Itro to category theory

A **category** $\mathcal{C}$ consists of :

1) a class of objects $Ob(\mathcal{C})$ "$A \in \mathcal{C}$"

2) $\forall$ $A, B \in Ob(\mathcal{C})$ a set

$Hom_{\mathcal{C}}(A, B)$ of morphisms "$A \xrightarrow{f} B$"

3) a composition law of morphisms :

$\forall$ $A, B, C \in \mathcal{C}$

$$Hom_{\mathcal{C}}(B, C) \times Hom_{\mathcal{C}}(A, B) \longrightarrow Hom_{\mathcal{C}}(A, C)$$

$$(g, f) \longmapsto g \circ f$$

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[rr, bend right, "g \circ f"'] & B \arrow[r, "g"] & C \end{tikzcd}$$

such that : ~~a) if $(C, D) \neq (A, B)$ $A, B, C, D \in \mathcal{C}$~~

~~$Hom_{\mathcal{C}}(C, D)$ disjoint $Hom_{\mathcal{C}}(A, B)$~~

b) $\forall$ $A \in \mathcal{C}$ $\exists$ $id_A : A \to A$ st

$$\forall \quad \begin{tikzcd} B \arrow[r, "f"] & A \arrow[r, "id_A"] & A \arrow[r, "g"] & C \end{tikzcd}$$

$f = id_A \circ f$ and $g =$

c) associatity of the composition.

<u>Def</u> $A \xrightarrow{f} B$ in $\mathcal{C}$ is an isomorphism if

$\exists$ $B \xrightarrow{g} A$ in $\mathcal{C}$ st

$$f \circ g = id_B, \quad g \circ f = id_A$$

Examples

Sets , Top , Rings , CRings , Grp , Ab

∀ ring R : R-Mod , Mod-R

∀ division ring D : Vect-D , D-Vect

∀ field k : Vect_k

- Recall
  - R-Mod = Vect_k
  - $\mathbb{Z}$-Mod = Ab

- Point of category theory : don't think only about objects, but also about "arrows"

- $\mathcal{C}$, $\mathcal{D}$ categories.

  a (covariant) functor $F: \mathcal{C} \to \mathcal{D}$ is a law assigning :

  1) $\forall \, C \in \mathcal{C}$ an object $F(C) \in \mathcal{D}$

  2) $\forall \, C \xrightarrow{f} C'$ a morphism in $\mathcal{D}$ $F(C) \xrightarrow{F(f)} F(C')$

  satisfying : ⓐ $F(id_C) = id_{F(C)}$

  ⓑ $F(f \circ g) = F(f) \circ F(g)$

- " a contravariant functor : - $F(C) \xleftarrow{F(f)} F(C')$
  - $F(f \circ g) = F(g) \circ F(f)$ "

- $\mathcal{C}^{op}$: $Ob(\mathcal{C}^{op}) = Ob(\mathcal{C})$

  $\text{Hom}_{\mathcal{C}^{op}}(A,B) = \text{Hom}_{\mathcal{C}}(B,A)$

- A contravariant functor $\mathcal{C} \to \mathcal{D}$ is a covariant functor $\mathcal{C}^{op} \to \mathcal{D}$

Example

Consider the following category $\mathcal{C}$

$$Ob(\mathcal{C}) = \{*\}$$

$$Hom_{\mathcal{C}}(*, *) = G \text{ a group.}$$

Then a linear representation of G over a field k is a functor

$$F : \mathcal{C} \longrightarrow \underline{Vect_k}$$

i.e. a choice of a v.s. V and a group homorphism $G \longrightarrow Aut(V)$

i.e. we fix a $k[G]$-module structure on V

(• Rmks : - every morphism of $\mathcal{C}$ is an isomorphism ($\forall f\ \exists g$ st $f \circ g = id_* = g \circ f$)
- any functor sends iso to iso)

Idea : : rather than studying directly G, we study its representations. Turns out that we can get a lot of info from those :
- normal subgroubs,
- #elements
- # elts in the conjugacy classes

TALK ABOUT NATURAL TRANSFORMATIONS

& the category Funct $(\mathcal{C}, \mathcal{D})$

commu → (R - linear)

Def A category $\mathcal{C}$ is called preadditive if $\mathrm{Hom}_{\mathcal{C}}(A, B)$ is an abelian group (R-module) and the composition is bilinear: (R-bilinear)

$$(f+f')\circ(g+g') = (f+g)\circ(f'+g')\circ(f+g')\circ(f'\circ g)$$

Ex
- R-Mod, Mod-R, $\mathcal{A}b$, Vect_k are pre-additive
- Ring, Sets, Top are not preadditive

$$\hookrightarrow \left((f+g)(1) = \underset{\overset{\|}{1}}{f(1)} + \underset{\overset{\|}{1}}{g(1)} \neq 1 \right)$$

Def $\mathcal{C}$ category.
- $Z \in \mathcal{C}$ is an initial object if $\mathrm{Hom}_{\mathcal{C}}(Z, C)$ consist of only one morphism $\forall\, C \in \mathcal{C}$
- a terminal object if the same holds for $\mathrm{Hom}_{\mathcal{C}}(C, Z)$ $\forall\, C \in \mathcal{C}$
- They are unique up to iso
- $0_{\mathcal{C}}$ both initial & terminal is a zero objects

Ex
- Sets: $\emptyset$ initial, $\{*\}$ is terminal, no zero-obj.
- $\mathcal{A}b$, R-Mod, Mod-R, … have a zero object
- Ring: $\mathbb{Z}$ is initial, no terminal object.

Rmk • If $\mathcal{C}$ has a zero-obj $0_{\mathcal{C}}$
then $\forall A, B \in \mathcal{C}$

$$\begin{tikzcd}
A \arrow[rr, dashed, "\beta \circ \alpha"] \arrow[dr, "\alpha"'] & & B \\
& 0_{\mathcal{C}} \arrow[ur, "\beta"'] &
\end{tikzcd}$$

$\exists$ a distinguished morph $\beta \circ \alpha = 0$,

• If moreover $\mathcal{C}$ is preadditive :

0_{AB} is the neutral elt of $\mathrm{Hom}_{\mathcal{C}}(A,B)$.

Let $\mathcal{C}$ with zero object $0_{\mathcal{C}}$; $A, B \in \mathcal{C}$
$A \xrightarrow{f} B$

Def • A kernel of f is

$$\begin{tikzcd}
K \arrow[r, "k"] & A \arrow[r, "f"] & B \\
K' \arrow[u, dashed, "m"] \arrow[ur, "k'"'] & &
\end{tikzcd}$$

st $f \circ k = 0$

and with $f \circ k' = 0$.

$\exists !\ m : K' \to K$ st $k \circ m = k'$.

• A cokernel of f is

$$\begin{tikzcd}
A \arrow[r, "f"] & B \arrow[r, "h"] \arrow[dr, "h'"'] & C \\
& & C' \arrow[u, dashed, "\exists ! m"'] \arrow[ul, phantom, "\circlearrowright"]
\end{tikzcd}$$

st $h \circ f = 0$

and

$\forall h'$: $\exists !\ m$ ...

$\mathcal{C}$ preadditive w/ $0_{\mathcal{C}}$; $A, B \in \mathcal{C}$

Def • A product of A and B is

$(\underset{\overset{\cap}{\mathcal{C}}}{A \prod B}, p_A, p_B)$ s.t.

$$\begin{tikzcd} & C \arrow[dl, "f"'] \arrow[dr, "g"] \arrow[d, dashed, "\exists! h"] & \\ A & A \prod B \arrow[l, "p_A"] \arrow[r, "p_B"'] & B \end{tikzcd}$$

∀ $\exists! \, h : C \to A \prod B$

$p_A h = f$ & $p_B h = g$

(If ∃ ⇒ it's unique)

• A coproduct of A and B is

$(A \amalg B, \varepsilon_A, \varepsilon_B)$ st

$$\begin{tikzcd} & C & \\ A \arrow[ur, "f"] \arrow[r, "\varepsilon_A"'] \arrow[r, phantom, shift left=4, "\circlearrowright"] & A \amalg B \arrow[u, dashed, "\exists! h"] & B \arrow[ul, "g"'] \arrow[l, "\varepsilon_B"] \arrow[l, phantom, shift left=4, "\circlearrowright"] \end{tikzcd}$$

∀

Prop If $(A \prod B, p_A, p_B)$ exists then the $(A \amalg B, \varepsilon_A, \varepsilon_B)$ exists and

$$A \prod B \cong A \amalg B \quad (\text{"} A \oplus B \text{"})$$

!!! this is true for an additive category

Cor : the same is true for finite products

$$\prod_{i=1}^{N} A_i \simeq \coprod_{i=1}^{N} A_i$$

but NOT for infinite products.

Examples :

	prod	coprod
Sets	" × "	" ⊍ "
Ab	" ∏ "	" ⊕ "

<u>Def</u> $\mathcal{C}$ is <u>additive</u> if :

- $\mathcal{C}$ is preadditive w/ $0_{\mathcal{C}}$
- $\forall\ A, B \in \mathcal{C} \quad \exists\ A \Pi B$

⊛ Talk about K-S. cat. ⑫

<u>Def</u> $\mathcal{C}$ is <u>abelian</u> if additive

1 every morphism has ker and coker

2 every morphism f can be factored as $f = \overset{\text{ker}}{\gamma} \circ \overset{\text{Coker}}{q}$

$\Updownarrow$

or 2 bis $\forall\ A \xrightarrow{f} B$ we have $\text{Coker}(\ker(f)) \xrightarrow{\sim} \underset{\text{Im} f}{\underset{\shortparallel}{\ker(\text{coker}(f))}}$

"1st Iso thm"

Skip to ⑨ if needed (~ 30-35 min)

<u>Def</u> $\mathcal{C}$ any category, $A \xrightarrow{f} B$ in $\mathcal{C}$

- f is a <u>monomorphism</u> if

$$\forall \quad C \overset{g}{\underset{h}{\rightrightarrows}} A \xrightarrow{f} B$$

st $\quad f \circ g = f \circ h \implies g = h$

- f is an <u>epimorphism</u> if

$$\forall \quad A \xrightarrow{f} B \overset{g}{\underset{h}{\rightrightarrows}} C$$

st $\quad g \circ f = h \circ f \implies g = h$

<u>Rmk</u> : f iso $\implies$ f mono + epi

$\not\Longleftarrow$ in general

<u>Ex</u> $\mathbb{Z} \overset{i}{\hookrightarrow} \mathbb{Q}$ in <u>Rings</u> : is mono + epi but not an iso

- mono : obvious
- epi : $\mathbb{Z} \overset{i}{\hookrightarrow} \mathbb{Q} \overset{\psi}{\underset{\varphi}{\rightrightarrows}} R$ st $\psi|_{\mathbb{Z}} = \varphi|_{\mathbb{Z}}$

$1_R = \varphi\left(n \frac{1}{n}\right) = n\, \varphi\left(\frac{1}{n}\right)$

$= \psi\left(n \frac{1}{n}\right) = n\, \psi\left(\frac{1}{n}\right)$

So $\varphi\left(\frac{1}{n}\right) = \psi\left(\frac{1}{n}\right)$ as they are the inverse of n in R.

$\implies \varphi\left(\frac{m}{n}\right) = \varphi(m)\, \varphi\left(\frac{1}{n}\right) = \psi(m)\, \psi\left(\frac{1}{n}\right) = \psi\left(\frac{m}{n}\right)$

skip to ⑨ if needed

Prop $\mathcal{C}$ abelian Then

a) mono + epi $\Leftrightarrow$ iso

b) mono = kernel

c) epi = cokernel

Examples :

① R-Mod, Mod-R are abelian.

In fact :

(Freyd-) **Mitchell's embedding theorem** :

every small abelian category is equivalent to a full sub-category of modules

small = $Ob(\mathcal{C})$ is a set

$\mathcal{F} \subseteq \mathcal{C}$ full : if $A, B \in \mathcal{F}$ then $Hom_{\mathcal{F}}(A,B) = Hom_{\mathcal{C}}(A,B)$

② Ab = $\mathbb{Z}$-Mod is abelian

$\cup$

$\mathcal{F}$:= torsion free abelian groups

then $\mathcal{F}$ is additive but not abelian

$\mathcal{T}$ = torsion ab groups

$\mathcal{T}$ is abelian

Def • $\longrightarrow C_{i-1} \xrightarrow{f_{i-1}} C_i \xrightarrow{f_i} C_{i+1} \longrightarrow$ sequence
is exact if $\forall i$ $\operatorname{Im} f_{i-1} = \ker f_i$

• $\mathcal{C}, \mathcal{D}$ ab. cat, $F: \mathcal{C} \longrightarrow \mathcal{D}$ a functor

1 F is exact if it sends exact seq's to exact seq's

2 F is left exact if

$\forall \quad 0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$ s.e.s

$\Longrightarrow \quad 0 \longrightarrow F(A) \xrightarrow{Ff} F(B) \xrightarrow{Fg} F(C)$ is exact

3 F is right exact ...

$\Longrightarrow \quad F(A) \xrightarrow{F(f)} F(B) \xrightarrow{F(g)} F(C) \longrightarrow 0$

Lemma F exact $\Longleftrightarrow$ F both left and right exact

Important functor

$\mathcal{C}^{op} \times \mathcal{C} \ni (C, D)$, $C, D \in \mathcal{C}$

$$\begin{tikzcd}(C, D) \arrow[d, "(f,g)"] \\ (C', D')\end{tikzcd} \quad \longleftrightarrow \quad \begin{matrix} f : C' \to C \\ g : D \to D' \end{matrix}$$

$$\begin{tikzcd}[column sep=large] \mathrm{Hom}_{\mathcal{C}}(-,-) : \mathcal{C}^{op} \times \mathcal{C} \arrow[r] & \mathrm{Sets} \\ (C, D) \arrow[r, mapsto] \arrow[d, "(f,g)"] & \mathrm{Hom}_{\mathcal{C}}(C, D) \\ {} \arrow[r, phantom, "\longmapsto"] & \left(\begin{matrix} \alpha \\ \downarrow \\ g \circ \alpha \circ f \end{matrix}\right) \\ (C', D') \arrow[r, mapsto] & \mathrm{Hom}_{\mathcal{C}}(C', D') \end{tikzcd}$$

Rmk $\mathrm{Hom}_{\mathcal{C}}(-,-)$ is covariant in both entries

Prop $\mathcal{C}$ abelian ; $\mathrm{Hom}_{\mathcal{C}}(-,-)$ is left exact in both entries.

Def $\mathcal{C}$ ab ;
- $P \in \mathcal{C}$ is projective if $\mathrm{Hom}_{\mathcal{C}}(P, -)$ is exact (i.e. also right exact)
- $I \in \mathcal{C}$ is injective if $\mathrm{Hom}_{\mathcal{C}}(-, I)$ is exact

- $\mathcal{C}$ has enough projectives if $\forall C \in \mathcal{C}$ $\exists$ $P \in \mathcal{C}$ projective and
$$P \xrightarrow{epi} C \to 0$$
- enough injective if $\forall C \in \mathcal{C}$ $\exists$ $I \in \mathcal{C}$ injective
$$0 \to C \xrightarrow{mono} I$$

Examples

- $\mathcal{C} = R\text{-Mod}$ has enough projectives and injectives
- $\mathcal{T}$ torsion ab groups:
  - $G_i \in Ob(\mathcal{T}) \Rightarrow \coprod G_i \in Ob(\mathcal{T})$

    but $\prod G_i$ is not torsion

    $\left(\prod_{n\geq 0} \mathbb{Z}/p^n\mathbb{Z} \quad : \quad (1_n \cdot)_n \text{ is not orsion} \right)$
- $\mathcal{T}$ has enough injectives
- $\mathcal{T}$ has not enough proj:

  $\ni P$ is projective $\Leftrightarrow P = 0$

Krull - Schmidt categories

A non-zero $C \in \mathcal{C}$ is indecomposable if:

$$C = C_1 \oplus C_2 \Rightarrow C_1 = 0_{\mathcal{C}} \text{ or } C_2 = 0_{\mathcal{C}}$$

Rmk : $\mathcal{C}$ preadditive , $C \in \mathcal{C}$ then

$\mathrm{End}_{\mathcal{C}}(C)$ is a ring

with $\circ$ as multiplication & 1_C as identity.

Def : An additive category is called a Krull-Schmidt cat if every object decomposes into a finite direct sum of object having local end. rings

Rmk In a K.S. cat: loc. end ring = indecomposable

only one max^{l} ideal

$\Updownarrow$

$\{$non-invertible elements$\}$ is additively closed.

Lemma : $\mathcal{C}$ additive if:

$$X = X_1 \oplus \dots \oplus X_r$$
$$X = Y_1 \oplus \dots \oplus Y_s$$

w/ X_i, Y_j with l. e. r

then $r = s$ and exist a permutation π s.t.

$$X_i \cong Y_{\pi(i)} \quad i = 1, \dots, s$$