# Cohen-Macaulay type of endomorphism rings of abelian varieties over finite fields

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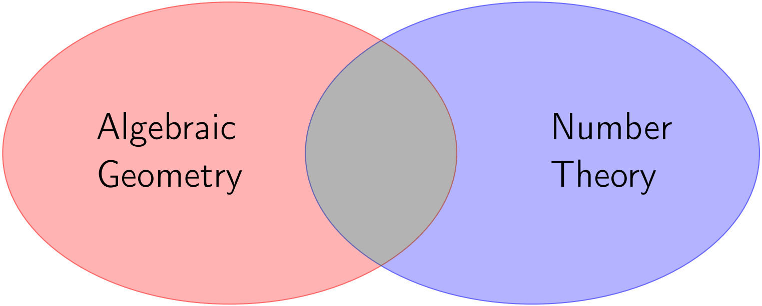
# What do I do for a living?



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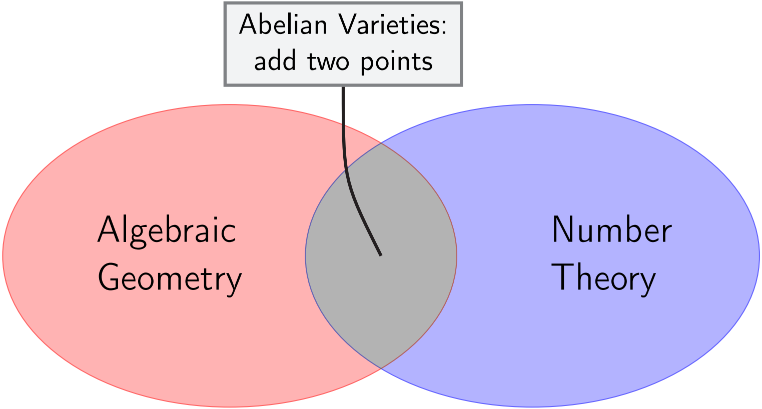

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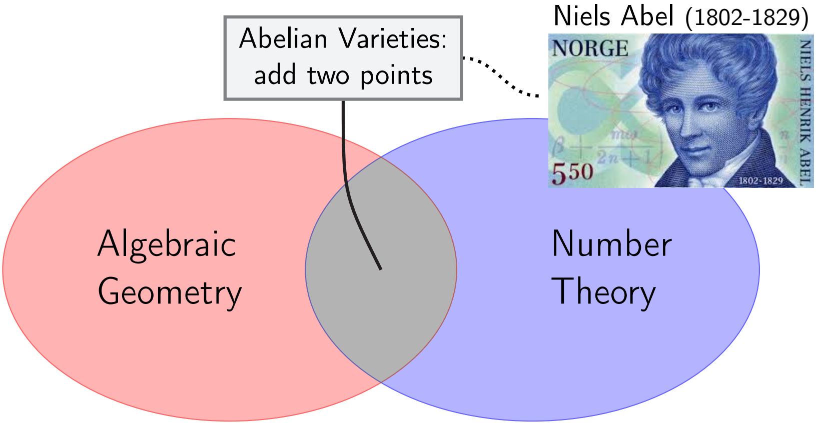

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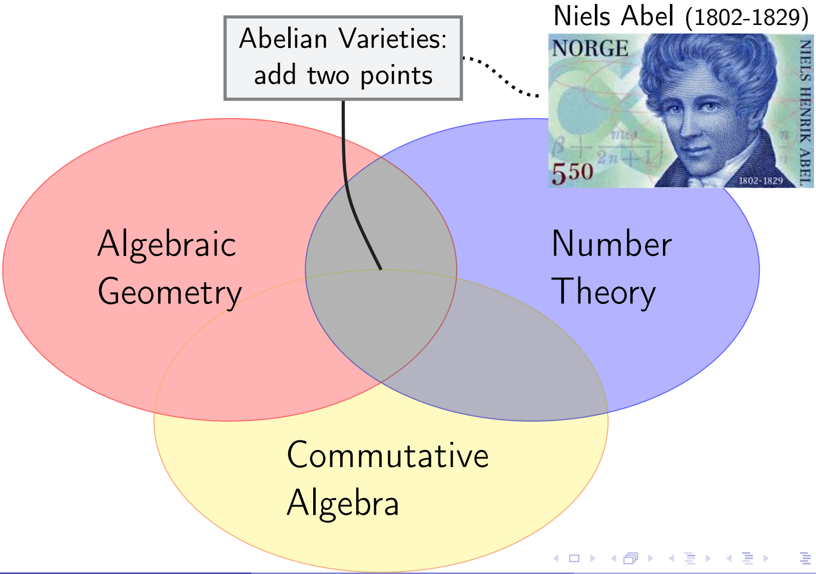

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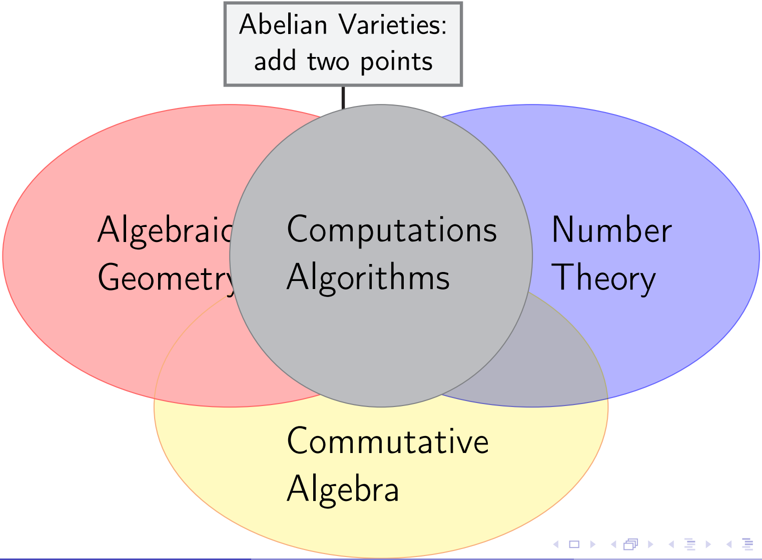

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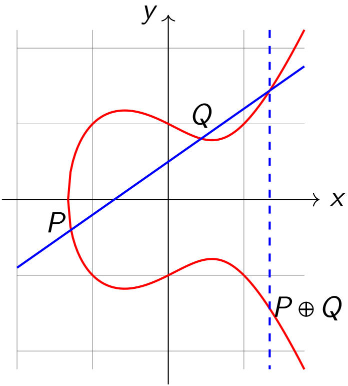

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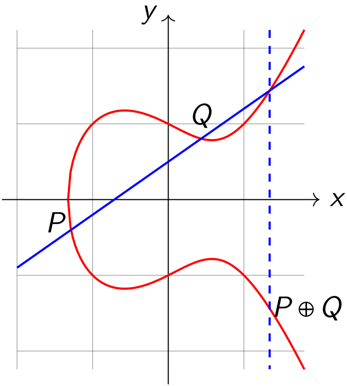

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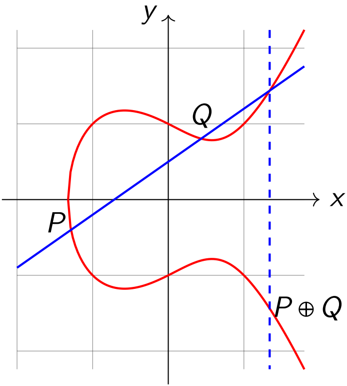
Equations are impractical in dim $\geq 2$.
We need a better way to represent them...



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- Nevertheless, as we will see later, over a finite field $\mathbb{F}_q$, we obtain analogous results if we restrict ourselves to certain **subcategories** of AVs.
- WARNING: all morphisms, endomorphisms, isogenies, etc. are defined over $\mathbb{F}_q$.

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  - the association

$$\text{isogeny class of } A \longmapsto h_A(x)$$

  allows us to enumerate all AVs up to isogeny.

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- Plan: study $A$ by studying some comm. algebra properties of $\mathrm{End}(A)$.

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- Def: The **(Cohen-Macaulay) type** of $R$ at a maximal ideal $\mathfrak{m}$ is

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- Ex: pick a prime $\ell \in \mathbb{Z}$. Then $\mathrm{type}_{\ell\mathcal{O}_K}(\mathbb{Z} + \ell\mathcal{O}_K) = \dim_{\mathbb{Q}} K - 1$.

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2. *If $\operatorname{type}_{\mathfrak{m}}(R) = 2$ then either $I_{\mathfrak{m}} \simeq R_{\mathfrak{m}}$ or $I_{\mathfrak{m}} \simeq R^t_{\mathfrak{m}}$ as $R_{\mathfrak{m}}$-modules.*

Part 1 is contained (in a much more general form) in the "Ubiquity" paper by H. Bass.
Part 2 is new, and we give a proof.

### Lemma

*Let $U, V, W$ be vectors spaces (over some field). Assume that $\dim W \geq 2$, and let $m : U \otimes V \to W$ be a surjective map. Then:*

1. *$\exists u \in U$ such that $\dim(m(u \otimes V)) \geq 2$, or*
2. *$\exists v \in V$ such that $\dim(m(U \otimes v)) \geq 2$.*

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- By the Lemma:
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  2. ...or, $\exists y \in I^t$ such that $U \otimes m(U \otimes (y + \mathfrak{m})I^t) = W$ implying $I^t_{\mathfrak{m}} \simeq R_{\mathfrak{m}} \iff I_{\mathfrak{m}} \simeq R^t_{\mathfrak{m}}$.

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References: Deligne, Howe, Centeleghe-Stix, Bergström-Karemaker-M.

# AVs: Isomorphism classes

- We get a bijection

$$\{ \text{ isom. classes of AVs in } \mathscr{I}_h \} \longleftrightarrow \{\text{isom. classes of fr. } \mathbb{Z}[\pi, q/\pi]\text{-ideals} \}$$
$$:= \mathrm{ICM}(\mathbb{Z}[\pi, q/\pi]) \text{ ideal class monoid}$$

- If $\mathbb{Z}[\pi, q/\pi] = \mathscr{O}_K$ is the maximal order then $\mathrm{ICM}(\mathbb{Z}[\pi, q/\pi]) = \mathrm{Pic}(\mathscr{O}_K)$ is a product of class groups of number fields and we are good.
- Problem: $\mathbb{Z}[\pi, q/\pi]$ might not be a Dedekind ring $\rightsquigarrow$ non-invertible ideals.

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$$1 \in (I : J)(J : I)$$ easy to check!

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- We have proven that:
  if the type of $S$ at $\mathfrak{m}$ is 1 then $(W_S(R))_{\mathfrak{m}} = \{[S_{\mathfrak{m}}]\}$, while
  if the type of $S$ at $\mathfrak{m}$ is 2 then $(W_S(R))_{\mathfrak{m}} = \{[S_{\mathfrak{m}}], [S^t_{\mathfrak{m}}]\}$

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*$\mathscr{I}_h$ and $K = \mathbb{Q}[\pi] = \mathbb{Q}[x]/h$ as before.*
*Let $R$ be an order in $K$ and $\mathfrak{m}$ a maximal ideal of $R$. Assume:*

$$R = \overline{R}, \quad \mathfrak{m} = \overline{\mathfrak{m}}, \quad \textit{and} \quad \mathrm{type}_{\mathfrak{m}}(R) = 2.$$

*Then for every $A \in \mathscr{I}_h$ such that $\mathrm{End}(A) = R$ we have that $A \not\simeq A^\vee$. In particular, such an $A$ cannot be principally polarized nor a Jacobian.*

Proof: Say that $A \mapsto I$. Hence $A^\vee \mapsto \overline{I}^t$.
By the Classification: either $I_\mathfrak{m} \simeq R_\mathfrak{m}$ or $I_\mathfrak{m} \simeq R^t_\mathfrak{m}$.
In the first case: $\overline{I}^t_\mathfrak{m} = \overline{I}^t_{\overline{\mathfrak{m}}} \simeq R^t_\mathfrak{m} \not\simeq R_\mathfrak{m}$.
Similarly, in the second: $\overline{I}^t_\mathfrak{m} = \overline{I}^t_{\overline{\mathfrak{m}}} \simeq R_\mathfrak{m} \not\simeq R^t_\mathfrak{m}$.
In both cases: $I \not\simeq \overline{I}^t \iff A \not\simeq A^\vee$.

# Some stats and refs

How often do the hypothesis of the previous theorem ($R = \overline{R}$, exists $\mathfrak{m} = \overline{\mathfrak{m}}$ with $\mathrm{type}_{\mathfrak{m}}(R) = 2$) do occur?
We computed the isomorphism classes of AVs$/\mathbb{F}_q$ (see LMFDB xyz) for 615.269 isogeny classes (for $1 \leq g \leq 5$ and various $q$).
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- 72.6% satisfy $R = \overline{R}$;
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- 7.4% satisfy $R = \overline{R}$, are non-Gorenstein and $\exists \mathfrak{m} = \overline{\mathfrak{m}}$ s.t. with $\mathrm{type}_{\mathfrak{m}}(R) = 2$.

Thank you!

# Thank you!

Main references:

- *Cohen-Macaulay type of orders, generators and ideal classes*
  `https://arxiv.org/abs/2206.03758`
- *Abelian varieties over finite fields and their groups of rational points*
  with Caleb Springer, to appear in Algebra&Number Theory
  `https://arxiv.org/abs/2211.15280`
- Magma package for étale $\mathbb{Q}$-algebras
  `https://github.com/stmar89/AlgEt` (also in Magma 2-28.1)