Modules over orders, conjugacy classes of integral matrices and abelian varieties over finite fields

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Thank you

Thank you

Back in Bristol. . . during the RUMP session

Welcome to your Linear Algebra 1 exam!

Don't forget to motivate your answers.
The use of the (Magma) calculator is allowed.

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- Let R be a commutative ring with unity.
- $A, B \in \mathrm{Mat}_{n\times n}(R)$ are R-**conjugate** ($A \sim_R B$) if $AP = PB$ for some $P \in \mathrm{GL}_n(R)$.
- The **minimal** polynomial $m(x)$ of $A \in \mathrm{Mat}_{n\times n}(R)$ is the polynomial of smallest degree such that $m(A) = O$ (the zero $n \times n$ matrix).
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Question 1: Are the following two matrices $\mathbb{Q}$-conjugate? Are they $\mathbb{Z}$-conjugate?

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Over $\mathbb{Z}$: no! Why?

Fix monic polynomials $m = m_1 \cdots m_n$ and $h = m_1^{s_1} \cdots m_n^{s_n}$ in $\mathbb{Z}[x]$ with

- each m_i irreducible and
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$$\left\{\text{matrices with min. poly. } m \text{ and char. poly. } h\right\}_{/\sim_{\mathbb{Z}}}$$

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- The induced map is well-defined and injective.
- For the 'surjectivity' part: take the $\mathbb{Z}$-span of 'algebraic eigenvectors'.

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1. Find a 'finite box' that contains representatives of all isomorphism classes.
2. (Use other people's work to) pick out a minimal set of representatives.

Set-up:

- $K_1, \ldots, K_n$ number fields, with ring of integers $\mathscr{O}_i \subset K_i$.
- $K = K_1 \times \ldots \times K_n$.
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- for an order R in K, set $\mathscr{L}(R,V) = \{R\text{-lattice in } V\}$.
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Let M be in $\mathcal{L}(\mathcal{O},V)$. Then there are fractional $\mathcal{O}_i$-ideals I_i and an $\mathcal{O}$-linear isomorphism

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The isomorphism class of M is uniquely determined by the isomorphism class of the fractional $\mathcal{O}$-ideal $I = I_1 \oplus \cdots \oplus I_n$.

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Proof:

By Steinintz: there are I_i's and an $\mathcal{O}$-isomorphism such that

$$\psi : M\mathcal{O} \to \bigoplus_{i=1}^n \left(\mathcal{O}_i^{\oplus(s_i-1)} \oplus I_i\right).$$

Set $M' = \psi(M)$. QED

- The previous theorem tells us that $M \in \mathscr{L}(R, V)$ admits an isomorphic copy M' among the lifts to V of the finitely many sub-R-modules of

$$\mathscr{Q}(I) = \frac{\mathscr{O}_1^{\oplus(s_1-1)} \oplus I_1 \oplus \cdots \oplus \mathscr{O}_n^{\oplus(s_n-1)} \oplus I_n}{\mathfrak{f}_1^{\oplus(s_1-1)} \oplus \mathfrak{f}_1 I_1 \oplus \cdots \oplus \mathfrak{f}_n^{\oplus(s_n-1)} \oplus \mathfrak{f}_n I_n}.$$

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- For each fractional $\mathscr{O}$-ideal $I = \oplus_i I_i$, we have an $\mathscr{O}$-isomorphism $\Psi_I : \mathcal{Q}(I) \to \mathcal{Q}(\mathscr{O})$ inducing a bijection between the sub-R-modules.

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- **Important**: there are algorithms `IsIsomorphic` that answer the following question: given $M, M' \in \mathscr{L}(R,V)$, is there an R-linear isomorphism $M \simeq M'$?
  See:
  - *Bley, Hofmann, Johnston. Computation of lattice isomor- phisms and the integral matrix similarity problem*, (2022), in Nemo/Hecke, or
  - *Eick, Hofmann, O'Brien. The conjugacy problem in* $\mathrm{GL}(n,\mathbb{Z})$, (2019), in Magma.

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6. Return $\sqcup_I \mathcal{L}_I$.

Example

Let

$$m_1 = x^2 - x + 3, \quad m_2 = x^2 + x + 3,$$
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- the $\mathbb{F}_3$-isomorphism classes of abelian varieties in the $\mathbb{F}_3$-isogeny class determined by the 3-Weil polynomial h are in bijection with $\mathscr{L}(R, V)/\simeq_R$: there is 2 of them.