# Computing isomorphism classes of abelian varieties over finite fields

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- Also, T admits a non-degenerate Riemann form  $\longleftrightarrow$  polarization.

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- Also, T admits a non-degenerate Riemann form  $\longleftrightarrow$  polarization.
- The functor  $A \mapsto A(\mathbb{C})$  induces an equivalence of categories:

$$\left\{ \text{abelian varieties } / \mathbb{C} \right\} \longleftrightarrow \left\{ \begin{matrix} \mathbb{C}^g / \Lambda \text{ with } \Lambda \simeq \mathbb{Z}^{2g} \text{ admitting} \\ \text{a Riemann form} \end{matrix} \right\}.$$

- In char. p > 0 such an equivalence cannot exist: there are (supersingular) elliptic curves with quaternionic endomorphism algebras.
- Nevertheless, over finite fields, we obtain analogous results if we restrict ourselves to certain subcategories of AVs.

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# Isogeny classification over $\mathbb{F}_q$

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# Isogeny classification over $\mathbb{F}_a$

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Frob<sub>A</sub>: 
$$T_{\ell}A \rightarrow T_{\ell}A$$
 for any  $\ell \neq p$ ,

where 
$$T_{\ell}(A) = \varprojlim A[\ell^n] \simeq \mathbb{Z}_{\ell}^{2g}$$
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•  $h_A(x) := \text{char}(\text{Frob}_A)$  is a *q*-Weil polynomial and **isogeny invariant**.

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is injective and allows us to enumerate all AVs up to isogeny.

• Also,  $h_A(x)$  is squarefree  $\iff$  End(A) is commutative.

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Recall:  $A/\mathbb{F}_q$  is **ordinary** if half of the *p*-adic roots of  $h_A$  are units.

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Theorem (Deligne [?])

Let  $q = p^r$ , with p a prime. There is an equivalence of categories:

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- Put  $T(A) := H_1(\mathscr{A}_{\operatorname{can}} \otimes \mathbb{C}, \mathbb{Z})$  and F(A) :=the induced Frobenius.

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- $\rightsquigarrow$  an isogeny class  $\mathscr{C}_h/\mathbb{F}_q$ .

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#### **Theorem**

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 \begin{aligned} & \left\{ \text{abelian varieties over } \mathbb{F}_q \text{ in } \mathscr{C}_h \right\}_{\simeq} \\ & \downarrow \\ & \left\{ \text{fractional ideals of } \mathbb{Z}[F,V] \subset K \right\}_{\simeq} \end{aligned}
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#### **Theorem**

• Problem:  $\mathbb{Z}[F, V]$  might not be maximal  $\rightsquigarrow$  non-invertible ideals.

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Let R be an **order** in an étale  $\mathbb{Q}$ -algebra K.

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• Hofmann-Sircana [?]: computation of over-orders.

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First, locally: Dade-Taussky-Zassenhaus [?].

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• weak equivalence:

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### simplify the problem

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• Let W(R) be the set of weak eq. classes... ...whose representatives can be found in

$$\left\{ \text{sub-}R\text{-modules of } \mathscr{O}_{K/\mathfrak{f}_{R}} \right\} \quad \begin{array}{l} \text{finite! and most of the} \\ \text{time not-too-big } \dots \end{array}$$

where  $f_R = (R : \mathcal{O}_K)$  is the conductor of R.

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Partition w.r.t. the multiplicator ring:

$$W(R) = \bigsqcup_{R \subseteq S \subseteq \mathcal{O}_K} W_S(R)$$

$$ICM(R) = \bigsqcup_{R \subseteq S \subseteq \mathcal{O}_K} ICM_S(R)$$

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#### Theorem ([?])

For every over-order S of R, Pic(S) acts freely on  $ICM_S(R)$  and

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#### Theorem ([?])

For every over-order S of R, Pic(S) acts freely on  $ICM_S(R)$  and

$$W_S(R) = ICM_S(R)/Pic(S)$$

Repeat for every  $R \subseteq S \subseteq \mathcal{O}_K$ :

$$\rightsquigarrow ICM(R)$$
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- To sum up:
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- To sum up:
- Given a **ordinary squarefree** *q*-Weil polynomial *h* ...
- ullet ...  $\leadsto$  algorithm to compute the isomorphism classes of AVs in  $\mathscr{C}_h$ .
- We can actually get a lot more!

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Howe [?]: dual varieties and polarizations on Deligne modules.

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Theorem ([?])

Let  $A \in \mathcal{C}_h$  with h ordinary and squarefree. If  $A \leftrightarrow I$ , then:

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Let  $A \in \mathcal{C}_h$  with h ordinary and squarefree. If  $A \leftrightarrow I$ , then:

•  $A^{\vee} \leftrightarrow \overline{I}^t := \{ \overline{x} \in K : \operatorname{Tr}(xI) \subseteq \mathbb{Z} \}.$ 

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- a polarization  $\mu$  of A corresponds to a  $\lambda \in K^{\times}$  such that
  - $\lambda I \subseteq \overline{I}^t$  (isogeny of  $\deg \mu = [\overline{I}^t : \lambda I]$ );
  - $\lambda$  is totally imaginary  $(\overline{\lambda} = -\lambda)$ ;
  - $\lambda$  is  $\Phi$ -positive  $(\Im \varphi(\lambda) > 0$  for all  $\varphi \in \Phi$ ),

where  $\Phi$  is a CM-type of K satisf. the Shimura-Taniyama formula.

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- if  $(A, \mu) \leftrightarrow (I, \lambda)$  is a princ. polarized ab. var. and S = (I:I) then  $\begin{cases} \text{non-isomorphic princ.} \\ \text{polarizations of } A \end{cases} \longleftrightarrow \frac{\{\text{totally positive } u \in S^{\times}\}}{\{v\overline{v}: v \in S^{\times}\}},$

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- if  $(A, \mu) \leftrightarrow (I, \lambda)$  is a princ. polarized ab. var. and S = (I:I) then  $\begin{cases} \text{non-isomorphic princ.} \\ \text{polarizations of } A \end{cases} \longleftrightarrow \frac{\{\text{totally positive } u \in S^\times\}}{\{v\overline{v}: v \in S^\times\}}, \text{statement for } \deg \mu > 1$

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Howe [?]: dual varieties and polarizations on Deligne modules.

#### Theorem ([?])

Let  $A \in \mathcal{C}_h$  with h ordinary and squarefree. If  $A \leftrightarrow I$ , then:

- $A^{\vee} \leftrightarrow \overline{I}^t := \{ \overline{x} \in K : \operatorname{Tr}(xI) \subseteq \mathbb{Z} \}.$
- a polarization  $\mu$  of A corresponds to a  $\lambda \in K^{\times}$  such that
  - $\lambda I \subseteq \overline{I}^t$  (isogeny of  $\deg \mu = [\overline{I}^t : \lambda I]$ );
  - $\lambda$  is totally imaginary  $(\overline{\lambda} = -\lambda)$ ;
  - $\lambda$  is  $\Phi$ -positive ( $\Im \varphi(\lambda) > 0$  for all  $\varphi \in \Phi$ ), where  $\Phi$  is a CM-type of K satisf. the Shimura-Taniyama formula.
- if  $(A, \mu) \leftrightarrow (I, \lambda)$  is a princ. polarized ab. var. and S = (I:I) then  $\begin{cases} \text{non-isomorphic princ.} \\ \text{polarizations of } A \end{cases} \longleftrightarrow \frac{\{\text{totally positive } u \in S^\times\}}{\{v\overline{v}: v \in S^\times\}}, \text{statement for } \deg \mu > 1$

• and  $Aut(A, \mu) = \{torsion \ units \ of \ S\}.$ 

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We have an **algorithm** to enumerate principal polarizations up to isomorphism:

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We have an **algorithm** to enumerate principal polarizations up to isomorphism:

- **1** Compute  $i_0$  such that  $i_0I = \overline{I}^t$ .
- $oldsymbol{2}$  Loop over the representatives u of the finite quotient

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- **3** If  $\lambda := i_0 u$  is totally imaginary and  $\Phi$ -positive ...
- 4 ... then we have one principal polarization.
- **5** By the previous Theorem, we have all princ. polarizations up to isom.

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• Let  $h(x) = x^8 - 5x^7 + 13x^6 - 25x^5 + 44x^4 - 75x^3 + 117x^2 - 135x + 81$ .



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- $\leadsto$  isogeny class of an simple ordinary abelian varieties over  $\mathbb{F}_3$  of dimension 4.

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- 5 are not invertible in their multiplicator ring.
- 8 classes admit principal polarizations.
- 10 isomorphism classes of princ. polarized AV.

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#### Concretely:

$$\begin{split} I_1 = & 2645633792595191\mathbb{Z} \oplus (F + 836920075614551)\mathbb{Z} \oplus (F^2 + 1474295643839839)\mathbb{Z} \oplus \\ & \oplus (F^3 + 1372829830503387)\mathbb{Z} \oplus (F^4 + 1072904687510)\mathbb{Z} \oplus \\ & \oplus \frac{1}{3}(F^5 + F^4 + F^3 + 2F^2 + 2F + 6704806986143610)\mathbb{Z} \oplus \\ & \oplus \frac{1}{9}(F^6 + F^5 + F^4 + 8F^3 + 2F^2 + 2991665243621169)\mathbb{Z} \oplus \\ & \oplus \frac{1}{27}(F^7 + F^6 + F^5 + 17F^4 + 20F^3 + 9F^2 + 68015312518722201)\mathbb{Z} \end{split}$$

principal polarizations:

$$x_{1,1} = \frac{1}{27} \left( -121922F^7 + 588604F^6 - 1422437F^5 + \right.$$

$$+ 1464239F^4 + 1196576F^3 - 7570722F^2 + 15316479F - 12821193 \right)$$

$$x_{1,2} = \frac{1}{27} \left( 3015467F^7 - 17689816F^6 + 35965592F^5 - \right.$$

$$- 64660346F^4 + 121230619F^3 - 191117052F^2 + 315021546F - 300025458 \right)$$

$$\operatorname{End}(I_1) = R$$

$$\# \operatorname{Aut}(I_1, x_{1,1}) = \# \operatorname{Aut}(I_1, x_{1,2}) = 2$$

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$$\begin{split} I_7 = & 2\mathbb{Z} \oplus (F+1)\mathbb{Z} \oplus (F^2+1)\mathbb{Z} \oplus (F^3+1)\mathbb{Z} \oplus (F^4+1)\mathbb{Z} \oplus \frac{1}{3}(F^5+F^4+F^3+2F^2+2F+3)\mathbb{Z} \oplus \\ & \oplus \frac{1}{36}(F^6+F^5+10F^4+26F^3+2F^2+27F+45)\mathbb{Z} \oplus \\ & \oplus \frac{1}{216}(F^7+4F^6+49F^5+200F^4+116F^3+105F^2+198F+351)\mathbb{Z} \end{split}$$

principal polarization:

$$x_{7,1} = \frac{1}{54} (20F^7 - 43F^6 + 155F^5 - 308F^4 + 580F^3 - 1116F^2 + 2205F - 1809)$$

$$\operatorname{End}(I_7) = \mathbb{Z} \oplus F \mathbb{Z} \oplus F^2 \mathbb{Z} \oplus F^3 \mathbb{Z} \oplus F^4 \mathbb{Z} \oplus \frac{1}{3} (F^5 + F^4 + F^3 + 2F^2 + 2F) \mathbb{Z} \oplus$$

$$\oplus \frac{1}{18} (F^6 + F^5 + 10F^4 + 8F^3 + 2F^2 + 9F + 9) \mathbb{Z} \oplus$$

$$\oplus \frac{1}{108} (F^7 + 4F^6 + 13F^5 + 56F^4 + 80F^3 + 33F^2 + 18F + 27) \mathbb{Z}$$
#Aut $(I_7, x_{7,1}) = 2$ 

 $I_1$  is invertible in R, but  $I_7$  is not invertible in  $\operatorname{End}(I_7)$ 

#### The Power-of-a-Bass case

• Another case we understand well:  $h = g^r$  for g square-free and ordinary.

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- Every A in  $\mathscr{C}_{g^r}$  is  $A \sim B^r$  for  $B \in \mathscr{C}_g$ .

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 $\left\{\text{abelian varieties in }\mathscr{C}_{g^r}\right\} \longleftrightarrow \left\{R\text{-modules }M\subseteq K_g^r\right\}.$ 

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- ... or equivalently  $ICM(R) = \coprod_{S} Pic(S)$ . (see [?])

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- Eg: quadratic orders are Bass  $\rightsquigarrow$  powers of ordinary elliptic curves  $E^r$ .
- If R is Bass, then M is isomorphic to a direct sum of frac. R-ideals.

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```
Theorem ([?])

If R = \mathbb{Z}[F, V] is Bass then

\{abelian \ varieties \ in \ \mathscr{C}_{g^r}\}_{/\simeq} \longleftrightarrow \{I_1 \oplus \ldots \oplus I_r : I_j \ a \ frac. \ R-ideal\}_{/\simeq}

we have a classification: \{S_1 \subseteq S_2 \subseteq \ldots \subseteq S_r, \ [I]_\simeq\}: I \ a \ frac. \ R-ideal\}_{with}

with \{I:I\} = S_r
```

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#### Corollary

If  $A \in \mathcal{C}_{g^r}$  then  $A \simeq C_1 \times ... \times C_r$ , for  $C_j \in \mathcal{C}_g$ .

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#### Corollary

If 
$$A \in \mathcal{C}_{g^r}$$
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- ... but computing them in general is harder!
- Solved for E<sup>r</sup> by Kirschmer-Narbonne-Ritzenthaler-Robert [?].

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## Outside of the ordinary...

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## Outside of the ordinary...

#### Theorem (Centeleghe-Stix [?])

There is an equivalence of categories:

```
{abelian varieties A over \mathbb{F}_p with h_A(\sqrt{p}) \neq 0}
\downarrow
pairs <math>(T, F), where T \simeq_{\mathbb{Z}} \mathbb{Z}^{2g} and T \xrightarrow{F} T s.t.
-F \otimes \mathbb{Q} \text{ is semisimple}
- the roots of <math>\operatorname{char}_{F \otimes \mathbb{Q}}(x) \text{ have abs. value } \sqrt{p}
- \operatorname{char}_{F}(\sqrt{p}) \neq 0
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## Outside of the ordinary...

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$$- \operatorname{char}_{F}(\sqrt{p}) \neq 0$$

$$-\exists V : T \to T \text{ such that } FV = VF = p$$$$$$

- Now,  $T(A) := \text{Hom}(A, A_w)$ , where  $A_w$  has minimal End among the varieties with Weil support w = w(A).
- *F*(*A*) is the induced Frobenius.

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- Everything I told so far about isomorphism classes works in the same way using the Centeleghe-Stix functor:
- both in the squarefree and Power-of-Bass cases, over  $\mathbb{F}_p$ .

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- both in the squarefree and Power-of-Bass cases, over  $\mathbb{F}_p$ .
- For polarizations, the results by Howe do not apply immediately to the Centeleghe-Strix case:
- in general we cannot lift canonically each abelian variety.

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• New strategy: jt. Jonas Bergström and Valentijn Karemaker [?].

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If  $(K,\Phi)$  satisfies the RRC then in  $\mathscr{C}_h$  there exists an abelian variety A admitting a canonical lifting  $\mathscr{A}$ .

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#### Theorem ([?])

If  $(K,\Phi)$  satisfies the RRC then in  $\mathscr{C}_h$  there exists an abelian variety A admitting a canonical lifting  $\mathscr{A}$ .

• If we understand the polarizations of A we can 'spread' them to the whole isogeny class.

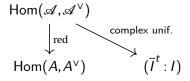
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Now  $\mathscr{C}_h$  over  $\mathbb{F}_p$ : let  $\mathscr{G}$  be the Centeleghe-Stix functor. Assume that there exists A admitting a canonical lifting  $\mathscr{A}$ .

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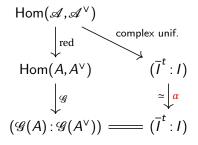
Now  $\mathscr{C}_h$  over  $\mathbb{F}_p$ : let  $\mathscr{G}$  be the Centeleghe-Stix functor. Assume that there exists A admitting a canonical lifting  $\mathscr{A}$ .



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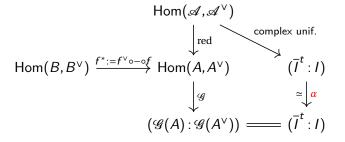
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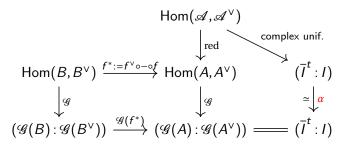
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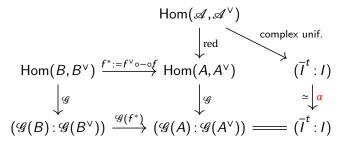
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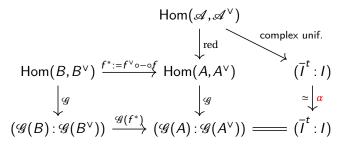
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Note that  $\mathcal{G}(f^*)$  is multiplication by the totally positive element  $\mathcal{G}(f)\mathcal{G}(f)$ :

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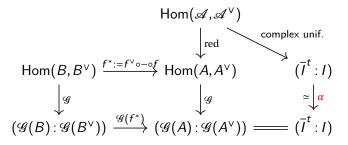
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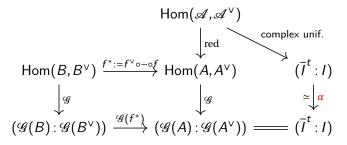
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• Base field extensions and twists (ordinary case) [?].

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- Magma implementations of the algorithms are on GitHub!
- Results of computations will appear on the LMFDB.

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	ordinary	$\mathbb{F}_p$ and no real roots	$\mathbb{F}_{p^k}$ or real roots
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functor		[?]	[?]	[ <b>?</b> ] new!
isomorphism classes	SQ	[?]		work in prog.
	PP	[?] (Bass)		?
	mixed	?	?	?

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		ordinary	$\mathbb{F}_p$ and	$\mathbb{F}_{p^k}$ or
			no real roots	real roots
functor		[?]	[?]	[ <b>?</b> ] new!
isomorphism classes	SQ	[?]		work in prog.
	PP	[?] (Bass)		?
	mixed	?	?	?
polarizations	SQ	[?]+[?]	[?]	?
	PP	[?] (E <sup>r</sup> ), [?] (descr. but no algorithm)	?	?
	mixed	?	?	?

#### More comments:

- in [?]: a functor for isogeny classes of the form  $E^r$ .
- in [?]+[?]: almost-ordinary SQ with polarizations .
- in [?]: they use  $\operatorname{Hom}_{\mathbb{F}_{n^k}}(-,A_w)$  as in [?], but  $A_w$  is more complicated.

## Thank you!

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