Computing isomorphism classes of abelian varieties over finite fields

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VaNTAGe

Abelian varieties over $\mathbb{C}$ vs $\mathbb{F}_q$

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- In char. $p > 0$ such an equivalence cannot exist : there are (supersingular) elliptic curves with quaternionic endomorphism algebras.
- Nevertheless, over finite fields, we obtain analogous results if we restrict ourselves to certain **subcategories** of AVs.

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is injective and allows us to enumerate all AVs up to isogeny.

- Also, $h_A(x)$ is squarefree $\iff$ $\mathrm{End}(A)$ is commutative.

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Let $q = p^r$, with p a prime. There is an equivalence of categories:

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- Problem: $\mathbb{Z}[F,V]$ might not be maximal $\rightsquigarrow$ non-invertible ideals.

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- Hofmann-Sircana [HS20]: computation of over-orders.

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  ...whose representatives can be found in

$$\left\{\text{sub-}R\text{-modules of } {}^{\mathscr{O}_K}\!\big/\!{}_{\mathfrak{f}_R}\right\} \quad \text{finite! and most of the time not-too-big ...}$$

  where $\mathfrak{f}_R = (R : \mathscr{O}_K)$ is the conductor of R.

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Theorem ([Mar20b])

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Repeat for every $R \subseteq S \subseteq \mathcal{O}_K$:

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- We can actually get a lot more!

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- *a polarization μ of A corresponds to a $\lambda \in K^\times$ such that*
  - $\lambda I \subseteq \overline{I}^t$ *(isogeny of* $\deg \mu = [\overline{I}^t : \lambda I]$*);*
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similar statement for $\deg \mu > 1$

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Howe [How95] : **dual** varieties and **polarizations** on Deligne modules.

Theorem ([Mar21])

Let $A \in \mathscr{C}_h$ with h ordinary and squarefree. If $A \leftrightarrow I$, then:

- $A^\vee \leftrightarrow \overline{I}^t := \{\overline{x} \in K : \mathrm{Tr}(xI) \subseteq \mathbb{Z}\}$.
- *a polarization μ of A corresponds to a $\lambda \in K^\times$ such that*
  - $\lambda I \subseteq \overline{I}^t$ *(isogeny of* $\deg\mu = [\overline{I}^t : \lambda I]$*);*
  - λ *is totally imaginary* $(\overline{\lambda} = -\lambda)$*;*
  - λ *is* Φ*-positive* $(\Im\varphi(\lambda) > 0$ *for all* $\varphi \in \Phi)$*,*

  where Φ *is a CM-type of* K *satisf. the Shimura-Taniyama formula.*
- *if* $(A,\mu) \leftrightarrow (I,\lambda)$ *is a princ. polarized ab. var. and* $S = (I : I)$ *then*

$$\left\{\begin{matrix}\text{non-isomorphic princ.}\\ \text{polarizations of } A\end{matrix}\right\} \longleftrightarrow \frac{\{\text{totally positive } u \in S^\times\}}{\{v\overline{v} : v \in S^\times\}},$$

similar statement for $\deg\mu > 1$

- *and* $\mathrm{Aut}(A,\mu) = \{$*torsion units of* $S\}$.

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4. ... then we have one principal polarization.
5. By the previous Theorem, we have all princ. polarizations up to isom.

Example

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- 10 isomorphism classes of princ. polarized AV.

Example

Concretely:

$$
\begin{aligned}
I_1 =& 2645633792595191\mathbb{Z} \oplus (F + 836920075614551)\mathbb{Z} \oplus (F^2 + 1474295643839839)\mathbb{Z} \oplus \\
& \oplus (F^3 + 1372829830503387)\mathbb{Z} \oplus (F^4 + 1072904687510)\mathbb{Z} \oplus \\
& \oplus \frac{1}{3}(F^5 + F^4 + F^3 + 2F^2 + 2F + 670480698614361 0)\mathbb{Z} \oplus \\
& \oplus \frac{1}{9}(F^6 + F^5 + F^4 + 8F^3 + 2F^2 + 2991665243621169)\mathbb{Z} \oplus \\
& \oplus \frac{1}{27}(F^7 + F^6 + F^5 + 17F^4 + 20F^3 + 9F^2 + 6801531251872 2201)\mathbb{Z}
\end{aligned}
$$

principal polarizations:

$$
\begin{aligned}
x_{1,1} =& \frac{1}{27}(-121922F^7 + 588604F^6 - 1422437F^5 + \\
& + 1464239F^4 + 1196576F^3 - 7570722F^2 + 15316479F - 12821193) \\
x_{1,2} =& \frac{1}{27}(3015467F^7 - 17689816F^6 + 35965592F^5 - \\
& - 64660346F^4 + 121230619F^3 - 191117052F^2 + 315021546F - 300025458)
\end{aligned}
$$

$\mathrm{End}(I_1) = R$

$\#\mathrm{Aut}(I_1, x_{1,1}) = \#\mathrm{Aut}(I_1, x_{1,2}) = 2$

Example

$$I_7 = 2\mathbb{Z} \oplus (F+1)\mathbb{Z} \oplus (F^2+1)\mathbb{Z} \oplus (F^3+1)\mathbb{Z} \oplus (F^4+1)\mathbb{Z} \oplus \frac{1}{3}(F^5+F^4+F^3+2F^2+2F+3)\mathbb{Z} \oplus$$
$$\oplus \frac{1}{36}(F^6+F^5+10F^4+26F^3+2F^2+27F+45)\mathbb{Z} \oplus$$
$$\oplus \frac{1}{216}(F^7+4F^6+49F^5+200F^4+116F^3+105F^2+198F+351)\mathbb{Z}$$

principal polarization:

$$x_{7,1} = \frac{1}{54}(20F^7 - 43F^6 + 155F^5 - 308F^4 + 580F^3 - 1116F^2 + 2205F - 1809)$$

$$\mathrm{End}(I_7) = \mathbb{Z} \oplus F\mathbb{Z} \oplus F^2\mathbb{Z} \oplus F^3\mathbb{Z} \oplus F^4\mathbb{Z} \oplus \frac{1}{3}(F^5+F^4+F^3+2F^2+2F)\mathbb{Z} \oplus$$
$$\oplus \frac{1}{18}(F^6+F^5+10F^4+8F^3+2F^2+9F+9)\mathbb{Z} \oplus$$
$$\oplus \frac{1}{108}(F^7+4F^6+13F^5+56F^4+80F^3+33F^2+18F+27)\mathbb{Z}$$

$$\#\mathrm{Aut}(I_7, x_{7,1}) = 2$$

I_1 is invertible in R, but I_7 is not invertible in $\mathrm{End}(I_7)$.

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- Eg: quadratic orders are Bass $\rightsquigarrow$ powers of ordinary elliptic curves E^r.
- If R is Bass, then M is isomorphic to a direct sum of frac.R-ideals.

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Theorem ([Mar19])

If $R = \mathbb{Z}[F, V]$ is Bass then

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we have a classification:

$$\longleftrightarrow \left\{ (S_1 \subseteq S_2 \subseteq \ldots \subseteq S_r,\ [I]_{\simeq}) : \begin{array}{l} R \subseteq S_j \text{ orders,} \\ I \text{ a frac. } R\text{-ideal} \\ \text{with } (I:I) = S_r \end{array} \right\}$$

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- ... but computing them in general is harder!
- Solved for E^r by Kirschmer-Narbonne-Ritzenthaler-Robert [KNRR21].

Outside of the ordinary...

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Theorem (Centeleghe-Stix [CS15])

There is an equivalence of categories:

$$\begin{array}{cc}
\{\text{abelian varieties } A \text{ over } \mathbb{F}_p \text{ with } \boldsymbol{h_A(\sqrt{p}) \neq 0}\} & A \\
\updownarrow & \downarrow \\
\left\{\begin{array}{l} \text{pairs } (T, F), \text{ where } T \simeq_{\mathbb{Z}} \mathbb{Z}^{2g} \text{ and } T \xrightarrow{F} T \text{ s.t.} \\ \text{- } F \otimes \mathbb{Q} \text{ is semisimple} \\ \text{- the roots of } \mathrm{char}_{F \otimes \mathbb{Q}}(x) \text{ have abs. value } \sqrt{p} \\ \text{- } \boldsymbol{\mathrm{char}_F(\sqrt{p}) \neq 0} \\ \text{- } \exists V : T \to T \text{ such that } FV = VF = p \end{array}\right\} & (T(A), F(A))
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- Now, $T(A) := \mathrm{Hom}(A, A_w)$, where A_w has minimal End among the varieties with Weil support $w = w(A)$.
- $F(A)$ is the induced Frobenius.

Outside of the ordinary...isomorphism classes

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- For polarizations, the results by Howe do not apply immediately to the Centeleghe-Strix case:
- in general we cannot lift canonically each abelian variety.

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- If we understand the polarizations of A we can 'spread' them to the whole isogeny class.

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(\mathscr{G}(A) : \mathscr{G}(A^\vee)) \arrow[r, equal] & (\bar{I}^t : I)
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We study when we can 'pretend' $\alpha = 1$.

Some related work

- Base field extensions and **twists** (ordinary case) [Mar20a].

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We group isogeny classes into:
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polarizations	SQ	[How95]+[Mar21]	[BKM21]	?
	PP	[KNRR21] (E^r), [Mar19] (descr. but no algorithm)	?	?
	mixed	?	?	?

More comments:
- in [JKP$^+$18]: a functor for isogeny classes of the form E^r.
- in [OS20]+[BKM21]: almost-ordinary SQ with polarizations .
- in [CS21]: they use $\mathrm{Hom}_{\mathbb{F}_{p^k}}(-, A_w)$ as in [CS15], but A_w is more complicated.

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Thank you!