Computing isomorphism classes of abelian varieties over finite fields

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VaNTAGe

Abelian varieties over $\mathbb{C}$ vs $\mathbb{F}_q$

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- In char. $p > 0$ such an equivalence cannot exist : there are (supersingular) elliptic curves with quaternionic endomorphism algebras.
- Nevertheless, over finite fields, we obtain analogous results if we restrict ourselves to certain **subcategories** of AVs.

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  is injective and allows us to enumerate all AVs up to isogeny.
- Also, $h_A(x)$ is squarefree $\iff$ End(A) is commutative.

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Let $q = p^r$, with p a prime. There is an equivalence of categories:

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- Put $T(A) := H_1(\mathscr{A}_{\mathrm{can}} \otimes \mathbb{C}, \mathbb{Z})$ and $F(A) :=$ the induced Frobenius.

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- Problem: $\mathbb{Z}[F,V]$ might not be maximal $\rightsquigarrow$ non-invertible ideals.

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- Hofmann-Sircana [?]: computation of over-orders.

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- Let $\mathscr{W}(R)$ be the set of weak eq. classes...
  ...whose representatives can be found in

$$\left\{\text{sub-}R\text{-modules of } {}^{\mathcal{O}_K}\!/\!_{\mathfrak{f}_R}\right\}$$ finite! and most of the time not-too-big ...

where $\mathfrak{f}_R = (R : \mathcal{O}_K)$ is the conductor of R.

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Repeat for every $R \subseteq S \subseteq \mathscr{O}_K$:

$$\rightsquigarrow \mathrm{ICM}(R).$$

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- We can actually get a lot more!

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  - $\lambda I \subseteq \overline{I}^t$ *(isogeny of* $\deg \mu = [\overline{I}^t : \lambda I]$*);*
  - λ *is totally imaginary* ($\overline{\lambda} = -\lambda$)*;*
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$$\left\{\begin{matrix}\text{non-isomorphic princ.}\\ \text{polarizations of } A\end{matrix}\right\} \longleftrightarrow \frac{\{\text{totally positive } u \in S^\times\}}{\{v\bar{v} : v \in S^\times\}},$$

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similar statement for $\deg \mu > 1$

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Let $A \in \mathscr{C}_h$ with h ordinary and squarefree. If $A \leftrightarrow I$, then:

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- *a polarization μ of A corresponds to a $\lambda \in K^\times$ such that*
  - $\lambda I \subseteq \overline{I}^t$ *(isogeny of* $\deg \mu = [\overline{I}^t : \lambda I]$*);*
  - λ *is totally imaginary* ($\overline{\lambda} = -\lambda$)*;*
  - λ *is* Φ*-positive* ($\Im\varphi(\lambda) > 0$ *for all* $\varphi \in \Phi$*),*

  where Φ *is a CM-type of* K *satisf. the Shimura-Taniyama formula.*
- *if* $(A,\mu) \leftrightarrow (I,\lambda)$ *is a princ. polarized ab. var. and* $S = (I:I)$ *then*

$$\left\{\begin{matrix}\text{non-isomorphic princ.}\\ \text{polarizations of } A\end{matrix}\right\} \longleftrightarrow \frac{\{\text{totally positive } u \in S^\times\}}{\{v\overline{v} : v \in S^\times\}},$$

similar statement for $\deg \mu > 1$

- *and* $\mathrm{Aut}(A,\mu) = \{$*torsion units of* $S\}$.

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4. ... then we have one principal polarization.
5. By the previous Theorem, we have all princ. polarizations up to isom.

Example

- Let $h(x) = x^8 - 5x^7 + 13x^6 - 25x^5 + 44x^4 - 75x^3 + 117x^2 - 135x + 81$.

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- 10 isomorphism classes of princ. polarized AV.

Example

Concretely:

$$
\begin{aligned}
I_1 = & 2645633792595191\mathbb{Z} \oplus (F + 836920075614551)\mathbb{Z} \oplus (F^2 + 1474295643839839)\mathbb{Z} \oplus \\
& \oplus (F^3 + 1372829830503387)\mathbb{Z} \oplus (F^4 + 1072904687510)\mathbb{Z} \oplus \\
& \oplus \frac{1}{3}(F^5 + F^4 + F^3 + 2F^2 + 2F + 6704806986143610)\mathbb{Z} \oplus \\
& \oplus \frac{1}{9}(F^6 + F^5 + F^4 + 8F^3 + 2F^2 + 2991665243621169)\mathbb{Z} \oplus \\
& \oplus \frac{1}{27}(F^7 + F^6 + F^5 + 17F^4 + 20F^3 + 9F^2 + 68015312518722201)\mathbb{Z}
\end{aligned}
$$

principal polarizations:

$$
\begin{aligned}
x_{1,1} = & \frac{1}{27}(-121922F^7 + 588604F^6 - 1422437F^5 + \\
& + 1464239F^4 + 1196576F^3 - 7570722F^2 + 15316479F - 12821193) \\
x_{1,2} = & \frac{1}{27}(3015467F^7 - 17689816F^6 + 35965592F^5 - \\
& - 64660346F^4 + 121230619F^3 - 191117052F^2 + 315021546F - 300025458) \\
& \mathrm{End}(I_1) = R \\
& \#\mathrm{Aut}(I_1, x_{1,1}) = \#\mathrm{Aut}(I_1, x_{1,2}) = 2
\end{aligned}
$$

Example

$$
\begin{aligned}
I_7 = & 2\mathbb{Z} \oplus (F+1)\mathbb{Z} \oplus (F^2+1)\mathbb{Z} \oplus (F^3+1)\mathbb{Z} \oplus (F^4+1)\mathbb{Z} \oplus \frac{1}{3}(F^5+F^4+F^3+2F^2+2F+3)\mathbb{Z}\oplus \\
& \oplus \frac{1}{36}(F^6+F^5+10F^4+26F^3+2F^2+27F+45)\mathbb{Z}\oplus \\
& \oplus \frac{1}{216}(F^7+4F^6+49F^5+200F^4+116F^3+105F^2+198F+351)\mathbb{Z}
\end{aligned}
$$

principal polarization:

$$x_{7,1} = \frac{1}{54}(20F^7-43F^6+155F^5-308F^4+580F^3-1116F^2+2205F-1809)$$

$$
\begin{aligned}
\mathrm{End}(I_7) = & \mathbb{Z} \oplus F\mathbb{Z} \oplus F^2\mathbb{Z} \oplus F^3\mathbb{Z} \oplus F^4\mathbb{Z} \oplus \frac{1}{3}(F^5+F^4+F^3+2F^2+2F)\mathbb{Z}\oplus \\
& \oplus \frac{1}{18}(F^6+F^5+10F^4+8F^3+2F^2+9F+9)\mathbb{Z}\oplus \\
& \oplus \frac{1}{108}(F^7+4F^6+13F^5+56F^4+80F^3+33F^2+18F+27)\mathbb{Z}
\end{aligned}
$$

$$\#\mathrm{Aut}(I_7, x_{7,1}) = 2$$

I_1 is invertible in R, but I_7 is not invertible in $\mathrm{End}(I_7)$.

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- Another case we understand well: $h = g^r$ for g square-free and ordinary.

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- Eg: quadratic orders are Bass $\rightsquigarrow$ powers of ordinary elliptic curves E^r.
- If R is Bass, then M is isomorphic to a direct sum of frac. R-ideals.

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Theorem ([?])

If $R = \mathbb{Z}[F,V]$ is Bass then

$$\left\{\text{abelian varieties in } \mathscr{C}_{g^r}\right\}_{/\simeq} \longleftrightarrow \left\{I_1 \oplus \ldots \oplus I_r \;:\; I_j \text{ a frac. } R\text{-ideal}\right\}_{/\simeq}$$

we have a classification:

$$\longleftrightarrow \left\{ (S_1 \subseteq S_2 \subseteq \ldots \subseteq S_r,\ [I]_{\simeq}) : \begin{array}{l} R \subseteq S_j \text{ orders,} \\ I \text{ a frac. } R\text{-ideal} \\ \text{with } (I:I) = S_r \end{array} \right\}$$

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- ... but computing them in general is harder!
- Solved for E^r by Kirschmer-Narbonne-Ritzenthaler-Robert [?].

Outside of the ordinary...

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Theorem (Centeleghe-Stix [?])

There is an equivalence of categories:

$$\begin{array}{cc}
\{\text{abelian varieties } A \text{ over } \mathbb{F}_p \text{ with } \boldsymbol{h_A(\sqrt{p}) \neq 0}\} & A \\
\updownarrow & \downarrow \\
\left\{\begin{array}{l}
\text{pairs } (T, F), \text{ where } T \simeq_{\mathbb{Z}} \mathbb{Z}^{2g} \text{ and } T \xrightarrow{F} T \text{ s.t.} \\
\text{- } F \otimes \mathbb{Q} \text{ is semisimple} \\
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- Now, $T(A) := \mathrm{Hom}(A, A_w)$, where A_w has minimal End among the varieties with Weil support $w = w(A)$.
- $F(A)$ is the induced Frobenius.

Outside of the ordinary...isomorphism classes

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- For polarizations, the results by Howe do not apply immediately to the Centeleghe-Strix case:
- in general we cannot lift canonically each abelian variety.

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- If we understand the polarizations of A we can 'spread' them to the whole isogeny class.

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More comments:
- in [?]: a functor for isogeny classes of the form E^r.
- in [?]+[?]: almost-ordinary SQ with polarizations .
- in [?]: they use $\mathrm{Hom}_{\mathbb{F}_{p^k}}(-, A_w)$ as in [?], but A_w is more complicated.

Thank you!