Isomorphism classes of abelian varieties over finite fields

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- in positive characteristic we don't have such equivalence.

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- $A/\mathbb{F}_{p^r}$ comes with a Frobenius endomorphism, that induces an action

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  $\mathrm{char}(\mathrm{Frob}_A)$ is a p^r-Weil polynomial.
- By Honda-Tate theory, the association

$$\text{isogeny class of } A \mapsto \mathrm{char}(\mathrm{Frob}_A)$$

  is injective and allows us to enumerate all AVs up to isogeny.

Deligne's equivalence

Theorem (Deligne '69)

Let $q = p^r$, with p a prime. There is an equivalence of categories:

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\updownarrow & \mapsto \\
\left\{ \begin{array}{l} \textit{pairs } (T, F), \textit{ where } T \simeq_{\mathbb{Z}} \mathbb{Z}^{2g} \textit{ and } T \xrightarrow{F} T \textit{ s.t.} \\ \textit{- } F \otimes \mathbb{Q} \textit{ is semisimple} \\ \textit{- the roots of } \mathrm{char}_{F\otimes\mathbb{Q}}(x) \textit{ have abs. value } \sqrt{q} \\ \textit{- } \textbf{\textit{half of them are }} p\textbf{\textit{-adic units}} \\ \textit{- } \exists V : T \to T \textit{ such that } FV = VF = q \end{array} \right\} & (T(A), F(A))
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Remark

- *If* $\dim(A) = g$ *then* $\mathrm{Rank}(T(A)) = 2g$;
- $\mathrm{Frob}(A) \rightsquigarrow F(A)$.

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Theorem (M.)

$$\left\{\text{abelian varieties over } \mathbb{F}_q \text{ in } \mathscr{C}_h\right\}_{/\simeq}$$
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- Problem: $\mathbb{Z}[F,q/F]$ might not be maximal $\rightsquigarrow$ non-invertible ideals.

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- ...and actually

$$\mathrm{ICM}(R) \supseteq \bigsqcup_{\substack{R \subseteq S \subseteq \mathcal{O}_K \\ \text{over-orders}}} \mathrm{Pic}(S) \qquad \text{with equality iff } R \text{ is Bass}$$

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- Let $\mathscr{W}(R)$ be the set of weak eq. classes...
  ...whose representatives can be found in

$$\left\{ \text{sub-}R\text{-modules of } {}^{\mathcal{O}_K}\!/\!{}_{\mathfrak{f}_R} \right\} \quad \text{finite! and most of the time not-too-big ...}$$

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Partition w.r.t. the multiplicator ring:

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Theorem (M.)

For every over-order S of R, $\mathrm{Pic}(S)$ acts freely on $\mathrm{ICM}_S(R)$ and

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Repeat for every $R \subseteq S \subseteq \mathscr{O}_K$:

$$\rightsquigarrow \mathrm{ICM}(R).$$

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- *a polarization* μ *of* A *corresponds to a* $\lambda \in K^\times$ *such that*
  - $\lambda I \subseteq \overline{I}^t$ *(isogeny);*
  - λ *is totally imaginary* $(\overline{\lambda} = -\lambda)$*;*
  - λ *is* Φ*-positive, where* Φ *is a specific CM-type of* K.

  Also: $\deg \mu = [\overline{I}^t : \lambda I]$.

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- *if $(A,\mu) \leftrightarrow (I,\lambda)$ and $S = (I : I)$ then*

$$\left\{ \begin{array}{l} \textit{non-isomorphic} \\ \textit{polarizations of } A \\ \textit{of degree} = \deg \lambda \end{array} \right\} \longleftrightarrow \frac{\{\textit{totally positive } u \in S^\times\}}{\{v\overline{v} : v \in S^\times\}}.$$

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- $\mathrm{Aut}(A,\mu) = \{\textit{torsion units of } S\}$.

Example

- Let $h(x) = x^8 - 5x^7 + 13x^6 - 25x^5 + 44x^4 - 75x^3 + 117x^2 - 135x + 81$.
- $\rightsquigarrow$ isogeny class of an simple ordinary abelian varieties over $\mathbb{F}_3$ of dimension 4.
- Let F be a root of $h(x)$ and put $R := \mathbb{Z}[F, 3/F] \subset \mathbb{Q}(F)$.
- 8 over-orders of R: two of them are not Gorenstein.
- $\#\mathrm{ICM}(R) = 18 \rightsquigarrow 18$ isom. classes of AV in the isogeny class.
- 5 are not invertible in their multiplicator ring.
- 8 classes admit principal polarizations.
- 10 isomorphism classes of princ. polarized AV.

Example

Concretely:

$$I_1 = 2645633792595191\mathbb{Z} \oplus (F + 836920075614551)\mathbb{Z} \oplus (F^2 + 1474295643839839)\mathbb{Z} \oplus$$
$$\oplus (F^3 + 1372829830503387)\mathbb{Z} \oplus (F^4 + 1072904687510)\mathbb{Z} \oplus$$
$$\oplus \frac{1}{3}(F^5 + F^4 + F^3 + 2F^2 + 2F + 670480698614361 0)\mathbb{Z} \oplus$$
$$\oplus \frac{1}{9}(F^6 + F^5 + F^4 + 8F^3 + 2F^2 + 2991665243621169)\mathbb{Z} \oplus$$
$$\oplus \frac{1}{27}(F^7 + F^6 + F^5 + 17F^4 + 20F^3 + 9F^2 + 6801531251872220 1)\mathbb{Z}$$

principal polarizations:

$$x_{1,1} = \frac{1}{27}(-121922F^7 + 588604F^6 - 1422437F^5 + $$
$$+ 1464239F^4 + 1196576F^3 - 7570722F^2 + 15316479F - 12821193)$$
$$x_{1,2} = \frac{1}{27}(3015467F^7 - 17689816F^6 + 35965592F^5 - $$
$$- 64660346F^4 + 121230619F^3 - 191117052F^2 + 315021546F - 300025458)$$

$\mathrm{End}(I_1) = R$

$\#\mathrm{Aut}(I_1, x_{1,1}) = \#\mathrm{Aut}(I_1, x_{1,2}) = 2$

Example

$$
\begin{aligned}
I_7 =& 2\mathbb{Z} \oplus (F+1)\mathbb{Z} \oplus (F^2+1)\mathbb{Z} \oplus (F^3+1)\mathbb{Z} \oplus (F^4+1)\mathbb{Z} \oplus \frac{1}{3}(F^5+F^4+F^3+2F^2+2F+3)\mathbb{Z} \oplus \\
&\oplus \frac{1}{36}(F^6+F^5+10F^4+26F^3+2F^2+27F+45)\mathbb{Z} \oplus \\
&\oplus \frac{1}{216}(F^7+4F^6+49F^5+200F^4+116F^3+105F^2+198F+351)\mathbb{Z}
\end{aligned}
$$

principal polarization:

$$x_{7,1} = \frac{1}{54}(20F^7-43F^6+155F^5-308F^4+580F^3-1116F^2+2205F-1809)$$

$$
\begin{aligned}
\mathrm{End}(I_7) = & \mathbb{Z} \oplus F\mathbb{Z} \oplus F^2\mathbb{Z} \oplus F^3\mathbb{Z} \oplus F^4\mathbb{Z} \oplus \frac{1}{3}(F^5+F^4+F^3+2F^2+2F)\mathbb{Z} \oplus \\
&\oplus \frac{1}{18}(F^6+F^5+10F^4+8F^3+2F^2+9F+9)\mathbb{Z} \oplus \\
&\oplus \frac{1}{108}(F^7+4F^6+13F^5+56F^4+80F^3+33F^2+18F+27)\mathbb{Z}
\end{aligned}
$$

$$\#\mathrm{Aut}(I_7, x_{7,1}) = 2$$

I_1 is invertible in R, but I_7 is not invertible in $\mathrm{End}(I_7)$.

Final remarks

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- base field extensions and twists (ordinary case) (soon on arXiv).
- period matrices (ordinary case) of the canonical lift.
- results of computations will appear on the LMFDB.

Thank you!