back to AV's: Dual variety/Polarization

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- a polarization μ of A corresponds to a $\lambda \in K^\times$ such that
  - $\lambda I \subseteq \bar{I}^t$ (isogeny);
  - λ is totally imaginary ($\bar{\lambda} = -\lambda$);
  - λ is Φ-positive, where Φ is a specific CM-type of K.

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  Also: $\deg \mu = [\overline{I}^t : I]$.
- if $A \leftrightarrow I$ and $S = (I : I)$ then

$$\left\{ \begin{matrix} \text{non-isomorphic} \\ \text{polarizations of } A \end{matrix} \right\} \longleftrightarrow \frac{\{\text{totally positive } u \in S^\times\}}{\{v\overline{v} : v \in S^\times\}}$$

  and $\mathrm{Aut}(A, \mu) = \{\text{torsion units of } S\}$

For elliptic curves the number of isomorphism classes can be expressed as a closed formula (Deuring, Waterhouse).
Let $h(x) = x^2 + \beta x + q$, with $q = p^r$ and β an integer coprime with p such that $\beta^2 < 4q$.
Put $F := x \bmod (h(x))$ in $K := \mathbb{Q}[x]/(h)$.
Then $\mathbb{Z}[F] = \mathbb{Z}[F, q/F]$ and

$$\mathrm{ICM}(\mathbb{Z}[F]) = \bigsqcup_{n|f} \mathrm{Pic}(\mathbb{Z} + n\mathscr{O}_K)$$

where $f := [\mathscr{O}_K : \mathbb{Z}[F]]$, which implies that

$$\#\left\{\begin{array}{l}\text{iso. classes of ell. curves}\\ \text{with } q-1+\beta \ \mathbb{F}_q\text{-points}\end{array}\right\} = \frac{\#\,\mathrm{Pic}(\mathscr{O}_K)}{\#\mathscr{O}_K^\times} \sum_{n|f} n \prod_{p|n} \left(1 - \frac{\Delta_K}{p}\frac{1}{p}\right)$$

Example

- Let $h(x) = x^8 - 5x^7 + 13x^6 - 25x^5 + 44x^4 - 75x^3 + 117x^2 - 135x + 81$;
- $\rightsquigarrow$ isogeny class of an simple ordinary abelian varieties over $\mathbb{F}_3$ of dimension 4;
- Let F be a root of $h(x)$ and put $R := \mathbb{Z}[F, 3/F] \subset \mathbb{Q}(F)$;
- 8 over-orders of R: two of them are not Gorenstein;
- $\#\mathrm{ICM}(R) = 18 \rightsquigarrow 18$ isom. classes of AV in the isogeny class;
- 5 are not invertible in their multiplicator ring;
- 8 classes admit principal polarizations;
- 10 isomorphism classes of princ. polarized AV.

Concretely:

$$
\begin{aligned}
I_1 =\, & 2645633792595191\mathbb{Z} \oplus (F + 836920075614551)\mathbb{Z} \oplus (F^2 + 1474295643839839)\mathbb{Z} \oplus \\
& \oplus (F^3 + 1372829830503387)\mathbb{Z} \oplus (F^4 + 1072904687510)\mathbb{Z} \oplus \\
& \oplus \frac{1}{3}(F^5 + F^4 + F^3 + 2F^2 + 2F + 6704806986143610)\mathbb{Z} \oplus \\
& \oplus \frac{1}{9}(F^6 + F^5 + F^4 + 8F^3 + 2F^2 + 2991665243621169)\mathbb{Z} \oplus \\
& \oplus \frac{1}{27}(F^7 + F^6 + F^5 + 17F^4 + 20F^3 + 9F^2 + 68015312518722201)\mathbb{Z}
\end{aligned}
$$

principal polarizations:

$$
\begin{aligned}
x_{1,1} = \frac{1}{27}(&-121922F^7 + 588604F^6 - 1422437F^5 + \\
& + 1464239F^4 + 1196576F^3 - 7570722F^2 + 15316479F - 12821193) \\
x_{1,2} = \frac{1}{27}(&3015467F^7 - 17689816F^6 + 35965592F^5 - \\
& - 64660346F^4 + 121230619F^3 - 191117052F^2 + 315021546F - 300025458)
\end{aligned}
$$

$$\mathrm{End}(I_1) = R$$

$$\#\mathrm{Aut}(I_1, x_{1,1}) = \#\mathrm{Aut}(I_1, x_{1,2}) = 2$$

Example

$$I_7 = 2\mathbb{Z} \oplus (F+1)\mathbb{Z} \oplus (F^2+1)\mathbb{Z} \oplus (F^3+1)\mathbb{Z} \oplus (F^4+1)\mathbb{Z} \oplus \frac{1}{3}(F^5+F^4+F^3+2F^2+2F+3)\mathbb{Z} \oplus$$
$$\oplus \frac{1}{36}(F^6+F^5+10F^4+26F^3+2F^2+27F+45)\mathbb{Z} \oplus$$
$$\oplus \frac{1}{216}(F^7+4F^6+49F^5+200F^4+116F^3+105F^2+198F+351)\mathbb{Z}$$

principal polarization:

$$x_{7,1} = \frac{1}{54}(20F^7-43F^6+155F^5-308F^4+580F^3-1116F^2+2205F-1809)$$

$$\mathrm{End}(I_7) = \mathbb{Z} \oplus F\mathbb{Z} \oplus F^2\mathbb{Z} \oplus F^3\mathbb{Z} \oplus F^4\mathbb{Z} \oplus \frac{1}{3}(F^5+F^4+F^3+2F^2+2F)\mathbb{Z} \oplus$$
$$\oplus \frac{1}{18}(F^6+F^5+10F^4+8F^3+2F^2+9F+9)\mathbb{Z} \oplus$$
$$\oplus \frac{1}{108}(F^7+4F^6+13F^5+56F^4+80F^3+33F^2+18F+27)\mathbb{Z}$$

$$\#\mathrm{Aut}(I_7, x_{7,1}) = 2$$

I_1 is invertible in R, but I_7 is not invertible in $\mathrm{End}(I_7)$.

Thank you!