PhD Seminar 04/05/2018

(1)

§1 In this talk I would like to compare $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$ and $\mathbb{Q} = \{\frac{n}{m} ; n, m \in \mathbb{Z}, m \neq 0\}$

Compare linear algebra over $\mathbb{Z}$ and over $\mathbb{Q}$

Ab Groups — $\mathbb{Q}$-Vector Space

These categories are instances of the same algebraic structure

- Mod_R : Modules over a (commutative) ring R.

$(M, +)$ additive group $M \times M \to M$, $(m, n) \mapsto m+n$ — associative, 0 elt, commutative, inverses

$\cdot : R \times M \to M$, $(r, m) \mapsto r \cdot m$ — linear: $r(m+n) = rm + rn$; ass: $(r \cdot s) m = r(sm)$, $(r+s)m = rm + sm$; id: $1 \cdot m = m$

$Mod_{\mathbb{Q}} = \mathbb{Q}$VectSpaces $\quad R = \mathbb{Q}$

$Mod_{\mathbb{Z}} =$ Abgroups $\quad R = \mathbb{Z}$

- I will assume fin. generated M

$\exists \underbrace{S = (m_1, \ldots, m_N)}_{\text{generating set}} \subseteq M$ st $\forall m \in M \ \exists z_1, \ldots, z_N \in R$ with $\underbrace{m = z_1 m_1 + \ldots + z_N m_N}_{R\text{-linear comb}}$

Ex Ab Groups

$\mathbb{Z},+$

$\mathbb{Q},+$ fin. gen

$\mathbb{Z}/6\mathbb{Z},+$

m.f.g.

$\mathbb{Q}$-Vect Spaces

$\left.\begin{array}{l}\mathbb{Q} \\ \mathbb{Q}^N \\ M_{n\times m}(\mathbb{Q})\end{array}\right\}$ f.g

$\mathbb{Q}[x] = \underbrace{\mathbb{Q}\oplus x\mathbb{Q}\oplus x^2\mathbb{Q}\oplus\ldots}_{\text{m.f.g}}$

§ 2

(" We all know what a basis of a $\mathbb{Q}$ v.s. is. "
" What does lin. indep. mean? "

Let V be a $\mathbb{Q}$ v.s.

$\mathcal{S} = \{v_1,\ldots,v_N\}$ is linearly dependent

if $v_j = \sum_{i\neq j} q_i v_i$ for $q_i \in \mathbb{Q}$

(" is this the right definition in general? "

$A = \mathbb{Z}$ ab. group

$\mathcal{S} = \{2,3\}$ is a generating set of A

(" but want A " to have dim 1 "

also $a2 + b3 = 0$ for $a = -3$, $b = 2$

Def In R-module M a set $\mathcal{S} = \{m_1,\ldots,m_N\}$ is linearly indep if $\sum_{i=1}^{N} r_i m_i = 0 \Rightarrow r_i = 0 \ \forall i=1,\ldots,N$

$\mathcal{S}$ is a basis if lin indep + gen. set

If M has a basis we say that M is free

" So lin. indep means no-non-trivial relations

" Over $\mathbb{Q}$ the 2 def are equiv., over $\mathbb{Z}$ this is not true

" This might look like a small difference but the consequence are very big which explains why the th. of fin. dim. vect. sp. is easy while groups are fucked up ".

<u>Examples</u> V is a $\mathbb{Q}$ v.sp. fin. gen.

1) $\{ \overset{0}{\overset{\#}{v}} \} \subseteq V$ is lin. indep.

but $\{ \bar{2} \} \subseteq \mathbb{Z}/7\mathbb{Z}$ is not : $7 \cdot \bar{2} = \bar{0}$

2) In (a f.g) V a maximal lin. ind. subset is a spanning set ($\Rightarrow$ basis) and a minimal generating set is lin. ind ($\Rightarrow$ basis).

but $\{2\} \subseteq \mathbb{Z}$ max l. in not a basis

$\{2,3\} \subseteq \mathbb{Z}$ min gen set not a basis

3) V. s. one can enlarge a set of lin ind vectors to a basis

ab. gr not true $\{2\} \subseteq \mathbb{Z}$

4) V, W v. sp. with $\dim V = \dim W$

$\varphi : V \to W$ $\mathbb{Q}$-linear

$\rightsquigarrow$ φ injective $\Leftrightarrow$ φ surjective

while,

$\varphi : \mathbb{Z} \to \mathbb{Z}$ injective, not onto

$x \mapsto 2x$

5) $A, B \in \mathcal{M}_{N\times N}(\mathbb{Q})$

$\chi_A^{(x)} \equiv \det(A - x I_N) \in \mathbb{Q}[x]$

A and B are conjugate if $\exists\, U \in GL_N(\mathbb{Q})$

st $A = U B U^{-1}$

Assume $\chi_A^{(x)}$ is irreducible

over $\mathbb{Q}$: A conj B $\Leftrightarrow$ $\chi_A = \chi_B$

over $\mathbb{Z}$: $\Rightarrow$

$\not\Leftarrow$

(" Why do we have such differences ?

(" First the answer , then the explaimation

- $\boxed{gl.\dim \mathbb{Z} = 1}$ $\quad$ $gl.\dim \mathbb{Q} = 0$

(" 1 what is it ? , 2 why " ~~implies~~ ex " explains the ?
3 howto compute it !

__Def__
$gl.\dim R$, R a PID.

M an R-module
build a free resolution of M:

$$0 \to F_N \to \quad \to F_2 \to F_1 \to F_0 \to \underset{\overset{\parallel}{F_{-1}}}{M} \to 0$$

each F_i is <u>free</u>

$\forall i$ $\quad F_i \xrightarrow{d_i} F_{i-1} \xrightarrow{d_{i+1}} F_{i-2}$

$\mathrm{Im}(d_i) = \mathrm{Ker}(d_{i+1})$ $\quad$ <u>exact</u>

P is proj if

$\forall$

$$\begin{tikzcd} & N \arrow[d] \\ P \arrow[r] \arrow[ur, "\exists h"] & M \end{tikzcd}$$

$pd(M)$ = minimal lenght of a free res of M
proj dim

$$gl.\dim(R) := \sup_M (pd(M))$$

$$\begin{tikzcd} 0 \arrow[r] & M \arrow[r] \arrow[d, equal] \arrow[dr, phantom, "\circlearrowright"] & L \arrow[r] \arrow[d, "\wr"] \arrow[dr, phantom, "\circlearrowright"] & N \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & M \arrow[r] & M \oplus N \arrow[r] & N \arrow[r] & 0 \end{tikzcd}$$

exact seq of R-modules

is split $\Leftrightarrow$

$m \mapsto (m,0)$

$(m',m) \mapsto m$

all the examples mentioned before ~~are~~ can be explained (more or less directly) in terms of some appropriate exact sequences of ab groups being non-split

Ex 4) $\varphi : \mathbb{Z} \to \mathbb{Z}$ inj not onto

$x \mapsto 2x$

$$\begin{tikzcd} 0 \arrow[r] & \mathbb{Z} \arrow[r, "\varphi"] \arrow[d, equal] & \mathbb{Z} \arrow[r] \arrow[d, "\not\cong"] & \mathbb{Z}/2\mathbb{Z} \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & \mathbb{Z} \arrow[r] & \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \arrow[r] & \mathbb{Z}/2\mathbb{Z} \arrow[r] & 0 \end{tikzcd}$$

Def $\mathrm{Ext}^1(M,N) = \left\{ 0 \to M \to L \to N \to 0 \text{ exact} \right\} / \text{equiv.}$

Baer sum

$0 \longleftrightarrow 0 \to M \to M \oplus N \to N \to 0$

$$\begin{tikzcd} 0 \arrow[r] & M \arrow[r] \arrow[d, equal] \arrow[dr, phantom, "\circlearrowright"] & L \arrow[r] \arrow[d, "\wr"] \arrow[dr, phantom, "\circlearrowright"] & N \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & M \arrow[r] & L' \arrow[r] & N \arrow[r] & 0 \end{tikzcd}$$

Thm (Weibel)

$\mathrm{gl.dim}\, R = 0 \implies \mathrm{Ext}^1(M,N) = \{0\} \quad \forall M, N$ (every exact seq splits)

$\mathrm{gl.dim}(R) = 1 \implies \mathrm{Ext}^1(M,N) \neq \{0\} \quad \exists M, N$

Let's compute that

$\Rightarrow$ $\quad gl.\dim \mathbb{Z} = 1$

Need an ab group A with $p.d.(A) = 1$

<u>Ex</u> $\quad A = \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}^2$

$3\mathbb{Z} \times (0) \times (0)$

$$0 \longrightarrow \underset{F_1}{\mathbb{Z}} \longrightarrow \underset{F_0}{\mathbb{Z}^3} \longrightarrow \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}^2 \longrightarrow 0$$

$1 \longmapsto (3,0,0)$

$(1,0,0) \longmapsto (\bar{1}, 0, 0)$

$(0,1,0) \longmapsto (0, 1, 0)$

$(0,0,1) \longmapsto (0, 0, 1)$

<u>free res</u>

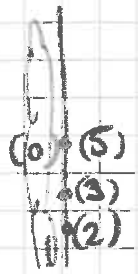
you cannot make it shorter !

$\Rightarrow pd(A) = 1$

<u>Also</u> $\forall A$ we have $pd(A) \leq 1$

$\boxed{\text{while}}$ over $\mathbb{Q}$ every v.sp has a basis
so it is <u>free</u>
$\Rightarrow gl.\dim \mathbb{Q} = 0.$

Some $\quad gl.\dim R = \text{Krull} \dim R$
nce : Noeth. local regular

Spec $\mathbb{Z}$ 

Spec $\mathbb{Q}$ = $\bullet$