N}\cdot\ln 2} \\
&\sim pdf\left[2^{t-\bar{N}_2} e^{2^{t-\bar{N}_2}} \ln 2\right]
\end{aligned}$$

$$\begin{aligned}
\log_2 I - \log_2 \bar{I} = \mathbb{L}/\ln 2 = \mathbb{L}_2 &\sim pdf\left[e^{x-e^{x}}\right]_{x=t\cdot\ln 2} \\
&\sim pdf\left[2^{t} e^{2^{t}} \ln 2\right]
\end{aligned}$$

$$\begin{aligned}
avg(\mathbb{L}_2) &= avg(\mathbb{L})/\ln 2 = -\gamma/\ln 2 \\
var(\mathbb{L}_2) &= var(\mathbb{L})/\ln^2 2 = \frac{\pi^2}{6}\frac{1}{\ln^2 2} \\
mse(\mathbb{L}_2) &= mse(\mathbb{L})/\ln^2 2 = \frac{1}{\ln^2 2}(\gamma^2 + \pi^2/6) = 4.1161
\end{aligned}$$

A.6 Deriving the PDF for SAR dispersion and contrast

The PDF for SAR dispersion can be easily derived from the PDF for the Log-distance given above as:

$$\ln I - avg(\ln I) = \mathbb{D} \sim pdf\left[\frac{e^{L[x+\psi^0(L)]-Le^{x+\psi^0(L)-\ln L}}}{\Gamma(L)}\right] \tag{A.53}$$

due to $d = 1$ and

$$\begin{aligned} \mathbb{D} &\sim \mathbb{L} - avg(\mathbb{L}) \\ avg(\mathbb{L}) &= \psi^0(L) - \ln L \\ \mathbb{L} &\sim pdf\left[\frac{L^L e^{Lt-Le^t}}{\Gamma(L)}\right] \end{aligned}$$

.

Setting $L = 1$ for Single-Look SAR we have

$$\mathbb{D} \sim pdf\left[e^{x-\gamma-e^{x-\gamma}}\right] \tag{A.54}$$

due to: $\psi^0(1) = -\gamma$ and $\Gamma(1) = 1$ with γ being the Euler Mascheroni Constant which equals 0.5772. In base-2 logarithm, the variable change theorem is invoked

$$\begin{aligned} \mathbb{D}_2 &= \log_2 I - avg(\log_2 I) = \mathbb{D}/\ln 2 \\ \mathbb{D}_2 &\sim pdf\left[e^{x-\gamma-e^{x-\gamma}} \cdot \frac{dx}{dt}\right]_{x=t\cdot\ln 2} \end{aligned}$$

Thus we have

$$\mathbb{D}_2 \sim pdf\left[e^{-(2^x e^{-\gamma})}(2^x e^{-\gamma})\ln 2\right] \tag{A.55}$$

which is consistent to the result in chapter 3.

Setting $d = 1$ into Eqn. for contrast result in

$$\ln I_1 - \ln I_2 = \mathbb{C} \sim \Delta(2L) \tag{A.56}$$

The characteristic function would then be

$$CF_{\mathbb{C}} = \frac{\Gamma(2L)B(L-it, L+it)}{\Gamma(L)^2} \tag{A.57}$$

Thus PDF can be written as

$$\mathbb{C} \sim pdf\left[\frac{\Gamma(2L)}{\Gamma(L)^2}\frac{e^{Lx}}{(1+e^x)^{2L}}\right] \tag{A.58}$$

due to

$$\begin{aligned}
CF_{\mathbb{C}}(x) &= \frac{\Gamma(2L)}{\Gamma(L)^2}B(1/(1+e^x), L-it, L+it) \\
&= \frac{\Gamma(2L)}{\Gamma(L)^2}\int_0^{1/(1+e^x)} z^{L-it-1}(1-z)^{L+it-1}dz \\
\frac{\partial}{\partial x}CF_{\mathbb{C}}(x) &= \frac{\partial CF_{\mathbb{C}}(x)}{\partial 1/(1+e^x)} \cdot \frac{\partial 1/(1+e^x)}{\partial x} \\
&= e^{itx}\frac{\Gamma(2L)}{\Gamma(L)^2}\frac{e^{Lx}}{(1+e^x)^{2L}}
\end{aligned}$$

Setting $L = 1$ into Eqn. A.58 we have the PDF for contrast of single-look SAR:

$$\mathbb{C} \sim pdf\left[\frac{e^x}{(1+e^x)^2}\right] \tag{A.59}$$

Converting to base-2 logarithm we have

$$\begin{aligned}
\mathbb{C}/\ln 2 = \mathbb{C}_2 &\sim pdf\left[\frac{e^x}{(1+e^x)^2} \cdot dx/dt\right]_{x=t\cdot\ln 2} \\
&\sim pdf\left[\ln 2\frac{2^t}{(1+2^t)^2}\right]
\end{aligned}$$

which is also consistent to the result in chapter 3.