Isomorphism classes of abelian varieties over finite fields

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Introduction

Today's plan:

- Introduction.
- Equivalence of categories.
- AVs A isogenous to B^r, for B square-free defined over $\mathbb{F}_q$.
- Isomorphism classes.
- Polarizations (only in the ordinary case).
- Computations of polarizations and period matrices (only ordinary and $r = 1$).

Also, all morphisms are defined over the field of definition!

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- in positive characteristic we don't have such equivalence.

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- Recall: $T_\ell(A) = \varprojlim A[\ell^n] \simeq \mathbb{Z}_\ell^{2d}$ for any $\ell \neq \mathrm{char}(k)$.

Classification up to isogeny : over $\mathbb{F}_q$

- $A/\mathbb{F}_q$ comes with a Frobenius endomorphism, that induces an action

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    is well defined and injective, and
  - there is a bijection between the set of simple abelian varieties over $\mathbb{F}_q$ up to isogeny and the set of q-Weil numbers (up to conjugacy).

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- the construction is functorial: $A_{\mathbb{C}}$ has a Frobenius.

Equivalences of categories 1

Theorem (Deligne 1969)

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A similar result holds for almost-ordinary AVs (Oswal-Shankar 2019).

Equivalences of categorie 2

Theorem (Centeleghe-Stix 2015)

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The functor : $A \longmapsto \mathrm{Hom}_{\mathbb{F}_p}(A, A_w) = T(A)$, where A_w is an abelian varieties satisfying certain minimality conditions.

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Observe: if $A \in \mathrm{AV}(g^r)$ then

$$A \sim (B_1 \times \ldots \times B_s)^r$$

with

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Main theorem

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Theorem (M.)

There are equivalences of categories

$$\mathrm{AV}(g^r) \overset{Del/CS}{\longleftrightarrow} \mathscr{M}(g^r) \longleftrightarrow \mathscr{B}(g^r)$$

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The category $\mathscr{B}(g^r)$ becomes more explicit and computable under certain assumptions on the order R.

Bass orders

Recall

- a fractional R-ideal I is a sub-R-module of K which is also a lattice
- a fractional R-ideal is invertible in R if $I(R:I) = R$.

Define

$$\mathrm{ICM}(R) = {\{\text{fractional } R\text{-ideals}\}}\big/_{\simeq_R} \qquad \text{ideal class monoid}$$

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Moreover, given $M = \bigoplus_{k=1}^r I_k$ and $M' = \bigoplus_{k=1}^r J_k$ we have that

$$M \simeq_R M' \iff \begin{cases} (I_k : I_k) = (J_k : J_k) \text{ for every } k \text{ (up to permutation), and} \\ \prod_{k=1}^r I_k \simeq_R \prod_{k=1}^r J_k \end{cases}$$

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$$M \simeq_R I_1 \oplus \dots \oplus I_r.$$

everything is a direct sum of fractional ideals

Moreover, given $M = \bigoplus_{k=1}^r I_k$ and $M' = \bigoplus_{k=1}^r J_k$ we have that

$$M \simeq_R M' \iff \begin{cases} (I_k : I_k) = (J_k : J_k) \text{ for every } k \text{ (up to permutation), and} \\ \prod_{k=1}^r I_k \simeq_R \prod_{k=1}^r J_k \end{cases}$$

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$\mathscr{B}(g^r)$ in the Bass case

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and $$\mathrm{Aut}_R(M) = \{A \in \mathrm{End}_R(M) \cap \mathrm{GL}_r(K) : A^{-1} \in \mathrm{End}_R(M)\}.$$

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$$\mathrm{End}(M_1) = \mathrm{Mat}_3(R) \text{ and } \mathrm{End}(M_2) = \begin{pmatrix} R & R & I \\ R & R & I \\ (R:I) & (R:I) & R \end{pmatrix}$$

Dual modules: only in the ordinary case

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$$\Lambda^\vee := \overline{\Lambda}^T$$

"Proof": Howe (1995) described dual modules in $\mathscr{M}^{\mathrm{ord}}(q)$. We translated this notion to $\mathscr{B}(g^r)$.

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Fix

$$\Phi := \{\varphi : K \to \mathbb{C} : v_p(\varphi(F)) > 0\}, \text{ tricky to compute!}$$

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"Proof": Howe (1995) put polarizations in Deligne's category $\mathscr{M}^{\mathrm{ord}}(q)$.
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Unfortunately

$$\mathrm{Pol}(M) / \mathrm{Aut}(M) \text{ is hard to understand if } r \geq 2$$

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  Also: $\deg \mu = [\overline{I}^t : \lambda I]$.
- if $(A, \mu) \leftrightarrow (I, \lambda)$, $\deg \mu = 1$ and $S = (I : I)$ then

$$\left\{\begin{array}{l}\text{non-isomorphic}\\ \text{polarizations of } A\end{array}\right\} \longleftrightarrow \frac{\{\text{totally positive } u \in S^\times\}}{\{v\overline{v} : v \in S^\times\}}$$

and $\mathrm{Aut}(A, \mu) = \{\text{torsion units of } S\}$

Example

- Let $h(x) = x^8 - 5x^7 + 13x^6 - 25x^5 + 44x^4 - 75x^3 + 117x^2 - 135x + 81$;
- $\rightsquigarrow$ isogeny class of an simple ordinary abelian varieties over $\mathbb{F}_3$ of dimension 4;
- Let F be a root of $h(x)$ and put $R := \mathbb{Z}[F, 3/F] \subset \mathbb{Q}(F)$;
- 8 over-orders of R: two of them are not Gorenstein;
- $\#\mathrm{ICM}(R) = 18 \rightsquigarrow 18$ isom. classes of AV in the isogeny class;
- 5 are not invertible in their multiplicator ring;
- 8 classes admit principal polarizations;
- 10 isomorphism classes of princ. polarized AV.

Example

Concretely:

$$\begin{aligned}I_1 =& 2645633792595191\mathbb{Z} \oplus (F + 836920075614551)\mathbb{Z} \oplus (F^2 + 1474295643839839)\mathbb{Z} \oplus \\ & \oplus (F^3 + 1372829830503387)\mathbb{Z} \oplus (F^4 + 1072904687510)\mathbb{Z} \oplus \\ & \oplus \frac{1}{3}(F^5 + F^4 + F^3 + 2F^2 + 2F + 670480698614361 0)\mathbb{Z} \oplus \\ & \oplus \frac{1}{9}(F^6 + F^5 + F^4 + 8F^3 + 2F^2 + 2991665243621169)\mathbb{Z} \oplus \\ & \oplus \frac{1}{27}(F^7 + F^6 + F^5 + 17F^4 + 20F^3 + 9F^2 + 6801531251872220 1)\mathbb{Z}\end{aligned}$$

principal polarizations:

$$\begin{aligned}x_{1,1} =& \frac{1}{27}(-121922F^7 + 588604F^6 - 1422437F^5 + \\ & + 1464239F^4 + 1196576F^3 - 7570722F^2 + 15316479F - 12821193) \\ x_{1,2} =& \frac{1}{27}(3015467F^7 - 17689816F^6 + 35965592F^5 - \\ & - 64660346F^4 + 121230619F^3 - 191117052F^2 + 315021546F - 300025458)\end{aligned}$$

$\mathrm{End}(I_1) = R$

$\#\mathrm{Aut}(I_1, x_{1,1}) = \#\mathrm{Aut}(I_1, x_{1,2}) = 2$

Example

$$
\begin{aligned}
I_7 =&2\mathbb{Z} \oplus (F+1)\mathbb{Z} \oplus (F^2+1)\mathbb{Z} \oplus (F^3+1)\mathbb{Z} \oplus (F^4+1)\mathbb{Z} \oplus \frac{1}{3}(F^5+F^4+F^3+2F^2+2F+3)\mathbb{Z} \oplus \\
&\oplus \frac{1}{36}(F^6+F^5+10F^4+26F^3+2F^2+27F+45)\mathbb{Z} \oplus \\
&\oplus \frac{1}{216}(F^7+4F^6+49F^5+200F^4+116F^3+105F^2+198F+351)\mathbb{Z}
\end{aligned}
$$

principal polarization:

$$x_{7,1} = \frac{1}{54}(20F^7 - 43F^6 + 155F^5 - 308F^4 + 580F^3 - 1116F^2 + 2205F - 1809)$$

$$
\begin{aligned}
\mathrm{End}(I_7) = &\mathbb{Z} \oplus F\mathbb{Z} \oplus F^2\mathbb{Z} \oplus F^3\mathbb{Z} \oplus F^4\mathbb{Z} \oplus \frac{1}{3}(F^5+F^4+F^3+2F^2+2F)\mathbb{Z} \oplus \\
&\oplus \frac{1}{18}(F^6+F^5+10F^4+8F^3+2F^2+9F+9)\mathbb{Z} \oplus \\
&\oplus \frac{1}{108}(F^7+4F^6+13F^5+56F^4+80F^3+33F^2+18F+27)\mathbb{Z}
\end{aligned}
$$

$$\#\mathrm{Aut}(I_7, x_{7,1}) = 2$$

I_1 is invertible in R, but I_7 is not invertible in $\mathrm{End}(I_7)$.

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We have an isomorphism of complex tori

$$A_{\mathbb{C}}(\mathbb{C}) \simeq {}^{\mathbb{C}^d}\!/_{\Phi(I)}, \qquad \Phi(I) = \langle(\varphi_1(\alpha_i), \ldots, \varphi_d(\alpha_i) : i = 1, \ldots, 2d\rangle .$$

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The Riemann form associated to λ is given by

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$$b(\gamma_i, \beta_i) = 1 \text{ for all } i, \text{ and}$$

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This is big period matrix of $(A_{\mathbb{C}}, \mu_{\mathbb{C}})$.

Period matrices - Example

Let $g = (x^4 - 4x^3 + 8x^2 - 12x + 9)(x^4 - 2x^3 + 2x^2 - 6x + 9)$.

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$$
\begin{aligned}
I=&\frac{1}{54}\left(432-549\alpha+441\alpha^2-331\alpha^3+186\alpha^4-81\alpha^5+33\alpha^6-7\alpha^7\right)\mathbb{Z}\oplus\\
&\oplus\frac{1}{6}\left(63-78\alpha+65\alpha^2-49\alpha^3+27\alpha^4-12\alpha^5+5\alpha^6-1\alpha^7\right)\mathbb{Z}\oplus\\
&\oplus\frac{1}{6}\left(81-99\alpha+84\alpha^2-61\alpha^3+33\alpha^4-15\alpha^5+6\alpha^6-1\alpha^7\right)\mathbb{Z}\oplus\\
&\oplus\frac{1}{18}\left(-63+96\alpha-86\alpha^2+68\alpha^3-39\alpha^4+18\alpha^5-8\alpha^6+2\alpha^7\right)\mathbb{Z}\oplus(-1)\mathbb{Z}\oplus\\
&\oplus(-\alpha)\mathbb{Z}\oplus(-\alpha^2)\mathbb{Z}\oplus\frac{1}{9}\left(81-96\alpha+81\alpha^2-64\alpha^3+33\alpha^4-15\alpha^5+6\alpha^6-\alpha^7\right)\mathbb{Z}\\
\lambda=&\frac{537}{80}-\frac{1343}{120}\alpha+\frac{1343}{144}\alpha^2-\frac{419}{60}\alpha^3+\frac{337}{80}\alpha^4-\frac{15}{8}\alpha^5+\frac{559}{720}\alpha^6-\frac{1}{5}\alpha^7
\end{aligned}
$$

$$
\Omega=\begin{pmatrix}
2.8-i & -2.8+0.6i & 0 & 0 & 1 & 1.7-0.3i & 0 & 0\\
-2.8+i & 2.8-3.4i & 0 & 0 & 1 & 0.3+1.7i & 0 & 0\\
0 & 0 & -1 & -0.4-0.2i & 0 & 0 & -1.6-0.6i & -0.2-0.2i\\
0 & 0 & -1 & -2.6+6.9i & 0 & 0 & 0.6-1.6i & -6.9+6.9i
\end{pmatrix}
$$

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Let $g = (x^4 - 4x^3 + 8x^2 - 12x + 9)(x^4 - 2x^3 + 2x^2 - 6x + 9)$. We compute the principally polarized abelian varieties and we find that 4 isomorphism classes admit a unique principal polarization. Here is one of them with the period matrix of the canonical lift.

$$
\begin{aligned}
I = {} & \frac{1}{54}\left(432 - 549\alpha + 441\alpha^2 - 331\alpha^3 + 186\alpha^4 - 81\alpha^5 + 33\alpha^6 - 7\alpha^7\right)\mathbb{Z}\oplus \\
& \oplus \frac{1}{6}\left(63 - 78\alpha + 65\alpha^2 - 49\alpha^3 + 27\alpha^4 - 12\alpha^5 + 5\alpha^6 - 1\alpha^7\right)\mathbb{Z}\oplus \\
& \oplus \frac{1}{6}\left(81 - 99\alpha + 84\alpha^2 - 61\alpha^3 + 33\alpha^4 - 15\alpha^5 + 6\alpha^6 - 1\alpha^7\right)\mathbb{Z}\oplus \\
& \oplus \frac{1}{18}\left(-63 + 96\alpha - 86\alpha^2 + 68\alpha^3 - 39\alpha^4 + 18\alpha^5 - 8\alpha^6 + 2\alpha^7\right)\mathbb{Z}\oplus(-1)\mathbb{Z}\oplus \\
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$$\begin{aligned} I = & \frac{1}{54}\left(432 - 549\alpha + 441\alpha^2 - 331\alpha^3 + 186\alpha^4 - 81\alpha^5 + 33\alpha^6 - 7\alpha^7\right)\mathbb{Z}\oplus \\ & \oplus \frac{1}{6}\left(63 - 78\alpha + 65\alpha^2 - 49\alpha^3 + 27\alpha^4 - 12\alpha^5 + 5\alpha^6 - 1\alpha^7\right)\mathbb{Z}\oplus \\ & \oplus \frac{1}{6}\left(81 - 99\alpha + 84\alpha^2 - 61\alpha^3 + 33\alpha^4 - 15\alpha^5 + 6\alpha^6 - 1\alpha^7\right)\mathbb{Z}\oplus \\ & \oplus \frac{1}{18}\left(-63 + 96\alpha - 86\alpha^2 + 68\alpha^3 - 39\alpha^4 + 18\alpha^5 - 8\alpha^6 + 2\alpha^7\right)\mathbb{Z}\oplus(-1)\mathbb{Z}\oplus \\ & \oplus(-\alpha)\mathbb{Z}\oplus(-\alpha^2)\mathbb{Z}\oplus\frac{1}{9}\left(81 - 96\alpha + 81\alpha^2 - 64\alpha^3 + 33\alpha^4 - 15\alpha^5 + 6\alpha^6 - \alpha^7\right)\mathbb{Z} \end{aligned}$$

$$\lambda = \frac{537}{80} - \frac{1343}{120}\alpha + \frac{1343}{144}\alpha^2 - \frac{419}{60}\alpha^3 + \frac{337}{80}\alpha^4 - \frac{15}{8}\alpha^5 + \frac{559}{720}\alpha^6 - \frac{1}{5}\alpha^7$$

$$\Omega = \begin{pmatrix} 2.8 - i & -2.8 + 0.6i & 0 & 0 & 1 & 1.7 - 0.3i & 0 & 0 \\ -2.8 + i & 2.8 - 3.4i & 0 & 0 & 1 & 0.3 + 1.7i & 0 & 0 \\ 0 & 0 & -1 & -0.4 - 0.2i & 0 & 0 & -1.6 - 0.6i & -0.2 - 0.2i \\ 0 & 0 & -1 & -2.6 + 6.9i & 0 & 0 & 0.6 - 1.6i & -6.9 + 6.9i \end{pmatrix}$$

Final remarks

- Compute base field extensions and twists (ordinary case) (soon on arXiv).
- Polarizations (and period matrices) in the Centeleghe-Stix case are work in progress.
- The Magma code is available on my webpage.
- Results of computations will appear on the LMFDB.

Thank you!