Introduction to

MODULAR FORMS

Prekolloquium 27/09/2017
for Gerard van der Geer

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Def The modular group is

$$\Gamma_1 = SL_2(\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a,b,c,d \in \mathbb{Z} \quad ad-bc=1 \right\}$$

- Γ_1 acts on $\mathcal{H} = \{ \tau \in \mathbb{C} : Im(\tau) > 0 \}$ by Moebius transformation:
  $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_1$, $\gamma(\tau) = \dfrac{a\tau+b}{c\tau+d} \in \mathcal{H} \quad \forall \tau \in \Gamma_1$

  (+ identity & composition) $\left(Im(\gamma(\tau)) = \dfrac{Im(\tau)}{|c\tau+d|^2} \right.$

Def Let k be an integer. A meromorphic function $f: \mathcal{H} \to \mathbb{C}$ is weakly modular of weight k (wrt Γ_1) if

"almost Γ_1 invariant" $\quad f(\gamma(\tau)) = (c\tau+d)^k f(\tau) \quad \forall \gamma \in \Gamma_1, \ \forall \tau \in \mathcal{H}$

- Γ_1 is generated by $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

f w. modular of w k iff $f(\tau+1) = f(\tau)$ and $f\left(\frac{-1}{\tau}\right) = \tau^k f(\tau)$

- "modular forms are w.m.f. which are holomorphic on $\mathcal{H}$ and at ∞"

Identify $\mathcal{H} \longleftrightarrow D_0 = \{ q \in \mathbb{C} : 0 < |q| < 1 \}$

$\tau \longmapsto e^{2\pi i \tau} = q$

and define

$$g_f : D_0 \to \mathbb{C} \qquad q \mapsto f\left(\frac{\log(q)}{2\pi i}\right)$$

Since $f(\tau+1) = f(\tau) \Rightarrow g_f$ is well defined!
If f is holomorphic on $\mathcal{H}$, then g_f is so on D_0 and hence admits a Laurent exp:

$$g_f(q) = \sum_{n \in \mathbb{Z}} a_n q^n , \quad q \in D_0$$

Since

$$q \to 0 \quad \text{iff} \quad \operatorname{Im}(\tau) \to \infty$$

We say that f is holomorphic at ∞ if g_f admits an holom. extension to $q = 0$

$\Rightarrow$

f has a Fourier expansion

$$f(\tau) = \sum_{n \in \mathbb{N}} a_n(f) q^n , \quad q = e^{2\pi i \tau}, \tau \in \mathcal{H}.$$

Def $k \in \mathbb{Z}$, $f : \mathcal{H} \to \mathbb{C}$ is a modular form of weight k if (wrt Γ_1)

a) f is holom on $\mathcal{H}$;
b) f is weakly mod. of weight k;
c) f is holom. at ∞.

$\rightsquigarrow$ set of those $\boxed{M_k(SL_2(\mathbb{Z}))}$.

"the sum of m.f. of weights ℓ and k is a m.f. of weight $k+\ell$"

$$\rightsquigarrow \quad M(SL_2(\mathbb{Z})) = \bigoplus_k M_k(SL_2(\mathbb{Z}))$$

2.1

Generalize the definition of modular form in many directions:

- Instead of Γ_1 put Γ a discrete subgroup of Γ.

  eg: $\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z}) \,;\, \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv_N \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \right\}$

  $\rightsquigarrow$ modular forms of weight k and level N

- Maass forms (relax holomorphic)
- Hilbert m. f. (many variables)
- Siegel m. f. (simplectic group)
- Automorphic f. (Lie group)

Some examples

- Eisenstein series of weight $k > 2$ + even

$$G_k(\tau) := \sum_{(c,d)\in\mathbb{Z}^2\setminus(0,0)} \frac{1}{(c\tau+d)^k} \qquad \tau \in \mathcal{H}$$

" it is a 2-dimsional analogue of $\zeta(k) = \sum_{d\geq 1} \frac{1}{d^k}$ "

" it is abs. convergent ; converges unif on compact subsets of $\mathcal{H}$ $\rightsquigarrow$ G_k is holomorphic on $\mathcal{H}$ "

Fourier exp:

$$G_k(\tau) = 2\zeta(k) + 2\frac{2\pi i}{(k-1)!} \sum_{m=1}^{\infty} \underbrace{\sigma_{k-1}(m)}_{\sigma_{k-1}(m) = \sum_{\substack{m>0 \\ m/m}} m^{k-1}} q^m, \quad q = e^{2\pi i\tau}$$

Pf Start with $\frac{1}{\tau} + \sum_{d=1}^{\infty} \left(\frac{1}{\tau - d} + \frac{1}{\tau+d} \right) = \pi \cot \pi\tau =$

$$= \pi i - 2\pi i \sum_{m=0}^{\infty} q^m, \qquad q = e^{2\pi i \tau}$$

differentiate $(k-1)$-time

$$\sum_{d\in\mathbb{Z}} \frac{1}{(\tau+d)^k} = \frac{(-2\pi i)^k}{(k-1)!} \sum_{m=1}^{\infty} m^{k-1} q^m, \quad k \geq 2$$

(k even)

$$G_k(\tau) = \sum_{d\neq 0} \frac{1}{d^k} + 2 \sum_{c=1}^{\infty} \left(\sum_{d\in\mathbb{Z}} \frac{1}{(c\tau+d)^k} \right)$$

$$= 2\zeta(k) + 2 \frac{(2\pi i)^k}{(k-1)!} \sum_{c=1}^{\infty} \sum_{m=1}^{\infty} m^{k-1} q^{cm}$$

$$= 2\zeta(k) + 2 \frac{(2\pi i)^k}{(k-1)!} \sum_{m=1}^{\infty} \sigma_{k-1}(m) q^m$$

Normalized E. series $E_k(\tau) = \frac{G_k(\tau)}{2\zeta(k)}$

Fact $\mathcal{M}(SL_2(\mathbb{Z}))$ is freely generated by E_4 and E_6

$\rightsquigarrow$ **Cor**

$$\dim \mathcal{M}_k(SL_2(\mathbb{Z})) = \begin{cases} [k/12] + 1 & \text{if } k \not\equiv 2 \mod 12 \\ [k/12] & \text{if } k \equiv 2 \mod 12 \end{cases}$$

Complex tori and Elliptic curves

- Let $\mathcal{L} = \{ \omega_1 \mathbb{Z} \oplus \omega_2 \mathbb{Z} = \Lambda \subset \mathbb{C} \; : \; \Lambda \otimes \mathbb{R} = \mathbb{C} \}$

  lattices in $\mathbb{C}$

- Λ homotetic to Λ' if $\exists \, c \in \mathbb{C}^*$ st $\Lambda = c \Lambda'$

  - w.l.o.g assume $\omega_1 / \omega_2 = \tau \in H$

$\therefore \; \forall \; \Lambda, \; \exists \, \tau \in H$ st $\Lambda \underset{h}{\approx} \Lambda_\tau = \tau \mathbb{Z} \oplus \mathbb{Z}$

i.e. $H \twoheadrightarrow \mathcal{L} / \mathbb{C}^*$

"What is the kernel?"

Lemma $\Lambda_{\tau_1} \underset{h}{\approx} \Lambda_{\tau_2} \iff \exists \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z})$ st

$$\tau_2 = \frac{a\tau_1 + b}{c\tau_1 + d}.$$

Cor

$$SL(2, \mathbb{Z}) \backslash H \longleftrightarrow \mathcal{L} / \mathbb{C}^*$$

- A complex torus is a quotient $\mathbb{C} / \Lambda$

  It is : - an abelian group

  - a Riemann surface.

$\rightsquigarrow$ morphisms : holomorphic group homomorphisms

$$\varphi : \mathbb{C}/\Lambda_1 \to \mathbb{C}/\Lambda_2 .$$

$\rightsquigarrow$ In part. $\mathbb{C}/\Lambda_1 \simeq \mathbb{C}/\Lambda_2 \iff \exists \, c \in \mathbb{C}^*$ st

$$\Lambda_1 = c \, \Lambda_2$$

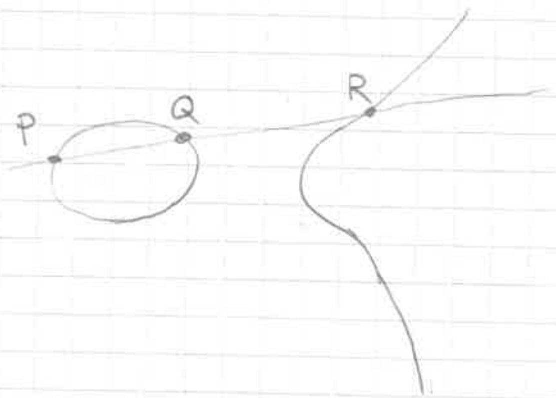
Prop $\mathcal{L}/_{\mathbb{C}^*} \longleftrightarrow \{\mathbb{C}\text{-tori}\}/_{\cong}$

Def Let k be a field (char $k \neq 2, 3$)
An <u>elliptic curve</u> E is the set of solutions of $y^2 = x^3 + Ax + B$ where $A, B \in k$

$$\Delta_E = -4A^3 - 27B^2 \neq 0$$

$\cup \{\infty\}$

- It is an abelian group with addition law given by "collinearity" an ∞ as neutral element



$P \oplus Q \oplus R = \infty$

$\rightsquigarrow$ formulas in the coordinates and A, B that make sense over any k.

Prop $\{\mathbb{C}\text{-tori}\}/_{\cong} \longleftrightarrow \{\text{Elliptic curves}/\mathbb{C}\}/_{\cong}$

Idea: ? $\mathbb{C}/_{\Lambda} \xrightarrow{?} E$

Def the $\wp$-Weierstrass function associated to Λ by

$$\wp(z) = \frac{1}{z^2} + \sum_{\omega \in \Lambda \setminus \{0,0\}} \left(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right)$$

$$\rightsquigarrow \wp'(z) = -2 \cdot \sum_{\omega \in \Lambda} \frac{1}{(z-\omega)^3}$$

are Λ periodic

and $(\wp, \wp')$ are solution to the non-singular eq:

$$y^2 = 4x^3 - 60\, G_4(\Lambda)\, x - 140\, G_6(\Lambda)$$

where $G_k(\Lambda) = \sum_{\omega \in \Lambda \setminus 0} \frac{1}{\omega^k}$ "it's an EC"

So

$SL(2,\mathbb{Z})\backslash\mathcal{H}$ is the moduli space of E.C. $/\mathbb{C}$

" But there is a deeper connection b/w E.C. and modular forms "

- Consider an elliptic curve over $\mathbb{Q}$

$E: \quad Y^2 = X^3 + AX + B \qquad \Delta_E = -4A^3 - 27B^2$

w.l.o.g $A, B \in \mathbb{Z}$

so it makes sense to reduce mod p,

If E mod p is an elliptic curve (eg $\Delta_E \not\equiv 0 \bmod p$)

then we count the points. We have:

$\# E(\mathbb{F}_p) = p + 1 - a_p \quad , \quad |a_p| < 2\sqrt{p}$ Hasse Weil Thm

" happens for a.a. primes, minimal model "

Def L- fct of E :

$$L_E(s) = \prod_{\text{bad } p} (1 - a_p p^{-s})^{-1} \cdot \prod_{\text{good } p} (1 - a_p p^{-s} + p^{1-2s})^{-1}$$

(converges for $\mathrm{Re}(s) > \frac{3}{2}$)

$$= \sum_{n=1}^{\infty} a_n \frac{1}{n^s} \quad \text{where} \quad a_n = \prod_j a_{p_j^{e_j}} \quad \text{for } n = \prod_j p_j^{e_j}.$$

So given $E \rightsquigarrow L_E(s) = \sum_{n=1}^{\infty} a_n \frac{1}{n^s}$

Def for $\tau \in \mathcal{H}$, $q = e^{2\pi i \tau}$

$$f_E(\tau) = \sum_{n=1}^{\infty} a_n q^n$$

" It reminds a bit of the Fourier exp of a modular form "

" One of the biggest achievement of ... "

Thm (Breuil, Conrad, Diamond, Taylor, Wiles)
Let $E/\mathbb{Q}$. Then there exists an integer N st $f_E(\tau)$ is a modular form of weight 2 and level N (Weil-Shimura-Taniyama conjecture)

Cor Last Fermat Theorem: $a^n + b^n = c^n$ has no non trival sol's. for $n \geq 3$

- First: enough to consider $n=4$ proved by Fermat and $n = \ell$ a prime with a,b,c coprime

Frey, Hellgouarch: $E^{\text{Frey}}: y^2 = x(x-a^\ell)(x+b^\ell)$

Thm (Ribet): E^{Frey} cannot be modular

Thm (Wiles, Taylor): All semistable ell. curves $/\mathbb{Q}$ are modular

" Also m.f. of higher weight appear in similar statements "