Abelian varieties over finite fields

Stefano Marseglia

UPF - Gaati Lab

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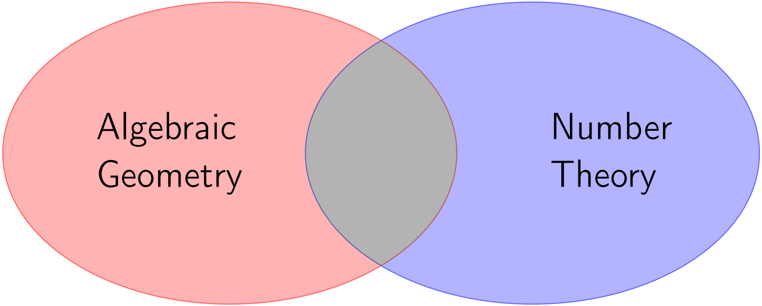
What do I do for a living?



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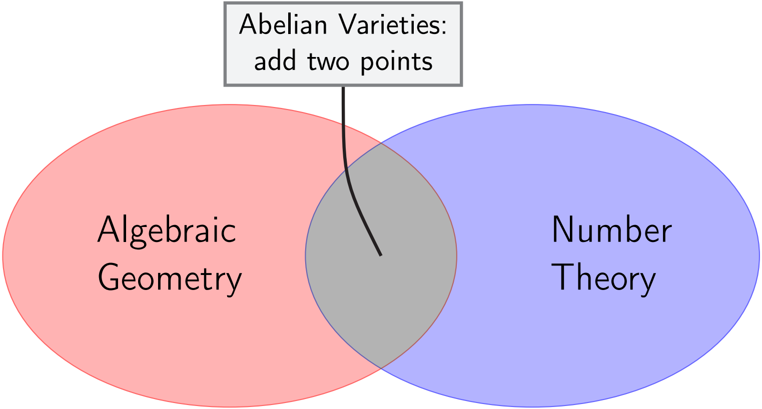

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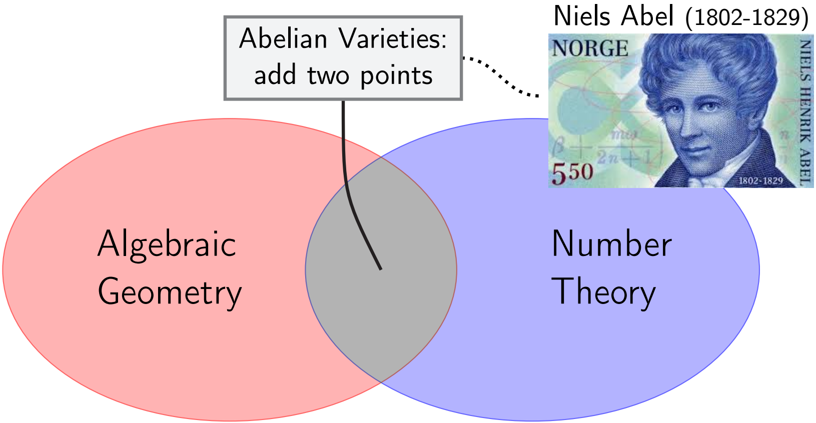

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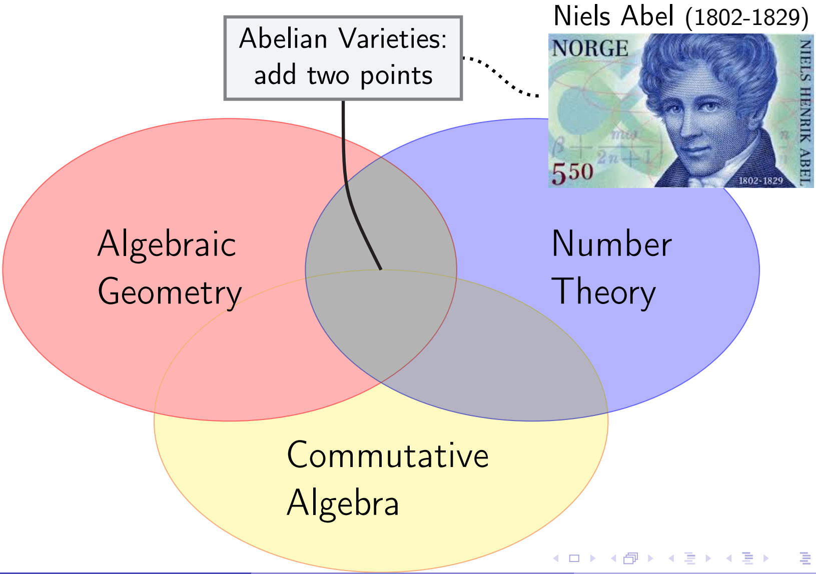

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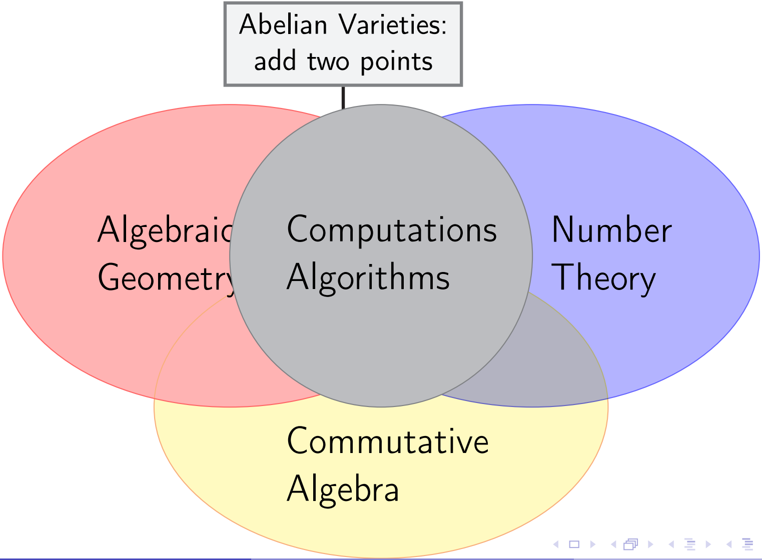

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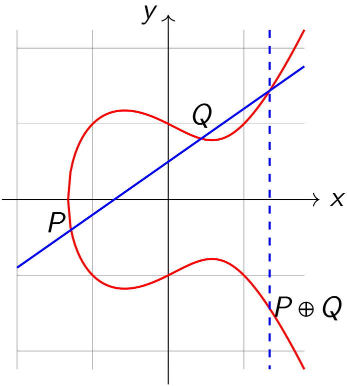

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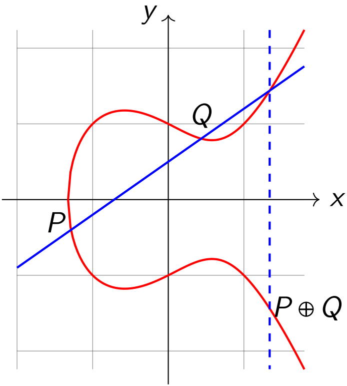

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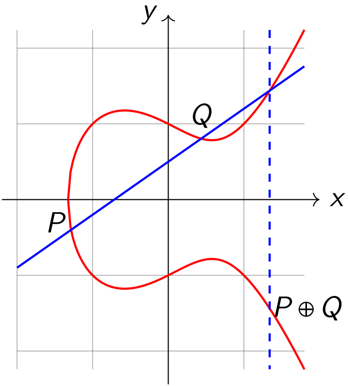
Equations are impractical in dim ≥ 2.
We need a better way to represent them...



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- ... which we are going to use to classify the AVs up to isomorphism.

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is injective and allows us to enumerate all AVs up to isogeny.
- Also, $h_A(x)$ is squarefree $\iff$ $\mathrm{End}(A)$ is commutative.

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- Problem: $\mathbb{Z}[F,V]$ might not be maximal $\rightsquigarrow$ non-invertible ideals.

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- Hofmann-Sircana '19: computation of over-orders.

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- Let $\mathscr{W}(R)$ be the set of weak eq. classes...
  ...whose representatives can be found in

$$\left\{ \text{sub-}R\text{-modules of } {}^{\mathcal{O}_K}\!/\!_{\mathfrak{f}_R} \right\} \quad \text{finite! and most of the time not-too-big ...}$$

where $\mathfrak{f}_R = (R : \mathcal{O}_K)$ is the conductor of R.

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Partition w.r.t. the multiplicator ring:

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Theorem (M.)

For every over-order S of R, $\mathrm{Pic}(S)$ acts freely on $\mathrm{ICM}_S(R)$ and

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Repeat for every $R \subseteq S \subseteq \mathscr{O}_K$:

$$\rightsquigarrow \mathrm{ICM}(R).$$

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Remark

Let $\mathscr{C}_h$ be a **squarefree** *isogeny classes over the* **prime field** $\mathbb{F}_p$*. Building on work by Centeleghe-Stix, we get a bijection between the isomorphism classes of AVs in $\mathscr{C}_h$ and the ideal class monoid of $\mathbb{Z}[F, V]$, as above. But the functor is completely different! (eg. It is contravariant)*

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- *a polarization μ of A corresponds to a $\lambda \in K^\times$ such that*
  - $\lambda I \subseteq \overline{I}^t$ *(isogeny);*
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  Also: $\deg \mu = [\overline{I}^t : \lambda I]$.
- *if $(A, \mu) \leftrightarrow (I, \lambda)$ is a princ. polarized ab. var. and $S = (I : I)$ then*

$$\left\{ \begin{matrix} \text{non-isomorphic princ.} \\ \text{polarizations of } A \end{matrix} \right\} \longleftrightarrow \frac{\{\text{totally positive } u \in S^\times\}}{\{v\overline{v} : v \in S^\times\}},$$

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Let $A \in \mathscr{C}_h$ with h ordinary and squarefree. If $A \leftrightarrow I$, then:

- $A^\vee \leftrightarrow \overline{I}^t := \{\overline{x} \in K : \mathrm{Tr}(xI) \subseteq \mathbb{Z}\}$.
- *a polarization μ of A corresponds to a $\lambda \in K^\times$ such that*
  - $\lambda I \subseteq \overline{I}^t$ *(isogeny);*
  - λ *is totally imaginary ($\overline{\lambda} = -\lambda$);*
  - λ *is Φ-positive, where Φ is a CM-type of K satisf. the Shimura-Taniyama formula.*

  Also: $\deg \mu = [\overline{I}^t : \lambda I]$.
- *if $(A,\mu) \leftrightarrow (I,\lambda)$ is a princ. polarized ab. var. and $S = (I:I)$ then*

$$\left\{ \begin{matrix} \text{non-isomorphic princ.} \\ \text{polarizations of } A \end{matrix} \right\} \longleftrightarrow \frac{\{\text{totally positive } u \in S^\times\}}{\{v\overline{v} : v \in S^\times\}},$$

- *and* $\mathrm{Aut}(A,\mu) = \{\text{torsion units of } S\}$.

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Can modify to compute polarizations of any degree.

Example

- Let $h(x) = x^8 - 5x^7 + 13x^6 - 25x^5 + 44x^4 - 75x^3 + 117x^2 - 135x + 81$, LMFDB label: 4.3.af_n_az_bs.

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- More info at

  `https://abvar.lmfdb.xyz/Variety/Abelian/Fq/4/3/af_n_az_bs`

Example

Concretely:

$$\begin{aligned}
I_1 =& 2645633792595191\mathbb{Z} \oplus (F + 836920075614551)\mathbb{Z} \oplus (F^2 + 1474295643839839)\mathbb{Z} \oplus \\
&\oplus (F^3 + 1372829830503387)\mathbb{Z} \oplus (F^4 + 1072904687510)\mathbb{Z} \oplus \\
&\oplus \frac{1}{3}(F^5 + F^4 + F^3 + 2F^2 + 2F + 6704806986143610)\mathbb{Z} \oplus \\
&\oplus \frac{1}{9}(F^6 + F^5 + F^4 + 8F^3 + 2F^2 + 2991665243621169)\mathbb{Z} \oplus \\
&\oplus \frac{1}{27}(F^7 + F^6 + F^5 + 17F^4 + 20F^3 + 9F^2 + 6801531251872201)\mathbb{Z}
\end{aligned}$$

principal polarizations:

$$\begin{aligned}
x_{1,1} =& \frac{1}{27}(-121922F^7 + 588604F^6 - 1422437F^5 + \\
&+ 1464239F^4 + 1196576F^3 - 7570722F^2 + 15316479F - 12821193) \\
x_{1,2} =& \frac{1}{27}(3015467F^7 - 17689816F^6 + 35965592F^5 - \\
&- 64660346F^4 + 121230619F^3 - 191117052F^2 + 315021546F - 300025458)
\end{aligned}$$

$$\mathrm{End}(I_1) = R$$

$$\#\mathrm{Aut}(I_1, x_{1,1}) = \#\mathrm{Aut}(I_1, x_{1,2}) = 2$$

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Corollary

If G is cyclic we can take A to be ordinary and squarefree.

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Then there is an ordinary $A/\mathbb{F}_q$ such that $G = A(\mathbb{F}_q)$.

Thank you!