Non Archimedean valuation

- $v : \mathbb{K} \longrightarrow \cancel{\mathbb{R}} \mathbb{Z} \cup \{\infty\}$

s.t. 1) $v(x) = \infty \iff x = 0$

2) $v(xy) = v(x) + v(y)$

3) $v(x+y) \geq \min\{v(x), v(y)\}$

Fix $\alpha \in (0,1) \subset \mathbb{R}$

def $|\cdot|_v = \alpha^{-v(\cdot)}$

~~1) $|x|_v = 0 \iff x = 0$~~

~~2) $|xy|_v = |x|_v |y|_v$~~

~~3) $|x+y|_v \leq \max\{|x|_v, |y|_v\}$~~

It's a norm on $\mathbb{K}$.

- Ex : $\underset{\neq 0}{x} \in \mathbb{Q}$, write $x = p^r \frac{a}{b}$, $p \nmid a,b$ $r \in \mathbb{Z}$

set $v_p(x) = r$ and $v_p(0) = +\infty$.

$$\mathbb{Q} \xrightarrow{\text{complete w.r.t } v_p(\cdot)} \mathbb{Q}_p = \frac{\{\text{Cauchy seq}\}}{\{\text{null-seq}\}} \quad \text{p-adic numbers}$$

$\mathbb{Z}_p := \{x \in \mathbb{Q}_p : v_p(x) \geq 0\}$ p-adic integers

$\mathbb{Z}_p$ it's a local ring with max ideal

$p\mathbb{Z}_p := \{x \in \mathbb{Q}_p : v_p(x) > 0\}$

- Cool thing : Every triangle in $(\mathbb{Q}^2, |\cdot|_p)$ is isosceles.

- L finite extension of $\mathbb{K}$. We can extend v to L by

$$\tilde{v}(x) := \frac{1}{[L:K]} \cdot v(\underset{\underset{\mathbb{K}}{\cap}}{N_{L/K}(x)})$$

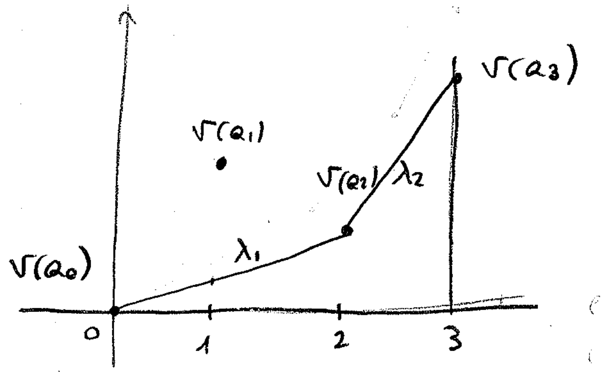
Note $\tilde{v}|_K = v$.

N.P of a polynomial

Let $\mathbb{K}$ field with a discrete valuation v,

$h(T) = \sum a_i T^i \in \mathbb{K}[T]$ a polynomial.

- Take the l.c.h of $(i, v(a_i))$.
  This is the N.P. of $h(T)$. $NP(h)$



$e(\lambda_1) = 2$

$e(\lambda_2) = 1$

$$\mathbb{Q} \longrightarrow \mathbb{Z}_{>0}$$
$$\lambda \longmapsto e(\lambda)$$

slope — multiplicity of the slope λ.

- $\#\{\alpha \in \overline{\mathbb{K}} : h(\alpha) = 0, v(\alpha) = -\lambda\} = e(\lambda)$
- $e(\lambda)\lambda \in \mathbb{Z}$ i.e. the "breaking pts" of the N.P. have integer coordinates.
- If $\lambda = \frac{a_\lambda}{b_\lambda}$, a_λ, b_λ coprime, then
  $$\sum_\lambda \underbrace{e(\lambda)/b_\lambda}_{= m(\lambda)} = \deg h$$
- It's enough to remember $(e(\lambda))_\lambda$ to recover $NP(h)$.
- p.o. rel on the N.P.'s:
  $\nu \prec \mu$ iff they have same end-pts and ν lies <u>above</u> μ

"smaller" strata $\subseteq$ closure of "larger" strata

N.P. of a curve

C smooth projective of genus g over $\mathbb{F}_q$, $q = p^n$ (3)

$$Z_{C/\mathbb{F}_q}(T) := \exp\left(\sum_{k\geq 1} \# C(\mathbb{F}_{q^k}) \frac{T^k}{k}\right)$$

Weil's conjecture (for a curve): (Deligne, Dwork, et al)

$$Z_{C/\mathbb{F}_q} \in \mathbb{Q}[[T]]$$
$$= \frac{L_{C/\mathbb{F}_q}(T)}{(1-T)(1-qT)}$$

and $L_{C/\mathbb{F}_q}(T) \in \mathbb{Z}[T]$

Over $\overline{\mathbb{Q}}$: $L_{C/\mathbb{F}_q}(T) = \prod_{1\leq j\leq 2g} (1-\alpha_j T)$

such that (eventually after reordering)

$$* \quad \alpha_j \alpha_{g+j} = q \quad \forall\, 1 \leq j \leq g.$$

- Each α_j has archimedean size q, i.e. $\forall\ i : \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$ we have $|i(\alpha_j)| = \sqrt{q}$!
- but the p-adic absolute values are not constant
- Normalize the p-adic valuation st $v(q) = 1$

and def $N.P.(C) := N.P.(L_{C/\mathbb{F}_q}(T))$

- $*$ implies that : A $e(\lambda) = 0$ if $\lambda \notin \mathbb{Q} \cap [0,1]$

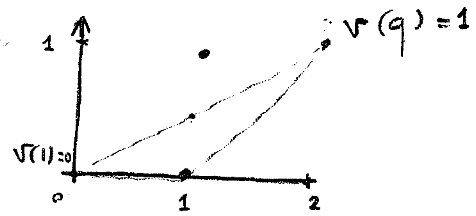
  B $e(\lambda) = e(1-\lambda)$

A N.P. satisfying A, B, $e(\lambda)\lambda \in \mathbb{Z}$ is called an admissable symmetric N.P. of height $\sum_\lambda \frac{e(\lambda)}{b_\lambda}$

Example

Let $E/\mathbb{F}_q$ be an elliptic curve, i.e. a projective smooth curve of genus 1. $\left(y^2 = x^3 + a_i x + b, \quad \Delta = -16(4a^3+27b^2) \neq 0 \right)$

Then $$Z(E/\mathbb{F}_q) = \frac{1 - aT + qT^2}{(1-T)(1-qT)}$$ for some $|a| \leq 2\sqrt{q}$. $a \in \mathbb{Z}$



1) $p \nmid a \Rightarrow v(a) = 0$ ordinary

2) $p \mid a \Rightarrow v(a) = 1$ supersingular

Ab. varieties

Def An ab. variety A over a field k is

- a group scheme over k : i.e. $\exists$ k-maps

$$m : A \times_k A \to A$$
$$inv : A \to A$$

and a k-point e that satisfies the group axioms or equivalently that $A(T)$ is a group $\forall$ k-scheme T

+ A of finite type (over k)
  (i.e. A has an finite affine open cover of fin. gen. k-alg)

+ geom. integral ($\mathcal{O}_X(U)$ int. domain $\forall$ U open in A + geom.)

+ Proper over k (i.e. $A \times_k B \to B$ is closed $\forall$ k-scheme B).

Facts
- A projective.
- The group law on A is commutative. (not because of this are called ab)
- For every smooth projective curve of genus g over k there $\exists$ an ab. var. J of dim. g over k s.t. $J(K) \cong \mathrm{Pic}^0(C_K)$ $\forall$ K/k st $C(K) \neq \phi$.
- $$\begin{aligned} \mathcal{M}_g &\longmapsto \mathcal{A}_g \\ [C] &\longmapsto [\mathrm{Jac}(C)] \end{aligned}$$
- If $\mathrm{char}\, k = p > 0$ and $N \geq 2$, $p \nmid N$ then $A[N]$ is étale and $A[N](\bar{k}) \cong (\mathbb{Z}/N\mathbb{Z})^{\oplus 2g}$.

Not true for $A[p]$:

$$A[p](\bar{k}) \simeq (\mathbb{Z}/p\mathbb{Z})^{\oplus f}$$

for some $0 \leq f \leq g$.

f is called the p-rank of A.

- Given such A, consider the associated p-divisible group $A[p^\infty] = \varinjlim_n A[p^n]$.

- There is a canonical way to associate a N.P. to every p-divisible group G over an alg. closed field $\bar{k}$

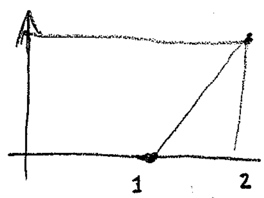
$$G \xrightarrow{\sim} \bigoplus_\lambda H_\lambda^{\oplus m_G(\lambda)} \quad \text{isogeny}$$

$$\text{If } \lambda = \frac{a_\lambda}{b_\lambda}, \text{ set } e_G(\lambda) = b_\lambda m_G(\lambda)$$

- Def the N.P. of A is the N.P. of $A[p^\infty]_{\bar{k}}$

- Note: "$NP(C) = NP(\mathrm{Jac}(C))$."

- "$e(0) = p$-rank"



- Newton stratification is finer than the p-rank one