Abelian varieties over finite fields isogenous to a power

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Introduction

Today's plan:

- Introduction.
- AV A isogenous to B^r, for B ordinary square-free defined over $\mathbb{F}_q$.
- Isomorphism classes.
- Polarizations.
- Computations of polarizations and period matrices ($r = 1$).

Also, all morphisms are defined over the field of definition!

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- in positive characteristic we don't have such equivalence (on the whole category).

Isogeny classes

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Theorem (Honda-Tate)

There is a bijection betweeen the set of simple abelian varieties over $\mathbb{F}_q$ up to isogeny and the set of q-Weil numbers up to conjugacy.

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Proposition

For B ordinary over $\mathbb{F}_q$:

$$h_B \text{ is irreducible} \iff B \text{ is simple}$$

Deligne's equivalence

Theorem (Deligne '69)

Let $q = p^d$, with p a prime. There is an equivalence of categories:

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$$\updownarrow$$

$$\mathscr{M}^{ord}(q) := \left\{ \begin{array}{l} \text{pairs } (T,F)\text{, where } T \simeq_{\mathbb{Z}} \mathbb{Z}^{2g} \text{ and } T \xrightarrow{F} T \text{ s.t.} \\ \text{- } F \otimes \mathbb{Q} \text{ is semisimple} \\ \text{- the roots of } \mathrm{char}_{F\otimes\mathbb{Q}}(x) \text{ have abs. value } \sqrt{q} \\ \text{- } \textbf{half of them are } p\textbf{-adic units} \\ \text{- } \exists V : T \to T \text{ such that } FV = VF = q \end{array} \right\}$$

Deligne's equivalence - the functor

- fix an embedding of $\varepsilon : W = W(\overline{\mathbb{F}}_p) \hookrightarrow \mathbb{C}$
- take $A \in \mathrm{AV}^{\mathrm{ord}}(q)$
- let A' be the canonical lift of A to W
- put $A_{\mathbb{C}} := A' \otimes_{\varepsilon} \mathbb{C}$
- finally, let $T(A) := H_1(A_{\mathbb{C}}, \mathbb{Z})$
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Observe if $\dim(A) = g$ then $\mathrm{Rank}(T(A)) = 2g$;

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Observe: if $A \in \mathrm{AV}(g^r)$ then

$$A \sim (B_1 \times \ldots \times B_s)^r$$

with

$$g = h_{B_1 \times \ldots \times B_s}$$

Main theorem

Consider the CM étale $\mathbb{Q}$-algebra

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Theorem (M.)

There are equivalences of categories

$$\mathrm{AV}(g^r) \overset{\textit{Deligne}}{\longleftrightarrow} \mathscr{M}(g^r) \longleftrightarrow \mathscr{B}(g^r)$$

The category $\mathscr{B}(g^r)$

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We can think of modules $M \in \mathscr{B}(g^r)$ as **embedded** in K^r.
The category $\mathscr{B}(g^r)$ becomes more explicit and computable under certain assumption on the order R.

Bass orders

Recall

- a fractional R-ideal I is a sub-R-module of K which is also a lattice
- a fractional R-ideal is invertible in R if $I(R:I) = R$.

Define

$$\mathrm{ICM}(R) = {}^{\{\text{fractional } R\text{-ideals}\}}\big/_{\simeq_R} \qquad \text{ideal class monoid}$$

and

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- $\mathrm{ICM}(R) = \bigsqcup_{R\subseteq S\subseteq \mathcal{O}_K} \mathrm{Pic}(S)$.

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generalization of Steinitz theory

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Corollary

Assume that R is Bass. Then for every $M \in \mathscr{B}(g^r)$ there are over orders $S_1 \subseteq \ldots \subseteq S_r$ of R and a fractional ideal I invertible in S_r such that

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and

$$\mathrm{Aut}_R(M) = \{A \in \mathrm{End}_R(M) \cap \mathrm{GL}_r(K) : A^{-1} \in \mathrm{End}_R(M)\}.$$

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  $$A \longleftrightarrow \bigoplus_k I_k \text{ and } B \longleftrightarrow \bigoplus_k J_k$$

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  Moreover, μ is an isogeny if and only if $\det(\Lambda) \in K^\times$

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Let I be a representatives of a generator of $\mathrm{Pic}(R)$.

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Define $K = \mathbb{Q}[x]/(g) = \mathbb{Q}(F)$ and $R = \mathbb{Z}[F, V]$.
The only over-order of R is the maximal order $\mathcal{O}_K$ of K and, since R is Gorenstein R is Bass.
Observe

$$\mathrm{Pic}(R) \simeq {}^{\mathbb{Z}}\!/\!{}_{3\mathbb{Z}} \text{ and } \mathrm{Pic}(\mathcal{O}_K) = \{1\}.$$

Let I be a representatives of a generator of $\mathrm{Pic}(R)$.
We now list the representatives of the isomorphism classes in $\mathrm{AV}(g^3)$:

$$\begin{array}{lll} M_1 = R \oplus R \oplus R & M_2 = R \oplus R \oplus I & M_3 = R \oplus R \oplus I^2 \\ M_4 = R \oplus R \oplus \mathcal{O}_K & M_5 = R \oplus \mathcal{O}_K \oplus \mathcal{O}_K & M_6 = \mathcal{O}_K \oplus \mathcal{O}_K \oplus \mathcal{O}_K \end{array}$$

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$$\mathrm{End}(M_1) = \mathrm{Mat}_3(R) \text{ and } \mathrm{End}(M_2) = \begin{pmatrix} R & R & I \\ R & R & I \\ (R:I) & (R:I) & R \end{pmatrix}$$

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Let $M \in \mathscr{B}(g^r)$ and let $\mathrm{Tr}: K^r \to \mathbb{Q}$ be the map induced by $\mathrm{Tr}_{K/\mathbb{Q}}$

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$$\Lambda^\vee := \overline{\Lambda}^T$$

"Proof": Howe (1995) described dual modules in $\mathscr{M}^{\mathrm{ord}}(q)$.

Polarizations

Fix

$$\Phi := \{\varphi : K \to \mathbb{C} : v_p(\varphi(F)) > 0\}, \text{ tricky to compute!}$$

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- $\Lambda = -\overline{\Lambda}^T$, *and*
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"Proof": Howe (1995) put polarizations in Deligne's category $\mathscr{M}^{\mathrm{ord}}(q)$.
We translated this notion to $\mathscr{B}(g^r)$.

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Unfortunately

$$\mathrm{Pol}(M) / \mathrm{Aut}(M) \text{ is hard to understand if } r \geq 2$$

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  Also: $\deg \mu = [\bar{I}^t : \lambda I]$.
- if $(A, \mu) \leftrightarrow (I, \lambda)$ and $S = (I : I)$ then

$$\left\{ \begin{matrix} \text{non-isomorphic} \\ \text{polarizations of } A \end{matrix} \right\} \longleftrightarrow \frac{\{\text{totally positive } u \in S^\times\}}{\{v\bar{v} : v \in S^\times\}}$$

  and $\mathrm{Aut}(A, \mu) = \{\text{torsion units of } S\}$

Example

- Let $h(x) = x^8 - 5x^7 + 13x^6 - 25x^5 + 44x^4 - 75x^3 + 117x^2 - 135x + 81$;
- $\rightsquigarrow$ isogeny class of an simple ordinary abelian varieties over $\mathbb{F}_3$ of dimension 4;
- Let F be a root of $h(x)$ and put $R := \mathbb{Z}[F, 3/F] \subset \mathbb{Q}(F)$;
- 8 over-orders of R: two of them are not Gorenstein;
- $\#\mathrm{ICM}(R) = 18 \rightsquigarrow 18$ isom. classes of AV in the isogeny class;
- 5 are not invertible in their multiplicator ring;
- 8 classes admit principal polarizations;
- 10 isomorphism classes of princ. polarized AV.

Example

Concretely:

$$
\begin{aligned}
I_1 =& 2645633792595191\mathbb{Z} \oplus (F + 836920075614551)\mathbb{Z} \oplus (F^2 + 1474295643839839)\mathbb{Z} \oplus \\
&\oplus (F^3 + 1372829830503387)\mathbb{Z} \oplus (F^4 + 1072904687510)\mathbb{Z} \oplus \\
&\oplus \frac{1}{3}(F^5 + F^4 + F^3 + 2F^2 + 2F + 6704806986143610)\mathbb{Z} \oplus \\
&\oplus \frac{1}{9}(F^6 + F^5 + F^4 + 8F^3 + 2F^2 + 2991665243621169)\mathbb{Z} \oplus \\
&\oplus \frac{1}{27}(F^7 + F^6 + F^5 + 17F^4 + 20F^3 + 9F^2 + 6801531251872201)\mathbb{Z}
\end{aligned}
$$

principal polarizations:

$$
\begin{aligned}
x_{1,1} = & \frac{1}{27}(-121922F^7 + 588604F^6 - 1422437F^5 + \\
& + 1464239F^4 + 1196576F^3 - 7570722F^2 + 15316479F - 12821193) \\
x_{1,2} = & \frac{1}{27}(3015467F^7 - 17689816F^6 + 35965592F^5 - \\
& - 64660346F^4 + 121230619F^3 - 191117052F^2 + 315021546F - 300025458)
\end{aligned}
$$

$\mathrm{End}(I_1) = R$

$\#\mathrm{Aut}(I_1, x_{1,1}) = \#\mathrm{Aut}(I_1, x_{1,2}) = 2$

Example

$$I_7 = 2\mathbb{Z} \oplus (F+1)\mathbb{Z} \oplus (F^2+1)\mathbb{Z} \oplus (F^3+1)\mathbb{Z} \oplus (F^4+1)\mathbb{Z} \oplus \frac{1}{3}(F^5+F^4+F^3+2F^2+2F+3)\mathbb{Z}\oplus$$
$$\oplus \frac{1}{36}(F^6+F^5+10F^4+26F^3+2F^2+27F+45)\mathbb{Z}\oplus$$
$$\oplus \frac{1}{216}(F^7+4F^6+49F^5+200F^4+116F^3+105F^2+198F+351)\mathbb{Z}$$

principal polarization:

$$x_{7,1} = \frac{1}{54}(20F^7-43F^6+155F^5-308F^4+580F^3-1116F^2+2205F-1809)$$
$$\mathrm{End}(I_7) = \mathbb{Z} \oplus F\mathbb{Z} \oplus F^2\mathbb{Z} \oplus F^3\mathbb{Z} \oplus F^4\mathbb{Z} \oplus \frac{1}{3}(F^5+F^4+F^3+2F^2+2F)\mathbb{Z}\oplus$$
$$\oplus \frac{1}{18}(F^6+F^5+10F^4+8F^3+2F^2+9F+9)\mathbb{Z}\oplus$$
$$\oplus \frac{1}{108}(F^7+4F^6+13F^5+56F^4+80F^3+33F^2+18F+27)\mathbb{Z}$$
$$\#\mathrm{Aut}(I_7, x_{7,1}) = 2$$

I_1 is invertible in R, but I_7 is not invertible in $\mathrm{End}(I_7)$.

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We have an isomorphism of complex tori

$$A'(\mathbb{C}) \simeq {}^{\mathbb{C}^g}\!/_{\Phi(I)}, \qquad \Phi(I) = \langle(\varphi_1(\alpha_i), \ldots, \varphi_g(\alpha_i) : i = 1, \ldots, 2g\rangle\,.$$

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$$b(\gamma_i, \beta_i) = 1 \text{ for all } i, \text{ and}$$
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This is big period matrix of (A',λ').

Period matrices - Example

Let $g = (x^4 - 4x^3 + 8x^2 - 12x + 9)(x^4 - 2x^3 + 2x^2 - 6x + 9)$.

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$$
\begin{aligned}
I = {} & \frac{1}{54}\left(432 - 549\alpha + 441\alpha^2 - 331\alpha^3 + 186\alpha^4 - 81\alpha^5 + 33\alpha^6 - 7\alpha^7\right)\mathbb{Z}\oplus \\
& \oplus \frac{1}{6}\left(63 - 78\alpha + 65\alpha^2 - 49\alpha^3 + 27\alpha^4 - 12\alpha^5 + 5\alpha^6 - 1\alpha^7\right)\mathbb{Z}\oplus \\
& \oplus \frac{1}{6}\left(81 - 99\alpha + 84\alpha^2 - 61\alpha^3 + 33\alpha^4 - 15\alpha^5 + 6\alpha^6 - 1\alpha^7\right)\mathbb{Z}\oplus \\
& \oplus \frac{1}{18}\left(-63 + 96\alpha - 86\alpha^2 + 68\alpha^3 - 39\alpha^4 + 18\alpha^5 - 8\alpha^6 + 2\alpha^7\right)\mathbb{Z}\oplus(-1)\mathbb{Z}\oplus \\
& \oplus(-\alpha)\mathbb{Z}\oplus(-\alpha^2)\mathbb{Z}\oplus\frac{1}{9}\left(81 - 96\alpha + 81\alpha^2 - 64\alpha^3 + 33\alpha^4 - 15\alpha^5 + 6\alpha^6 - \alpha^7\right)\mathbb{Z} \\
\lambda = {} & \frac{537}{80} - \frac{1343}{120}\alpha + \frac{1343}{144}\alpha^2 - \frac{419}{60}\alpha^3 + \frac{337}{80}\alpha^4 - \frac{15}{8}\alpha^5 + \frac{559}{720}\alpha^6 - \frac{1}{5}\alpha^7
\end{aligned}
$$

$$
\Omega = \begin{pmatrix}
2.8 - i & -2.8 + 0.59i & 0 & 0 & 1 & 1.7 - 0.29i & 0 & 0 \\
-2.8 + i & 2.8 - 3.4i & 0 & 0 & 1 & 0.29 + 1.7i & 0 & 0 \\
0 & 0 & -1 & -0.38 - 0.15i & 0 & 0 & -1.6 - 0.62i & -0.15 - 0.15i \\
0 & 0 & -1 & -2.6 + 6.9i & 0 & 0 & 0.62 - 1.6i & -6.9 + 6.9i
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$$\oplus\frac{1}{6}\left(63-78\alpha+65\alpha^2-49\alpha^3+27\alpha^4-12\alpha^5+5\alpha^6-1\alpha^7\right)\mathbb{Z}\oplus$$
$$\oplus\frac{1}{6}\left(81-99\alpha+84\alpha^2-61\alpha^3+33\alpha^4-15\alpha^5+6\alpha^6-1\alpha^7\right)\mathbb{Z}\oplus$$
$$\oplus\frac{1}{18}\left(-63+96\alpha-86\alpha^2+68\alpha^3-39\alpha^4+18\alpha^5-8\alpha^6+2\alpha^7\right)\mathbb{Z}\oplus(-1)\mathbb{Z}\oplus$$
$$\oplus(-\alpha)\mathbb{Z}\oplus(-\alpha^2)\mathbb{Z}\oplus\frac{1}{9}\left(81-96\alpha+81\alpha^2-64\alpha^3+33\alpha^4-15\alpha^5+6\alpha^6-\alpha^7\right)\mathbb{Z}$$
$$\lambda=\frac{537}{80}-\frac{1343}{120}\alpha+\frac{1343}{144}\alpha^2-\frac{419}{60}\alpha^3+\frac{337}{80}\alpha^4-\frac{15}{8}\alpha^5+\frac{559}{720}\alpha^6-\frac{1}{5}\alpha^7$$

Period matrices - Example

Let $g = (x^4 - 4x^3 + 8x^2 - 12x + 9)(x^4 - 2x^3 + 2x^2 - 6x + 9)$. We compute the principally polarized abelian varieties and we find that 4 isomorphism classes admit a unique principal polarization. Here is one of them with the period matrix of the canonical lift.

$$
\begin{aligned}
I = &\frac{1}{54}\left(432 - 549\alpha + 441\alpha^2 - 331\alpha^3 + 186\alpha^4 - 81\alpha^5 + 33\alpha^6 - 7\alpha^7\right)\mathbb{Z}\oplus\\
&\oplus \frac{1}{6}\left(63 - 78\alpha + 65\alpha^2 - 49\alpha^3 + 27\alpha^4 - 12\alpha^5 + 5\alpha^6 - 1\alpha^7\right)\mathbb{Z}\oplus\\
&\oplus \frac{1}{6}\left(81 - 99\alpha + 84\alpha^2 - 61\alpha^3 + 33\alpha^4 - 15\alpha^5 + 6\alpha^6 - 1\alpha^7\right)\mathbb{Z}\oplus\\
&\oplus \frac{1}{18}\left(-63 + 96\alpha - 86\alpha^2 + 68\alpha^3 - 39\alpha^4 + 18\alpha^5 - 8\alpha^6 + 2\alpha^7\right)\mathbb{Z}\oplus(-1)\mathbb{Z}\oplus\\
&\oplus(-\alpha)\mathbb{Z}\oplus(-\alpha^2)\mathbb{Z}\oplus\frac{1}{9}\left(81 - 96\alpha + 81\alpha^2 - 64\alpha^3 + 33\alpha^4 - 15\alpha^5 + 6\alpha^6 - \alpha^7\right)\mathbb{Z}
\end{aligned}
$$

$$
\lambda = \frac{537}{80} - \frac{1343}{120}\alpha + \frac{1343}{144}\alpha^2 - \frac{419}{60}\alpha^3 + \frac{337}{80}\alpha^4 - \frac{15}{8}\alpha^5 + \frac{559}{720}\alpha^6 - \frac{1}{5}\alpha^7
$$

$$
\Omega = \begin{pmatrix}
2.8 - i & -2.8 + 0.59i & 0 & 0 & 1 & 1.7 - 0.29i & 0 & 0 \\
-2.8 + i & 2.8 - 3.4i & 0 & 0 & 1 & 0.29 + 1.7i & 0 & 0 \\
0 & 0 & -1 & -0.38 - 0.15i & 0 & 0 & -1.6 - 0.62i & -0.15 - 0.15i \\
0 & 0 & -1 & -2.6 + 6.9i & 0 & 0 & 0.62 - 1.6i & -6.9 + 6.9i
\end{pmatrix}
$$

Final remarks

- Computations of isomorphism classes can be done in the same way replacing "ordinary over $\mathbb{F}_q$" with "over $\mathbb{F}_p$, away from real primes", by using Centeleghe-Stix (2015)...
- ...but polarizations (and period matrices) are still work in progress.
- The Magma code is avalable on my webpage.

Thank you!