Polarizations of abelian varieties over finite fields via canonical liftings

Stefano Marseglia

Utrecht University

UGC Seminar - 29 March 2022

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joint work with
Jonas Bergström and **Valentijn Karemaker**.

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  if $Q = (x_Q, y_Q) \neq \ominus P$ then $P \oplus Q = (x_R, y_R)$ where

$$x_R = \lambda^2 - x_P - x_Q, \quad y_R = y_P + \lambda(x_R - x_P),$$

  where

$$\lambda = \begin{cases} \frac{3x_P^2 + B}{2A} & \text{if } P = Q \\ \frac{y_P - y_Q}{x_P - x_Q} & \text{if } P \neq Q \end{cases}$$

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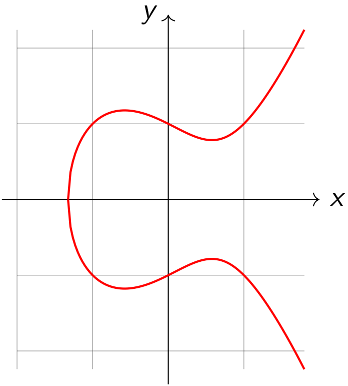

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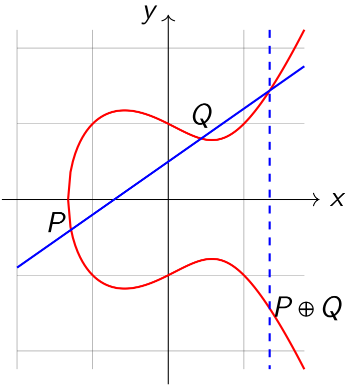
Addition law: $P, Q \rightsquigarrow P \oplus Q$



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  2. proper smooth curve $C/k \rightsquigarrow \mathrm{Pic}^0_C =: \mathrm{Jac}(C)$ a PPAV.

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  induced by $A \mapsto A(\mathbb{C})$ is an equivalence of categories.
- In char. $p > 0$ such an equivalence cannot exist : there are (supersingular) elliptic curves with quaternionic endomorphism algebras.

Canonical Liftings

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A **canonical lifting** of A_0 is an abelian scheme over a normal local domain $\mathscr{R}$ of characteristic zero with residue field $\mathbb{F}_q$ with:

1. special fiber A_0, and
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- Non-example: supersingular EC (quaternions).

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$$A_{\mathrm{can}}(\mathbb{C}) \simeq {\mathbb{C}^g}/{\Phi(I)}$$

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- In particular: $\mathscr{H}(\mathrm{Hom}(A_{\text{can}}, A^\vee_{\text{can}})) = (\bar{I}^t : I) = \left\{x \in L : xI \subseteq \bar{I}^t\right\}$.

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- One can prove that $h_A(x)$ is squarefree $\iff$ $\mathrm{End}(A)$ is commutative.

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Let $\mathrm{AV}_h(p)$ *be the isogeny class over the* **prime field** $\mathbb{F}_p$ *determined by a* **squarefree** *characteristic polynomial of Frobenius* h*.*
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Isomorphism classification over $\mathbb{F}_p$

Theorem (Centeleghe-Stix)

Let $\mathrm{AV}_h(p)$ *be the isogeny class over the* **prime field** $\mathbb{F}_p$ *determined by a* **squarefree** *characteristic polynomial of Frobenius* h.
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- In particular:

$$\mathscr{G}(\mathrm{Hom}(B_0, B_0^\vee)) = (\mathscr{G}(B_0) : \overline{\mathscr{G}(B_0)}^t).$$

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Comparison : Polarizations

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Theorem ("lift and spread")

Let $\mu : B_0 \to B_0^\vee$ be an isogeny. Then

$$\mu \text{ is a } \textit{polarization} \iff \alpha^{-1}\mathscr{G}(\mu) \text{ is totally imaginary and } \Phi\text{-positive}$$

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- If μ and μ' are principal polarizations of B_0 then $(B_0, \mu) \simeq (B_0, \mu')$ (as PPAVs) if and only if there is $v \in T^*$ such that $\mathscr{G}(\mu) = v\overline{v}\mathscr{G}(\mu')$.

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$$\mathscr{P}^\alpha_\Phi(J) := \{ i_0 \cdot u : u \in \mathscr{T} \text{ s.t. } \alpha^{-1} i_0 u \text{ is tot. imaginary and } \Phi\text{-positive} \}$$

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  is a set or representatives of the PPs of B_0 up to isomorphism.
- It depends on α!

Effective Results : when can we ignore α?

Assume A_0 admits a canonical lifting. Put $S := \mathrm{End}(A_0)$
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If $S = \mathbb{Z}[F, V]$ (eg. $\mathrm{AV}_h(p)$ is ordinary or almost-ordinary) then we can ignore α.

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Effective Results II

Theorem (2)

Assume that there are r isomorphism classes of abelian varieties in $\mathrm{AV}_h(p)$ with endomorphism ring T, represented under $\mathscr{G}$ by the fractional ideals $I_1, \ldots, I_r$. For any CM-type Φ', we put

$$\mathscr{P}^1_{\Phi'}(I_i) = \{ i_0 \cdot u : u \in \mathscr{T} \text{ such that } i_0 u \text{ is totally imaginary and } \Phi'\text{-positive} \}.$$

If there exists a non-negative integer N such that for every CM-type Φ' we have

$$|\mathscr{P}^1_{\Phi'}(I_1)| + \ldots + |\mathscr{P}^1_{\Phi'}(I_r)| = N$$

then there are exactly N isomorphism classes of principally polarized abelian varieties with endomorphism ring T.

Proof.

- Consider the association $\Phi' \mapsto b$ where $b \in L^*$ is tot. imaginary and Φ'-positive.
- We can go back: for every b tot. imaginary there exists a unique CM-type Φ_b s.t. b is Φ_b-positive.
- Hence the totally real elements of L^* acts on the set of CM-types.
- If $\Phi = \Phi_b$ is the CM-type for which we have a canonical lift (as before) then $\mathscr{P}^{\alpha}_{\Phi_b}(I_i) \longleftrightarrow \mathscr{P}^{1}_{\Phi_{\alpha b}}(I_i)$.
- If the we get the 'same sum' (over the I_i's) for every CM-type we know that the result must be the correct one!

□

Note: even if the sum is not the same for all Φ''s then we know that one of the outputs is the correct one!

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Definition (Chai-Conrad-Oort)

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2. *Let E be the reflex field attached to (L,Φ), and let v be the induced p-adic place of E. Then the* **residue field** *k_v of $\mathcal{O}_{E,v}$ can be realized as a* **subfield** *of $\mathbb{F}_q$.*

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Then we can canonically lift an abelian variety A_0 *with* $\mathcal{O}_L = \mathrm{End}(A_0)$*.*

When can we lift up to isogeny?

Theorem (Chai-Conrad-Oort)

Assume that (L,Φ) satisfies the **Residual Reflex Condition** *w.r.t. F, that is,*

1. *Φ satisfies the Shimura-Taniyama formula for F, and*
2. *the reflex field E has residue field $k_E \subseteq \mathbb{F}_q$.*

Then we can canonically lift an abelian variety A_0 with $\mathcal{O}_L = \mathrm{End}(A_0)$.

- If there is a separable isogeny $A_0 \to A_0'$ then A_0' admits a canonical lifting (useful in combination with Thm 1).

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total			185	621	2863	7847
ordinary			82	390	2280	6700
almost ordinary			58	170	474	996
p-rank 1	no RRC		0	0	0	0
	yes RRC	Thm 1 yes	20	26	76	118
		Thm 1 no	4	16	12	8
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Among the 45 isogeny classes which we cannot 'handle' with Thm 1, we can compute the number of PPAV for 32 of them using Thm 2. For the remaining 13 (all over $\mathbb{F}_2$ and $\mathbb{F}_3$) we only get partial info.

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squarefree dimension 4			$p=2$	$p=3$
total			1431	10453
ordinary			656	6742
almost ordinary			392	2506
p-rank 2	no RRC		0	0
	yes RRC	Thm 1 yes	149	500
		Thm 1 no	49	312
p-rank 1	no RRC		6	36
	yes RRC	Thm 1 yes	80	184
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Thank you!