# Computing isomorphism classes and polarisations of abelian varieties over finite fields.

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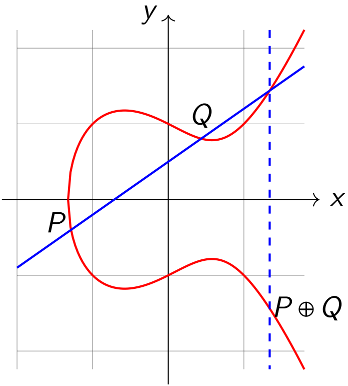

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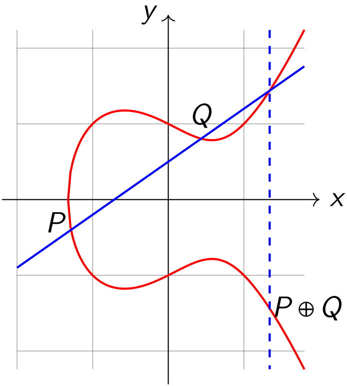

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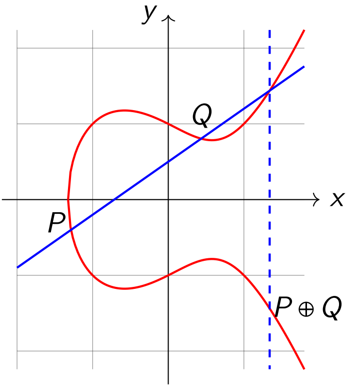
Equations are impractical in dim $\geq 2$.
We need a better way to represent them...



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- Nevertheless, over finite fields, we obtain analogous results if we restrict ourselves to certain **subcategories** of AVs.

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- Also, $h_A(x)$ is squarefree $\iff$ End$(A)$ is commutative.

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- Problem: $\mathbb{Z}[F,V]$ might not be maximal $\rightsquigarrow$ non-invertible ideals.

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- Hofmann-Sircana : computation of over-orders.

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  ...whose representatives can be found in

$$\left\{ \text{sub-}R\text{-modules of } {}^{\mathscr{O}_K}\!\diagup_{\mathfrak{f}_R} \right\} \quad \text{finite! and most of the time not-too-big ...}$$

where $\mathfrak{f}_R = (R:\mathscr{O}_K)$ is the conductor of $R$.

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Theorem (M.)

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*Repeat for every* $R \subseteq S \subseteq \mathcal{O}_K$*:*

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- What about polarizations?

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- $A^\vee \leftrightarrow \overline{I}^t := \{\overline{x} \in K : \mathrm{Tr}(xI) \subseteq \mathbb{Z}\}$.
- *a polarization $\mu$ of $A$ corresponds to a $\lambda \in K^\times$ such that*
  - $\lambda I \subseteq \overline{I}^t$ *(isogeny of $\deg \mu = [\overline{I}^t : \lambda I]$);*
  - $\lambda$ *is totally imaginary ($\overline{\lambda} = -\lambda$);*
  - $\lambda$ *is $\Phi$-positive ($\Im \varphi(\lambda) > 0$ for all $\varphi \in \Phi$),*

  *where $\Phi$ is a CM-type of $K$ satisf. the Shimura-Taniyama formula.*

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Howe : **dual** varieties and **polarizations** on Deligne modules.

Theorem (M.)

*Let $A \in \mathscr{C}_h$ with $h$ ordinary and squarefree. If $A \leftrightarrow I$, set $S = (I : I)$, then:*

- $A^\vee \leftrightarrow \overline{I}^t := \{\overline{x} \in K : \mathrm{Tr}(xI) \subseteq \mathbb{Z}\}$.
- *a polarization $\mu$ of $A$ corresponds to a $\lambda \in K^\times$ such that*
  - $\lambda I \subseteq \overline{I}^t$ *(isogeny of* $\deg\mu = [\overline{I}^t : \lambda I]$*);*
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4. ... then we have one principal polarization.
5. By the previous Theorem, we have all princ. polarizations up to isom.

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- 8 classes admit principal polarizations.
- 10 isomorphism classes of princ. polarized AV.

## Example

Concretely:

$$\begin{aligned}
I_1 =& 2645633792595191\mathbb{Z} \oplus (F + 836920075614551)\mathbb{Z} \oplus (F^2 + 1474295643839839)\mathbb{Z} \oplus \\
&\oplus (F^3 + 1372829830503387)\mathbb{Z} \oplus (F^4 + 1072904687510)\mathbb{Z} \oplus \\
&\oplus \frac{1}{3}(F^5 + F^4 + F^3 + 2F^2 + 2F + 670480698614361 0)\mathbb{Z} \oplus \\
&\oplus \frac{1}{9}(F^6 + F^5 + F^4 + 8F^3 + 2F^2 + 299166524362116 9)\mathbb{Z} \oplus \\
&\oplus \frac{1}{27}(F^7 + F^6 + F^5 + 17F^4 + 20F^3 + 9F^2 + 6801531251872220 1)\mathbb{Z}
\end{aligned}$$

principal polarizations:

$$\begin{aligned}
x_{1,1} = \frac{1}{27}(&-121922F^7 + 588604F^6 - 1422437F^5 + \\
&+ 1464239F^4 + 1196576F^3 - 7570722F^2 + 15316479F - 12821193) \\
x_{1,2} = \frac{1}{27}(&3015467F^7 - 17689816F^6 + 35965592F^5 - \\
&- 64660346F^4 + 121230619F^3 - 191117052F^2 + 315021546F - 300025458)
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$\mathrm{End}(I_1) = R$

$\#\mathrm{Aut}(I_1, x_{1,1}) = \#\mathrm{Aut}(I_1, x_{1,2}) = 2$

# Example

$$\begin{aligned} I_7 = & 2\mathbb{Z} \oplus (F+1)\mathbb{Z} \oplus (F^2+1)\mathbb{Z} \oplus (F^3+1)\mathbb{Z} \oplus (F^4+1)\mathbb{Z} \oplus \frac{1}{3}(F^5+F^4+F^3+2F^2+2F+3)\mathbb{Z} \oplus \\ & \oplus \frac{1}{36}(F^6+F^5+10F^4+26F^3+2F^2+27F+45)\mathbb{Z} \oplus \\ & \oplus \frac{1}{216}(F^7+4F^6+49F^5+200F^4+116F^3+105F^2+198F+351)\mathbb{Z} \end{aligned}$$

principal polarization:

$$x_{7,1} = \frac{1}{54}(20F^7 - 43F^6 + 155F^5 - 308F^4 + 580F^3 - 1116F^2 + 2205F - 1809)$$

$$\begin{aligned} \mathrm{End}(I_7) = & \mathbb{Z} \oplus F\mathbb{Z} \oplus F^2\mathbb{Z} \oplus F^3\mathbb{Z} \oplus F^4\mathbb{Z} \oplus \frac{1}{3}(F^5+F^4+F^3+2F^2+2F)\mathbb{Z} \oplus \\ & \oplus \frac{1}{18}(F^6+F^5+10F^4+8F^3+2F^2+9F+9)\mathbb{Z} \oplus \\ & \oplus \frac{1}{108}(F^7+4F^6+13F^5+56F^4+80F^3+33F^2+18F+27)\mathbb{Z} \end{aligned}$$

$$\#\mathrm{Aut}(I_7, x_{7,1}) = 2$$

$I_1$ is invertible in $R$, but $I_7$ is not invertible in $\mathrm{End}(I_7)$.

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**Theorem (Centeleghe-Stix '15)**

*There is an equivalence of categories:*

$$\left\{\text{abelian varieties } A \text{ over } \mathbb{F}_p \text{ with } \boldsymbol{h_A(\sqrt{p}) \neq 0}\right\} \qquad\qquad A$$

$$\updownarrow \qquad\qquad\qquad\qquad \downarrow$$

$$\left\{\begin{array}{l} \text{pairs } (T,F)\text{, where } T \simeq_{\mathbb{Z}} \mathbb{Z}^{2g} \text{ and } T \xrightarrow{F} T \text{ s.t.} \\ \text{- } F \otimes \mathbb{Q} \text{ is semisimple} \\ \text{- the roots of } \mathrm{char}_{F\otimes\mathbb{Q}}(x) \text{ have abs. value } \sqrt{p} \\ \text{- } \boldsymbol{\mathrm{char}_F(\sqrt{p}) \neq 0} \\ \text{- } \exists V : T \to T \text{ such that } FV = VF = p \end{array}\right\} \qquad (T(A), F(A))$$

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- Now, $T(A) := \mathrm{Hom}(A, A_w)$, where $A_w$ has minimal End among the varieties with Weil support $w = w(A)$.
- $F(A)$ is the induced Frobenius.

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- For polarizations, the results by Howe do not apply immediately to the Centeleghe-Strix case:
- in general we cannot lift canonically each abelian variety.

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**Theorem (Chai-Conrad-Oort)**

*If $(K, \Phi)$ satisfies the RRC then in $\mathscr{C}_h$ there exists an abelian variety $A$ admitting a canonical lifting $\mathscr{A}$.*

- If we understand the polarizations of $A$ we can 'spread' them to the whole isogeny class.

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Note that $\mathscr{G}(f^*)$ is multiplication by the totally positive element $\overline{\mathscr{G}(f)}\mathscr{G}(f)$: it sends totally imaginary elements to totally imaginary elements and $\Phi$-positive elements to $\Phi$-positive elements. The only 'issue' is the $\alpha$.
We study when we can 'pretend' $\alpha = 1$.

# Effective Results : when can we ignore $\alpha$?

Assume $A$ admits a canonical lifting. Put $S := \mathrm{End}(A)$
Let $B$ be isogenous to $A$. Put $T = \mathrm{End}(B)$. The previous diagram tells us that the princ. polarisations of $B$ (up-to-iso) are in bijections with

$$\mathscr{P}^{\alpha}_{\Phi}(B) := \{ i_0 \cdot u : u \in T^{\times} / \left\{ v\overline{v} : v \in T^{\times} \right\} \text{ s.t. } \alpha^{-1} i_0 u \text{ is tot. img. and } \Phi\text{-pos.} \}$$

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Among the 45 isogeny classes which we cannot 'handle' with Thm 1, we can compute the number of PPAV for 32 of them using other techniques. For the remaining 13 (all over $\mathbb{F}_2$ and $\mathbb{F}_3$) we only get partial info.

| squarefree dimension 4 | | | $p=2$ | $p=3$ |
|---|---|---|---|---|
| total | | | 1431 | 10453 |
| ordinary | | | 656 | 6742 |
| almost ordinary | | | 392 | 2506 |
| $p$-rank 2 | no RRC | | 0 | 0 |
| | yes RRC | Thm 1 yes | 149 | 500 |
| | | Thm 1 no | 49 | 312 |
| $p$-rank 1 | no RRC | | 6 | 36 |
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Thm 1 ($S^*_{\mathbb{R}} \subseteq T^*_{\mathbb{R}}$) doesn't handle $72/\mathbb{F}_2$ and $391/\mathbb{F}_3$. Out of these, we can use other techniques for $20/\mathbb{F}_2$ and $214/\mathbb{F}_3$. For the remaining $52/\mathbb{F}_2$ and $171/\mathbb{F}_3$ we can only get information about certain endomorphism rings (723 out of 946 and 3481 out of 4636, respectively).

Thank you!

# Effective Results II

**Theorem (2)**

*Assume that there are $r$ isomorphism classes of abelian varieties in $\mathrm{AV}_h(p)$ with endomorphism ring $T$, represented under $\mathscr{G}$ by the fractional ideals $I_1, \ldots, I_r$. For any CM-type $\Phi'$, we put*

$$\mathscr{P}^1_{\Phi'}(I_i) = \{ i_0 \cdot u : u \in \mathscr{T} \text{ such that } i_0 u \text{ is totally imaginary and } \Phi'\text{-positive} \}.$$

*If there exists a non-negative integer $N$ such that for every CM-type $\Phi'$ we have*

$$|\mathscr{P}^1_{\Phi'}(I_1)| + \ldots + |\mathscr{P}^1_{\Phi'}(I_r)| = N$$

*then there are exactly $N$ isomorphism classes of principally polarized abelian varieties with endomorphism ring $T$.*

Proof.

- Consider the association $\Phi' \mapsto b$ where $b \in L^*$ is tot. imaginary and $\Phi'$-positive.
- We can go back: for every $b$ tot. imaginary there exists a unique CM-type $\Phi_b$ s.t. $b$ is $\Phi_b$-positive.
- Hence the totally real elements of $L^*$ acts on the set of CM-types.
- If $\Phi = \Phi_b$ is the CM-type for which we have a canonical lift (as before) then $\mathscr{P}^{\alpha}_{\Phi_b}(I_i) \longleftrightarrow \mathscr{P}^{1}_{\Phi_{\alpha b}}(I_i)$.
- If the we get the 'same sum' (over the $I_i$'s) for every CM-type we know that the result must be the correct one!

$\square$

Note: even if the sum is not the same for all $\Phi'$'s then we know that one of the outputs is the correct one!