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The use of the (Magma) calculator is allowed.

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Over $\mathbb{Z}$: no! Every such a P must have even determinant.

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Answer: ....check the arXiv in the next couple of days....

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Congrats: you passed the exam!