

# Intro to category theory

A category  $\mathcal{C}$  consists of :

- 1) a class of objects  $Ob(\mathcal{C})$  " $A \in \mathcal{C}$ "
- 2)  $\forall A, B \in Ob(\mathcal{C})$  a set  $Hom_{\mathcal{C}}(A, B)$  of morphisms " $A \xrightarrow{f} B$ "
- 3) a composition law of morphisms:

$$\forall A, B, C \in \mathcal{C}$$

$$Hom_{\mathcal{C}}(B, C) \times Hom_{\mathcal{C}}(A, B) \rightarrow Hom_{\mathcal{C}}(A, C)$$

$$(g, f) \longmapsto g \circ f$$

$$A \xrightarrow{f} B \xrightarrow{g} C$$

$$\quad \quad \quad \searrow \quad \quad \quad \nearrow$$

$$\quad \quad \quad g \circ f$$

such that :

$$\left( \begin{array}{l} \text{a) if } (C, D) \neq (A, B) \quad A, B, C, D \in \mathcal{C} \\ \quad \quad \quad Hom_{\mathcal{C}}(C, D) \text{ disjoint } Hom_{\mathcal{C}}(A, B) \end{array} \right)$$

$$\text{b) } \forall A \in \mathcal{C} \quad \exists id_A : A \rightarrow A \text{ st}$$

$$\forall B \xrightarrow{f} A \xrightarrow{id_A} A \xrightarrow{g} C$$

$$f = id_A \circ f \quad \text{and} \quad g =$$

c) associativity of the composition.

Def  $A \xrightarrow{f} B$  in  $\mathcal{C}$  is an isomorphism if

$$\exists B \xrightarrow{g} A \text{ in } \mathcal{C} \text{ st}$$

$$f \circ g = id_B, \quad g \circ f = id_A$$

## Examples

(2)

Sets , Top , Rings , CRings , Grp , Ab

$\forall$  ring  $R$  :  $R\text{-Mod}$  ,  $\text{Mod-}R$

$\forall$  division ring  $D$  :  $\text{Vect-}D$  ,  $D\text{-Vect}$

$\forall$  field  $k$  :  $\text{Vect}_k$       • Recall -  $R\text{-Mod}$  =  $\text{Vect}_R$   
-  $\mathbb{Z}\text{-Mod}$  =  $\text{Ab}$

- Point of category theory: don't think only about objects, but also about "arrows"

- $\mathcal{C}$  ,  $\mathcal{D}$  categories.

a (covariant) functor  $F: \mathcal{C} \rightarrow \mathcal{D}$  is a law assigning:

1)  $\forall c \in \mathcal{C}$  an object  $F(c) \in \mathcal{D}$

2)  $\forall c \xrightarrow{f} c'$  a morphism in  $\mathcal{C}$   
$$F(c) \xrightarrow{F(f)} F(c')$$

satisfying: (a)  $F(\text{id}_c) = \text{id}_{F(c)}$

(b)  $F(f \circ g) = F(f) \circ F(g)$

- "a contravariant functor: -  $F(c) \xleftarrow{F(f)} F(c')$  "  
-  $F(f \circ g) = F(g) \circ F(f)$

- $\mathcal{C}^{\text{op}}$  :  $\text{Ob}(\mathcal{C}^{\text{op}}) = \text{Ob}(\mathcal{C})$

$$\text{Hom}_{\mathcal{C}^{\text{op}}}(A, B) = \text{Hom}_{\mathcal{C}}(B, A)$$

- A contravariant functor  $\mathcal{C} \rightarrow \mathcal{D}$  is a covariant functor  $\mathcal{C}^{\text{op}} \rightarrow \mathcal{D}$

## Example

(3)

Consider the following category  $\mathcal{C}$

$$\text{Ob}(\mathcal{C}) = \{*\}$$

$$\text{Hom}_{\mathcal{C}}(*, *) = G \text{ a group.}$$

Then a linear representation of  $G$  over a field  $k$  is a functor

$$F: \mathcal{C} \longrightarrow \text{Vect}_k$$

i.e. a choice of a v.s.  $V$  and a group homomorphism  $G \longrightarrow \text{Aut}(V)$

i.e. we fix a  $k[G]$ -module structure on  $V$

- (• Rmks: - every morphism of  $\mathcal{C}$  is an isomorphism  
( $\forall f \exists g \text{ st } f \circ g = \text{id}_* = g \circ f$ )  
- any functor sends iso to iso)

Idea: : rather than studying directly  $G$ , we study its representations.

Turns out that we can get a lot of info from those:

- normal subgroups,
- #elements
- #elts in the conjugacy classes

TALK ABOUT NATURAL TRANSFORMATIONS

& the category  $\text{Funct}(\mathcal{C}, \mathcal{D})$

Def A category  $\mathcal{C}$  is called preadditive if  $\text{Hom}_{\mathcal{C}}(A, B)$  is an abelian group and the composition is bilinear:  $(R\text{-module})$   
 $(R\text{-bilinear})$

$$(f + f') \circ (g + g') = (f + g) \circ (f' + g') = (f + g') \circ (f' + g)$$

Ex -  $R\text{-Mod}$ ,  $\text{Mod-}R$ ,  $\text{Ab}$ ,  $\text{Vect}_K$  are pre-additive

-  $\text{Ring}$ ,  $\text{Sets}$ ,  $\text{Top}$  are not preadditive

$$\rightarrow ((f+g)(1) = \underset{\parallel}{f}(1) + \underset{\parallel}{g}(1) \neq 1)$$

Def  $\mathcal{C}$  category.

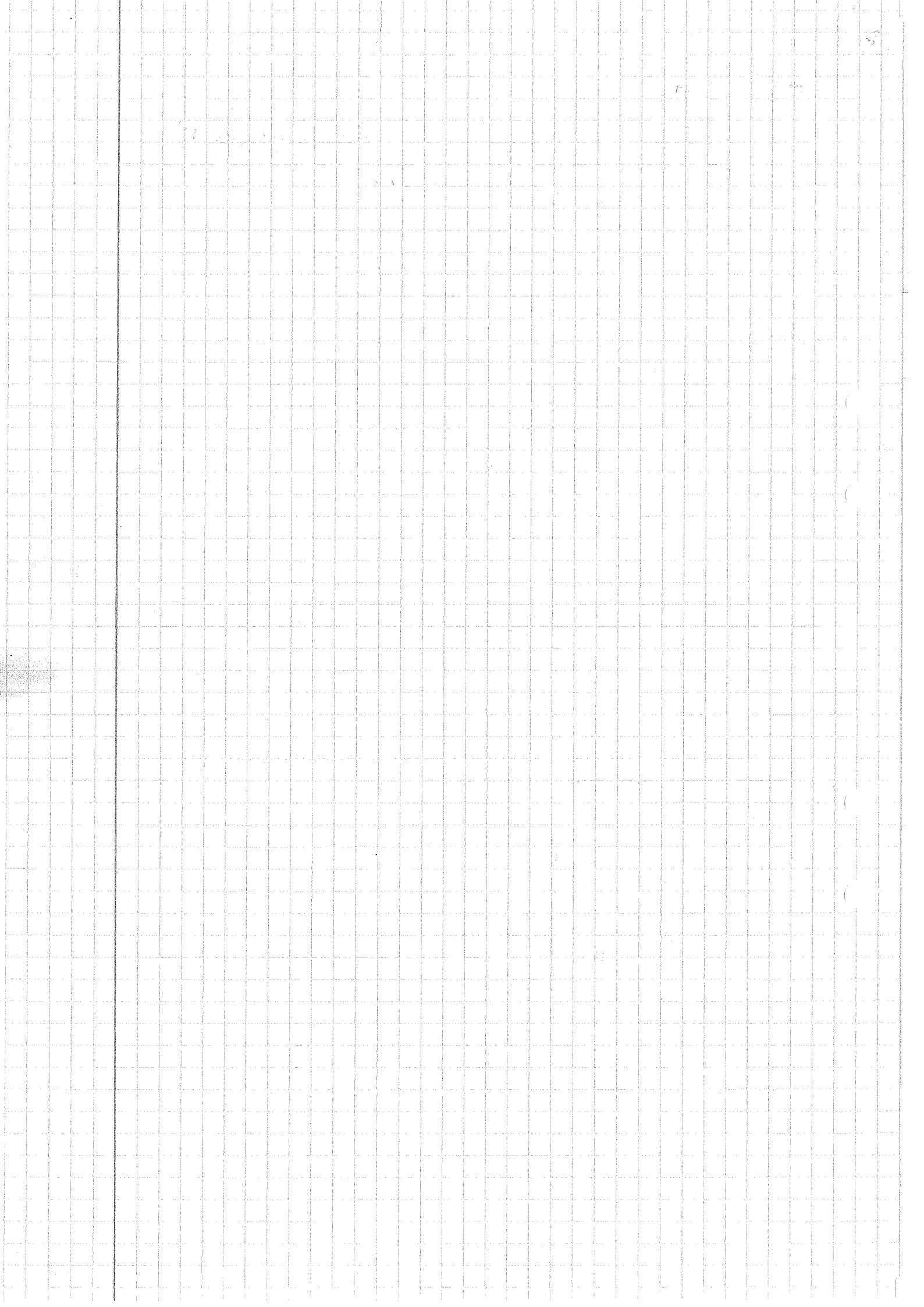
- $Z \in \mathcal{C}$  is an initial object if  $\text{Hom}_{\mathcal{C}}(Z, C)$  consist of only one morphism  $\forall C \in \mathcal{C}$
- a terminal object if the same holds for  $\text{Hom}_{\mathcal{C}}(C, Z)$   $\forall C \in \mathcal{C}$
- they are unique up to iso
- $O_{\mathcal{C}}$  both initial & terminal is a zero objects

Ex - Sets:

$\emptyset$  initial,  $\{*\}$  is terminal, no zero-dbj.

- Ab,  $R\text{-Mod}$ ,  $\text{Mod-}R$ , ... have a zero object

- Ring:  $\mathbb{Z}$  is initial, no terminal object.



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Rmk • If  $\mathcal{C}$  has a zero-obj  $0_{\mathcal{C}}$

then  $\forall A, B \in \mathcal{C}$

$$\begin{array}{ccc} A & \xrightarrow{\beta \circ \alpha} & B \\ \alpha \searrow & & \nearrow \beta \\ & 0_{\mathcal{C}} & \end{array}$$

$\exists$  a distinguished morph  $\beta \circ \alpha = 0$

• If moreover  $\mathcal{C}$  is preadditive:

$0_{AB}$  is the neutral elt. of  $\text{Hom}_{\mathcal{C}}(A, B)$ .

Let  $\mathcal{C}$  with zero object  $0_{\mathcal{C}}$ ;  $A, B \in \mathcal{C}$

$$A \xrightarrow{f} B$$

Def • A kernel of  $f$  is

$$K \xrightarrow{k} A \xrightarrow{f} B \quad \text{st} \quad f \circ k = 0$$

and

$$\begin{array}{ccc} K & \xrightarrow{k} & A \\ \uparrow m & \nearrow k' & \\ K' & & \end{array}$$

with  $f \circ k' = 0$ .

$\exists! m: K' \rightarrow K$  st  $k \circ m = k'$ .

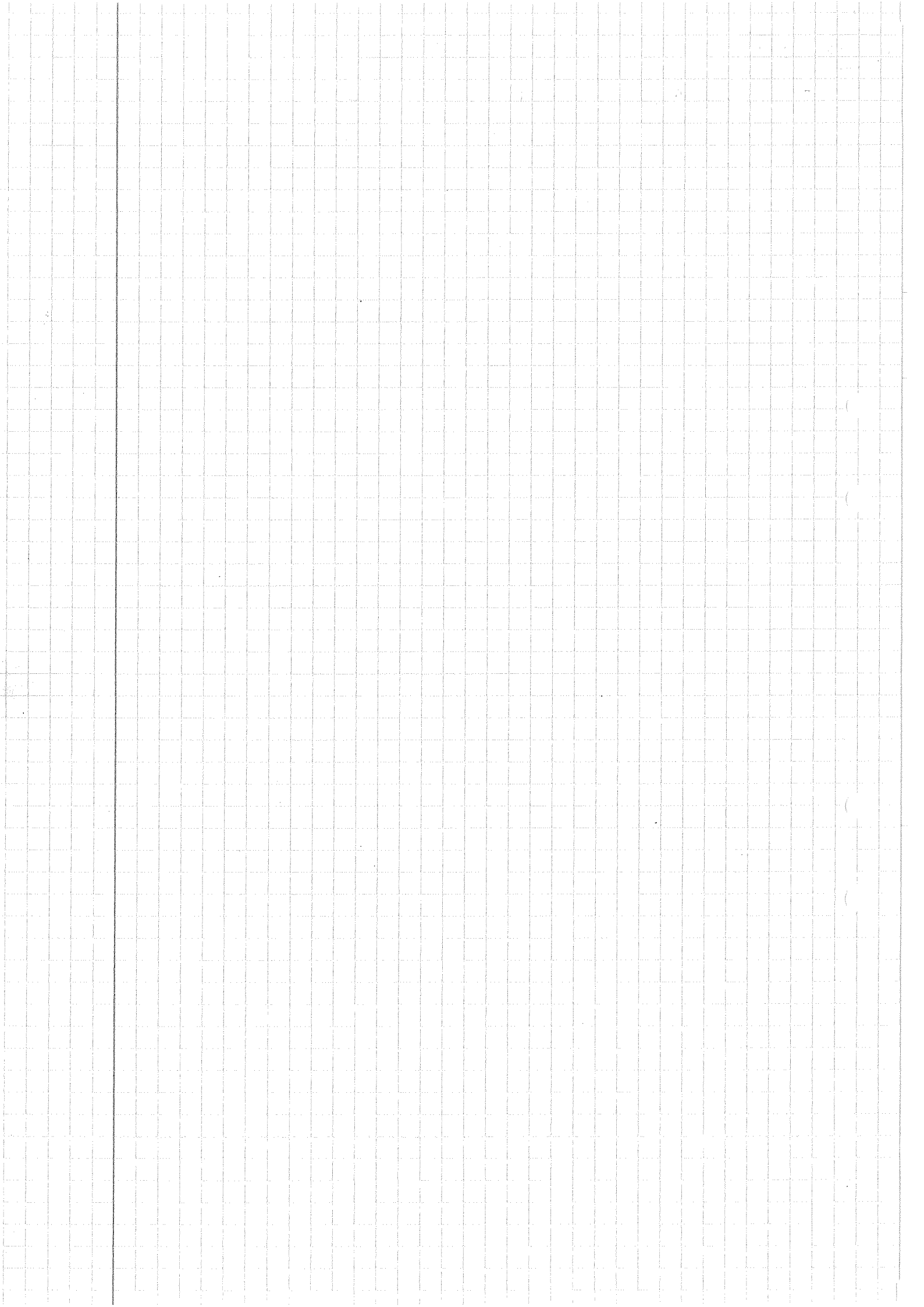
• A cokernel of  $f$  is

$$A \xrightarrow{f} B \xrightarrow{h} C \quad \text{st} \quad h \circ f = 0$$

and

$$\begin{array}{ccc} B & \xrightarrow{h} & C \\ \searrow h' & \downarrow \eta & \nearrow \exists! m \\ & C' & \end{array}$$

$\forall h': \exists! m$

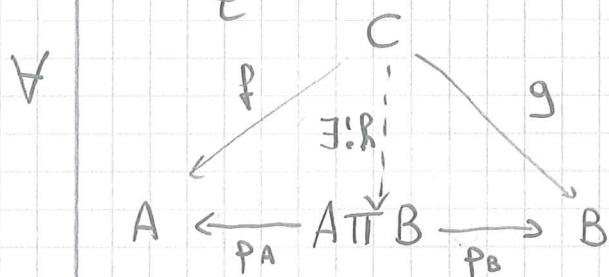




$\mathcal{C}$  preadditive w/  $0_{\mathcal{C}}$ ;  $A, B \in \mathcal{C}$

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Def • A product of  $A$  and  $B$  is  
 $(A \prod B, p_A, p_B)$  s.t.

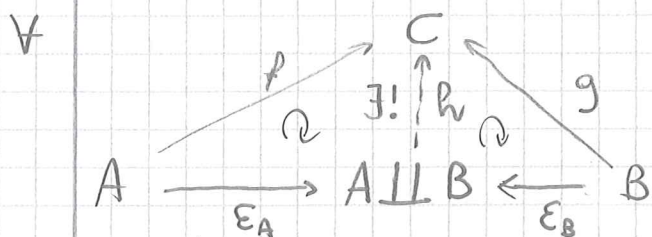


$$\exists! h: C \rightarrow A \prod B$$

$$p_A h = f \quad \& \quad p_B h = g$$

(if  $\exists \Rightarrow$  it's unique)

• A coproduct of  $A$  and  $B$  is  
 $(A \sqcup B, \epsilon_A, \epsilon_B)$  s.t.



Prop If  $(A \prod B, p_A, p_B)$  exists then  $(A \sqcup B, \epsilon_A, \epsilon_B)$  exists and !!!  
this is true  
for an additive  
category

$$A \prod B \cong A \sqcup B \quad ("A \oplus B")$$

Cor: the same is true for finite products

$$\prod_{i=1}^N A_i \cong \coprod_{i=1}^N A_i$$

but NOT for infinite products.

Examples:

Sets

Ab

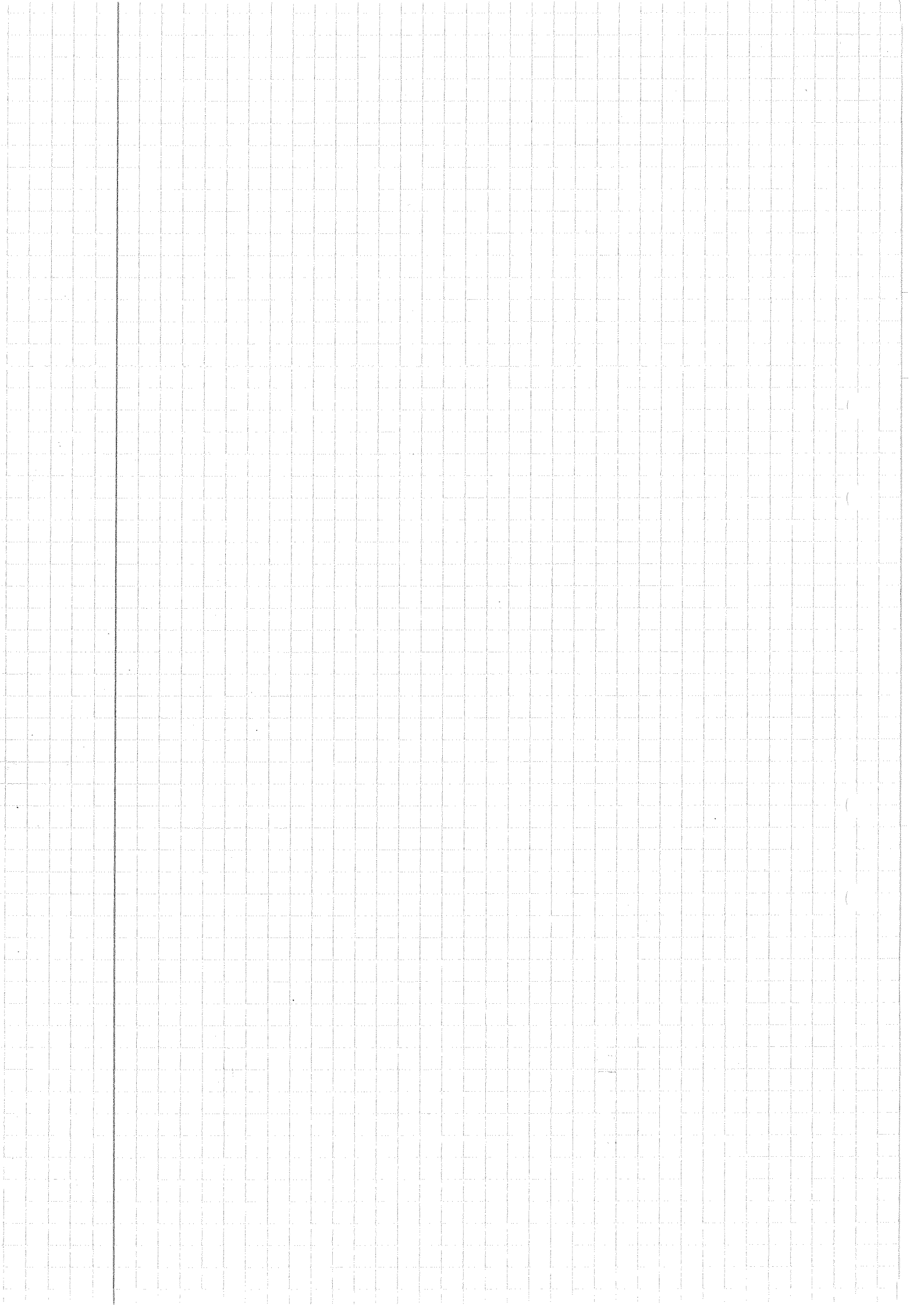
prod  
"  $\times$  "

"  $\prod$  "

coprod

"  $\sqcup$  "

"  $\oplus$  "



Def  $\mathcal{C}$  is additive if:

(7)

- $\mathcal{C}$  is preadditive w/  $0_{\mathcal{C}}$
- $\forall A, B \in \mathcal{C} \quad \exists A \oplus B$

⊗ Talk about k-s. cat. (12)

Def  $\mathcal{C}$  is abelian if additive

- 1 every morphism has ker and coker
  - 2 every morphism  $f$  can be factored as  $f = \gamma \circ \eta$ 

$\begin{matrix} \text{ker} & & \text{Coker} \\ \downarrow & & \downarrow \end{matrix}$
- or 2 bis  $\forall A \xrightarrow{f} B$  we have  $\text{Coker}(\text{ker}(f)) \xrightarrow{\sim} \text{ker}(\text{Coker}(f))$   
 "1st Iso Thm" ii  $\text{Im } f$

Def  $\mathcal{C}$  any category,  $A \xrightarrow{f} B$  in  $\mathcal{C}$

-  $f$  is a monomorphism if

$$\forall C \begin{matrix} \xrightarrow{g} \\ \xrightarrow{h} \end{matrix} A \xrightarrow{f} B$$

st  $f \circ g = f \circ h \Rightarrow g = h$

-  $f$  is an epimorphism if

$$\forall A \xrightarrow{f} B \begin{matrix} \xrightarrow{g} \\ \xrightarrow{h} \end{matrix} C$$

st  $g \circ f = h \circ f \Rightarrow g = h$

Rmk:  $f$  iso  $\Rightarrow f$  mono + epi

~~iff~~ in general

Ex  $\mathbb{Z} \xrightarrow{i} \mathbb{Q}$  in Rings: is mono + epi but not an iso

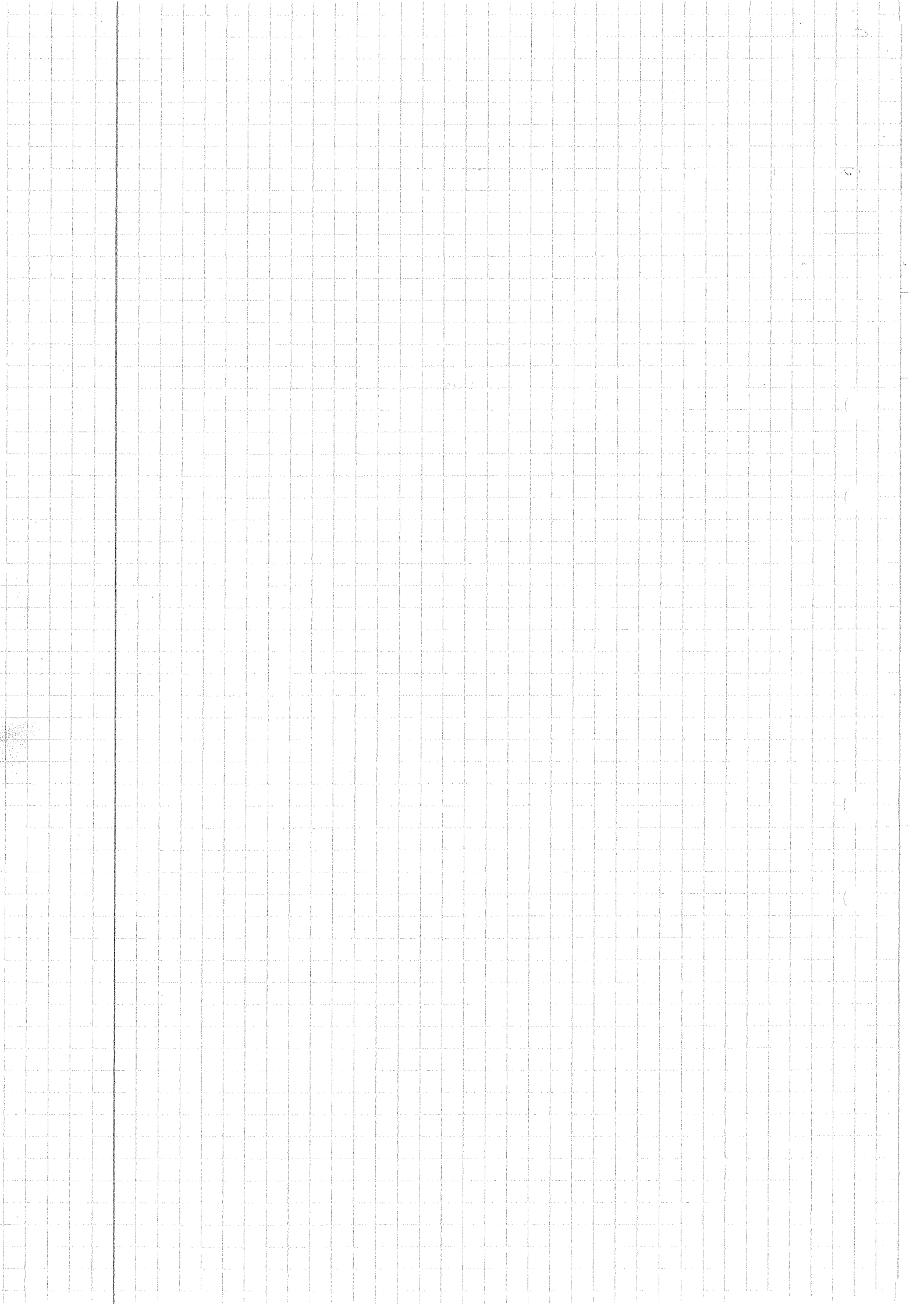
- mono: obvious

- epi:  $\mathbb{Z} \xrightarrow{i} \mathbb{Q} \xrightarrow{\varphi} R$  st  $\varphi|_{\mathbb{Z}} = \text{id}_{\mathbb{Z}}$

$$\begin{aligned} 1_R &= \varphi\left(n \frac{1}{n}\right) = n \varphi\left(\frac{1}{n}\right) \\ &= \varphi\left(n \frac{1}{n}\right) = n \varphi\left(\frac{1}{n}\right) \end{aligned}$$

So  $\varphi\left(\frac{1}{n}\right) = \varphi\left(\frac{1}{n}\right)$  as they are the inverse of  $n$  in  $R$ .

$$\Rightarrow \varphi\left(\frac{m}{n}\right) = \varphi(m) \varphi\left(\frac{1}{n}\right) = \varphi(m) \varphi\left(\frac{1}{n}\right) = \varphi\left(\frac{m}{n}\right)$$





Prop  $\mathcal{C}$  abelian. Then

skip to ⑨ if  
needed

⑧

- a) mono + epi  $\Leftrightarrow$  iso
- b) mono = kernel
- c) epi = cokernel

Examples:

①  $R\text{-Mod}$ ,  $\text{Mod-}R$  are abelian.

Im fact:  
(Freyd-)

Mitchell's embedding theorem:

every small abelian category is equivalent to

a full sub-category of modules

$\mathcal{A} = \text{ob}(\mathcal{C})$  is a set

$\mathcal{A} \subseteq \mathcal{C}$

full: if  $A, B \in \mathcal{A}$

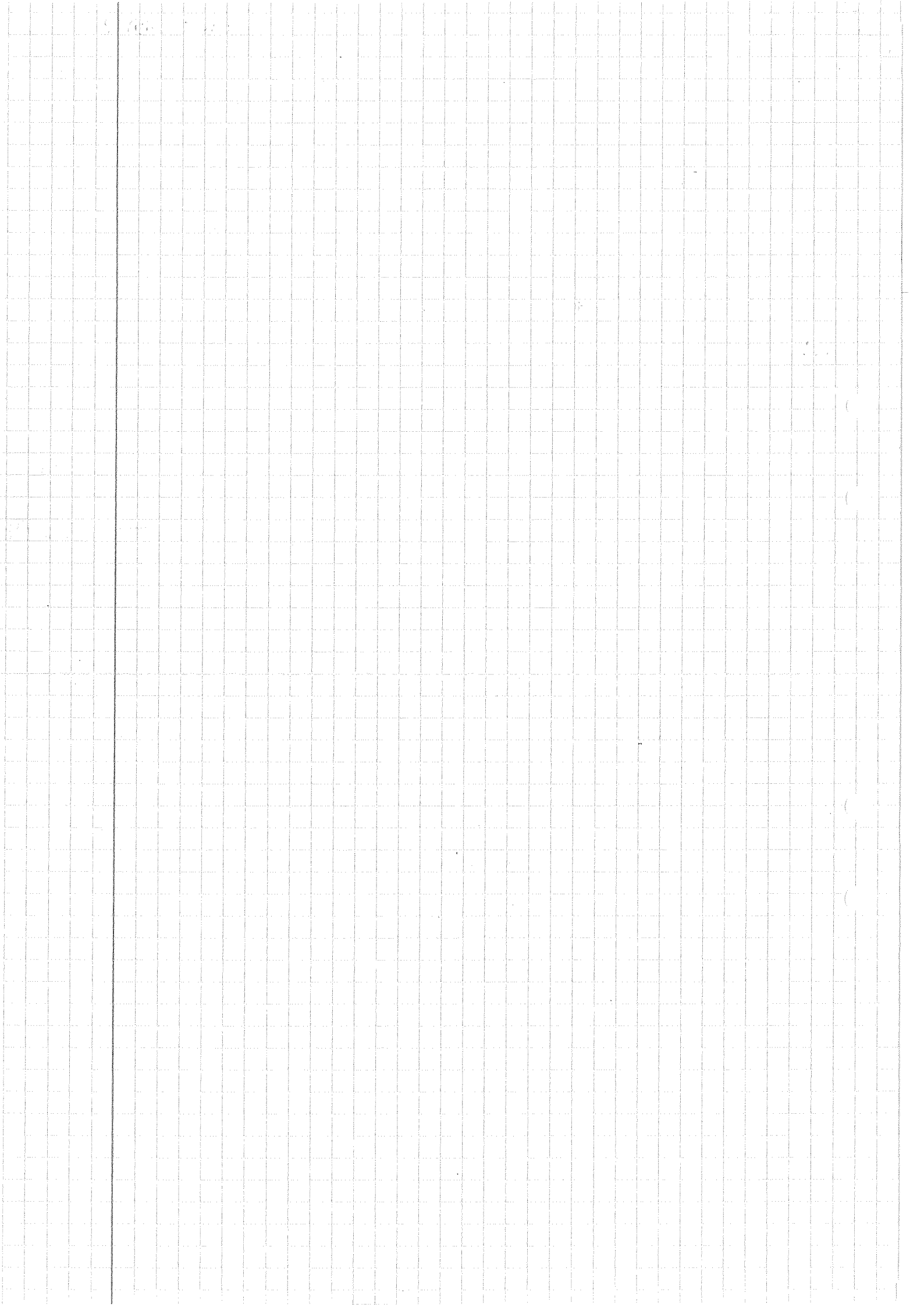
then  $\text{Hom}_{\mathcal{A}}(A, B) = \text{Hom}_{\mathcal{C}}(A, B)$

②  $\frac{\text{Ab}}{\mathcal{U}} = \underline{\mathbb{Z}\text{-Mod}}$  is abelian

$\mathcal{A} :=$  torsion free abelian groups

then  $\mathcal{A}$  is additive but not abelian

$\mathcal{U} =$  torsion ab groups  
 $\mathcal{U}$  is abelian



Def.  $\cdots \rightarrow C_{i-1} \xrightarrow{f_{i-1}} C_i \xrightarrow{f_i} C_{i+1} \rightarrow \cdots$  sequence  
is exact if  $\forall i \quad \text{Im } f_{i-1} = \text{Ker } f_i$

•  $\mathcal{C}, \mathcal{D}$  ab. cat,  $F: \mathcal{C} \rightarrow \mathcal{D}$  a functor

1  $F$  is exact if it sends exact seq's to exact seq's

2  $F$  is left exact if

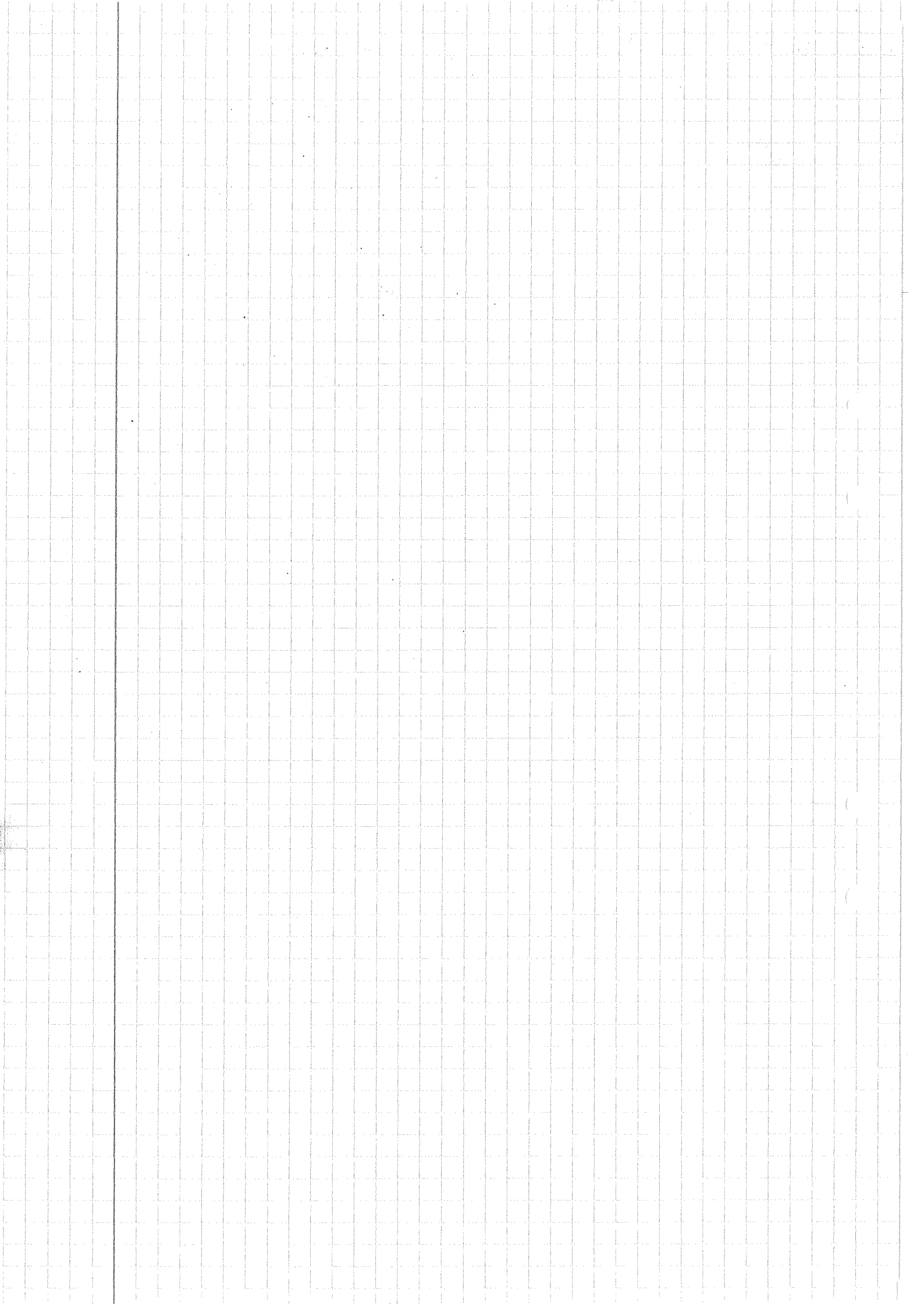
$$\forall \quad 0 \rightarrow A \xrightarrow{f} B \xrightarrow{g} C \rightarrow 0 \quad \text{s.e.s}$$

$$\Rightarrow 0 \rightarrow F(A) \xrightarrow{F(f)} F(B) \xrightarrow{F(g)} F(C) \quad \text{is exact}$$

3  $F$  is right exact

$$\Rightarrow F(A) \xrightarrow{F(f)} F(B) \xrightarrow{F(g)} F(C) \rightarrow 0$$

Lemma  $F$  exact  $\Leftrightarrow F$  both left and right exact





## Important functor

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$$\begin{array}{ccc} \mathcal{C}^{\text{op}} \times \mathcal{C} & \ni & (C, D) \\ \downarrow (f, g) & \iff & \begin{array}{l} f: C' \rightarrow C \\ g: D \rightarrow D' \end{array} \\ (C', D') & & \end{array}$$

$$\begin{array}{ccc} \text{Hom}_{\mathcal{C}}(-, -) : \mathcal{C}^{\text{op}} \times \mathcal{C} & \longrightarrow & \text{Sets} \\ (C, D) & \longmapsto & \text{Hom}_{\mathcal{C}}(C, D) \\ \downarrow (f, g) & \longmapsto & \begin{array}{c} \downarrow \alpha \\ g \circ \alpha \circ f \end{array} \\ (C', D') & \longmapsto & \text{Hom}_{\mathcal{C}}(C', D') \end{array}$$

Remark  $\text{Hom}_{\mathcal{C}}(-, -)$  is covariant in both entries

Prop  $\mathcal{C}$  abelian;  $\text{Hom}_{\mathcal{C}}(-, -)$  is left exact in both entries.

Def  $\mathcal{C}$  ab; •  $P \in \mathcal{C}$  is projective if  $\text{Hom}_{\mathcal{C}}(P, -)$  is exact (i.e. also right exact)

•  $I \in \mathcal{C}$  is injective if

$\text{Hom}_{\mathcal{C}}(-, I)$  is exact

•  $\mathcal{C}$  has enough projectives if  $\forall C \in \mathcal{C}$   
 $\exists P \in \mathcal{C}$  projective and

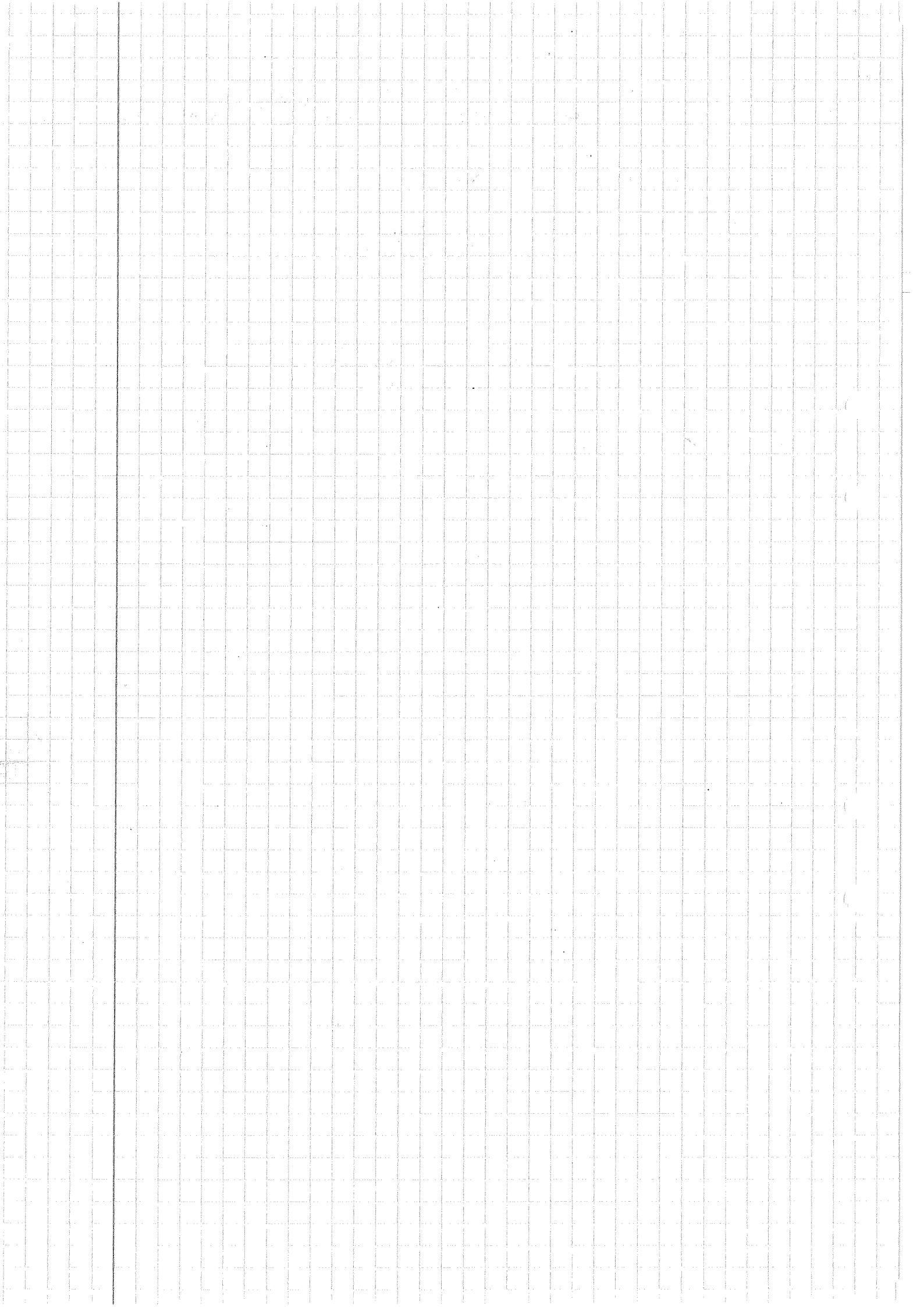
$$P \xrightarrow{\text{epi}} C \rightarrow 0$$

• enough injective if  $\forall C \in \mathcal{C}$

$\exists I \in \mathcal{C}$

injective

$$0 \rightarrow C \xrightarrow{\text{mono}} I$$



## Examples

(11)

- $\mathcal{C} = R\text{-Mod}$  has enough projectives and injectives

- $\mathcal{T}$  torsion ab groups:

- $G_i \in \text{Ob}(\mathcal{T}) \Rightarrow \coprod G_i \in \text{Ob}(\mathcal{T})$

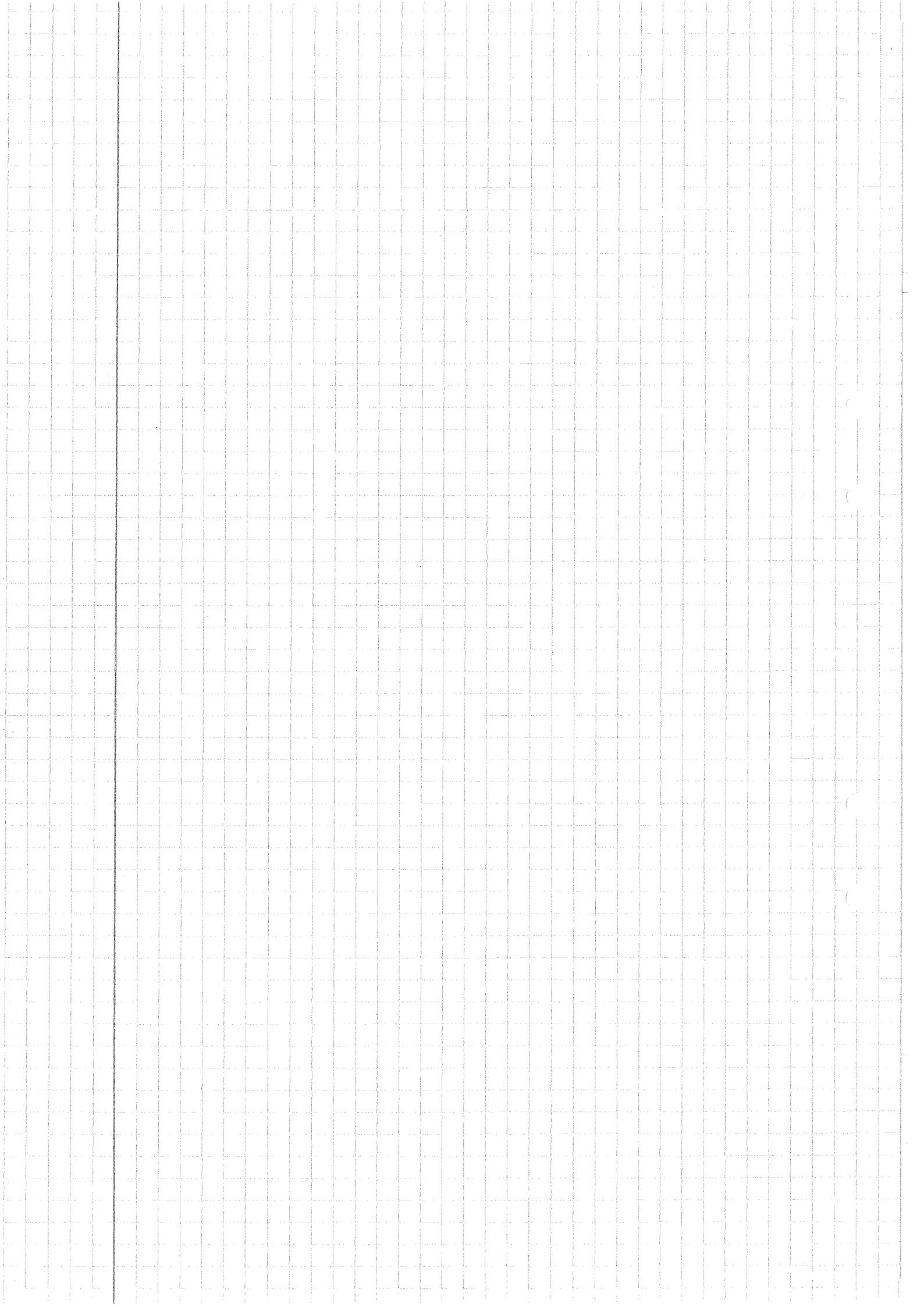
- but  $\prod G_i$  is not torsion

- $\left( \prod_{n \geq 0} \mathbb{Z}/p^n\mathbb{Z} : (1_n)_n \text{ is not torsion} \right)$

- $\mathcal{T}$  has enough injectives

- $\mathcal{T}$  has not enough proj:

- $\exists P \text{ is projective} \Leftrightarrow P = 0$



# Krull - Schmidt categories

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A non-zero  $C \in \mathcal{C}$  is indecomposable if:

$$C = C_1 \oplus C_2 \Rightarrow C_1 = 0_{\mathcal{C}} \text{ or } C_2 = 0_{\mathcal{C}}$$

Rmk:  $\mathcal{C}$  preadditive,  $C \in \mathcal{C}$  then

$\text{End}_{\mathcal{C}}(C)$  is a ring  
with  $\circ$  as multiplication &  $1_C$  as identity.

Def: An additive category is called a Krull-Schmidt cat if every object decomposes into a finite direct sum of object having local end. rings

Rmk In a k.s. cat: loc. end ring = indecomposable  
only one max<sup>e</sup> ideal  
 $\Downarrow$   
{non-invertible elements} is additively closed.

Lemma:  $\mathcal{C}$  additive if:

$$X = X_1 \oplus \dots \oplus X_r$$

$$X = Y_1 \oplus \dots \oplus Y_s \quad \text{w/ } X_i, Y_j \text{ with l.e.r}$$

then  $r = s$  and exist a permutation  $\pi$  s.t.

$$X_i \cong Y_{\pi(i)} \quad i = 1, \dots, s$$

