Cohen-Macaulay type of endomorphism rings of abelian varieties over finite fields

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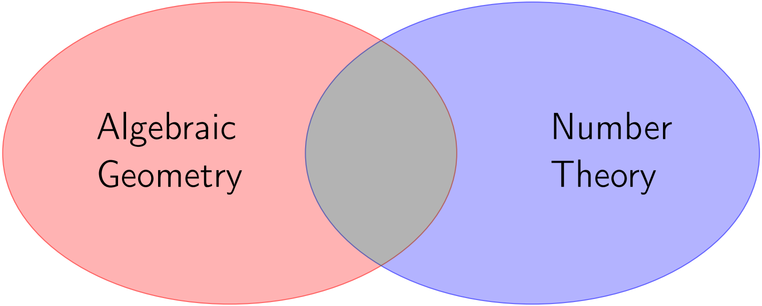
What do I do for a living?



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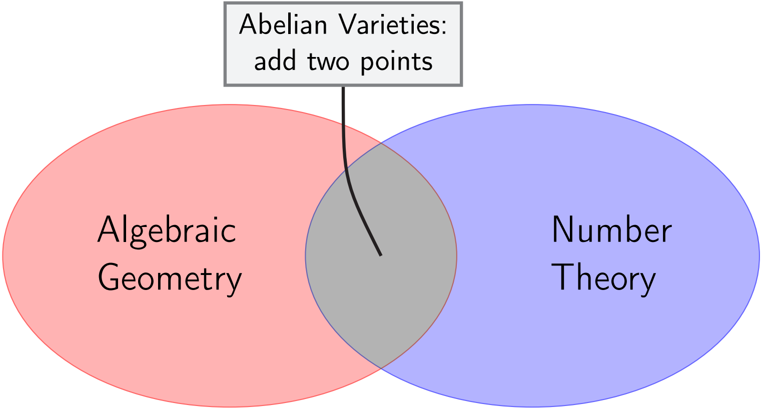

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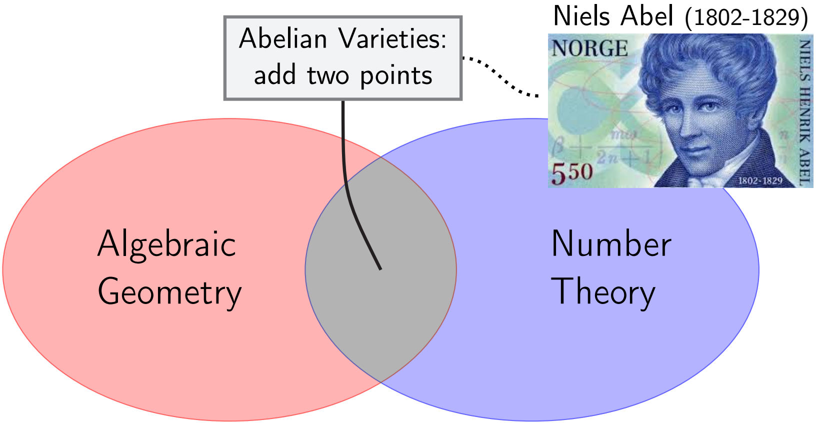

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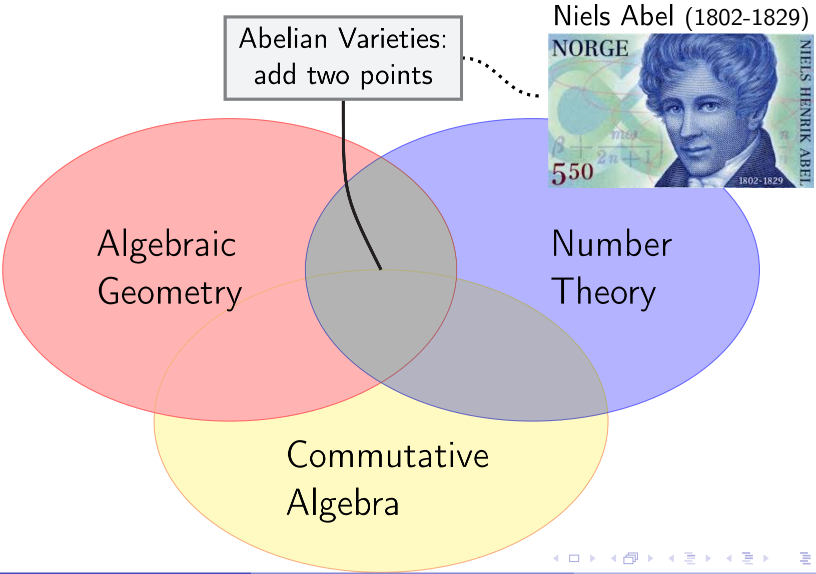

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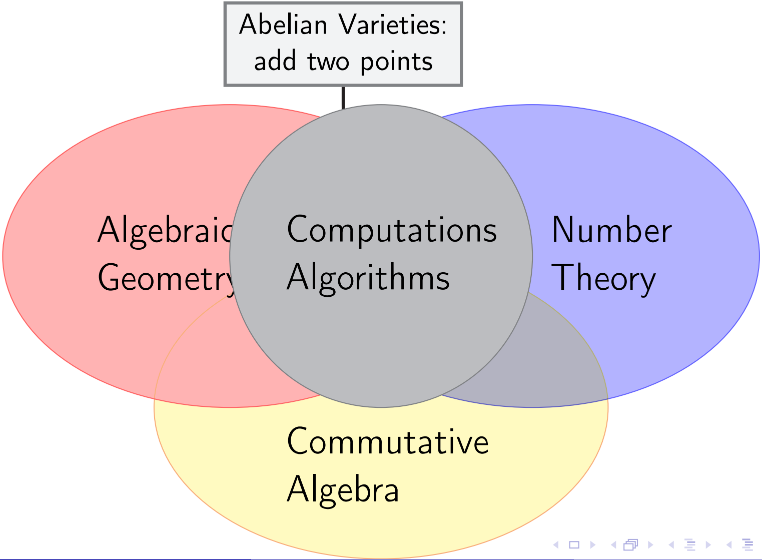

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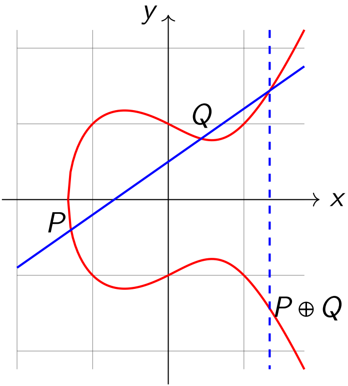

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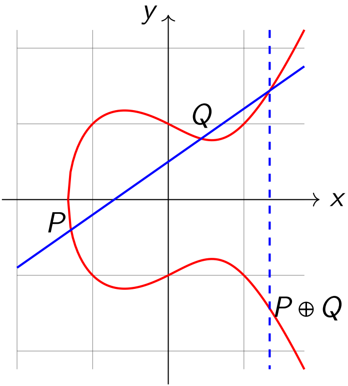

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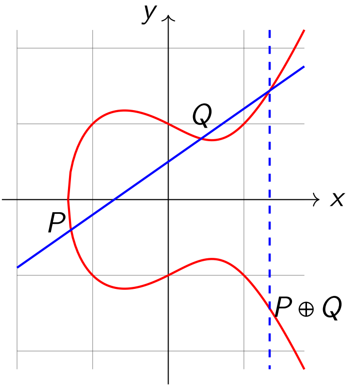
Equations are impractical in dim ≥ 2.
We need a better way to represent them...



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- Nevertheless, as we will see later, over a finite field $\mathbb{F}_q$, we obtain analogous results if we restrict ourselves to certain **subcategories** of AVs.
- WARNING: all morphisms, endomorphisms, isogenies, etc. are defined over $\mathbb{F}_q$.

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  - the association

$$\text{isogeny class of } A \longmapsto h_A(x)$$

  allows us to enumerate all AVs up to isogeny.

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- Plan: study A by studying some comm. algebra properties of $\mathrm{End}(A)$.

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- If I is invertible, then $(I:I) = R$.

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- Ex: pick a prime $\ell \in \mathbb{Z}$. Then $\mathrm{type}_{\ell\mathcal{O}_K}(\mathbb{Z} + \ell\mathcal{O}_K) = \dim_{\mathbb{Q}} K - 1$.

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References: Deligne, Howe, Centeleghe-Stix, Bergström-Karemaker-M.

- We get a bijection

$$\{\text{ isom. classes of AVs in } \mathscr{I}_h \} \longleftrightarrow \{\text{isom. classes of fr. } \mathbb{Z}[\pi, q/\pi]\text{-ideals} \}$$
$$:= \mathrm{ICM}(\mathbb{Z}[\pi, q/\pi]) \text{ ideal class monoid}$$

- To **classify** the AVs in $\mathscr{I}_h$ we need to compute the ICM.
- If $\mathbb{Z}[\pi, q/\pi] = \mathscr{O}_K$ is the maximal order then $\mathrm{ICM}(\mathbb{Z}[\pi, q/\pi]) = \mathrm{Pic}(\mathscr{O}_K)$ is a product of class groups of number fields and we are good.
- Problem: $\mathbb{Z}[\pi, q/\pi]$ might not be a Dedekind ring $\rightsquigarrow$ non-invertible ideals.

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- We have proven that:
  if the type of S at $\mathfrak{m}$ is 1 then $(\mathcal{W}_S(R))_{\mathfrak{m}} = \{[S_{\mathfrak{m}}]\}$, while
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In both cases: $I \not\simeq \overline{I}^t \iff A \not\simeq A^\vee$.

Some stats and refs

How often do the hypothesis of the previous theorem ($R = \overline{R}$, exists $\mathfrak{m} = \overline{\mathfrak{m}}$ with $\text{type}_{\mathfrak{m}}(R) = 2$) do occur?
We computed the isomorphism classes of AVs/$\mathbb{F}_q$ (see LMFDB xyz) for 615.269 isogeny classes (for $1 \leq g \leq 5$ and various q).
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- 72.6% satisfy $R = \overline{R}$;
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- 7.4% satisfy $R = \overline{R}$, are non-Gorenstein and $\exists \mathfrak{m} = \overline{\mathfrak{m}}$ s.t. with $\mathrm{type}_{\mathfrak{m}}(R) = 2$.

Thank you!

Main references:

- *Cohen-Macaulay type of orders, generators and ideal classes*
  to appear in Journal of Algebra
  https://arxiv.org/abs/2206.03758
- *Abelian varieties over finite fields and their groups of rational points*
  with Caleb Springer, to appear in Algebra&Number Theory
  https://arxiv.org/abs/2211.15280
- Magma package for étale $\mathbb{Q}$-algebras
  https://github.com/stmar89/AlgEt (also in Magma 2-28.1)