Introduction to stabilizer theory

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1 Stabilizer definition

In quantum mechanics, a **state** is given by a **vector**, and an **operator** is given by a **matrix**. The state of N qubits is a 1×2^N vector, and an operation on the state is a $2^N \times 2^N$ matrix.

For example, a state of 1 qubit could be

$$|+\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

and an operation on it could be

$$X = \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

In linear algebra, if for a matrix M there is a **vector V** and a **scalar** v such that

$$\mathrm{M}.\mathbf{V} = v.\mathbf{V}$$

then **V** is an **eigenvector** of M and v is the associated **eigenvalue**.

The vector **V** can be the eigenvector of more than one matrix. The vector **V** can be **fully defined** by the set $\{\mathrm{M}, v\}$ of matrices that **V** is an eigenvector of.

The **stabilizers** of a state $|\psi\rangle$ are the operators $\frac{1}{v}.\mathrm{M}$ where $|\psi\rangle$ is an eigenvector of M with eigenvalue v.

We see that

$$X|+\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = |+\rangle$$

so X is a stabilizer of $|+\rangle$.

Now look at

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad , \quad |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

These are eigenstates of

$$Z = \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Their eigenvalues are 1 and -1. The stabilizer of $|0\rangle$ is Z and the stabilizer of $|1\rangle$ is $-Z$.

A state of N qubits has N stabilizers. For example, a 2-qubit state could have the stabilizers X_1X_2 and Z_1Z_2, where

$$X_1X_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

$$Z_1Z_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

2 Properties of stabilizers

All stabilizers of a given state **commute**. The operators X and Z **do not** commute:

$$X_1Z_1|\psi\rangle = -Z_1X_1|\psi\rangle$$

However, if X and Z stabilize **different** qubits then they **do** commute:

$$X_1Z_2|\phi\rangle = Z_2X_1|\phi\rangle$$

The stabilizers X_1X_2 and Z_1Z_2 **do** commute:

$$(X_1X_2)(Z_1Z_2)|\phi\rangle = (X_1Z_1)(X_2Z_2)|\phi\rangle = (-Z_1X_1)(-Z_2X_2)|\phi\rangle = (Z_1Z_2)(X_1X_2)|\phi\rangle$$

All operators **commute with themselves**.

3 Calculating with stabilizers

There are two ways that stabilizers change:

- **Operating** on the state with an operator
- **Measuring** one or more qubits

3.1 Operating on stabilizers

An operation M changes the state $|\psi\rangle$:

$$\mathrm{M}|\psi\rangle = |\psi'\rangle$$

The stabilizers of the state change by **operator multiplication by M**:

$$[\text{stabilizers of } |\psi'\rangle] = \mathrm{M}.[\text{stabilizers of } |\psi\rangle].\mathrm{M}$$

For example, we start with the state stabilized by X_1X_2 and Z_1Z_2. We operator on this state by X_1. Then the final stabilizers are

$$X_1X_2 \longrightarrow (X_1)(X_1X_2)(X_1) = (X_1X_1X_1)X_2 = X_1X_2$$

and

$$Z_1Z_2 \longrightarrow (X_1)(Z_1Z_2)(X_1) = (X_1Z_1X_1)Z_2 = (-X_1X_1Z_1)Z_2 = -Z_1Z_2$$

where we have used the fact that

$$X.X = Z.Z = \mathbb{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

3.2 Measuring the state

When we **measure** a state, two things happen:

- The set of stabilizers changes
- We get a measurement outcome (0 or 1)

For example, take the set of stabilizers on **3 qubits**:

$$\begin{aligned} Z_1Z_2Z_3 \\ X_1X_2 \\ X_2X_3 \end{aligned}$$

Now we measure the operator Z_2. We get the outcome $t = 0$ or $t = 1$. Then we make the new stabilizers:

1. The **first new stabilizer** is the operator we measured and the outcome:

$$(-1)^t Z_2$$

2. The **next new stabilizers** are all the old stabilizers that **commute with the measurement operator**. Here, that is

$$Z_1Z_2Z_3$$

3. Now take the **first old stabilizer that does not commute with the measurement operator**. Here that is X_1X_2 Then multiply each of the remaining old stabilizers by this one, to get the **rest of the new stabilizers**. Here that is

$$(X_1X_2)(X_2X_3) = X_1(X_2X_2)(X_3) = X_1X_3$$

The new stabilizers for this state are then

$$\begin{aligned} Z_2 \\ Z_1Z_2Z_3 \\ X_1X_3 \end{aligned}$$

4 Stabilizers and the density operator

The **density operator** is another way of writing the quantum state of a system. It is the matrix ρ formed from the state vector $|\psi\rangle$:

$$\rho = |\psi\rangle\langle\psi|$$

For example, if the state vector is $|+\rangle$ then the the density matrix is

$$\rho_+ = \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}(1,1) \;=\; \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

The numbers down the diagonal are the probabilities of measuring the system in the $|0\rangle$ and $|1\rangle$ state respectively.

If we have a set of stabilizers then we can reconstruct the density matrix. We can write any unitary matrix as a weighted sum of **Pauli group matrices** of the same dimension. The Pauli group is formed from the tensor product of the Pauli matrices $P_0 = \mathbb{1}$, $P_1 = X$, $P_2 = Z$ and $P_3 = Y = XZ$. In other words, the Pauli group matrices form a **basis** for any unitary matrix, including the density operator. For example, for a two-qubit system.

$$\rho = \sum_{i,j=0}^{3} a_{ij}P_{ij}$$

The key is now to find which values of a_{ij} are nonzero. For density matrices, $a_{0\ldots0}$ is always $1/2^N$, where N is the number of qubits. Now let us look again at the definition of a stabilizer S:

$$S|\psi\rangle = |\psi\rangle \Rightarrow S|\psi\rangle\langle\psi|S^\dagger = S\rho S^\dagger$$

where $S^\dagger$ is the **adjoint matrix** of S. If we write ρ as a sum of matrices that all mutually commute, then

$$S(A+B+C+D)S^\dagger = (A+B+C+D) \Rightarrow S = A \text{ or } B \text{ or } C \text{ or } D$$

That is, the by adding all the stabilizers of the system together we get the density matrix.

We note that not all stabilizers of a system are generally given when defining a state. All quantum states have $\mathbb{1} \otimes \ldots \otimes \mathbb{1}$ as a stabilizer. Also, any **product of stabilizers is also a stabilizer**. We need to add all of these to get the density matrix, and then normalize.

For example, suppose we are told that the stabilizers of a state are XX and ZZ. Then density matrix is then

$$\begin{aligned}
\rho &= \frac{1}{2}\left(\mathbb{1} \otimes \mathbb{1} + XX + ZZ + (XX)(ZZ)\right) \\
&= \frac{1}{2}\left(\mathbb{1} \otimes \mathbb{1} + \sigma_x \otimes \sigma_x + \sigma_z \otimes \sigma_z + \sigma_y \otimes \sigma_y\right) \\
&= \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}
\end{aligned}$$

which is indeed the density matrix for the state $\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$.

5 Note

The examples here are all **Clifford-group** stabilizers. If we operate on a set of stabilizers with a **complicated operator M**, then they can become **sums of operators** with **different co-efficients**.

For example, a state can have the **non-Clifford-group** stabilizer

$$\alpha(Z_1 Z_2, X_1 X_2) + \beta(-Z_1 Z_2, X_1 X_2)$$