

Polarizations of abelian varieties over finite fields via canonical liftings

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AGCCT 2021 - 1 June 2021

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joint work with

Jonas Bergström and **Valentijn Karemaker**.

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Definition

A **canonical lifting** of A_0 is an abelian scheme over a normal local domain \mathcal{R} of characteristic zero with residue field \mathbb{F}_q with:

- 1 special fiber A_0 , and
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- Example: ordinary abelian variety; almost-ordinary abelian variety.
 - Non-example: supersingular EC (quaternions).

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$$A_{\text{can}}(\mathbb{C}) \simeq \mathbb{C}^g / \Phi(I) \quad \begin{array}{l} - I : \text{a fractional } \mathbb{Z}[F, V]\text{-ideal in } L := \mathbb{Q}[F], \\ - \Phi : \text{a CM-type of } L. \end{array}$$

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- In particular: $\mathcal{H}(\text{Hom}(A_{\text{can}}, A_{\text{can}}^\vee)) = (\bar{I}^t : I) = \{x \in L : xI \subseteq \bar{I}^t\}$.

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- We have:

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Theorem (Centeleghe-Stix)

Let $AV_h(p)$ be the isogeny class over the **prime field** \mathbb{F}_p determined by a **squarefree** characteristic polynomial of Frobenius h .

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- In particular:

$$\mathcal{G}(\text{Hom}(B_0, B_0^\vee)) = (\mathcal{G}(B_0) : \overline{\mathcal{G}(B_0)}^t).$$

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it sends totally imaginary elements to totally imaginary elements and Φ -positive elements to Φ -positive elements.

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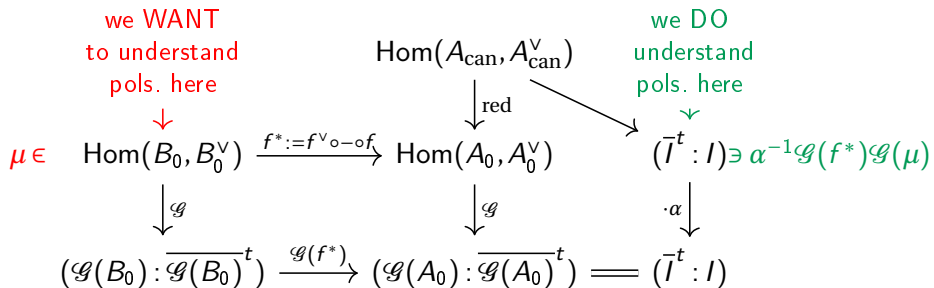
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we WANT to understand pols. here

we DO understand pols. here

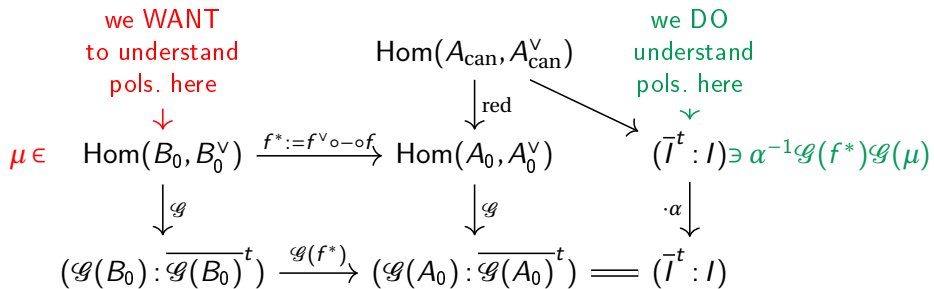
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Theorem ("lift and spread")

Let $\mu : B_0 \rightarrow B_0^{\vee}$ be an isogeny. Then

μ is a **polarization** $\iff \alpha^{-1} \mathcal{G}(\mu)$ is **totally imaginary** and **Φ -positive**

Principal Polarizations up to isomorphism

- Let $B_0 \in AV_h(p)$. Put $T = \text{End}(B_0)$ and $\mathcal{G}(B_0) = J$.

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- If μ and μ' are principal polarizations of B_0 then $(B_0, \mu) \simeq (B_0, \mu')$ (as PPAVs) if and only if there is $v \in T^*$ such that $\mathcal{G}(\mu) = v \bar{v} \mathcal{G}(\mu')$.

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- Let \mathcal{T} be a transversal of $T^* / \langle v \bar{v} : v \in T^* \rangle$.
- Then

$$\mathcal{P}_\Phi^\alpha(J) := \{i_0 \cdot u : u \in \mathcal{T} \text{ s.t. } \alpha^{-1} i_0 u \text{ is tot. imaginary and } \Phi\text{-positive}\}$$

is a set of representatives of the PPs of B_0 up to isomorphism.

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- Let $B_0 \in AV_h(p)$. Put $T = \text{End}(B_0)$ and $\mathcal{G}(B_0) = J$.
- Assume that $B_0 \simeq B_0^\vee$, i.e. $J = i_0 \bar{J}^t$ for some $i_0 \in L^*$.
- If μ and μ' are principal polarizations of B_0 then $(B_0, \mu) \simeq (B_0, \mu')$ (as PPAVs) if and only if there is $v \in T^*$ such that $\mathcal{G}(\mu) = v \bar{v} \mathcal{G}(\mu')$.
- Let \mathcal{T} be a transversal of $T^* / \langle v \bar{v} : v \in T^* \rangle$.
- Then

$$\mathcal{P}_\Phi^\alpha(J) := \{i_0 \cdot u : u \in \mathcal{T} \text{ s.t. } \alpha^{-1} i_0 u \text{ is tot. imaginary and } \Phi\text{-positive}\}$$

is a set of representatives of the PPs of B_0 up to isomorphism.

- It depends on α !

Effective Results : when can we ignore α ?

Assume A_0 admits a canonical lifting. Put $S := \text{End}(A_0)$

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If $S = \mathbb{Z}[F, V]$ (eg. $AV_h(p)$ is ordinary or almost-ordinary) then we can ignore α .

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Corollary

If $S = \mathbb{Z}[F, V]$ (eg. $AV_h(p)$ is ordinary or almost-ordinary) then we can ignore α . *We recover Deligne+Howe and Oswal-Shankar*

When can we lift up to isogeny?

Theorem (Chai-Conrad-Oort)

Assume that (L, Φ) satisfies the **Residual Reflex Condition** w.r.t. F , that is,

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- If there is a separable isogeny $A_0 \rightarrow A'_0$ then A'_0 admits a canonical lifting (useful in combination with Thm 1).

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total			1431	10453
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almost ordinary			392	2506
p -rank 2	no RRC		0	0
	yes RRC	Thm 1 yes	149	500
		Thm 1 no	49	312
p -rank 1	no RRC		6	36
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Thank you!

Effective Results II

Theorem (2)

Assume that there are r isomorphism classes of abelian varieties in $AV_h(p)$ with endomorphism ring T , represented under \mathcal{G} by the fractional ideals I_1, \dots, I_r . For any CM-type Φ' , we put

$$\mathcal{P}_{\Phi'}^1(I_i) = \{i_0 \cdot u : u \in \mathcal{T} \text{ such that } i_0 u \text{ is totally imaginary and } \Phi' \text{-positive} \}.$$

If there exists a non-negative integer N such that for every CM-type Φ' we have

$$|\mathcal{P}_{\Phi'}^1(I_1)| + \dots + |\mathcal{P}_{\Phi'}^1(I_r)| = N$$

then there are exactly N isomorphism classes of principally polarized abelian varieties with endomorphism ring T .

Proof.

- Consider the association $\Phi' \mapsto b$ where $b \in L^*$ is tot. imaginary and Φ' -positive.
- We can go back: for every b tot. imaginary there exists a unique CM-type Φ_b s.t. b is Φ_b -positive.
- Hence the totally real elements of L^* acts on the set of CM-types.
- If $\Phi = \Phi_b$ is the CM-type for which we have a canonical lift (as before) then $\mathcal{P}_{\Phi_b}^\alpha(l_i) \longleftrightarrow \mathcal{P}_{\Phi_{ab}}^1(l_i)$.
- If the we get the 'same sum' (over the l_i 's) for every CM-type we know that the result must be the correct one!



Note: even if the sum is not the same for all Φ' 's then we know that one of the outputs is the correct one!