SUPER-MULTIPLICATIVITY OF IDEAL NORMS IN NUMBER FIELDS

Let R be a commutative ring, I an R-ideal. The norm of I is defined as $N(I) = \#(R/I) = [R : I]$. We say that the norm is super-multiplicative on R (briefly R is S.M.) if for every pair of ideals I, J s.t. $[R : IJ] < \infty$ we have $N(IJ) \geq N(I)N(J)$. Observe that if the norm of an order R satisfies the other inequality for every maximal ideal $\mathfrak{p}$, that is $N(\mathfrak{p}^2) \leq N(\mathfrak{p})^2$, then $\dim_{(R/\mathfrak{p})}(\mathfrak{p}/\mathfrak{p}^2) = 1$, i.e. the order is Dedekind and the ideal norm is actually multiplicative. The main goal of my thesis is to give a characterization for a number ring, that is a subfield of a number field, of being S.M. in terms of the minimal number of generators of its ideals. I proved:

Theorem. *Let R be a number ring. Then the following statements are equivalent:*

1. *every ideal of R can be generated by 3 elements;*
2. *for every ring extension $R \subseteq R' \subseteq \tilde{R}$, where $\tilde{R}$ is the normalization of R, we have that the norm is super-multiplicative.*

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