Abelian varieties over finite fields

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UPF - Gaati Lab

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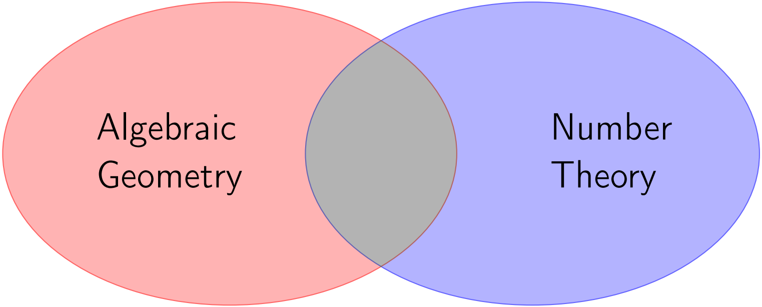
What do I do for a living?



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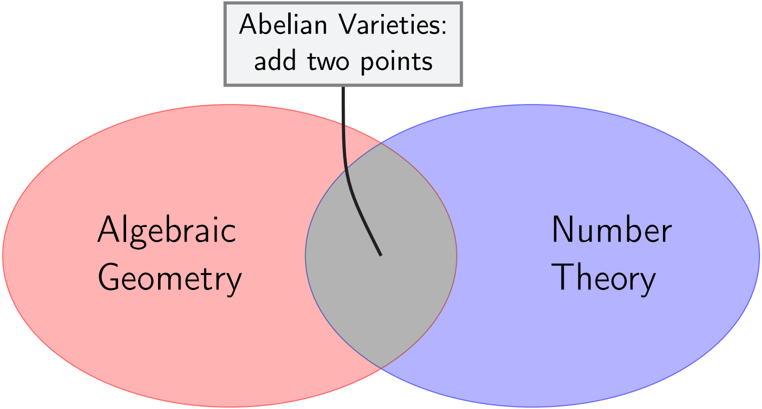

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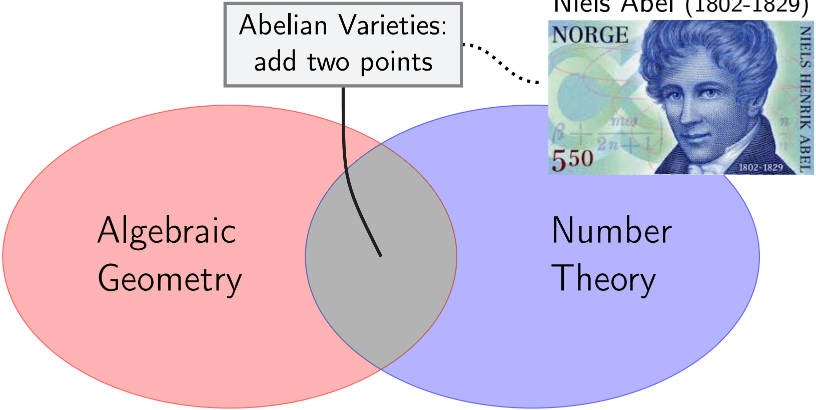

Abelian Varieties: add two points

Niels Abel (1802-1829)

Algebraic Geometry

Number Theory

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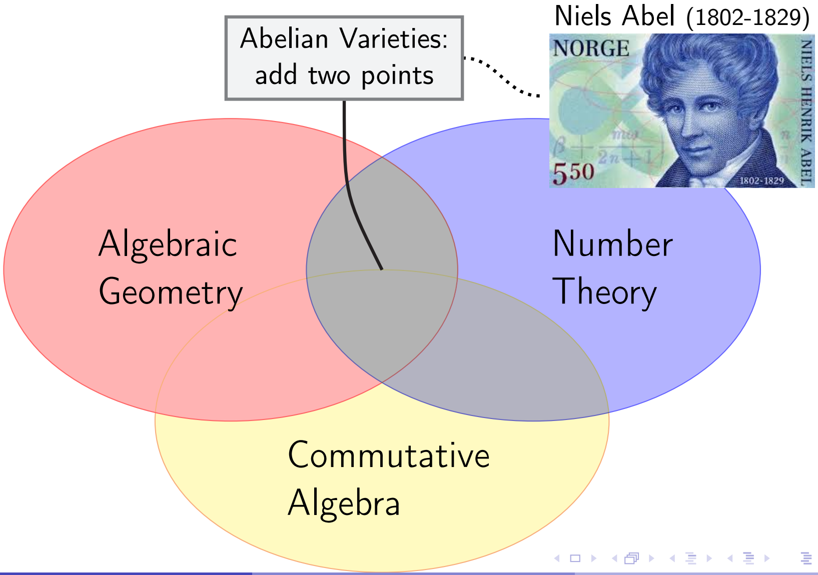

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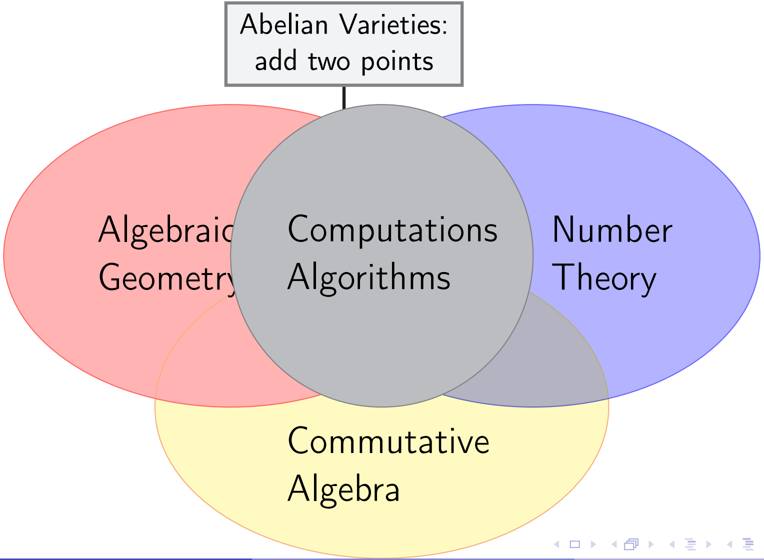

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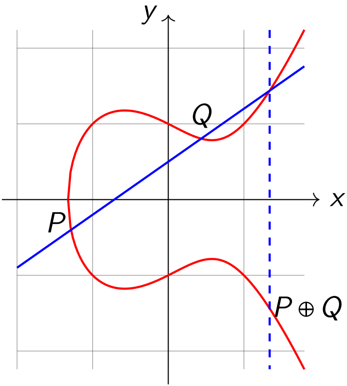

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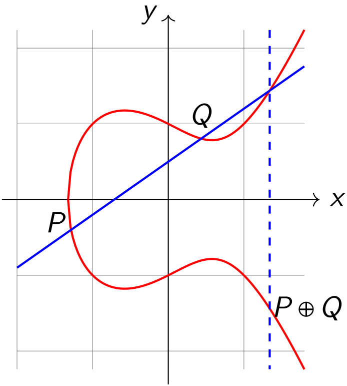

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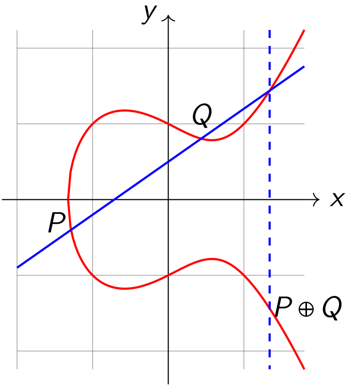
Equations are impractical in dim ≥ 2. We need a better way to represent them...



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- Nevertheless, over finite fields, we obtain analogous results if we restrict ourselves to certain **subcategories** of AVs...
- ... which we are going to use to classify the AVs up to isomorphism.

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- Also, $h_A(x)$ is squarefree $\iff$ $\mathrm{End}(A)$ is commutative.

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- Problem: $\mathbb{Z}[F,V]$ might not be maximal $\rightsquigarrow$ non-invertible ideals.

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- Hofmann-Sircana '19: computation of over-orders.

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  ...whose representatives can be found in

$$\left\{ \text{sub-}R\text{-modules of } {}^{\mathscr{O}_K}\!\diagup_{\mathfrak{f}_R} \right\}$$ finite! and most of the time not-too-big ...

  where $\mathfrak{f}_R = (R : \mathscr{O}_K)$ is the conductor of R.

Compute ICM(R)

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Partition w.r.t. the multiplicator ring:

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Theorem (M.)

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Repeat for every $R \subseteq S \subseteq \mathscr{O}_K$:

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Remark

Let $\mathscr{C}_h$ be a **squarefree** *isogeny classes over the* **prime field** $\mathbb{F}_p$*. Building on work by Centeleghe-Stix, we get a bijection between the isomorphism classes of AVs in $\mathscr{C}_h$ and the ideal class monoid of $\mathbb{Z}[F, V]$, as above. But the functor is completely different! (eg. It is contravariant)*

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  - λ is Φ-positive, where Φ is a CM-type of K satisfying the Shimura-Taniyama formula.
- *if $(A,\mu) \leftrightarrow (I,\lambda)$ is a princ. polarized ab. var. and $S = (I : I)$ then*

$$\left\{ \begin{array}{l} \text{non-isomorphic princ.} \\ \text{polarizations of } A \end{array} \right\} \longleftrightarrow \frac{\{\text{totally positive } u \in S^\times\}}{\{v\overline{v} : v \in S^\times\}},$$

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Let $A \in \mathscr{C}_h$ with h ordinary and squarefree. If $A \leftrightarrow I$, then:

- $A^\vee \leftrightarrow \overline{I}^t := \{\overline{x} \in K : \mathrm{Tr}(xI) \subseteq \mathbb{Z}\}$.
- *if μ is an isogeny $A \to A^\vee$ then $\mu \leftrightarrow \lambda \in K^\times$ with $\lambda I \subseteq \overline{I}^t$.*
- *μ is a polarization if and only if*
  - λ is totally imaginary ($\overline{\lambda} = -\lambda$);
  - λ is Φ-positive, where Φ is a CM-type of K satisfying the Shimura-Taniyama formula.
- *if $(A,\mu) \leftrightarrow (I,\lambda)$ is a princ. polarized ab. var. and $S = (I:I)$ then*

$$\left\{ \begin{matrix} \text{non-isomorphic princ.} \\ \text{polarizations of } A \end{matrix} \right\} \longleftrightarrow \frac{\{\text{totally positive } u \in S^\times\}}{\{v\overline{v} : v \in S^\times\}},$$

- and $\mathrm{Aut}(A,\mu) = \{\text{torsion units of } S\}$.

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Can modify to compute polarizations of any degree.

Example

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- More info at

  `https://abvar.lmfdb.xyz/Variety/Abelian/Fq/4/3/af_n_az_bs`

Example

Concretely:

$$\begin{aligned} I_1 =& 2645633792595191\mathbb{Z} \oplus (F + 836920075614551)\mathbb{Z} \oplus (F^2 + 1474295643839839)\mathbb{Z} \oplus \\ &\oplus (F^3 + 1372829830503387)\mathbb{Z} \oplus (F^4 + 1072904687510)\mathbb{Z} \oplus \\ &\oplus \frac{1}{3}(F^5 + F^4 + F^3 + 2F^2 + 2F + 6704806986143610)\mathbb{Z} \oplus \\ &\oplus \frac{1}{9}(F^6 + F^5 + F^4 + 8F^3 + 2F^2 + 2991665243621169)\mathbb{Z} \oplus \\ &\oplus \frac{1}{27}(F^7 + F^6 + F^5 + 17F^4 + 20F^3 + 9F^2 + 6801531251872201)\mathbb{Z} \end{aligned}$$

principal polarizations:

$$\begin{aligned} x_{1,1} = \frac{1}{27}(&-121922F^7 + 588604F^6 - 1422437F^5 + \\ &+ 1464239F^4 + 1196576F^3 - 7570722F^2 + 15316479F - 12821193) \\ x_{1,2} = \frac{1}{27}(&3015467F^7 - 17689816F^6 + 35965592F^5 - \\ &- 64660346F^4 + 121230619F^3 - 191117052F^2 + 315021546F - 300025458) \end{aligned}$$

$\mathrm{End}(I_1) = R$

$\#\mathrm{Aut}(I_1, x_{1,1}) = \#\mathrm{Aut}(I_1, x_{1,2}) = 2$

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They use extremely clever constructions that allows them to construct characteristic polynomials h_A such that $h_A(1) = m$.

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Corollary

If G is cyclic we can take A to be ordinary and squarefree.

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Then there is an ordinary $A/\mathbb{F}_q$ such that $G = A(\mathbb{F}_q)$.

Thank you!