Computing isomorphism classes of abelian varieties over finite fields.

Stefano Marseglia

Utrecht University

Curves over Finite Fields:
Past, Present and Future

Abelian varieties over $\mathbb{C}$ vs $\mathbb{F}_q$

- Let $A/\mathbb{C}$ be an abelian variety of dimension g.
- Then $A(\mathbb{C})$ is a **torus**: $T := \mathbb{C}^g/\Lambda$, where $\Lambda \simeq_{\mathbb{Z}} \mathbb{Z}^{2g}$.
- Moreover, T admits a non-degenerate **Riemann form** $\longleftrightarrow$ polarization.
- Actually, $A \mapsto A(\mathbb{C})$ induces an equivalence of categories:

$$\{\text{abelian varieties } /\mathbb{C}\} \longleftrightarrow \left\{\begin{array}{c}\mathbb{C}^g/\Lambda \text{ with } \Lambda \simeq \mathbb{Z}^{2g} \text{ admitting} \\ \text{a Riemann form}\end{array}\right\}.$$

- In char. $p > 0$ such an equivalence cannot exist : there are (supersingular) elliptic curves with quaternionic endomorphism algebras.
- Nevertheless, over finite fields, we obtain analogous results if we restrict ourselves to certain **subcategories** of AVs.

Isogeny classification over $\mathbb{F}_q$

- $A/\mathbb{F}_q$ comes with a **Frobenius** endomorphism, that induces an action

$$\mathrm{Frob}_A : T_\ell A \to T_\ell A \text{ for any } \ell \neq p,$$

  where $T_\ell(A) = \varprojlim A[\ell^n] \simeq \mathbb{Z}_\ell^{2g}$.
- $h_A(x) := \mathrm{char}(\mathrm{Frob}_A)$ is a q-Weil polynomial and **isogeny invariant**.
- By **Honda-Tate** theory, the association

$$\text{isogeny class of } A \longmapsto h_A(x)$$

  is injective and allows us to enumerate all AVs up to isogeny.
- Also, $h_A(x)$ is squarefree $\iff$ $\mathrm{End}(A)$ is commutative.

Deligne's equivalence

Recall: $A/\mathbb{F}_q$ is **ordinary** if half of the p-adic roots of h_A are units.

Theorem (Deligne '69)

Let $q = p^r$, with p a prime. There is an equivalence of categories:

$$\begin{array}{ccc} \{\ \textbf{Ordinary}\ \textit{abelian varieties over}\ \mathbb{F}_q\} & & A \\ \updownarrow & & \downarrow \\ \left\{\begin{array}{l} \textit{pairs}\ (T,F),\ \textit{where}\ T \simeq_{\mathbb{Z}} \mathbb{Z}^{2g}\ \textit{and}\ T \xrightarrow{F} T\ \textit{s.t.} \\ \text{- } F\otimes\mathbb{Q}\ \textit{is semisimple} \\ \text{- the roots of } \mathrm{char}_{F\otimes\mathbb{Q}}(x)\ \textit{have abs. value } \sqrt{q} \\ \text{- } \textbf{\textit{half of them are } p\textit{-adic units}} \\ \text{- } \exists V : T \to T\ \textit{such that}\ FV = VF = q \end{array}\right\} & & (T(A), F(A)) \end{array}$$

- Ordinary $A/\mathbb{F}_q$ can be canonically lifted: $\rightsquigarrow \mathscr{A}_{\mathrm{can}}/\mathrm{Witt}(\mathbb{F}_q)$...
- ... characterized by: $\mathrm{End}_{\mathbb{F}_q}(A) = \mathrm{End}_W(\mathscr{A}_{\mathrm{can}})$.
- Put $T(A) := H_1(\mathscr{A}_{\mathrm{can}} \otimes \mathbb{C}, \mathbb{Z})$ and $F(A) :=$ the induced Frobenius.

Squarefree case

- Fix an **ordinary squarefree** q-Weil polynomial h :
- $\rightsquigarrow$ an isogeny class $\mathscr{C}_h/\mathbb{F}_q$.
- Put $K := \mathbb{Q}[x]/(h) = \mathbb{Q}[F]$, an étale algebra = product of number fields.
- Put $V = q/F$. Deligne's equivalence induces:

Theorem

$$\{\textit{abelian varieties over } \mathbb{F}_q \textit{ in } \mathscr{C}_h\}_{/\simeq}$$
$$\updownarrow$$
$$\{\textit{fractional ideals of } \mathbb{Z}[F,V] \subset K\}_{/\simeq} \quad =: \mathrm{ICM}(\mathbb{Z}[F,V])$$
$$\textit{ideal class monoid}$$

- Problem: $\mathbb{Z}[F,V]$ might not be maximal $\rightsquigarrow$ non-invertible ideals.

ICM : Ideal Class Monoid

Let R be an **order** in an étale $\mathbb{Q}$-algebra K.

- Recall: for **fractional R-ideals** I and J

$$I \simeq_R J \iff \exists x \in K^\times \text{ s.t. } xI = J$$

- We have

$$\mathrm{ICM}(R) \supseteq \mathrm{Pic}(R) = {}^{\{\text{invertible fractional } R\text{-ideals}\}}\!/_{\simeq_R}$$

with equality $\updownarrow$ iff $R = \mathscr{O}_K$

- ...and actually

$$\mathrm{ICM}(R) \supseteq \bigsqcup_{\substack{R \subseteq S \subseteq \mathscr{O}_K \\ \text{over-orders}}} \mathrm{Pic}(S) \qquad \text{with equality iff } R \text{ is Bass}$$

- Hofmann-Sircana '19: computation of over-orders.

simplify the problem

Study the isomorphism problem locally: (Dade, Taussky, Zassenhaus '62)

- **weak equivalence**:

$$I_{\mathfrak{p}} \simeq_{R_{\mathfrak{p}}} J_{\mathfrak{p}} \text{ for every } \mathfrak{p} \in \mathrm{mSpec}(R)$$

$$\Updownarrow$$

$$1 \in (I : J)(J : I) \quad \text{easy to check!}$$

- Let $\mathscr{W}(R)$ be the set of weak eq. classes...
  ...whose representatives can be found in

$$\left\{\text{sub-}R\text{-modules of } \mathscr{O}_K/\mathfrak{f}_R\right\} \quad \text{finite! and most of the time not-too-big ...}$$

  where $\mathfrak{f}_R = (R : \mathscr{O}_K)$ is the conductor of R.

Compute $\mathrm{ICM}(R)$

Partition w.r.t. the multiplicator ring:

$$\mathscr{W}(R) = \bigsqcup_{R \subseteq S \subseteq \mathscr{O}_K} \mathscr{W}_S(R)$$

$$\mathrm{ICM}(R) = \bigsqcup_{R \subseteq S \subseteq \mathscr{O}_K} \mathrm{ICM}_S(R)$$

the "pedix" $-_S$ means "only classes with multiplicator ring S"

Theorem (M.)

For every over-order S of R, $\mathrm{Pic}(S)$ acts freely on $\mathrm{ICM}_S(R)$ and

$$\mathscr{W}_S(R) = \mathrm{ICM}_S(R)/\mathrm{Pic}(S)$$

Repeat for every $R \subseteq S \subseteq \mathscr{O}_K$:

$$\rightsquigarrow \mathrm{ICM}(R).$$

To sum up:

- To sum up:
- Given a **ordinary squarefree** q-Weil polynomial h ...
- ... $\rightsquigarrow$ algorithm to compute the isomorphism classes of AVs in $\mathscr{C}_h$.
- We can actually get a lot more!

Dual varieties and Polarizations

Howe described **dual** varieties and **polarizations** on Deligne modules.

Theorem

Let $A \in \mathscr{C}_h$ with h ordinary and squarefree. If $A \leftrightarrow I$, then:

- $A^\vee \leftrightarrow \overline{I}^t := \{\overline{x} \in K : \mathrm{Tr}(xI) \subseteq \mathbb{Z}\}$.
- *a polarization μ of A corresponds to a $\lambda \in K^\times$ such that*
  - $\lambda I \subseteq \overline{I}^t$ (isogeny);
  - λ is totally imaginary ($\overline{\lambda} = -\lambda$);
  - λ is Φ-positive, where Φ is a CM-type of K satisf. the Shimura-Taniyama formula.
  Also: $\deg \mu = [\overline{I}^t : \lambda I]$.
- *if $(A, \mu) \leftrightarrow (I, \lambda)$ is a princ. polarized ab. var. and $S = (I : I)$ then*

$$\left\{\begin{matrix}\text{non-isomorphic princ.} \\ \text{polarizations of } A\end{matrix}\right\} \longleftrightarrow \frac{\{\text{totally positive } u \in S^\times\}}{\{v\overline{v} : v \in S^\times\}},$$

- *and* $\mathrm{Aut}(A, \mu) = \{\text{torsion units of } S\}$.

Principal Polarizations

We have an **algorithm** to enumerate principal polarizations up to isomorphism:

1. Compute i_0 such that $i_0 I = \overline{I}^t$.
2. Loop over the representatives u of the finite quotient

$$\frac{S^\times}{\{v\overline{v} : v \in S^\times\}}.$$

3. If $\lambda := i_0 u$ is totally imaginary and Φ-positive ...
4. ... then we have one principal polarization.
5. By the previous Theorem, we have all princ. polarizations up to isom.

Can modify to compute polarizations of any degree.

Example

- Let $h(x) = x^8 - 5x^7 + 13x^6 - 25x^5 + 44x^4 - 75x^3 + 117x^2 - 135x + 81$.
- $\rightsquigarrow$ isogeny class of an simple ordinary abelian varieties over $\mathbb{F}_3$ of dimension 4.
- Let F be a root of $h(x)$ and put $R := \mathbb{Z}[F, 3/F] \subset \mathbb{Q}(F)$.
- 8 over-orders of R: two of them are not Gorenstein.
- $\#\mathrm{ICM}(R) = 18 \rightsquigarrow 18$ isom. classes of AV in the isogeny class.
- 5 are not invertible in their multiplicator ring.
- 8 classes admit principal polarizations.
- 10 isomorphism classes of princ. polarized AV.

Example

Concretely:

$$\begin{aligned} I_1 =& 2645633792595191\mathbb{Z} \oplus (F + 836920075614551)\mathbb{Z} \oplus (F^2 + 1474295643839839)\mathbb{Z} \oplus \\ & \oplus (F^3 + 1372829830503387)\mathbb{Z} \oplus (F^4 + 1072904687510)\mathbb{Z} \oplus \\ & \oplus \frac{1}{3}(F^5 + F^4 + F^3 + 2F^2 + 2F + 6704806986143610)\mathbb{Z} \oplus \\ & \oplus \frac{1}{9}(F^6 + F^5 + F^4 + 8F^3 + 2F^2 + 2991665243621169)\mathbb{Z} \oplus \\ & \oplus \frac{1}{27}(F^7 + F^6 + F^5 + 17F^4 + 20F^3 + 9F^2 + 6801531251872201)\mathbb{Z} \end{aligned}$$

principal polarizations:

$$\begin{aligned} x_{1,1} = \frac{1}{27}(&-121922F^7 + 588604F^6 - 1422437F^5 + \\ &+ 1464239F^4 + 1196576F^3 - 7570722F^2 + 15316479F - 12821193) \end{aligned}$$

$$\begin{aligned} x_{1,2} = \frac{1}{27}(&3015467F^7 - 17689816F^6 + 35965592F^5 - \\ &- 64660346F^4 + 121230619F^3 - 191117052F^2 + 315021546F - 300025458) \end{aligned}$$

$\mathrm{End}(I_1) = R$

$\#\mathrm{Aut}(I_1, x_{1,1}) = \#\mathrm{Aut}(I_1, x_{1,2}) = 2$

Example

$$I_7 = 2\mathbb{Z} \oplus (F+1)\mathbb{Z} \oplus (F^2+1)\mathbb{Z} \oplus (F^3+1)\mathbb{Z} \oplus (F^4+1)\mathbb{Z} \oplus \frac{1}{3}(F^5+F^4+F^3+2F^2+2F+3)\mathbb{Z} \oplus$$
$$\oplus \frac{1}{36}(F^6+F^5+10F^4+26F^3+2F^2+27F+45)\mathbb{Z} \oplus$$
$$\oplus \frac{1}{216}(F^7+4F^6+49F^5+200F^4+116F^3+105F^2+198F+351)\mathbb{Z}$$

principal polarization:

$$x_{7,1} = \frac{1}{54}(20F^7-43F^6+155F^5-308F^4+580F^3-1116F^2+2205F-1809)$$

$$\mathrm{End}(I_7) = \mathbb{Z} \oplus F\mathbb{Z} \oplus F^2\mathbb{Z} \oplus F^3\mathbb{Z} \oplus F^4\mathbb{Z} \oplus \frac{1}{3}(F^5+F^4+F^3+2F^2+2F)\mathbb{Z} \oplus$$
$$\oplus \frac{1}{18}(F^6+F^5+10F^4+8F^3+2F^2+9F+9)\mathbb{Z} \oplus$$
$$\oplus \frac{1}{108}(F^7+4F^6+13F^5+56F^4+80F^3+33F^2+18F+27)\mathbb{Z}$$

$$\#\mathrm{Aut}(I_7, x_{7,1}) = 2$$

I_1 is invertible in R, but I_7 is not invertible in $\mathrm{End}(I_7)$.

The Power-of-a-Bass case

- Another case we understand well: $h = g^r$ for g square-free and ordinary.
- Every A in $\mathscr{C}_{g^r}$ is $A \sim B^r$ for $B \in \mathscr{C}_g$.
- Put $R := \mathbb{Z}[F, V] \subset K_g := \mathbb{Q}[x]/(g) = \mathbb{Q}[F]$.
- Under these assumption, Deligne's theorem induces:

$$\{\text{abelian varieties in } \mathscr{C}_{g^r}\} \longleftrightarrow \left\{R\text{-modules } M \subseteq K_g^r\right\}.$$

- Recall: an order R is **Bass** if all its over-orders S are **Gorenstein**, ...
- ... or equivalently $\mathrm{ICM}(R) = \bigsqcup_S \mathrm{Pic}(S)$.
- Eg: quadratic orders are Bass $\rightsquigarrow$ powers of ordinary elliptic curves E^r.
- If R is Bass, then M is isomorphic to a direct sum of frac.R-ideals.

The Power-of-a-Bass case

Theorem

$$\{\text{abelian varieties in } \mathscr{C}_{g^r}\}_{/\simeq} \longleftrightarrow \{I_1 \oplus \ldots \oplus I_r \ : \ I_j \text{ a frac. } R\text{-ideal}\}_{/\simeq}$$

$$\text{we have a classification:} \longleftrightarrow \left\{ (S_1 \subseteq S_2 \subseteq \ldots \subseteq S_r,\ [I]_{\simeq}) : \begin{array}{l} R \subseteq S_j \text{ orders,} \\ I \text{ a frac. } R\text{-ideal} \\ \text{with } (I:I) = S_r \end{array} \right\}$$

Corollary

If $A \in \mathscr{C}_{g^r}$ then $A \simeq C_1 \times \ldots \times C_r$, for $C_j \in \mathscr{C}_g$. *everything is a product!*

- Howe's results on polarizations carry over ...
- ... but computing them in general is harder!
- Solved for E^r by Kirschmer-Narbonne-Ritzenthaler-Robert '20. next talk!

Example

Let $g = x^6 - 3x^5 + 6x^4 - 10x^3 + 18x^2 - 27x + 27$. Note $\mathscr{C}(g)$ is an isogeny class of simple ordinary abelian varieties over $\mathbb{F}_3$. Define $K = \mathbb{Q}[x]/(g) = \mathbb{Q}(F)$ and $R = \mathbb{Z}[F, V]$. The only over-order of R is the maximal order $\mathscr{O}_K$ of K and, since R is Gorenstein R is Bass. Observe

$$\mathrm{Pic}(R) \simeq {}^{\mathbb{Z}}\!/_{3\mathbb{Z}} \text{ and } \mathrm{Pic}(\mathscr{O}_K) = \{1\}.$$

Let I be a representatives of a generator of $\mathrm{Pic}(R)$. We now list the representatives of the isomorphism classes in $\mathscr{C}(g^3)$:

$$M_1 = R \oplus R \oplus R \qquad M_2 = R \oplus R \oplus I \qquad M_3 = R \oplus R \oplus I^2$$
$$M_4 = R \oplus R \oplus \mathscr{O}_K \qquad M_5 = R \oplus \mathscr{O}_K \oplus \mathscr{O}_K \qquad M_6 = \mathscr{O}_K \oplus \mathscr{O}_K \oplus \mathscr{O}_K$$

$$\mathrm{End}(M_1) = \mathrm{Mat}_3(R) \text{ and } \mathrm{End}(M_2) = \begin{pmatrix} R & R & I \\ R & R & I \\ (R:I) & (R:I) & R \end{pmatrix}$$

Outside of the ordinary...

Theorem (Centeleghe-Stix '15)

There is an equivalence of categories:

$$\begin{array}{ccc} \{\text{abelian varieties } A \text{ over } \mathbb{F}_p \text{ with } \boldsymbol{h_A(\sqrt{p}) \neq 0}\} & & A \\ \updownarrow & & \downarrow \\ \left\{\begin{array}{l} \text{pairs } (T,F), \text{ where } T \simeq_{\mathbb{Z}} \mathbb{Z}^{2g} \text{ and } T \xrightarrow{F} T \text{ s.t.} \\ \text{- } F \otimes \mathbb{Q} \text{ is semisimple} \\ \text{- the roots of } \mathrm{char}_{F\otimes\mathbb{Q}}(x) \text{ have abs. value } \sqrt{p} \\ \text{- } \boldsymbol{\mathrm{char}_F(\sqrt{p}) \neq 0} \\ \text{- } \exists V : T \to T \text{ such that } FV = VF = p \end{array}\right\} & & (T(A), F(A)) \end{array}$$

- Now, $T(A) := \mathrm{Hom}(A, A_w)$, where A_w has minimal End among the varieties with Weil support $w = w(A)$.
- $F(A)$ is the induced Frobenius.

Outside of the ordinary...isomorphism classes

- Everything I told so far about isomorphism classes works in the same way using the Centeleghe-Stix functor:
- both in the squarefree and Power-of-Bass cases, over $\mathbb{F}_p$.
- (There are there are other functors:
  Oswal-Shankar for almost-ordinary abelian varieties, together with generalzations by Bergström-Karemaker-M.;
  Serre and the work of Kani, Jordan-Keeton-Poonen-Rains--Shepherd-Barron-Tate for E^r; Isabel Vogt's talk
  ... but we will not use them in this talk.)
- For polarizations, the results by Howe do not apply immediately to the Centeleghe-Strix case:
- in general we cannot lift canonically each abelian variety.

Outside of the ordinary...polarizations

- New strategy: with Jonas Bergström and Valentijn Karemaker '21.
- Consider $\mathscr{C}_h$ with h squarefree $/\mathbb{F}_q \rightsquigarrow K = \mathbb{Q}[F]$.
- Chai-Conrad-Oort: A (p-adic) CM-type (K, Φ) satisfies the **Residual Reflex Condition** if:
  1. the Shimura-Taniyama formula holds for Φ.
  2. the residuel field k_E of the reflex field E of (K, Φ) satisfies: $k_E \subseteq \mathbb{F}_q$.

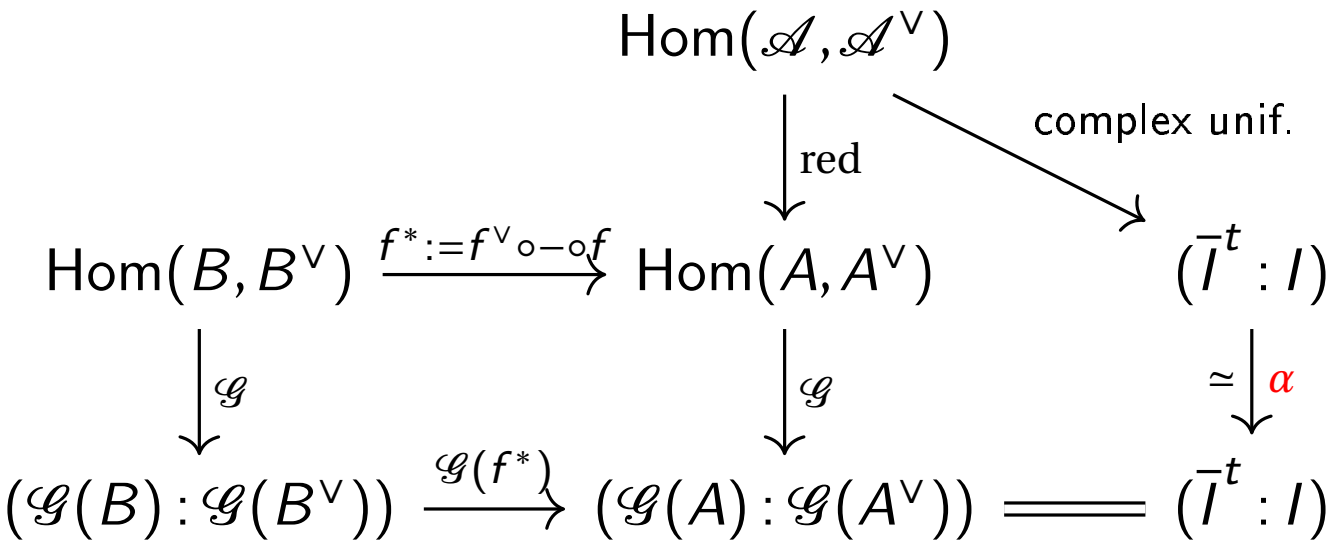
Theorem (Chai-Conrad-Oort)

If (K, Φ) satisfies the RRC then in $\mathscr{C}_h$ there exists an abelian variety A admitting a canonical lifting $\mathscr{A}$.

- If we understand the polarizations of A we can 'spread' them to the whole isogeny class.

Outside of the ordinary...polarizations

Now $\mathscr{C}_h$ over $\mathbb{F}_p$: let $\mathscr{G}$ be the Centeleghe-Stix functor.
Assume that there exists A admitting a canonical lifting $\mathscr{A}$.
Let $f : A \to B$ be an isogeny.



Note that $\mathscr{G}(f^*)$ is multiplication by the totally positive element $\overline{\mathscr{G}(f)}\mathscr{G}(f)$: it sends totally imaginary elements to totally imaginary elements and Φ-positive elements to Φ-positive elements. The only 'issue' is the α.
We study when we can 'pretend' $\alpha = 1$.

Final remarks

- Base field extensions and **twists** (ordinary case).
- **Period matrices** of the canonical lift (ordinary case).
- with Caleb Springer '21, building on Howe-Kedlaya '21, we proved that every finite abelian group occur as the **group of points** of an ordinary AV over $\mathbb{F}_2$. check Wanlin Li's talk!
- Magma implementations of the algorithms are on GitHub!
- Results of computations will appear on the LMFDB.

Thank you!