CDO, Copula and Correlation

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Main Topics

In this lecture we will:

1. Briefly review Collateralized Debt Obligation(CDO) from the perspective of pricing problem
2. Derive pricing formula for a synthetic Collateralized Debt Obligation
3. Introduce the basic concept and mathematical properties of copula models
4. Simulate default times using Gaussian copula
5. Introduce one factor model for credit risk modelling
6. Derive analytic solution for large homogenous portfolio based on one factor model
7. Analyse the impact of default correlation under asymptotic one factor model

Take Away

By the end of this lecture you will be able to

- Understand basic concept and mathematics of copula model
- How to implement Gaussian and / or t copula model to generate correlated default times
- Price Collateralized Debt Obligationand general credit derivatives using copula models
- Understand what roles default correlation play in credit risk models
- Apply factor models to price Collateralized Debt Obligationor general credit derives
- Be able to derive analytical solution for credit derives based on asymptotic one factor model

What is CDO?

CDO is a type of asset backed security, it is an investment on a pool of diversified assets (bonds, loans, CDS etc) in the form of tranched securities. Instead of directly selling the asset pool, the sponsor of CDO (banks, non-financial institutions, asset management companies) creates an independent legal entity called Special Purpose Vehicle(SPV) who repacks the asset pool and slices it into tranches according to the underlying asset pool's credit risk.

Typical CDO tranche

The tranches of a typical CDO are

- Senior tranche(AAA)
- Mezzanine tranche(AA to BB)
- Equity tranche(unrated)

The position of each tranche within the CDO capital structure is determined by its attachment and detachment points.

How CDO Works

- At good times, tranche Investors will receive premium (returns from asset pool) in turn periodically
- At bad times, when the total pool loss reaches the attachment point, investors in the tranche start to lose their capital, and when the total pool loss reaches the detachment point, the investors in the tranche lose all their capital and no further loss can occur to them
- Loss will be applied in reverse order of seniority
- The senior tranche is protected by the Mezzanine tranche and equity tranche; while mezzanine tranche is protected only by equity tranche

Motivation

First issued in the late 1980s, CDO emerged a decade later as the fastest growing sector in financial market.Banks that are actively trading CDO are motivated by two reasons, they attempt to

- explore arbitrage opportunities. The majority of CDO are arbitrage motivated
- off load credit risk from their loan book, and hence to reduce their regulatory capital requirement and improve return on risk capital

Cash CDO

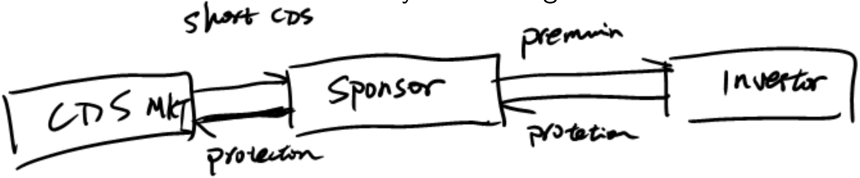
CDO is a broad term that can refer to several different types of products, it can be categorized in several ways. From pricing point of view we are interested in distinguish CDO with respect of its funding,i.e., cash CDO or synthetic CDO.

Cash CDO involves a portfolio of cash assets, such as corporate bonds, loans, etc. Ownership of the assets is transferred to a SPV who is the issuer of CDO.

Synthetic CDO

If the SPV of a CDO does not own the physical asset pool, instead obtaining the credit risk exposure by selling CDSs on the reference portfolio, the CDO is referred to as a synthetic CDO.

Synthetic CDO can be unfunded, which means investors only pay when their tranches are affected by defaults. In this case, counterparty default risk must be taken into account by risk managers.



CDO Market

Overall, the CDO market consists of

- an illiquid segment, for example cash CDO "buy-to-hold" investors whose investment decisions are mainly based on ratings and yields
- an actively traded segment in which the underlying credit portfolio is based on the standardized credit default swapindex (CDX) such as the iTraxx (European) or CDX (North American) index. This market are mostly traded by the correlation desks of hedge funds and banks

Because the net cash flows of index tranches are the same as synthetic CDO tranches and these tranches can be priced the same way as a synthetic CDO.

CDO Valuation and its Risk Management I

In the illiquid segment, CDO pricing and risk measurement are difficult.

- Due to lack of liquidity, hedging and market-to-market are generally unavailable
- A cash CDO often has complex waterfall structure so that the cashflow of its tranches are highly path-dependent, and it is generally actively managed, so structural and cashflow analysis are complicated. Plus key parameters like default correlation is difficult to estimate, so market-to-model is very weak
- Traditionally investors rely on rating agencies for their valuation and risk assessment, but since credit crunch 2007, rating agencies credibility has been severely undermined

CDO Valuation and its Risk management II

However in the synthetic CDO market

- the cashflows of synthetic CDO tranches are very simple and not path-dependent;
- market participants can easily take long or short positions in the CDX and index tranches, and the underlying single name CDS
- because of strong arbitrage relationships among them, the basis between indices, tranches or single names CDS in the synthetic market tends to stay within a reasonable range

Therefore the valuation and risk management in this market are more feasible. In this lecture in order to help us focus on credit risk modelling we will price synthetic CDO. In particular, we will treat CDO as a derivative on a portfolio of CDS contracts.

Notation

Let's get familiar with some notations before deriving the pricing equation

- Survival time for reference name i: τ_i
- Loss given default for reference name i: LGD_i
- Exposure at default for reference name i: EAD_i
- Tranche: $[D, U]$
- Settlement date: t_j
- Payment frequency: $\Delta = t_{j+1} - t_j$
- Maturity date: T_M
- Discount factor: $Z(t, T)$

Loss Function

1. the loss for reference name i by time t is $L_i(t)$,
2. the total reference pool loss by time t is $L(t)$,
3. the loss for tranche $[d, u]$ by time t is $L(t; d, u)$.

By definition

$$
\begin{aligned}
L_i(t) &= LGD_i * EAD_i * I\{\tau_i < t\} \\
L(t) &= \sum_{i=1}^{N} L_i(t) \quad / \text{Total EAD} = \sum_{i=1}^{n} EAD \\
\boxed{L(t; u, d) = \max\left[\min\left(L(t), u\right) - d, 0\right]}
\end{aligned}
$$

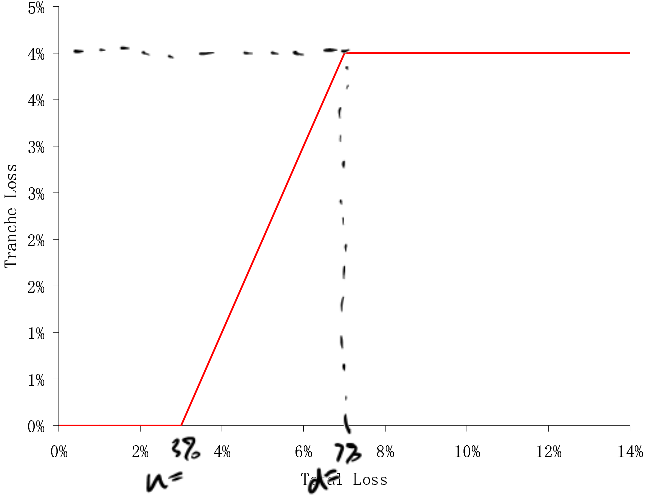
L(t) portfolio. loss Dis^n

$\Rightarrow$ L(t; d, u) Dis^n

Mezzanine Tranche Payoff

4m

u - d
= 7% - 3%
= 4%.



Synthetic CDO Pricing I

The present value of protection leg (assuming paid in arrear) is

$$\sum_{j=1}^{M} Z(0,t_j)\left[\overbrace{L(t_j;d,u) - L(t_{j-1};d,u)}^{\text{marginal loss}}\right], \tag{1}$$

Assuming the fair tranche spread is s, so the present value of premium leg is

$$s\Delta \sum_{j=1}^{M} Z(0,t_j)\left[(u-d) - L(t_j;d,u)\right], \tag{2}$$

s - spread ?

Synthetic CDO Pricing II

Same principle as pricing interest rate swap or credit default swap, present value of both legs must be equal at inception of the contract.

Now take expectation of present value of both legs and equate them, the fair spread s for tranche (d, u) is

$$s = \frac{\mathbb{E}\left\{\sum_{j=1}^{M} Z(0, t_j)\left[L(t_j; d, u) - L(t_{j-1}; d, u)\right]\right\}}{\Delta\mathbb{E}\left\{\sum_{j=1}^{M} Z(0, t_j)\left[(u - d) - L(t_j; d, u)\right]\right\}} \tag{3}$$

Portfolio Loss Distribution

From equation (3), one can see that the key input to the pricing formula is the expected tranche loss which in turn depends on the distribution of entire reference portfolio. To derive this loss distribution, we need know the following:

1. Joint distribution of default times
2. Specification of other parameters for each obligor, i.e. EAD and LGD

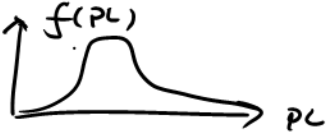
Suppose for synthetic CDO, LGD and EAD are provided for every name, so the remaining task to is to derive the joint distribution function for default times.

What is the Dis^n of $L(t)$

$L(t) = \sum L_i(t) = \sum \mathbb{I}\{\tau_i < t\}$ EAD LGD

?

pdf $f_{L(t)}$



Marginal and Joint Distribution

By definition marginal distribution of a random variable X is

$$F(x) = \Pr(X \leq x),$$

The joint distribution function of two random variables X and Y is

$$F(x, y) = \Pr(X \leq x, Y \leq y).$$

We can model default risk of a credit portfolio if we know joint default distribution function

$$* \quad \boxed{F(t_1, t_2, \cdots, t_n) = \Pr(\tau_1 \leq t_1, \tau_2 \leq t_2, \cdots, \tau_n \leq t_n).}$$

Problem with Joint Distribution

Directly work on joint distribution is inconvenient, because

- Marginal distribution are different, conventional joint distribution only accepts homogenous marginal distribution
- Extension to higher dimension maybe difficult
- Measures of dependence may appear in marginal distribution

Copula Approach

Instead of directly working on joint distribution which must be horrendous, the better way is to use copula function. Unlike joint distribution function, copula function can separate marginal distribution and their association completely, as a result, by copula, one can conveniently mix marginal distributions together with certain dependence structure to become a joint distribution.

Let's see how one can do that.

Definition of Copula

Definition [Copula function]: For k uniform random variables $(U_1, U_2, \cdots, U_k)$, the joint distribution function

$$C(u_1, u_2, \cdots, u_k; \rho)$$

is called a copula function. This definition shows that C is a multivariate distribution function of a bunch of uniform distributed margins.

$U_i \sim$ uniform$[0, 1]$

$u_i = F_i(x_i)$

implies $F_i(x_i) \sim$ uniform$[0, 1]$

X r.v. can any dis^{b}

$F(x) \to$ uniform

Copula Links to Joint Distribution

Since the distribution function of a random variable is uniformed distributed, so copula function can be used to link marginal distribution with a joint distribution.

Theorem: Suppose there are random variables $X_1, X_2, \cdots, X_k$, with distributions $F_1, F_2, \cdots, F_k$, then

$$\boxed{C(F_1(x_1), F_2(x_2), \cdots, F_k(x_k)) = F(x_1, x_2, \cdots, x_k)}$$

.

Sklar Theorem

Sklar proved converse version of above, that is any joint distribution can be written in the form of a copula function, and if the joint distribution is continuous then the copula function is unique.

Sklar's theorem shows that copula function can be used to model dependence structure. For any joint distribution function, the marginal distribution and the dependence structure can be isolated, with the latter completely described by copula.

Copula Density Function

By Sklar's theorem, one can derive the density function of a multivariate copula function.

$$\frac{\partial F(x_1, \cdots x_n)}{\partial x_1 \cdots \partial x_n}$$

$$\begin{aligned} f(x_1, \cdots, x_n) &= \frac{\partial^n \left[C(F_1(x_1), \cdots, F_n(x_n))\right]}{\partial F_1(x_1), \cdots, \partial F_n(x_n)} \prod_{i=1}^{n} f_i(x_i) \\ &= c\left(F_1(x_1), \cdots, F_n(x_n)\right) \prod_{i=1}^{n} f_i(x_i) \end{aligned}$$

So the density function of copula is

$$\boxed{c\left(F_1(x_1), \cdots, F_n(x_n)\right) = \frac{f(x_1, \cdots, x_n)}{\prod_{i=1}^{n} f_i(x_i)}}$$

With the copula density function it is possible to calibrate its parameters to market data via maximum likelihood estimation.

Classification of Copula

There are several families of copula functions, those are often used in quantitative finance due to their tractability are

- Elliptical copulae
  - Gaussian copula
  - t copula
- Archimedean copulae
  - Gumbel Copula
  - Clayton Copula
  - Frank copula

In the application of CDO, we will pay great attentions to Elliptical family, particularly for Gaussian copula.

Multivariate Gaussian Copula

Definition [Gaussian copula] Let Φ_n be the normalized multivariate standard normal distribution function and Φ be univariate standard normal distribution function, the multivariate Gaussian copula function is

$$\boxed{C(u_1, u_2, \cdots, u_n) = \Phi_n\left(\Phi^{-1}(u_1), \Phi^{-1}(u_2), \cdots, \Phi^{-1}(u_n); \Sigma\right)}$$

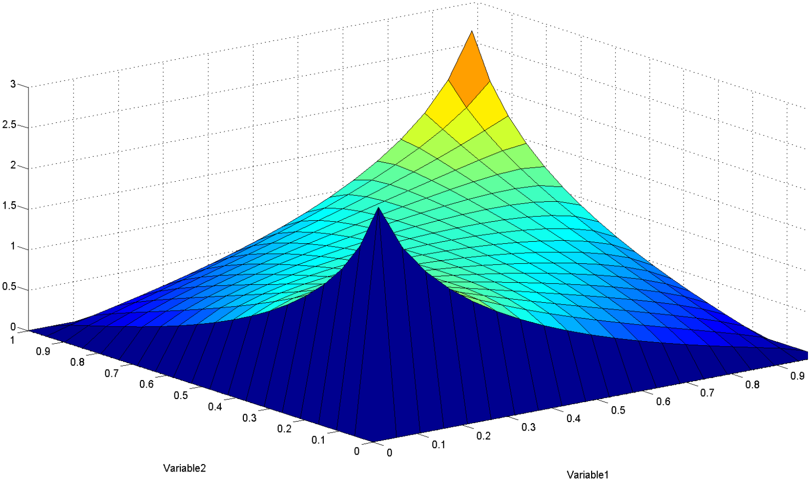
Its density function is

$$c\left(\phi(x_1), \cdots, \phi(x_n)\right) = \frac{\frac{1}{\sqrt{2\pi^n|\Sigma|}}\exp\left(-\frac{1}{2}X'\Sigma^{-1}X\right)}{\prod_{i=1}^{n}\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{1}{2}x_i^2\right)}$$

or in terms of marginal

$$c(u_1, \cdots, u_2) = \frac{1}{\sqrt{|\Sigma|}}\exp\left(-\frac{1}{2}\Phi^{-1}(U')(\Sigma^{-1} - I)\Phi^{-1}(U)\right)$$

Bivariate Gaussian Copula Density ($\rho = 0.5$)



Multivariate Student's t Copula

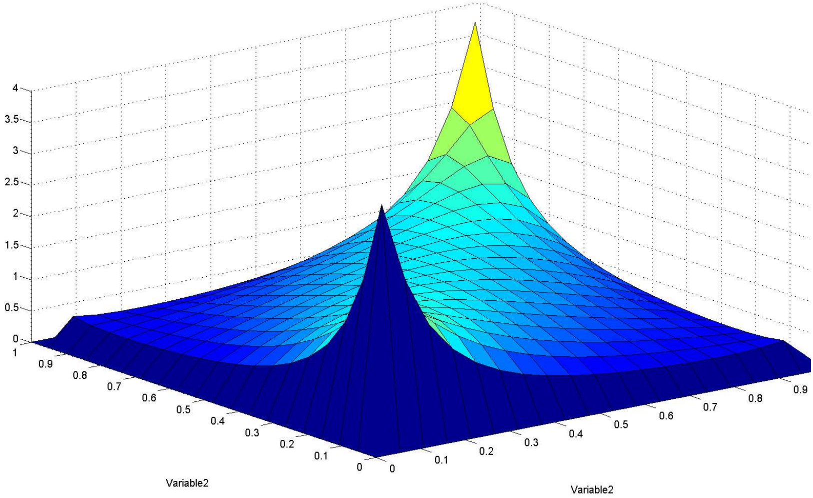
Definition [Student's t Copula]: Let T_v be the normalized multivariate Student's t distribution function with v degrees of freedom, and t_v is the normalized univariate t distribution function also with v degrees of freedom, then the multivariate Student's t Copula is

$$C(u_1, u_2, \cdots, u_n) = T_v\left(t_v^{-1}(u_1), t_v^{-1}(u_2), \cdots, t_v^{-1}(u_n); \Sigma\right)$$

The Student's t copula density function is

$$c(u_1, \cdots, u_2) = \frac{1}{\sqrt{|\Sigma|}} \frac{\Gamma\left(\frac{v+n}{2}\right)}{\Gamma\left(\frac{v}{2}\right)} \left(\frac{\Gamma\left(\frac{v}{2}\right)}{\Gamma\left(\frac{v+1}{2}\right)}\right)^n \frac{\left(1 + \frac{T_v^{-1}(U')\Sigma^{-1} T_v^{-1}(U)}{v}\right)^{-\frac{v+n}{2}}}{\prod_{i=1}^n \left(1 + \frac{T_v^{-1}(u_i)^2}{v}\right)^{-\frac{v+1}{2}}}$$

Bivariate Student's t copula density $\rho = 0.5$, $v = 3$



Gaussian Copula - Simulate Correlated Default Times

In our example of n assets portfolio, we assume, for each name i, the marginal distribution is $F_i(\tau_i)$, and suppose the correlation matrix is successfully estimated, so the normal copula for the joint survival time is

$$\boxed{C\left(F_1(\tau_i), \cdots, F_n(\tau_n)\right) = \Phi_n\left(\Phi^{-1}(F_1(\tau_1)), \cdots, \Phi^{-1}(F_n(\tau_n)); \Sigma\right),}$$

in which we define

$$x_i = \Phi^{-1}(F_i(\tau_i)).$$

We need to find out what τ_i is, $\forall i = 1, 2, \cdots, n$ through normal copula function. The first step is to generate correlated random variables x_is.

David X Li on default correlation
A Copula approach.

How to Simulate Correlated Multivariate Normal Distribution

Short answer is, instead of generating correlated multivariate normal variables directly, one can generate independent normal variables and then convert them into correlated ones according to predetermined correlation matrix.

Let's see how to do it mathematically.

Notation

Let's denote

$$Z^T = (z_1, z_2, \cdots, z_d)$$

be an independent d-dimensional standard normal vector.

The most effective method to create correlated normal vector is just by linearly combining independent normal vector. So introduce a $n \times d$ matrix

$$A = \begin{pmatrix} a_{11} & \dots & a_{1d} \\ \vdots & \ddots & \vdots \\ a_{n1} & \dots & a_{nd} \end{pmatrix},$$

where each row represents a vector of weights allocated to elements in Z.

Correlated Normal

Define a new vector

$$X^T = (x_1, \cdots, x_n),$$

such that

$$X = A\,Z,$$

where

$$x_i = \sum_{j=1}^{d} a_{ij} z_j$$

Vector X is then a linear combination of independent normal vector Z. Note the dimension is changed from d to n.

Covariance Matrix

Define the covariance matrix of X be Σ, then by definition it is,

$$\Sigma = E\left[\begin{pmatrix} x_1^2 & x_1 x_2 & \dots & x_1 x_n \\ \vdots & \dots & \dots & \vdots \\ x_n x_1 & x_n x_2 & \cdots & x_n^2 \end{pmatrix}\right] = E\left[\mathrm{X}\mathrm{X}^T\right]$$

Plug $\mathrm{X} = \mathrm{A}\,\mathrm{Z}$ into above equation come up with

$$\Sigma = E[\mathrm{A}\mathrm{Z}\mathrm{Z}^T\mathrm{A}^T] = \mathrm{A}\mathrm{A}^T$$

So given Covariance matrix, if one can find A, then by multiply independent vector Z by A end up with correlated vector X.

Matrix Factorization

The act of Finding matrix A is called Matrix Factorization or Decomposition.

We all know that non-negative numbers have real square root, whereas negative number doesn't.

Similar result holds for matrices. Any symmetric at least semi-positive definite matrix, like Σ can be factorized. But the solution is not unique.

Methods to decompose covariance matrix

We are going to introduce two popular methods of decomposing correlation matrix,

- Cholesky Factorization
- Spectral Decomposition PCA

Cholesky Factorization

The basic ideal of Cholesky Factorization is very easy, it claims that any symmetric positive definite matrix can be factorized in the form of triangular matrices.

Two Dimension Example I

The best way to see this via looking example. Let's suppose that Σ is a two-dimensional matrix:

$$\Sigma = \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{pmatrix}$$

Cholesky Factorization takes the form:

$$\begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{pmatrix} = \begin{pmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{pmatrix} \times \begin{pmatrix} A_{11} & A_{21} \\ 0 & A_{22} \end{pmatrix}$$

2D Example II

$$\begin{pmatrix} A_{11}^2 & A_{11}A_{21} \\ A_{21}A_{11} & A_{21}^2 + A_{22}^2 \end{pmatrix} = \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{pmatrix}$$

It will end up with 3 equations for 3 unknowns like this:

$$\begin{cases} A_{11}^2 = \sigma_{11} \\ A_{21}A_{11} = \sigma_{12} \\ A_{21}^2 + A_{22}^2 = \sigma_{22} \end{cases}$$

One can solve for A_{ij} sequentially, the answer is.

$$\begin{pmatrix} \sigma_1 & 0 \\ \sigma_2\rho & \sigma_2\sqrt{1-\rho^2} \end{pmatrix}$$

General Algorithm for Multi Dimension

For the case of a d-dimension covariance matrix Σ, we need to solve

$$\begin{pmatrix} A_{11} & & & \\ A_{21} & A_{22} & & \\ \vdots & \vdots & \ddots & \\ A_{d1} & A_{d2} & \cdots & A_{dd} \end{pmatrix} \begin{pmatrix} A_{11} & A_{21} & \cdots & A_{d1} \\ & A_{22} & \cdots & A_{d2} \\ & & \ddots & \vdots \\ & & & A_{dd} \end{pmatrix} =$$

$$\begin{pmatrix} \sigma_{11} & \sigma_{12} & \cdots & \sigma_{1d} \\ & \sigma_{22} & \cdots & \sigma_{2d} \\ & & \ddots & \vdots \\ & & & \sigma_{dd} \end{pmatrix}$$

General algorithm for multi dimension continue 2

Traversing the σ_{ij} by looping over i and then j produces,

$$\begin{aligned}
A_{11}^2 &= \sigma_{11} \\
A_{11}A_{21} &= \sigma_{12} \\
&\vdots \\
A_{11}A_{d1} &= \sigma_{1d} \\
A_{21}^2 + A_{22}^2 &= \sigma_{22} \\
&\vdots \\
A_{21}A_{d1} + A_{22}A_{d2} &= \sigma_{2d}
\end{aligned}$$

Exactly one new entry of the A matrix appears in each equation, making it possible to solve for the individual entries sequentially.

General Algorithm for Multi Dimension Continue III

More compactly from,

$$\sigma_{ij} = \sum_{k=1}^{i} A_{ik} A_{jk} \quad j \geq i,$$

We get have basic identity

$$A_{ji} = \left(\sigma_{ij} - \sum_{k=1}^{i-1} A_{ik} A_{jk} \right) / A_{ii} \quad j \geq i,$$

and

$$A_{ii} = \sqrt{\sigma_{ii} - \sum_{k=1}^{i-1} A_{ik}^2} \quad j = i$$

This formulae make a simply recursion to find Cholesky factor.

Spectral Decomposition I

spectral decomposition mainly relies on the fact that eigenvector of a symmetric matrix is orthogonal to each other.

Based on linear algebra, the spectral decomposition of of a symmetric matrix takes form

$$\Sigma = V \Lambda V^T$$

V is a matrix which collects eigenvectors in its column, and Λ is a diagonal matrix with its diagonal elements are eigenvalues of Σ.

Spectral Decomposition II

If Σ is positive semi-definite, it can be expressed as

$$\Sigma = V\Lambda^{\frac{1}{2}}\Lambda^{\frac{1}{2}}V' = \left(V\Lambda^{\frac{1}{2}}\right)\left(V\Lambda^{\frac{1}{2}}\right)'$$

Thus

$$A = V\Lambda^{\frac{1}{2}}$$

Pros and Cons

Cholesky Factorization has particular structure providing a computational advantage. Spectral Decomposition doesn't have it and hence isn't faster than Cholesky.

In addition to Cholesky's inability to deal with semi-definite matrix, Spectral Decomposition do however have a statistical interpretation that is occasionally useful, that is related to Principal Component Analysis (PCA).

Gaussian Copula Simulation Procedure

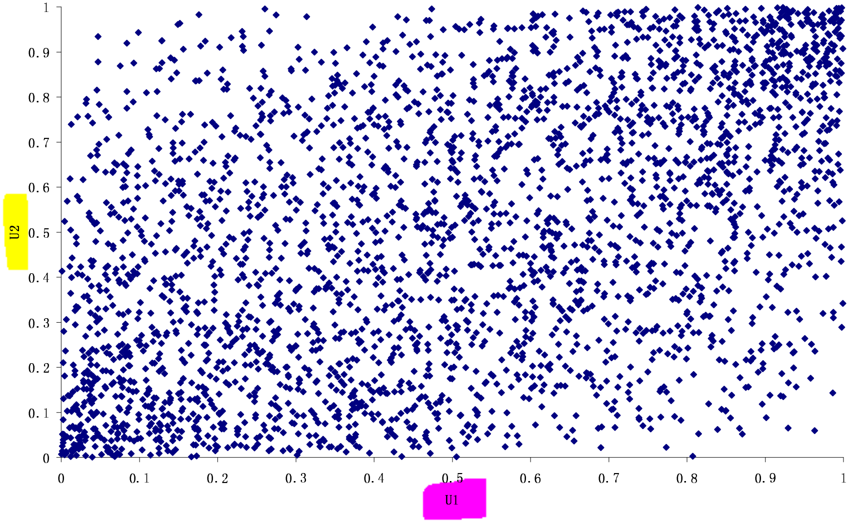
The procedure for generating random default times from normal copula with correlation maxtrix Σ proceeds as follows

1. Find a suitable (e.g. Cholesky) decomposition A from Σ, such that $\Sigma = AA'$
2. Draw a N-dimensional independent standard normal vector $Z = (Z_1, Z_2, \cdots, Z_n)'$
3. Let $X = AZ$ to obtain correlated normal vector
4. Calculate default time use the following equation

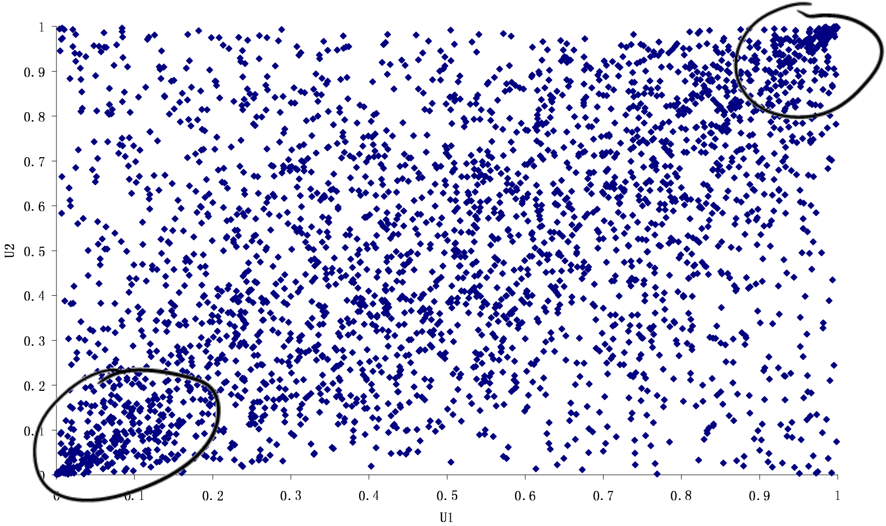
$$\tau_i = F^{-1}\left(\Phi(x_i)\right) \tag{4}$$

5. repeat 2 to 4 many times.

Simulated Bivariate Gaussian Copula ($\rho = 0.5$)



Simulated Bivariate Student's t copula ($\rho = 0.5$, $v = 2$)



Factor Model

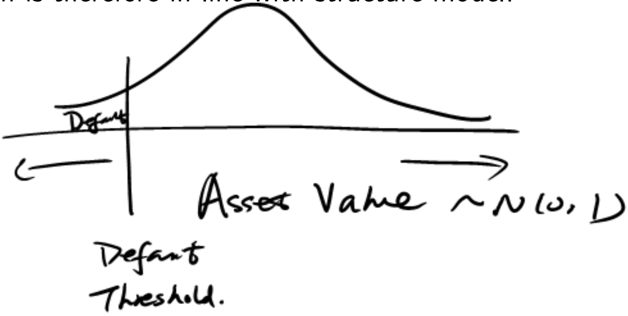
For the time being we have seen copula models which are explicitly based on copula function. Actually there exits another type of copula model which has factor representation, these models have its advantages and are widely used by financial industry.

Factor model does the same job like explicit copula model to obtain joint distribution by mixing marginal distributions with measure of dependence. However, it is more easy to understand and less computation effort.

Asset Value Approach

One way to obtain loss distribution is asset value approach which was initially developed by CreditMetrics in 1990s.

The heart of this method is the assumption that, for each obligor, there exists an latent variable which determines the occurrence of default event. Since the the latent variable is commonly viewed as proxy to asset value, this approach is therefore in line with structure model.



Latent Variable

For each obligor i, there exists a latent variable A_i and its associated threshold d_i such that

$$\begin{aligned} \text{Obligor i default} &\iff A_i \le d_i \\ \text{Obligor i not default} &\iff A_i > d_i \end{aligned} \tag{5}$$

Where

$$A_i = w_i Z + \sqrt{1 - w_i^2}\varepsilon_i \tag{6}$$

$$\text{cov}(\varepsilon_i, \varepsilon_j) = 0, i \neq j; \quad \text{cov}(Z, \varepsilon_i) = 0, \forall i$$

Where Z and ε_i are standard normal variables. So by construction A_i is also standard normal. w_i is called factor loading or sensitivity which ultimately links to correlation.

Default Correlation

Instead of directly imposing correlation structure on default themselves, the asset value approach represents it by imposing structure on latent variables.

Asset correlation

$$\rho_{ij} = \mathrm{cov}(A_i, A_j) = w_i\, w_j \tag{7}$$

Above can be easily shown by following:

$$\begin{aligned}
\mathrm{cov}(A_i, A_j) &= \mathrm{cov}(w_i Z + \cancel{\sqrt{1-w_i^2}\varepsilon_i},\ w_j Z + \cancel{\sqrt{1-w_j^2}\varepsilon_j}) \\
&= \mathrm{cov}(w_i Z,\ w_j Z) = w_i w_j \mathrm{var}(Z) \\
&= w_i w_j \\
&= \rho_A
\end{aligned}$$

$Pr(Z_1 < t, Z_2 < t)$

$= Pr(A_1 < d_1, A_2 < t)$

$= \Phi_2(d_1, d; \rho_A)$

Implementation:Monte Carlo simulation

One can easily implement asset value approach to obtain loss distribution by Monte Carlo simulation

1. Using (6), randomly draw latent variable for each obligor
2. For each obligor check if it defaulted according to (5). If yes, determine individual loss
3. Aggregate individual losses into portfolio loss
4. Repeat steps 1 to 3 many times to arrive at portfolio loss distribution

Homogenous portfolio

A portfolio is homogenous if each name in the portfolio shares the same default probability, recovery rate, correlation and notional principal.

The loss distribution for homogenous portfolio has closed form solution.

Default probability

Recall the one factor normal copula model, the asset value of a firm i is driven by

$$A_i = w_i Z + \sqrt{1 - w_i^2}\,\varepsilon_i. \tag{8}$$

We can drop subscript off since the reference portfolio is homogenous. The default probability for any i is

$$F(t) = \Pr[\tau < t] = \Pr[A < d(t)], = \Phi(d).$$

where $d(t)$ is default threshold.
Because of the assumption of normality across all i

$$F(t) = \Phi\left(d(t)\right),$$

from where the default threshold can be derived

$$\boxed{d(t) = \Phi^{-1}(F(t)).}$$

Conditional default probability

Conditional on common factor Z, the default probability will be

$$
\begin{aligned}
F(t|Z) &= \Pr\left[wZ + \sqrt{1-w^2}\,\overset{?}{\varepsilon} < d(t) \mid Z\right] \\
&= \Pr\left[\varepsilon < \frac{d(t) - w\,Z}{\sqrt{1-w^2}} \mid Z\right] \\
&= \Phi\left(\frac{d(t) - w\,Z}{\sqrt{1-w^2}}\right).
\end{aligned}
$$

F(t)
unconditional

Define $\rho = w^2$, then

$$\boxed{F(t|Z) = \Phi\left(\frac{d(t) - \sqrt{\rho}\,Z}{\sqrt{1-\rho}}\right).} \tag{9}$$

Conditional independence

Conditioning on Z, the default of an obligor will be independent of any other obligor. As a result, the number of default conditional on Z follows binomial distribution.

Let N the the size of the reference portfolio, and K be the number of default occurred before time t, so

$$K \sim \underline{\text{Binomial}\left(N, F(t|Z)\right)},$$

hence

$$\boxed{\Pr[K = k|Z] = \binom{N}{k} F(t|Z)^{k} \left(1 - F(t|Z)\right)^{N-k}.} \tag{10}$$

$$A_i = \boxed{\cdots Z} + \cdots \boxed{\varepsilon_i} \rightarrow \text{independent}$$
$$A_j = \boxed{\cdots Z} + \cdots \boxed{\varepsilon_j}$$

Unconditional Distribution

The unconditional distribution function of K can be derived by iterated expectation,

$$\begin{aligned}\Pr[K=k] &= \mathbb{E}\left(\Pr[K=k|Z]\right) \\ &= \int_{-\infty}^{\infty} \binom{N}{k} F(t|Z)^{k} \left(1-F(t|Z)\right)^{N-k} \, d\,\Phi(Z).\end{aligned} \tag{11}$$

Note, this integral must be solved numerically.

Φ CDF
of $N(0,1)$.

Loss distribution

Equation (11) is the distribution function for the number of default, we need to translate it to portfolio loss distribution.

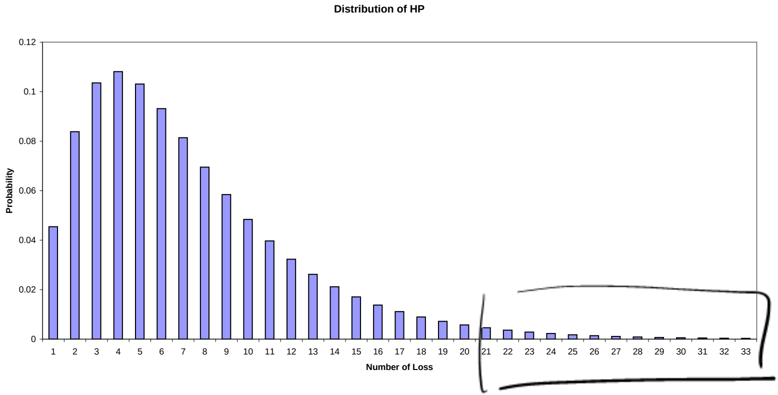
Denote $L(t)$ the percentage of loss for the reference portfolio by time t, and θ is the recovery rate. So when there is K defaults in the portfolio, the loss will be (assume unit NP):

$$L(t) = \frac{K(1-\theta)}{N}.$$

Thus the loss distribution function is

$$\Pr\left[L(t) = \frac{k(1-\theta)}{N}\right] = \Pr[K = k].$$

K Dis$^{\text{n}}$ $\Rightarrow$ L dis$^{\text{n}}$


Distribution of HP
Probability
0.12
0.1
0.08
0.06
0.04
0.02
0
1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25 26 27 28 29 30 31 32 33
Number of Loss

Large Homogenous Portfolio

If the number of reference names N of the underlying portfolio becomes reasonably large (as it effectively is for a typical CDO), the distribution function for the portfolio loss can be further simplified. This allows for computation of the portfolio loss distribution without resorting to either Monte Carlo simulationnor numerical schemes to solve integral (11).

Fraction of default

Define the fraction of default as

$$Y = \frac{K}{N}.$$

$E(K) = N F(t|z)$

$Var(K) = N F(1-F)$

According to CLT, conditional on Z, Y is approximately normal with

$$\begin{aligned}
\mathbb{E}[Y|Z] &= \frac{\mathbb{E}(K|Z)}{N} = F(t|Z) \\
\mathrm{Var}(Y|Z) &= \frac{\mathrm{Var}(K|Z)}{N^2} = \frac{F(t|Z)(1-F(t|Z))}{N}. \to 0.
\end{aligned}$$

So

$$\boxed{\lim_{N\to\infty} Y = F(t|Z).}$$

Loss distribution for LHP

Define the cumulative distribution function of $Y(z)$ is G, then

$$
\begin{aligned}
G(y) &= \Pr[Y(z) \leq y] = \Pr[F(t|Z) \leq y] \\
&= \Pr\left[\Phi\left(\frac{d(t) - \sqrt{\rho}\, Z}{\sqrt{1-\rho}}\right) \leq y\right] \\
&= \Pr\left[Z \leq \frac{\sqrt{1-\rho}\Phi^{-1}(y) - d(t)}{\sqrt{\rho}}\right] \\
&= \Phi\left(\frac{\sqrt{1-\rho}\Phi^{-1}(y) - d(t)}{\sqrt{\rho}}\right).
\end{aligned}
$$

Loss distribution for LHP

To derive ultimate loss distribution use the fact that percentage of loss equals to fraction of loss multiply by loss given default, i.e.,

$$\boxed{L(t)} = Y(1-\theta),$$

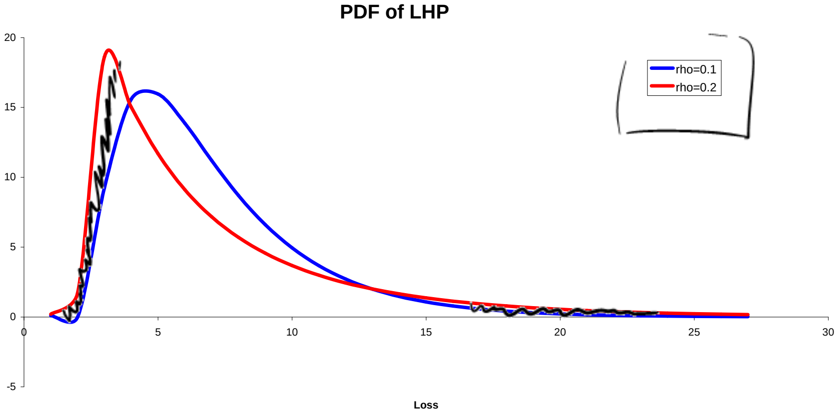
$Pr(L \le \ell)$
$= Pr((1-\theta)Y \le \ell)$
$= Pr(Y \le \frac{\ell}{1-\theta})$
$= G(\frac{\ell}{1-\theta})$

So the distribution function of $L(t)$ will be

$$\Pr[L(t) \le l] = G\left(\frac{l}{1-\theta}\right).$$

The PDF of the loss distribution of LHP will be

$$\boxed{f(y) = \sqrt{\frac{1-\rho}{\rho}} \exp\left(-\frac{1}{2\rho}\left(\sqrt{1-\rho}\,\Phi^{-1}(y) - d(t)\right)^2 + \frac{1}{2}\left(\Phi^{-1}(y)\right)^2\right).}$$



Vasicek Dis^n

Expected base tranche Loss

By using analytic distribution for LHP, one can use to calculate the expected base tranche loss(tranche with no subordination), for example the equity tranche loss function $L(t;0,l)$.

$$\begin{aligned}\mathbb{E}\left[L(t;0,l)\right] &= \mathbb{E}\left[L(t)\mathrm{I}\{L(t)<l\}+l\,\mathrm{I}\{L(t)\geq l\}\right]\\ &= \mathbb{E}\left[(1-\theta)F(t|Z)\mathrm{I}\{L(t)<l\}\right]+l\,\Phi(-a)\\ &= (1-\theta)\mathbb{E}\left[F(t|Z)\mathrm{I}\{Z>-a\}\right]+l\,\Phi(-a)\\ &= (1-\theta)\Phi_2\left(d(t),a,-\sqrt{\rho}\right)+l\,\Phi(-a),\end{aligned}$$

where

$$a=\frac{\sqrt{1-\rho}\Phi^{-1}(\frac{l}{1-\theta})-d(t)}{\sqrt{\rho}},$$

and $\Phi_2(x,y;r)$ is bivariate normal distribution with correlation parameter r.

Expected Loss for general tranche

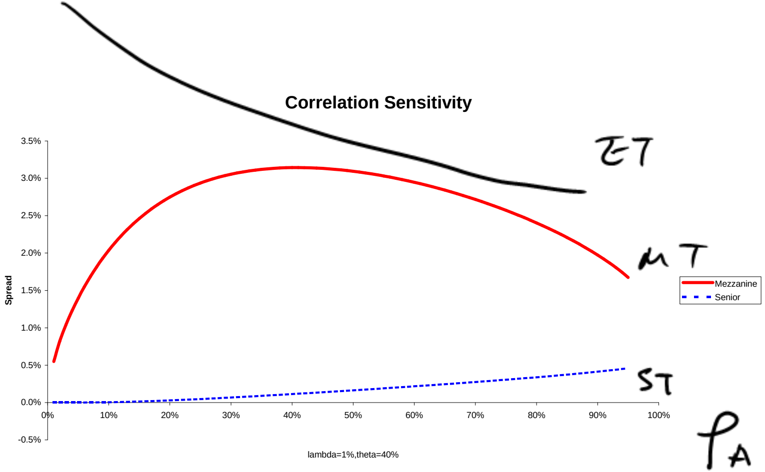
One can use the expected base tranche loss to calculate general tranche loss like this

$$\mathbb{E}\left[L(t; l_1, l_2)\right] = \mathbb{E}\left[L(t; 0, l_2)\right] - \mathbb{E}\left[L(t; 0, l_1)\right].$$

Therefore the closed for solution for tranche spread can be obtained by using this formula

$$s = \frac{\sum_{j=1}^{M} P(0, t_j)\left[\mathbb{E}[L(t_j; d, u)] - \mathbb{E}[L(t_{j-1}; d, u)]\right]}{\Delta \sum_{j=1}^{M} P(0, t_j)\left[1 - \mathbb{E}[L(t_j; d, u)]\right]}. \tag{12}$$

Note:$L(t_j; u, d)$ is redefined to be percentage of tranche loss.


Correlation Sensitivity
3.5%
3.0%
2.5%
2.0%
1.5%
1.0%
0.5%
0.0%
-0.5%
Spread
0%
10%
20%
30%
40%
50%
60%
70%
80%
90%
100%
lambda=1%,theta=40%
Mezzanine
Senior
ET
MT
ST
ρ_A

Please take away the following important ideas

1. Joint default distribution is a key input for modeling default risk in a portfolio
2. Copula is a powerful method to model joint default
3. One factor model is an intuitively appealing and easy approach to portfolio default risk embedded
4. Both copula and factor model can be employed to price general multi-name credit derivatives
5. If it is LHP, one factor model may lead to analytical solution
6. Portfolio default risk is drive by default correlation among other key risk parameters, which would impact the valuation and risk assessment for credit derivatives