Computing isomorphism classes of abelian varieties over finite fields.

Stefano Marseglia

Utrecht University

Curves over Finite Fields: Past, Present and Future

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$$\left\{ \text{abelian varieties } / \mathbb{C} \right\} \longleftrightarrow \left\{ \begin{matrix} \mathbb{C}^g / \Lambda \text{ with } \Lambda \simeq \mathbb{Z}^{2g} \text{ admitting} \\ \text{a Riemann form} \end{matrix} \right\}.$$

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- In char. p>0 such an equivalence cannot exist: there are (supersingular) elliptic curves with quaternionic endomorphism algebras.
- Nevertheless, over finite fields, we obtain analogous results if we restrict ourselves to certain subcategories of AVs.

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• $h_A(x) := \text{char}(\text{Frob}_A)$ is a q-Weil polynomial and isogeny invariant.

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- By Honda-Tate theory, the association

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• Also, $h_A(x)$ is squarefree \iff End(A) is commutative.

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Theorem (Deligne '69)

Let $q = p^r$, with p a prime. There is an equivalence of categories:

 $\{ \text{ Ordinary abelian varieties over } \mathbb{F}_q \}$

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• Ordinary A/\mathbb{F}_q can be canonically lifted: $\rightsquigarrow \mathscr{A}_{\operatorname{can}}/\operatorname{Witt}(\mathbb{F}_q)...$

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- Put $T(A) := H_1(\mathscr{A}_{\operatorname{can}} \otimes \mathbb{C}, \mathbb{Z})$ and $F(A) := \operatorname{the induced Frobenius}$.

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- \rightsquigarrow an isogeny class $\mathscr{C}_h/\mathbb{F}_q$.

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{abelian varieties over \mathbb{F}_q in \mathscr{C}_h}_{\simeq}

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{fractional ideals of \mathbb{Z}[F,V]\subset K}_{\simeq}
=: ICM(\mathbb{Z}[F,V])
ideal class monoid
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• Problem: $\mathbb{Z}[F, V]$ might not be maximal \rightsquigarrow non-invertible ideals.

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Let R be an **order** in an étale \mathbb{Q} -algebra K.

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• Recall: for fractional R-ideals I and J

$$I \simeq_R J \Longleftrightarrow \exists x \in K^\times \text{ s.t. } xI = J$$

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• Hofmann-Sircana '19: computation of over-orders.

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Study the isomorphism problem locally: (Dade, Taussky, Zassenhaus '62)

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• weak equivalence:

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simplify the problem

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• Let W(R) be the set of weak eq. classes... ...whose representatives can be found in

$$\left\{ \text{sub-}R\text{-modules of } \mathscr{O}_{K/\mathfrak{f}_{R}} \right\} \quad \begin{array}{l} \text{finite! and most of the} \\ \text{time not-too-big } \dots \end{array}$$

where $f_R = (R : \mathcal{O}_K)$ is the conductor of R.

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Partition w.r.t. the multiplicator ring:

$$W(R) = \bigsqcup_{R \subseteq S \subseteq \mathcal{O}_K} W_S(R)$$
$$ICM(R) = \bigsqcup_{R \subseteq S \subseteq \mathcal{O}_K} ICM_S(R)$$

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Theorem (M.)

For every over-order S of R, Pic(S) acts freely on $ICM_S(R)$ and

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Repeat for every $R \subseteq S \subseteq \mathcal{O}_K$:

$$\rightsquigarrow ICM(R)$$
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- We can actually get a lot more!

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Howe described dual varieties and polarizations on Deligne modules.

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Theorem

Let $A \in \mathcal{C}_h$ with h ordinary and squarefree. If $A \leftrightarrow I$, then:

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- a polarization μ of A corresponds to a $\lambda \in K^{\times}$ such that
 - $\lambda I \subseteq \overline{I}^t$ (isogeny);
 - λ is totally imaginary $(\overline{\lambda} = -\lambda)$;
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Also: $\deg \mu = [\overline{I}^t : \lambda I].$

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• if $(A, \mu) \leftrightarrow (I, \lambda)$ is a princ. polarized ab. var. and S = (I:I) then $\begin{cases} \text{non-isomorphic princ.} \\ \text{polarizations of } A \end{cases} \longleftrightarrow \frac{\{\text{totally positive } u \in S^{\times}\}}{\{v\overline{v}: v \in S^{\times}\}},$

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- and $Aut(A, \mu) = \{torsion \ units \ of \ S\}.$

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- Loop over the representatives u of the finite quotient

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3 If $\lambda := i_0 u$ is totally imaginary and Φ-positive ...

We have an **algorithm** to enumerate principal polarizations up to isomorphism:

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Can modify to compute polarizations of any degree.

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• Let $h(x) = x^8 - 5x^7 + 13x^6 - 25x^5 + 44x^4 - 75x^3 + 117x^2 - 135x + 81$.



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- Let $h(x) = x^8 5x^7 + 13x^6 25x^5 + 44x^4 75x^3 + 117x^2 135x + 81$.
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- 10 isomorphism classes of princ. polarized AV.

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Concretely:

$$\begin{split} I_1 = & 2645633792595191\mathbb{Z} \oplus (F + 836920075614551)\mathbb{Z} \oplus (F^2 + 1474295643839839)\mathbb{Z} \oplus \\ & \oplus (F^3 + 1372829830503387)\mathbb{Z} \oplus (F^4 + 1072904687510)\mathbb{Z} \oplus \\ & \oplus \frac{1}{3}(F^5 + F^4 + F^3 + 2F^2 + 2F + 6704806986143610)\mathbb{Z} \oplus \\ & \oplus \frac{1}{9}(F^6 + F^5 + F^4 + 8F^3 + 2F^2 + 2991665243621169)\mathbb{Z} \oplus \\ & \oplus \frac{1}{27}(F^7 + F^6 + F^5 + 17F^4 + 20F^3 + 9F^2 + 68015312518722201)\mathbb{Z} \end{split}$$

principal polarizations:

$$x_{1,1} = \frac{1}{27} \left(-121922F^7 + 588604F^6 - 1422437F^5 + \right.$$

$$+ 1464239F^4 + 1196576F^3 - 7570722F^2 + 15316479F - 12821193 \right)$$

$$x_{1,2} = \frac{1}{27} \left(3015467F^7 - 17689816F^6 + 35965592F^5 - \right.$$

$$- 64660346F^4 + 121230619F^3 - 191117052F^2 + 315021546F - 300025458 \right)$$

$$\operatorname{End}(I_1) = R$$

$$\# \operatorname{Aut}(I_{1,|X_1|}) = \# \operatorname{Aut}(I_{1,|X_1|}) = 2$$

$$\begin{split} I_7 = & 2\mathbb{Z} \oplus (F+1)\mathbb{Z} \oplus (F^2+1)\mathbb{Z} \oplus (F^3+1)\mathbb{Z} \oplus (F^4+1)\mathbb{Z} \oplus \frac{1}{3}(F^5+F^4+F^3+2F^2+2F+3)\mathbb{Z} \oplus \\ & \oplus \frac{1}{36}(F^6+F^5+10F^4+26F^3+2F^2+27F+45)\mathbb{Z} \oplus \\ & \oplus \frac{1}{216}(F^7+4F^6+49F^5+200F^4+116F^3+105F^2+198F+351)\mathbb{Z} \end{split}$$

principal polarization:

$$x_{7,1} = \frac{1}{54} (20F^7 - 43F^6 + 155F^5 - 308F^4 + 580F^3 - 1116F^2 + 2205F - 1809)$$

$$\text{End}(I_7) = \mathbb{Z} \oplus F \mathbb{Z} \oplus F^2 \mathbb{Z} \oplus F^3 \mathbb{Z} \oplus F^4 \mathbb{Z} \oplus \frac{1}{3} (F^5 + F^4 + F^3 + 2F^2 + 2F) \mathbb{Z} \oplus$$

$$\oplus \frac{1}{18} (F^6 + F^5 + 10F^4 + 8F^3 + 2F^2 + 9F + 9) \mathbb{Z} \oplus$$

$$\oplus \frac{1}{108} (F^7 + 4F^6 + 13F^5 + 56F^4 + 80F^3 + 33F^2 + 18F + 27) \mathbb{Z}$$
#Aut $(I_7, x_{7,1}) = 2$

 I_1 is invertible in R, but I_7 is not invertible in $\operatorname{End}(I_7)$

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The Power-of-a-Bass case

• Another case we understand well: $h = g^r$ for g square-free and ordinary.

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- ... or equivalently $ICM(R) = \coprod_S Pic(S)$.
- Eg: quadratic orders are Bass \rightsquigarrow powers of ordinary elliptic curves E^r .
- If R is Bass, then M is isomorphic to a direct sum of frac. R-ideals.

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Theorem

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If $A \in \mathcal{C}_{g^r}$ then $A \simeq C_1 \times ... \times C_r$, for $C_j \in \mathcal{C}_g$.

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If
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Theorem

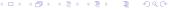
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- Solved for E^r by Kirschmer-Narbonne-Ritzenthaler-Robert '20.

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Let $g = x^6 - 3x^5 + 6x^4 - 10x^3 + 18x^2 - 27x + 27$. Note $\mathscr{C}(g)$ is an isogeny class of simple ordinary abelian varieties over \mathbb{F}_3 . Define $K = \mathbb{Q}[x]/(g) = \mathbb{Q}(F)$ and $R = \mathbb{Z}[F, V]$.



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$$\operatorname{Pic}(R) \simeq \mathbb{Z}/_{3\mathbb{Z}} \text{ and } \operatorname{Pic}(\mathscr{O}_K) = \{1\}.$$

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$$\begin{aligned} M_1 &= R \oplus R \oplus R & M_2 &= R \oplus R \oplus I & M_3 &= R \oplus R \oplus I^2 \\ M_4 &= R \oplus R \oplus \mathcal{O}_K & M_5 &= R \oplus \mathcal{O}_K \oplus \mathcal{O}_K & M_6 &= \mathcal{O}_K \oplus \mathcal{O}_K \oplus \mathcal{O}_K \end{aligned}$$

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$$\operatorname{End}(M_1) = \operatorname{Mat}_3(R) \text{ and } \operatorname{End}(M_2) = \begin{pmatrix} R & R & I \\ R & R & I \\ (R:I) & (R:I) & R \end{pmatrix}$$

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Outside of the ordinary...

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Outside of the ordinary...

Theorem (Centeleghe-Stix '15)

There is an equivalence of categories:

```
{abelian varieties A over \mathbb{F}_p with h_A(\sqrt{p}) \neq 0}
\downarrow \qquad \qquad \downarrow \qquad \qquad \downarrow
\begin{cases} pairs (T,F), \text{ where } T \simeq_{\mathbb{Z}} \mathbb{Z}^{2g} \text{ and } T \xrightarrow{F} T \text{ s.t.} \\ -F \otimes \mathbb{Q} \text{ is semisimple} \\ -\text{ the roots of } \text{char}_{F \otimes \mathbb{Q}}(x) \text{ have abs. value } \sqrt{p} \\ -\text{char}_{F}(\sqrt{p}) \neq 0 \\ -\exists V: T \to T \text{ such that } FV = VF = p \end{cases} 
(T(A), F(A))
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$$- \operatorname{char}_{F}(\sqrt{p}) \neq 0$$

$$-\exists V : T \to T \text{ such that } FV = VF = p$$$$$$

- Now, $T(A) := \text{Hom}(A, A_w)$, where A_w has minimal End among the varieties with Weil support w = w(A).
- F(A) is the induced Frobenius.

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- Everything I told so far about isomorphism classes works in the same way using the Centeleghe-Stix functor:
- both in the squarefree and Power-of-Bass cases, over \mathbb{F}_p .

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- For polarizations, the results by Howe do not apply immediately to the Centeleghe-Strix case:
- in general we cannot lift canonically each abelian variety.

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• New strategy: with Jonas Bergström and Valentijn Karemaker '21.

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- Consider \mathscr{C}_h with h squarefree $/\mathbb{F}_q \rightsquigarrow K = \mathbb{Q}[F]$.

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Theorem (Chai-Conrad-Oort)

If (K,Φ) satisfies the RRC then in \mathscr{C}_h there exists an abelian variety A admitting a canonical lifting \mathscr{A} .

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- New strategy: with Jonas Bergström and Valentijn Karemaker '21.
- Consider \mathscr{C}_h with h squarefree $/\mathbb{F}_q \rightsquigarrow K = \mathbb{Q}[F]$.
- Chai-Conrad-Oort: A (p-adic) CM-type (K,Φ) satisfies the **Residual** Reflex Condition if:
 - \bullet the Shimura-Taniyama formula holds for Φ .
 - **1** the residuel field k_E of the reflex field E of (K,Φ) satisfies: $k_E \subseteq \mathbb{F}_q$.

Theorem (Chai-Conrad-Oort)

If (K,Φ) satisfies the RRC then in \mathscr{C}_h there exists an abelian variety A admitting a canonical lifting \mathscr{A} .

• If we understand the polarizations of A we can 'spread' them to the whole isogeny class.

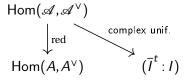
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Now \mathscr{C}_h over \mathbb{F}_p : let \mathscr{G} be the Centeleghe-Stix functor. Assume that there exists A admitting a canonical lifting \mathscr{A} .



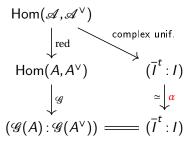
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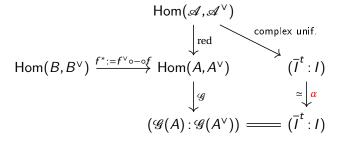
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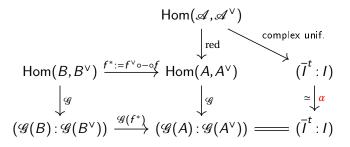
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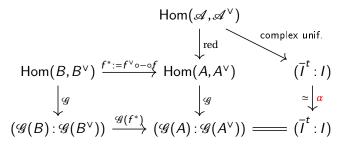
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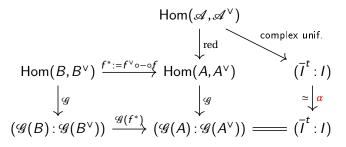
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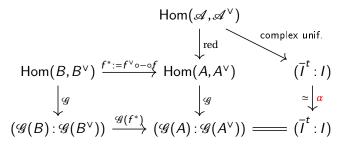
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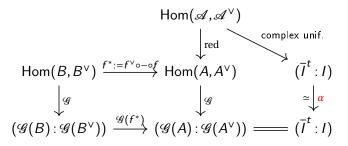
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• Base field extensions and twists (ordinary case).



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Thank you!

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