Cohen-Macaulay type of endomorphism rings of abelian varieties over finite fields

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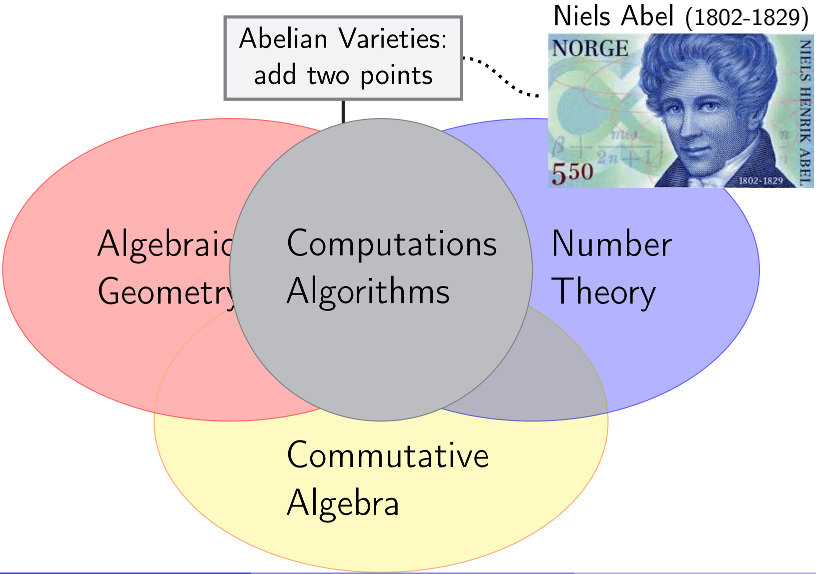
What do I do for a living?



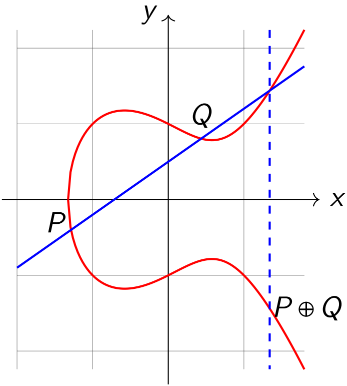
Abelian varieties: what are they ?

Abelian varieties are connected projective group varieties.

Abelian varieties of dim. 1 are called **elliptic curves**.
Eg: over $\mathbb{R}$, $y^2 = x^3 - x + 1$

We can add points:
$P, Q \rightsquigarrow P \oplus Q$

Equations are impractical in dim ≥ 2.
We need a better way to represent them...



Abelian varieties over $\mathbb{C}$ vs $\mathbb{F}_q$

- Let $A/\mathbb{C}$ be an abelian variety of dimension g.
- Then $A(\mathbb{C})$ is a **torus**: $T := \mathbb{C}^g/\Lambda$, where $\Lambda \simeq_{\mathbb{Z}} \mathbb{Z}^{2g}$.
- T admits a non-degenerate Riemann form $\longleftrightarrow$ polarization.
- In fact, $A \mapsto A(\mathbb{C})$ induces an equivalence of categories:

$$\{\text{abelian varieties } /\mathbb{C}\} \longleftrightarrow \left\{ \begin{matrix} \mathbb{C}^g/\Lambda \text{ with } \Lambda \simeq \mathbb{Z}^{2g} \text{ admitting} \\ \text{a Riemann form} \end{matrix} \right\}.$$

- In char. $p > 0$ such an equivalence cannot exist : there are (supersingular) elliptic curves with quaternionic endomorphism algebras.
- Nevertheless, as we will see later, over a finite field $\mathbb{F}_q$, we obtain analogous results if we restrict ourselves to certain **subcategories** of AVs.
- WARNING: all morphisms, endomorphisms, isogenies, etc. are defined over $\mathbb{F}_q$.

Isogeny classification over $\mathbb{F}_q$

- An **isogeny** $A \to B$ is a surjective morphism with finite kernel.
- $A/\mathbb{F}_q$ comes with a **Frobenius** endomorphism, that induces an action

$$\mathrm{Frob}_A : T_\ell A \to T_\ell A \text{ for any } \ell \neq p,$$

  where $T_\ell(A) = \varprojlim A[\ell^n] \simeq \mathbb{Z}_\ell^{2g}$.
- $h_A(x) := \mathrm{char}(\mathrm{Frob}_A)$ is a q-Weil polynomial.
- **Honda-Tate** theory:
  - $h_A(x)$ is *the* **isogeny invariant**

$$A \sim_{\mathbb{F}_q} B \text{ iff } h_A(x) = h_B(x),$$

  - the association

$$\text{isogeny class of } A \longmapsto h_A(x)$$

  allows us to enumerate all AVs up to isogeny.

Endomorphism rings

- $\mathrm{End}(A)$ is a free $\mathbb{Z}$-module of finite rank ...
- ... $\mathrm{End}(A) \subset \mathrm{End}(A) \otimes_{\mathbb{Z}} \mathbb{Q}$.
- Denote by $\pi_A \in \mathrm{End}(A)$ the Frobenius endomorphism of A.
- Tate: $h_A(x)$ is squarefree $\iff$ $\mathrm{End}(A)$ is commutative.
  (We will assume this for the rest of the talk.)
- Set $K = \mathbb{Q}[x]/(h_A) = \mathbb{Q}[\pi]$. It is an **étale $\mathbb{Q}$-algebra**
  (i.e. a finite product of number fields).
- The association $\pi_A \mapsto \pi$ allows us to identify $\mathrm{End}(A)$ with a special kind of subring of K:
- $\mathbb{Z}[\pi, q/\pi] \subseteq \mathrm{End}(A) \subseteq \mathcal{O}_K$ are orders in K
  (an **order** R in K is a subring $R \subset K$ such that $R \simeq_{\mathbb{Z}} \mathbb{Z}^{\dim_{\mathbb{Q}} K}$).
- Plan: study A by studying some comm. algebra properties of $\mathrm{End}(A)$.

Orders and fractional ideals in étale $\mathbb{Q}$-algebras

- Let R be an order in a étale $\mathbb{Q}$-algebra K.
- A **fractional** R**-ideal** is a sub-R-module $I \subset K$ such that $I \simeq_{\mathbb{Z}} \mathbb{Z}^{\dim_{\mathbb{Q}} K}$.
- Given fr. R-ideals I, J then

$$(I : J) = \{a \in K : aJ \subseteq I\} \quad \text{and} \quad I^t = \left\{a \in K : \mathrm{Tr}_{K/\mathbb{Q}}(aI) \subseteq \mathbb{Z}\right\}$$

  are also fr. R-ideals.
- The order $(I : I)$ is the **multiplicator ring** of I and satisfies:

$$(I : I)^t = I \cdot I^t.$$

- A fr. R-ideal I is invertible if $I(R : I) = R$ ...
- ... or, equivalently, $I_{\mathfrak{m}} \simeq R_{\mathfrak{m}}$ as $R_{\mathfrak{m}}$-modules for every $\mathfrak{m}$ maximal R-ideal.
  ($R_{\mathfrak{m}}$ is the completion of R at $\mathfrak{m}$)
- If I is invertible, then $(I : I) = R$.

Cohen-Macaulay type and Gorenstein orders

- Def: The **(Cohen-Macaulay) type** of R at a maximal ideal $\mathfrak{m}$ is

$$\mathrm{type}_{\mathfrak{m}}(R) := \dim_{R/\mathfrak{m}} \frac{R^t}{\mathfrak{m}R^t}.$$

- Def: R is **Gorenstein** at $\mathfrak{m}$ if $\mathrm{type}_{\mathfrak{m}}(R) = 1$.
- Remark: these definitions coincides with the 'usual' ones.
- Ex: monogenic $\mathbb{Z}[\alpha]$ and maximal $\mathcal{O}_K$ orders are Gorenstein. (also $\mathbb{Z}[\pi, q/\pi]$ for AVs).
- Ex: pick a prime $\ell \in \mathbb{Z}$. Then $\mathrm{type}_{\ell\mathcal{O}_K}(\mathbb{Z} + \ell\mathcal{O}_K) = \dim_{\mathbb{Q}} K - 1$.

Classification for orders of type ≤ 2

Theorem

Let $\mathfrak{m}$ be a maximal ideal of R, and I a fr. R-ideal with $(I:I)=R$.

1. *If $\mathrm{type}_{\mathfrak{m}}(R)=1$ (Gorenstein) then $I_{\mathfrak{m}} \simeq R_{\mathfrak{m}}$ as $R_{\mathfrak{m}}$-modules.*
2. *If $\mathrm{type}_{\mathfrak{m}}(R)=2$ then either $I_{\mathfrak{m}} \simeq R_{\mathfrak{m}}$ or $I_{\mathfrak{m}} \simeq R_{\mathfrak{m}}^t$ as $R_{\mathfrak{m}}$-modules.*

Part 1 is contained (in a much more general form) in the "Ubiquity" paper by H. Bass.
Part 2 is new, and we give a proof.

Lemma

Let U, V, W be vectors spaces (over some field). Assume that $\dim W \geq 2$, and let $m : U \otimes V \to W$ be a surjective map. Then:

1. *$\exists u \in U$ such that $\dim(m(u \otimes V)) \geq 2$, or*
2. *$\exists v \in V$ such that $\dim(m(U \otimes v)) \geq 2$.*

Proof of Part 2

- Put $U = I/\mathfrak{m}I$, $V = I^t/\mathfrak{m}I^t$ and $W = R^t/\mathfrak{m}R^t$.
- By assumption $R^t = I \cdot I^t$, so the map $m : U \otimes V \to W$ induced by multiplication $I \times I^t \to R^t$ is surjective.
- Moreover, $\dim W = 2$ (because of the assumption on the type).
- By the Lemma:
  1. $\exists x \in I$ such that $m((x + \mathfrak{m}I) \otimes V) = \frac{xI^t + \mathfrak{m}R^t}{\mathfrak{m}R^t}$ equals W.
     By Nakayama's lemma: $I^t_{\mathfrak{m}} \simeq R^t_{\mathfrak{m}} \iff R_{\mathfrak{m}} \simeq I_{\mathfrak{m}}$,...
  2. ...or, $\exists y \in I^t$ such that $U \otimes m(U \otimes (y + \mathfrak{m})I^t) = W$ implying $I^t_{\mathfrak{m}} \simeq R_{\mathfrak{m}} \iff I_{\mathfrak{m}} \simeq R^t_{\mathfrak{m}}$.

Back to AVs: Categorical equivalence(s)

Fix a squarefree characteristic poly $h(x)$ of Frobenius π over $\mathbb{F}_q$.
Put $K = \mathbb{Q}[x]/h = \mathbb{Q}[\pi]$. Let $\mathscr{I}_h$ be the corresponding isogeny class.

Theorem

If $q = p$ is prime or that $\mathscr{I}_h$ is ordinary (coeff. of x^g in $h(x)$ is $\not\equiv 0 \bmod p$) then there is an **equivalence** *of categories*

$$\left\{ \mathscr{I}_h \text{ with } \mathbb{F}_q\text{–morphisms} \right\}$$
$$\updownarrow$$
$$\{\text{fr. } \mathbb{Z}[\pi, q/\pi]\text{-ideals with linear morphisms}\}$$

Moreover, if $A \mapsto I$ then $A^\vee \mapsto \overline{I}^t$, where $\bar{\cdot}$ is defined by $\overline{\pi} = q/\pi$ (the CM-involution).

References: Deligne, Howe, Centeleghe-Stix, Bergström-Karemaker-M.

AVs: Isomorphism classes

- We get a bijection

$$\{ \text{ isom. classes of AVs in } \mathscr{I}_h \} \longleftrightarrow \{ \text{isom. classes of fr. } \mathbb{Z}[\pi, q/\pi]\text{-ideals} \}$$
$$:= \mathrm{ICM}(\mathbb{Z}[\pi, q/\pi]) \text{ ideal class monoid}$$

- If $\mathbb{Z}[\pi, q/\pi] = \mathscr{O}_K$ is the maximal order then $\mathrm{ICM}(\mathbb{Z}[\pi, q/\pi]) = \mathrm{Pic}(\mathscr{O}_K)$ is a product of class groups of number fields and we are good.
- Problem: $\mathbb{Z}[\pi, q/\pi]$ might not be a Dedekind ring $\rightsquigarrow$ non-invertible ideals.

ICM : Ideal Class Monoid

Let R be an **order** in an étale $\mathbb{Q}$-algebra K.

- Recall: for **fractional R-ideals** I and J

$$I \simeq_R J \Longleftrightarrow \exists x \in K^\times \text{ s.t. } xI = J.$$

- We have

$$\mathrm{ICM}(R) \supseteq \mathrm{Pic}(R) = {}^{\{\text{invertible fractional } R\text{-ideals}\}}\!\big/_{\simeq_R}$$

with equality $\updownarrow$ iff $R = \mathcal{O}_K$

- Simplify the problem by localizing: **weak equivalence** (Dade, Taussky, Zassenhaus '62)

$$I_{\mathfrak{m}} \simeq_{R_{\mathfrak{m}}} J_{\mathfrak{m}} \text{ for every } \mathfrak{m} \in \mathrm{mSpec}(R)$$

$$\Updownarrow$$

$$1 \in (I : J)(J : I) \quad \text{easy to check!}$$

Compute $\mathrm{ICM}(R)$

Let $\mathscr{W}(R)$ be the set of weak eq. classes. Partition w.r.t. the multiplicator ring:

$$\mathscr{W}(R) = \bigsqcup_{R\subseteq S\subseteq \mathscr{O}_K} \mathscr{W}_S(R)$$

$$\mathrm{ICM}(R) = \bigsqcup_{R\subseteq S\subseteq \mathscr{O}_K} \mathrm{ICM}_S(R)$$

the "pedix" $-_S$ means "only classes with multiplicator ring S"

Theorem (M.)

For every over-order S of R, $\mathrm{Pic}(S)$ acts freely on $\mathrm{ICM}_S(R)$ and

$$\mathscr{W}_S(R) = \mathrm{ICM}_S(R)/\mathrm{Pic}(S).$$

Repeat for every $R \subseteq S \subseteq \mathscr{O}_K \rightsquigarrow \mathrm{ICM}(R)$.

Compute ICM(R)

- To compute the overorders: see Hoffman-Sircana.
- To compute Pic(S): see Klüners-Pauli.
- To compute $W(R) = \sqcup W_S(R)$:
- all representatives can be found in

$$\left\{ \text{sub-}R\text{-modules of } {}^{\mathscr{O}_K}\!/\!_{\mathfrak{f}_R} \right\} \quad \text{finite}$$

  where $\mathfrak{f}_R = (R : \mathscr{O}_K)$ is the conductor of R.
- Can we use the type? Write $W_S(R) = \prod_{\mathfrak{m} \subset S} (W_S(R))_{\mathfrak{m}}$.
- We have proven that:
  if the type of S at $\mathfrak{m}$ is 1 then $(W_S(R))_{\mathfrak{m}} = \{[S_{\mathfrak{m}}]\}$, while
  if the type of S at $\mathfrak{m}$ is 2 then $(W_S(R))_{\mathfrak{m}} = \{[S_{\mathfrak{m}}], [S^t_{\mathfrak{m}}]\}$

AVs: Group of rational points

Theorem (Springer-M.)

$\mathscr{I}_h$ and $K = \mathbb{Q}[\pi] = \mathbb{Q}[x]/h$ as before.
Let R be an order in K and $\mathfrak{m}$ a maximal ideal of R (possibly but not necessarily above p). Assume:

$$\mathrm{type}_{\mathfrak{m}}(R) \leq 2 \textit{ for every } \mathfrak{m} \supseteq (1-\pi)R.$$

Then for every $A \in \mathscr{I}_h$ such that $\mathrm{End}(A) = R$ we have that $A(\mathbb{F}_q) \simeq_{\mathbb{Z}} R/(1-\pi)R$.

Proof: Say that $A \mapsto I$. Then $A(\mathbb{F}_q) = \ker(1-\pi_A) = \frac{I}{(1-\pi)I} =: M$.
M is finite: $M = \oplus_{\mathfrak{m}\supset(1-\pi)R} M_{\mathfrak{m}}$.
If $I_{\mathfrak{m}} \simeq R_{\mathfrak{m}}$ then $M_{\mathfrak{m}} \simeq_R \frac{R_{\mathfrak{m}}}{(1-\pi)R_{\mathfrak{m}}}$.
If $I_{\mathfrak{m}} \simeq R^t_{\mathfrak{m}}$ then $M_{\mathfrak{m}} \simeq_R \frac{R^t_{\mathfrak{m}}}{(1-\pi)R^t_{\mathfrak{m}}} \simeq_{\mathbb{Z}} \frac{R_{\mathfrak{m}}}{(1-\pi)R_{\mathfrak{m}}}$.

AVs: self-duality

Theorem (Springer-M.)

$\mathscr{I}_h$ and $K = \mathbb{Q}[\pi] = \mathbb{Q}[x]/h$ as before.
Let R be an order in K and $\mathfrak{m}$ a maximal ideal of R. Assume:

$$R = \overline{R}, \quad \mathfrak{m} = \overline{\mathfrak{m}}, \quad \textit{and} \quad \mathrm{type}_{\mathfrak{m}}(R) = 2.$$

Then for every $A \in \mathscr{I}_h$ such that $\mathrm{End}(A) = R$ we have that $A \not\simeq A^\vee$. In particular, such an A cannot be principally polarized nor a Jacobian.

Proof: Say that $A \mapsto I$. Hence $A^\vee \mapsto \overline{I}^t$.
By the Classification: either $I_{\mathfrak{m}} \simeq R_{\mathfrak{m}}$ or $I_{\mathfrak{m}} \simeq R^t_{\mathfrak{m}}$.
In the first case: $\overline{I}^t_{\mathfrak{m}} = \overline{I}^t_{\overline{\mathfrak{m}}} \simeq R^t_{\mathfrak{m}} \not\simeq R_{\mathfrak{m}}$.
Similarly, in the second: $\overline{I}^t_{\mathfrak{m}} = \overline{I}^t_{\overline{\mathfrak{m}}} \simeq R_{\mathfrak{m}} \not\simeq R^t_{\mathfrak{m}}$.
In both cases: $I \not\simeq \overline{I}^t \iff A \not\simeq A^\vee$.

Some stats and refs

How often do the hypothesis of the previous theorem ($R = \overline{R}$, exists $\mathfrak{m} = \overline{\mathfrak{m}}$ with $\mathrm{type}_{\mathfrak{m}}(R) = 2$) do occur?
We computed the isomorphism classes of AVs$/\mathbb{F}_q$ (see LMFDB xyz) for 615.269 isogeny classes (for $1 \leq g \leq 5$ and various q).
We encountered

- 3.914.908 commutative endomorphism rings, of which:
- 72.6% satisfy $R = \overline{R}$;
- 10.3% satisfy $R = \overline{R}$ and are non-Gorenstein;
- 7.4% satisfy $R = \overline{R}$, are non-Gorenstein and $\exists \mathfrak{m} = \overline{\mathfrak{m}}$ s.t. with $\mathrm{type}_{\mathfrak{m}}(R) = 2$.

Thank you!

Main references:

- *Cohen-Macaulay type of orders, generators and ideal classes*
  https://arxiv.org/abs/2206.03758
- *Abelian varieties over finite fields and their groups of rational points*
  with Caleb Springer, to appear in Algebra&Number Theory
  https://arxiv.org/abs/2211.15280
- Magma package for étale $\mathbb{Q}$-algebras
  https://github.com/stmar89/AlgEt (also in Magma 2-28.1)