A Proofs of Lemmas

Lemma 1. *The* future-reach $fr(\varphi)$ *of an STL formula* φ *is defined by:*

$$fr(p) = 0 \qquad fr(\varphi \wedge \varphi') = \max(fr(\varphi), fr(\varphi'))$$
$$fr(\neg\varphi) = fr(\varphi) \quad fr(\varphi \, \mathbf{U}_I \, \varphi') = \sup(I) + \max(fr(\varphi), fr(\varphi')).$$

For any bound $\tau > t + fr(\varphi)$*,* $\sigma, t \models_\tau \varphi$ *iff* $\sigma, t \models_\infty \varphi$*, where* $\mathrm{dom}(\sigma) \supseteq [0, \tau)$*.*

Proof. The proof is by structural induction on φ.
– ($\varphi = p$): $\sigma, t \models_\tau p$ iff $t < \tau$ and $p(\sigma(t)) = \mathit{true}$ iff, because $t = t + fr(p) < \tau$, $t < \infty$ and $p(\sigma(t)) = \mathit{true}$ iff $\sigma, t \models_\infty p$.
– ($\varphi = \neg\phi$): $\sigma, t \models_\tau \neg\phi$ iff $\sigma, t \not\models_\tau \phi$ iff, because $\tau > t + fr(\neg\phi) = t + fr(\phi)$, by induction hypothesis, iff $\sigma, t \not\models_\infty \phi$ iff $\sigma, t \models_\infty \neg\phi$.
– ($\varphi = \phi \wedge \phi'$): Since $\tau > t + fr(\phi \wedge \phi') = t + \max(fr(\phi), fr(\phi'))$, $\tau > t + fr(\phi)$ and $\tau > t + fr(\phi')$. Therefore, $\sigma, t \models_\tau \phi \wedge \phi'$ iff $\sigma, t \models_\tau \phi$ and $\sigma, t \models_\tau \phi'$ iff, by induction hypothesis, $\sigma, t \models_\infty \phi$ and $\sigma, t \models_\infty \phi'$ iff $\sigma, t \models_\infty \phi \wedge \phi'$.
– ($\varphi = \phi \, \mathbf{U}_I \, \phi'$): Since $\tau > t + fr(\phi \, \mathbf{U}_I \, \phi') = t + \sup(I) + \max(fr(\phi), fr(\phi'))$, for any $u \leq t + \sup(I)$, $\tau > u + fr(\phi)$ and $\tau > u + fr(\phi')$. Let $t' - t \in I$. Clearly, $t' \leq t + \sup(I)$, and for $t'' \in [t, t']$, $t'' \leq t + \sup(I)$. Therefore, $\sigma, t \models_\tau \phi \, \mathbf{U}_I \, \phi'$ iff $(\exists t' \geq t)\ \ t' < \tau$, $t' - t \in I$, $\sigma, t' \models_\tau \phi'$, $(\forall t'' \in [t, t'])\ \sigma, t'' \models_\tau \phi$ iff, by the above observation and induction hypothesis, $(\exists t' \geq t)\ \ t' < \infty$, $t' - t \in I$, $\sigma, t' \models_\infty \phi'$, $(\forall t'' \in [t, t'])\ \sigma, t'' \models_\infty \phi$ iff $\sigma, t \models_\infty \phi \, \mathbf{U}_I \, \phi'$. □

Lemma 2. *Given a signal* σ*, a proposition* p*, and an interval* $J \subseteq \mathrm{dom}(\sigma)$*: (1) If* J *is open and* p *is stable for* J*, then there is no variable point in* J*. (2) If there is no variable point in* J*, then* p *is stable for* J*.*

Proof. (1) Each time point $t \in J$ is not a variable point, because t has a stable open neighborhood in J, namely, J itself.
(2) Suppose that p is not stable for J. Then, there exist different time points $u < u'$ in J such that $p(\sigma(u)) \neq p(\sigma(u'))$. Let

$$K_u = \{t \in J \mid p(\sigma(t)) = p(\sigma(u))\}.$$

Observe that $u \leq \sup(K_u) \leq u'$. Therefore, $\sup(K_u) \in J$. Furthermore, for any $t \in J$ with $t > \sup(K_u)$, $p(\sigma(t)) = p(\sigma(u'))$, but any open neighborhood of $\sup(K_u)$ intersects with K_u. Therefore, $\sup(K_u)$ is not a variable point. □

Lemma 3. *For a signal* σ *and an interval* $J \subseteq \mathrm{dom}(\sigma)$*: (1)* φ *is stable for* J *iff* $\neg\varphi$ *is stable for* J*. (2) If* φ *and* φ' *are stable for* J*, then* $\varphi \wedge \varphi'$ *is stable for* J*.*

Proof. (1) φ is stable for J, iff $\sigma, u \models_\tau \varphi \iff \sigma, u' \models_\tau \varphi$ for any $u, u' \in J$, iff $\sigma, u \not\models_\tau \varphi \iff \sigma, u' \not\models_\tau \varphi$ for any $u, u' \in J$, iff $\sigma, u \models_\tau \neg\varphi \iff \sigma, u' \models_\tau \neg\varphi$ for any $u, u' \in J$, iff $\neg\varphi$ is stable for J.
(2) Suppose that φ and φ' are stable for J. Then, $\sigma, u \models_\tau \varphi \iff \sigma, u' \models_\tau \varphi$, and $\sigma, u \models_\tau \varphi' \iff \sigma, u' \models_\tau \varphi'$, for any $u, u' \in J$. Therefore, $\sigma, u \models_\tau \varphi$ and $\sigma, u \models_\tau \varphi'$ iff $\sigma, u' \models_\tau \varphi$ and $\sigma, u' \models_\tau \varphi'$. Consequently, $\sigma, u \models_\tau \varphi \wedge \varphi'$ iff $\sigma, u' \models_\tau \varphi \wedge \varphi'$, for any $u, u' \in J$. □

Lemma 12. *[4, Lemma 4.3] For intervals I and J of nonnegative real numbers, $t \in J - I$ iff $(\exists t' \geq t)\ t' \in J$ and $t' \in t + I$.*

Proof. $(\Rightarrow)$ By definition, $t \in J - I$ iff $t = j - i$ for some $j \in J$ and $i \in I$, where $i, j \geq 0$. Observe that $j \geq t$ and $j = t + i \in t + I$.
$(\Leftarrow)$ Since $t' \in t+I$, $t' = t+i$ for some $i \in I$. Since $t' \in J$, $t = t' - i \in J - I$. $\square$

Lemma 13. *For intervals K, J, and I, if either $K \subseteq J - I$ or $K \subseteq (J - I)^{\complement}$ holds, then for any $u_1, u_2 \in K$, $u_1 \in J - I$ iff $u_2 \in J - I$.*

Proof. Suppose $u_1 \in J - I$. Since $u_1 \in K$, $K \cap (J - I) \neq \emptyset$. Because either $K \subseteq J - I$ or $K \subseteq (J - I)^{\complement}$ holds, we must have $K \subseteq J - I$. As a consequence, $u_2 \in K \subseteq J - I$. The other direction is exactly the same. $\square$

Lemma 4. *An STL formula $\varphi \, \mathbf{U}_I \, \varphi'$ is stable for a partition $\mathcal{Q} = \{K_j\}_{j \in [M]}$ of time domain $[0, \tau)$, if: (i) φ and φ' are stable for a partition $\mathcal{P} = \{J_i\}_{i \in [N]}$, (ii) either $K_j \subseteq J_i - I$ or $K_j \subseteq (J_i - I)^{\complement}$ for any K_j and J_i, and (iii) $\mathcal{Q} \sqsubseteq \mathcal{P}$.*

Proof. Let $u_1, u_2 \in K_j$. By definition, $\sigma, u_1 \models_\tau \varphi \, \mathbf{U}_I \, \varphi'$ iff $(\exists t_1' \geq u_1)\ t_1' < \tau$, $t_1' \in u_1 + I$, $\sigma, t_1' \models_\tau \varphi'$, and $(\forall t_1'' \in [u_1, t_1'])\ \sigma, t_1'' \models_\tau \varphi$. Suppose $t_1' \in J_i$. Then:

$$\begin{aligned} & (\exists t_1' \geq u_1)\ t_1' < \tau,\ t_1' \in u_1 + I, \text{ and } t_1' \in J_i \\ \text{iff} \ \ & (\exists t_1' \geq u_1)\ t_1' \in J_i \cap [0, \tau) \text{ and } t_1' \in u_1 + I \\ \text{iff} \ \ & u_1 \in (J_i \cap [0, \tau)) - I \ \text{ iff } \ u_1 \in J_i - I, \end{aligned}$$

because of Lemma 12 and $J_i \subseteq [0, \tau)$. By Lemma 13, $u_2 \in J_i - I$. Similarly, $u_2 \in J_i - I$ iff $(\exists t_2' \geq u_2)\ t_2' < \tau$, $t_2' \in u_2 + I$, and $t_2' \in J_i$. Thus, $t_1', t_2' \in J_i$. Since φ and φ' are stable for $\mathcal{P}$, $\sigma, t_1' \models_\tau \varphi'$ iff $\sigma, t_2' \models_\tau \varphi'$. Since $\mathcal{Q} \sqsubseteq \mathcal{P}$, u_1 and u_2 are elements of the same interval in $\mathcal{P}$. Thus, $(\forall t_1'' \in [u_1, t_1'])\ \sigma, t_1'' \models_\tau \varphi$ iff $(\forall t_2'' \in [u_2, t_2'])\ \sigma, t_2'' \models_\tau \varphi$. Therefore, $\sigma, u_1 \models_\tau \varphi \, \mathbf{U}_I \, \varphi'$ iff $\sigma, u_2 \models_\tau \varphi \, \mathbf{U}_I \, \varphi'$. $\square$

Lemma 5. *[4, Lemma 4.18] For a finite set $T \subseteq [0, \tau)$ and an interval I, let $T_I = \bigcup_{u \in T \cup \{\tau\}} \{u - v \mid v \in e(I)\} \cap [0, \tau)$. For partitions $\mathcal{P}_\tau(T) = \{J_i\}_{i \in [N]}$ and $\mathcal{P}_\tau(T_I) = \{K_j\}_{j \in [M]}$, $K_j \subseteq J_i - I$ or $K_j \subseteq (J_i - I)^{\complement}$ holds for any K_j and J_i.*

Proof. Let $T_I \cup \{0, \tau\} = \{u_0, \ldots, u_{m+1}\}$, where $0 = u_0 < \cdots < u_m < u_{m+1} = \tau$. Suppose that the lemma does not hold for K_j in $\mathcal{P}_\tau(T_I)$ and J_i in $\mathcal{P}_\tau(T)$; that is, K_j intersects both $J_i - I$ and $(J_i - I)^{\complement}$. Then, K_j must include an endpoint of $J_i - I$. Observe that $T_I \cup \{0, \tau\}$ contains all such endpoints. By definition, K_j is either $\{u_i\}$ or (u_i, u_{i+1}), and only $\{u_i\}$ can include an endpoint in $T_I \cup \{0, \tau\}$. However, $\{u_i\} \subseteq J_i - I$ and $\{u_i\} \subseteq (J_i - I)^{\complement}$ cannot hold at the same time. $\square$

Lemma 6. *Given a signal σ, an STL formula φ, and a finite set $T \subseteq [0, \tau)$, if φ is stable for $\mathcal{P}_\tau(T)$, there exists a minimal subset $min^\sigma_\varphi(T) \subseteq T$ such that for any strict subset $U \subset min^\sigma_\varphi(T)$, φ is not stable for $\mathcal{P}_\tau(U)$.*

Proof. Let $T \cup \{0, \tau\} = \{u_0, \ldots, u_{m+1}\}$, $0 = u_0 < \cdots < u_m < u_{m+1} = \tau$. Let $min^\sigma_\varphi(T) = \{u_i \in T \mid \sigma, u_i \models_\tau \varphi$ is not equal to $\sigma, v_i \models_\tau \varphi$ or $\sigma, v_{i-1} \models_\tau \varphi\}$, where $v_i = (u_i + u_{i+1})/2$, $0 \leq i \leq m$. Observe that φ is stable for $\mathcal{P}_\tau(min^\sigma_\varphi(T))$ and for any strict subset $U \subset min^\sigma_\varphi(T)$, φ cannot be stable for $\mathcal{P}_\tau(U)$. $\square$

Lemma 7. *[4, Proposition 3.10] For intervals K_1 and K_2, where $K_1 \cap K_2 = \emptyset$ and $\sup(K_1) = \inf(K_2)$, $\varphi\,\mathbf{U}_I^{K_1 \cup K_2}\,\varphi' \;\equiv\; \varphi\,\mathbf{U}_I^{K_1}\,\varphi' \,\vee\, (\Box_{\geq 0}^{K_1}\,\varphi \wedge \varphi\,\mathbf{U}_I^{K_2}\,\varphi')$.*

Proof. Let σ be a signal, $t \geq 0$, and $K = K_1 \cup K_2$. Then, $\sigma, t \models_\tau \varphi\,\mathbf{U}_I^K\,\varphi'$ iff $(\exists t' \geq t)\; t' \in [0,\tau) \cap K, t' \in t + I, \sigma, t' \models_\tau \varphi', (\forall t'' \in [t,t'] \cap K)\; \sigma, t'' \models_\tau \varphi$. Because $t' \in [0,\tau) \cap K$ iff $t' \in [0,\tau) \cap (K_1 \cup K_2)$ iff either $t' \in [0,\tau) \cap K_1$ or $t' \in [0,\tau) \cap K_2$, the statement can be equivalently rewritten as:

$$\begin{aligned} &(\exists t' \geq t)\; t' \in [0,\tau) \cap K_1, t' \in t + I, \sigma, t' \models_\tau \varphi', (\forall t'' \in [t,t'] \cap K)\; \sigma, t'' \models_\tau \varphi \\ \text{or}\;\; &(\exists t' \geq t)\; t' \in [0,\tau) \cap K_2, t' \in t + I, \sigma, t' \models_\tau \varphi', (\forall t'' \in [t,t'] \cap K)\; \sigma, t'' \models_\tau \varphi \end{aligned}$$

When $t' \in K_1$, because $t' \leq \inf(K_2)$ and $K_1 \cap K_2 = \emptyset$, $[t,t'] \cap K = [t,t'] \cap K_1$. When $t' \in K_2$, because $\sup(K_1) \leq t' < \tau$, $[t,t'] \cap K = ([t,t'] \cap K_2) \cup ([t,\tau) \cap K_1)$. Therefore, the above statement can be equivalently rewritten as:

$$\begin{aligned} &(\exists t' \geq t)\; t' \in [0,\tau) \cap K_1, t' \in t + I, \sigma, t' \models_\tau \varphi', (\forall t'' \in [t,t'] \cap K_1)\; \sigma, t'' \models_\tau \varphi \\ \text{or}\;\; &(\exists t' \geq t)\; t' \in [0,\tau) \cap K_2, t' \in t + I, \sigma, t' \models_\tau \varphi', (\forall t'' \in [t,t'] \cap K_2)\; \sigma, t'' \models_\tau \varphi, \\ &\quad (\forall t'' \in [t,\tau) \cap K_1)\; \sigma, t'' \models_\tau \varphi \end{aligned}$$

Observe that $(\forall t'' \in [t,\tau) \cap K_1)\; \sigma, t'' \models_\tau \varphi$ iff $\sigma, t \models_\tau \Box_{\geq 0}^{K_1}\,\varphi$. Therefore, the above is equivalent to: $\sigma, t \models_\tau \varphi\,\mathbf{U}_I^{K_1}\,\varphi'$ or $(\sigma, t \models_\tau \varphi\,\mathbf{U}_I^{K_2}\,\varphi'$ and $\sigma, t \models_\tau \Box_{\geq 0}^{K_1}\,\varphi)$. Consequently, $\sigma, t \models_\tau \varphi\,\mathbf{U}_I^{K}\,\varphi'$ iff $\sigma, t \models_\tau \varphi\,\mathbf{U}_I^{K_1}\,\varphi' \,\vee\, (\Box_{\geq 0}^{K_1}\,\varphi \wedge \varphi\,\mathbf{U}_I^{K_2}\,\varphi')$. □

Lemma 8. *For a partition $\mathcal{P} = \{J_i\}_{i \in [N]}$ and its time sample $\{v_i\}_{i \in [N]}$, where φ and φ' are stable for $\mathcal{P}$, let $L_k = \bigcup_{j=k}^N J_k$ for $k \in [N]$. (1) $\sigma, v_i \models_\tau \varphi\,\mathbf{U}_I^{L_k}\,\varphi'$ iff $\sigma, v_i \models_\tau \varphi\,\mathbf{U}_I^{J_k}\,\varphi' \vee (\Box_{\geq 0}^{J_k}\,\varphi \wedge \varphi\,\mathbf{U}_I^{L_{k+1}}\,\varphi')$. (2) $\sigma, v_i \models_\tau \varphi\,\mathbf{U}_I^{J_k}\,\varphi'$ iff $v_i \in J_k - I$, $\sigma, v_k \models_\tau \varphi$, and $\sigma, v_k \models_\tau \varphi'$. (3) $\sigma, v_i \models_\tau \Box_{\geq 0}^{J_k}\,\varphi$ iff $\sigma, v_k \models_\tau \varphi$.*

Proof. (1) Because $L_k = J_k \cup L_{k+1}$ for $k \in [N]$, by Lemma 7, $\sigma, v_i \models_\tau \varphi\,\mathbf{U}_I^{L_k}\,\varphi'$ iff $\sigma, v_i \models_\tau \varphi\,\mathbf{U}_I^{J_k}\,\varphi' \,\vee\, (\Box_{\geq 0}^{J_k}\,\varphi \wedge \varphi\,\mathbf{U}_I^{L_{k+1}}\,\varphi')$.
(2) $\sigma, v_i \models_\tau \varphi\,\mathbf{U}_I^{J_k}\,\varphi'$ iff $(\exists t' \geq v_i)\; t' \in [0,\tau) \cap J_k, t' \in v_i + I, \sigma, t' \models_\tau \varphi'$, $(\forall t'' \in [v_i,t'] \cap K)\; \sigma, t'' \models_\tau \varphi$ iff, because $J_k \subseteq [0,\tau)$, and because φ and φ' are stable for J_k, by Lemma 12, $v_i \in J_k - I$, $\sigma, v_k \models_\tau \varphi$, and $\sigma, v_k \models_\tau \varphi'$.
(3) $\sigma, v_i \models_\tau \Box_{\geq 0}^{J_k}\,\varphi$ iff $\neg((\exists t' \geq v_i)\; t' \in [0,\tau) \cap J_k, t' \in v_i + [0,\infty), \sigma, t' \models_\tau \neg\varphi)$ iff, since φ is stable for J_k and $v_i \in [0,\tau) \cap J_k - [0,\infty)$, by Lemma 12, $\sigma, v_k \models_\tau \varphi$. □

Lemma 14. *Suppose that φ and φ' are stable for $\mathcal{P} = \{J_i\}_{i \in [N]}$, $\chi_\varphi^k = \top$ iff $\sigma, v_k \models_\tau \varphi$, $\chi_{\varphi'}^k = \top$ iff $\sigma, v_k \models_\tau \varphi'$, and $L_k = \bigcup_{j=k}^N J_k$, for each $k \in [N]$. For $\mathcal{P}$'s time sample $\{v_i\}_{i \in [N]}$, for $i \leq k \leq N$, $tr_i(\varphi\,\mathbf{U}_I\,\varphi', k) = \top$ iff $\sigma, v_i \models_\tau \varphi\,\mathbf{U}_I^{L_k}\,\varphi'$.*

Proof. The proof is by induction on $N - k$. When $k = N$, $tr_i(\varphi\,\mathbf{U}_I\,\varphi', N) = \top$ iff $(v_i \in J_N - I \wedge \chi_\varphi^N \wedge \chi_{\varphi'}^N) \vee (\chi_\varphi^N \wedge \bot)$ iff $v_i \in J_N - I \wedge (\sigma, v_N \models_\tau \varphi) \wedge (\sigma, v_N \models_\tau \varphi')$ iff, by Lemma 8, $\sigma, v_i \models_\tau \varphi\,\mathbf{U}_I^{J_N}\,\varphi'$ iff $\sigma, v_i \models_\tau \varphi\,\mathbf{U}_I^{L_N}\,\varphi'$. When $k = n + 1$, $tr_i(\varphi\,\mathbf{U}_I\,\varphi', n) = \top$ iff $(v_i \in J_n - I \wedge \chi_\varphi^n \wedge \chi_{\varphi'}^n) \vee (\chi_\varphi^n \wedge tr_i(\varphi\,\mathbf{U}_I\,\varphi', n+1))$ iff, by Lemma 8 and induction hypothesis, $(v_i \in J_n - I \wedge \sigma, v_n \models_\tau \varphi \wedge \sigma, v_n \models_\tau \varphi')$ $\vee\, (\sigma, v_n \models_\tau \varphi \wedge (\sigma, v_i \models_\tau \varphi\,\mathbf{U}_I^{L_{n+1}}\,\varphi'))$ iff, by Lemma 8, $\sigma, v_i \models_\tau \varphi\,\mathbf{U}_I^{J_n}\,\varphi' \,\vee$ $(\sigma, v_i \models_\tau \Box_{\geq 0}^{J_n}\,\varphi \wedge (\sigma, v_i \models_\tau \varphi\,\mathbf{U}_I^{L_{n+1}}\,\varphi'))$ iff, by Lemma 8, $\sigma, v_i \models_\tau \varphi\,\mathbf{U}_I^{L_n}\,\varphi'$. □

Lemma 9. *Suppose that each $\psi \in \mathrm{sub}(\varphi)$ of an STL formula φ is stable for a partition $\mathcal{P} = \{J_i\}_{i\in[N]}$. The following conditions are equivalent: (i) $\chi^i_\psi = \top$ iff $(\forall t \in J_i)\ \sigma, t \models_\tau \psi$. (ii) every condition in $\Gamma_\varphi(\mathcal{P}, \mathcal{X}) \cup \Gamma_\Pi(\sigma, \mathcal{S}_\mathcal{P}, \mathcal{X})$ holds.*

Proof. Let $\{v_i\}_{i\in[N]}$ be a time sample of $\mathcal{P}$, and $L_k = \bigcup_{j=k}^{N} J_k$ for each $k \in [N]$. First, $\sigma, v_i \models_\tau \psi$ iff $(\forall t \in J_i)\ \sigma, t \models_\tau \psi$, because each subformula $\psi \in \mathrm{sub}(\varphi)$ is stable for $\mathcal{P}$. The proof is by structural induction on φ.

– ($\varphi = p$): ($\Rightarrow$) $\chi^i_p = \top$ iff $\sigma, v_i \models_\tau p$ iff $v_i < \tau \wedge p(\sigma(v_i)) = \top$ iff, because $v_i < \tau, p(\sigma(v_i)) = \top$. Therefore, the condition $\chi^i_p = p(\sigma(v_i))$ holds. ($\Leftarrow$) Suppose $\chi^i_p = p(\sigma(v_i))$ holds. Then, $\chi^i_p = \top$ iff $p(\sigma(v_i)) = \top$ iff $(\forall t \in J_i)\ \sigma, t \models_\tau p$.

– ($\varphi = \neg\phi$): ($\Rightarrow$) $\chi^i_{\neg\phi} = \top$ iff $\sigma, v_i \models_\tau \neg\phi$ iff $\sigma, v_i \not\models_\tau \phi$ iff, by induction hypothesis, $\chi^i_\phi = \bot$. Thus, $\chi^i_{\neg\phi} = \neg\chi^i_\phi$. ($\Leftarrow$) Suppose $\chi^i_{\neg\phi} = \neg\chi^i_\phi$. Then, $\chi^i_{\neg\phi} = \top$ iff $\chi^i_\phi = \bot$ iff, by induction hypothesis, $\sigma, v_i \not\models_\tau \phi$ iff $(\forall t \in J_i)\ \sigma, t \models_\tau \neg\phi$.

– ($\varphi = \phi \wedge \phi'$): ($\Rightarrow$) $\chi^i_{\phi\wedge\phi'} = \top$ iff $\sigma, v_i \models_\tau \phi \wedge \phi'$ iff $\sigma, v_i \models_\tau \phi$ and $\sigma, v_i \models_\tau \phi'$ iff, by induction hypothesis, $\chi^i_\phi = \top$ and $\chi^i_{\phi'} = \top$. Therefore, $\chi^i_{\phi\wedge\phi'} = \chi^i_\phi \wedge \chi^i_{\phi'}$. ($\Leftarrow$) Suppose $\chi^i_{\phi\wedge\phi'} = \chi^i_\phi \wedge \chi^i_{\phi'}$. Then, $\chi^i_{\phi\wedge\phi'} = \top$ iff $\chi^i_\phi \wedge \chi^i_{\phi'} = \top$ iff, by induction hypothesis, $\sigma, v_i \models_\tau \phi$ and $\sigma, v_i \models_\tau \phi'$ iff $\sigma, v_i \models_\tau \phi\wedge\phi'$ iff $(\forall t \in J_i)\ \sigma, t \models_\tau \phi\wedge\phi'$.

– ($\varphi = \phi \,\mathbf{U}_I\, \phi'$): ($\Rightarrow$) $\chi^i_{\phi \mathbf{U}_I \phi'} = \top$ iff $\sigma, v_i \models_\tau \phi \,\mathbf{U}_I\, \phi'$ iff, because $v_i \in L_i$, $\sigma, v_i \models_\tau \phi \,\mathbf{U}_I^{L_i}\, \phi'$ iff, by induction hypothesis and Lemma 14, $tr_i(\phi \,\mathbf{U}_I\, \phi', i) = \top$. Thus, $\chi^i_{\phi \mathbf{U}_I \phi'} = tr_i(\phi \,\mathbf{U}_I\, \phi', i)$ holds. ($\Leftarrow$) Suppose $\chi^i_{\phi \mathbf{U}_I \phi'} = tr_i(\phi \,\mathbf{U}_I\, \phi', i)$ holds. Then, $\chi^i_{\phi \mathbf{U}_I \phi'} = \top$ iff $tr_i(\phi \,\mathbf{U}_I\, \phi', i) = \top$ iff, by induction hypothesis and Lemma 14, $\sigma, v_i \models_\tau \phi \,\mathbf{U}_I^{L_i}\, \phi'$ iff $\sigma, v_i \models_\tau \phi \,\mathbf{U}_I\, \phi'$ iff $(\forall t \in J_i)\ \sigma, t \models_\tau \phi \,\mathbf{U}_I\, \phi'$. $\square$

Lemma 10. *For a formula φ and a finite set $V \subseteq [0, \tau)$, if $V = \mathcal{T}^\sigma_\varphi(T)$ for a set of variable points T, then every condition in $\Omega_\varphi(V, \mathcal{X}, \sigma)$ holds.*

Proof. By Theorem 1, each subformula of φ is stable for $\mathcal{T}^\sigma_\varphi(T)$. By Lemma 9, $\Gamma_\varphi(\mathcal{P}_\tau(V), \sigma[\mathcal{S}_{\mathcal{P}_\tau(V)}]))$ holds for $V = \mathcal{T}^\sigma_\varphi(T)$, and $\chi^i_\psi = \top$ iff $(\forall t \in J_i)\ \sigma, t \models_\tau \psi$ for each $\psi \in \mathrm{sub}(\varphi)$. Suppose $k = n+1 \vee \bigvee_{j=1,2}(\chi^{2k}_{\phi_j} \neq \chi^{2k+2}_{\phi_j})$. Then, $t_k = \tau$ or the truth of ϕ_j changes at time t_k; i.e., for $U = min^\sigma_\phi(\mathcal{I}(\phi)) \cup min^\sigma_{\phi'}(\mathcal{I}(\phi')) \cup \{\tau\}$ in Def. 9, $t_k \in U$. By construction, $\bigcup_{u\in U}\{u - v \mid v \in e(I)\} \cap [0, \tau) \subseteq V$. Thus, the condition $v = 0 \vee t_k - v < 0 \vee \bigvee_{j=1}^{k-1}(t_k - v = t_j)$ holds. $\square$

Lemma 11. *If every condition in $\Omega_\varphi(V, \mathcal{X}, \sigma)$ holds, then each $\psi \in \mathrm{sub}(\varphi)$ is stable for $\mathcal{P}_\tau(V)$, provided that each $p \in \mathrm{Props}(\varphi)$ is stable for $\mathcal{P}_\tau(V)$.*

Proof. By structural induction on φ. The case of $\varphi = p$ is by assumption. The cases of $\varphi = \neg\phi$ and $\varphi = \phi\wedge\phi'$ are immediate by induction hypothesis. Consider $\varphi = \phi \,\mathbf{U}_I\, \phi'$. By assumption, the condition $(k = n+1 \vee \bigvee_{j=1,2}(\chi^{2k}_{\phi_j} \neq \chi^{2k+2}_{\phi_j})) \rightarrow (v = 0 \vee t_k - v < 0 \vee \bigvee_{j=1}^{k-1}(t_k - v = t_j))$ in $\Delta_\varphi(V)$ holds for each $t_k \in V \cup \{0, \tau\}$ and $v \in e(I)$. By induction hypothesis, any subformula of ϕ and ϕ' is stable for $\mathcal{P}_\tau(V)$. By Lemma 9, $\chi^k_{\phi_j} = \top$ iff $(\forall t \in J_k)\ \sigma, t \models_\tau \phi_j$ for $j = 1, 2$. Thus, the premise of the above condition holds if either $t_k = \tau$ or the truth value of ϕ_j changes at time t_k, and in that case, $t_k - v \in V$, provided $t_k - v \geq 0$. Therefore, by Lemma 4, Lemma 5, and Corollary 1, $\phi \,\mathbf{U}_I\, \phi'$ is stable for $\mathcal{P}_\tau(V)$. $\square$