Computing isomorphism classes of abelian varieties over finite fields.

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Curves over Finite Fields:
Past, Present and Future

Abelian varieties over $\mathbb{C}$ vs $\mathbb{F}_q$

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- In char. $p > 0$ such an equivalence cannot exist : there are (supersingular) elliptic curves with quaternionic endomorphism algebras.
- Nevertheless, over finite fields, we obtain analogous results if we restrict ourselves to certain **subcategories** of AVs.

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  is injective and allows us to enumerate all AVs up to isogeny.
- Also, $h_A(x)$ is squarefree $\iff$ $\mathrm{End}(A)$ is commutative.

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- Problem: $\mathbb{Z}[F,V]$ might not be maximal $\rightsquigarrow$ non-invertible ideals.

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- Hofmann-Sircana '19: computation of over-orders.

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  ...whose representatives can be found in

$$\left\{\text{sub-}R\text{-modules of } {}^{\mathscr{O}_K}\!\big/_{\mathfrak{f}_R}\right\}$$ finite! and most of the time not-too-big ...

where $\mathfrak{f}_R = (R:\mathscr{O}_K)$ is the conductor of R.

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Theorem (M.)

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Repeat for every $R \subseteq S \subseteq \mathscr{O}_K$:

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- We can actually get a lot more!

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- *and* $\mathrm{Aut}(A, \mu) = \{\textit{torsion units of } S\}$.

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Can modify to compute polarizations of any degree.

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- 8 classes admit principal polarizations.
- 10 isomorphism classes of princ. polarized AV.

Example

Concretely:

$$I_1 = 2645633792595191\mathbb{Z} \oplus (F + 836920075614551)\mathbb{Z} \oplus (F^2 + 1474295643839839)\mathbb{Z} \oplus$$
$$\oplus (F^3 + 1372829830503387)\mathbb{Z} \oplus (F^4 + 1072904687510)\mathbb{Z} \oplus$$
$$\oplus \frac{1}{3}(F^5 + F^4 + F^3 + 2F^2 + 2F + 670480698614361 0)\mathbb{Z} \oplus$$
$$\oplus \frac{1}{9}(F^6 + F^5 + F^4 + 8F^3 + 2F^2 + 2991665243621169)\mathbb{Z} \oplus$$
$$\oplus \frac{1}{27}(F^7 + F^6 + F^5 + 17F^4 + 20F^3 + 9F^2 + 6801531251872220 1)\mathbb{Z}$$

principal polarizations:

$$x_{1,1} = \frac{1}{27}(-121922F^7 + 588604F^6 - 1422437F^5 +$$
$$+ 1464239F^4 + 1196576F^3 - 7570722F^2 + 15316479F - 12821193)$$
$$x_{1,2} = \frac{1}{27}(3015467F^7 - 17689816F^6 + 35965592F^5 -$$
$$- 64660346F^4 + 121230619F^3 - 191117052F^2 + 315021546F - 300025458)$$
$$\mathrm{End}(I_1) = R$$
$$\#\mathrm{Aut}(I_1, x_{1,1}) = \#\mathrm{Aut}(I_1, x_{1,2}) = 2$$

Example

$$I_7 = 2\mathbb{Z} \oplus (F+1)\mathbb{Z} \oplus (F^2+1)\mathbb{Z} \oplus (F^3+1)\mathbb{Z} \oplus (F^4+1)\mathbb{Z} \oplus \frac{1}{3}(F^5+F^4+F^3+2F^2+2F+3)\mathbb{Z}\oplus$$
$$\oplus \frac{1}{36}(F^6+F^5+10F^4+26F^3+2F^2+27F+45)\mathbb{Z}\oplus$$
$$\oplus \frac{1}{216}(F^7+4F^6+49F^5+200F^4+116F^3+105F^2+198F+351)\mathbb{Z}$$

principal polarization:

$$x_{7,1} = \frac{1}{54}(20F^7-43F^6+155F^5-308F^4+580F^3-1116F^2+2205F-1809)$$
$$\mathrm{End}(I_7) = \mathbb{Z} \oplus F\mathbb{Z} \oplus F^2\mathbb{Z} \oplus F^3\mathbb{Z} \oplus F^4\mathbb{Z} \oplus \frac{1}{3}(F^5+F^4+F^3+2F^2+2F)\mathbb{Z}\oplus$$
$$\oplus \frac{1}{18}(F^6+F^5+10F^4+8F^3+2F^2+9F+9)\mathbb{Z}\oplus$$
$$\oplus \frac{1}{108}(F^7+4F^6+13F^5+56F^4+80F^3+33F^2+18F+27)\mathbb{Z}$$
$$\#\mathrm{Aut}(I_7, x_{7,1}) = 2$$

I_1 is invertible in R, but I_7 is not invertible in $\mathrm{End}(I_7)$.

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- If R is Bass, then M is isomorphic to a direct sum of frac.R-ideals.

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Theorem

$$\left\{\textit{abelian varieties in } \mathscr{C}_{g^r}\right\}_{\big/\simeq} \longleftrightarrow \left\{I_1 \oplus \ldots \oplus I_r \ : \ I_j \textit{ a frac. } R\textit{-ideal}\right\}_{\big/\simeq}$$

$$\textit{we have a classification:} \longleftrightarrow \left\{ (S_1 \subseteq S_2 \subseteq \ldots \subseteq S_r,\ [I]_\simeq) : \begin{array}{l} R \subseteq S_j \textit{ orders,} \\ I \textit{ a frac. } R\textit{-ideal} \\ \textit{with } (I:I) = S_r \end{array} \right\}$$

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- Solved for E^r by Kirschmer-Narbonne-Ritzenthaler-Robert '20. next talk!

Example

Let $g = x^6 - 3x^5 + 6x^4 - 10x^3 + 18x^2 - 27x + 27$. Note $\mathscr{C}(g)$ is an isogeny class of simple ordinary abelian varieties over $\mathbb{F}_3$. Define
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$$\mathrm{Pic}(R) \simeq {}^{\mathbb{Z}}\!/_{3\mathbb{Z}} \text{ and } \mathrm{Pic}(\mathscr{O}_K) = \{1\}.$$

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$$\mathrm{End}(M_1) = \mathrm{Mat}_3(R) \text{ and } \mathrm{End}(M_2) = \begin{pmatrix} R & R & I \\ R & R & I \\ (R:I) & (R:I) & R \end{pmatrix}$$

Outside of the ordinary...

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Theorem (Centeleghe-Stix '15)

There is an equivalence of categories:

$$\left\{ \textit{abelian varieties } A \textit{ over } \mathbb{F}_p \textit{ with } \boldsymbol{h_A(\sqrt{p}) \neq 0} \right\} \qquad\qquad A$$

$$\updownarrow \qquad\qquad\qquad\qquad \downarrow$$

$$\left\{ \begin{array}{l} \textit{pairs } (T,F), \textit{ where } T \simeq_{\mathbb{Z}} \mathbb{Z}^{2g} \textit{ and } T \xrightarrow{F} T \textit{ s.t.} \\ \textit{- } F \otimes \mathbb{Q} \textit{ is semisimple} \\ \textit{- the roots of } \mathrm{char}_{F\otimes\mathbb{Q}}(x) \textit{ have abs. value } \sqrt{p} \\ \textit{- } \boldsymbol{\mathrm{char}_F(\sqrt{p}) \neq 0} \\ \textit{- } \exists V : T \to T \textit{ such that } FV = VF = p \end{array} \right\} \qquad (T(A), F(A))$$

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- Now, $T(A) := \mathrm{Hom}(A, A_w)$, where A_w has minimal End among the varieties with Weil support $w = w(A)$.
- $F(A)$ is the induced Frobenius.

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  Oswal-Shankar for almost-ordinary abelian varieties, together with generalzations by Bergström-Karemaker-M.;
  Serre and the work of Kani, Jordan-Keeton-Poonen-Rains-
  -Shepherd-Barron-Tate for E^r; Isabel Vogt's talk
  ... but we will not use them in this talk.)
- For polarizations, the results by Howe do not apply immediately to the Centeleghe-Strix case:
- in general we cannot lift canonically each abelian variety.

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Theorem (Chai-Conrad-Oort)

If (K,Φ) satisfies the RRC then in $\mathscr{C}_h$ there exists an abelian variety A admitting a canonical lifting $\mathscr{A}$.

- If we understand the polarizations of A we can 'spread' them to the whole isogeny class.

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We study when we can 'pretend' $\alpha = 1$.

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- Results of computations will appear on the LMFDB.

Thank you!