Computing isomorphism classes of abelian varieties over finite fields

The $4th$ mini symposium of the RNTA

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$$\{\text{abelian varieties } /\mathbb{C}\} \longleftrightarrow \left\{ \begin{matrix} \mathbb{C}^g/L \text{ with } L \simeq \mathbb{Z}^{2g} \\ + \text{ Riemann form} \end{matrix} \right\}.$$

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- in positive characteristic we don't have such equivalence.

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\left\{\begin{array}{l}
\textit{pairs}\ (T, F),\ \textit{where}\ T \simeq_{\mathbb{Z}} \mathbb{Z}^{2g}\ \textit{and}\ T \xrightarrow{F} T\ \textit{s.t.} \\
\text{-}\ F \otimes \mathbb{Q}\ \textit{is semisimple} \\
\text{-}\ \textit{the roots of}\ \mathrm{char}_{F\otimes\mathbb{Q}}(x)\ \textit{have abs. value}\ \sqrt{q} \\
\text{-}\ \textbf{\textit{half of them are p-adic units}} \\
\text{-}\ \exists V : T \to T\ \textit{such that}\ FV = VF = q
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Remark

- *If* $\dim(A) = g$ *then* $\mathrm{Rank}(T(A)) = 2g$;
- $\mathrm{Frob}(A) \rightsquigarrow F(A)$.

Fix a **ordinary square-free** characteristic q-Weil polynomial h.

$\rightsquigarrow$ an isogeny class $\mathscr{C}_h$ (by Honda-Tate).

Deligne's equivalence: square-free case

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Deligne's equivalence induces:

Theorem

$$\left\{ \textit{Ordinary abelian varieties over } \mathbb{F}_q \textit{ in } \mathscr{C}_h \right\}_{/\simeq}$$

$$\updownarrow$$

$$\left\{ \textit{fractional ideals of } \mathbb{Z}[F, q/F] \subset K \right\}_{/\simeq} \quad =: \mathrm{ICM}(\mathbb{Z}[F, q/F])$$

ideal class monoid

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- ...actually
$$\mathrm{ICM}(R) \supseteq \bigsqcup_{R \subseteq S \subseteq \mathscr{O}_K} \mathrm{Pic}(S).$$

simplify the problem

Study the isomorphism problem locally: (Dade, Taussky, Zassenhaus '62)

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Let $\mathscr{W}(R)$ be the set of weak eq. classes...
...whose representatives can be found in

$$\left\{ \text{sub-}R\text{-modules of } {}^{\mathscr{O}_K}\!/\!_{\mathfrak{f}_R} \right\}$$ finite! and most of the time not-too-big ...

Compute ICM(R)

Partition w.r.t. the multiplicator ring:

$$\mathscr{W}(R) = \bigsqcup_{R \subseteq S \subseteq \mathscr{O}_K} \overline{\mathscr{W}}(S)$$

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Theorem

For every over-order S of R, $\mathrm{Pic}(S)$ acts freely on $\overline{\mathrm{ICM}(S)}$ and

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Repeat for every $R \subseteq S \subseteq \mathscr{O}_K$:

$$\rightsquigarrow \mathrm{ICM}(R).$$

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- a polarization μ of A corresponds to a $\lambda \in K^\times$ such that
  - $\lambda I \subseteq \bar{I}^t$ (isogeny);
  - λ is totally imaginary ($\bar{\lambda} = -\lambda$);
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  Also: $\deg \mu = [\bar{I}^t : I]$.
- if $A \leftrightarrow I$ and $S = (I : I)$ then

$$\left\{ \begin{matrix} \text{non-isomorphic} \\ \text{polarizations of } A \end{matrix} \right\} \longleftrightarrow \frac{\{\text{totally positive } u \in S^\times\}}{\{v\bar{v} : v \in S^\times\}}$$

and $\mathrm{Aut}(A, \mu) = \{\text{torsion units of } S\}$

- Let
  $h(x) = x^8 - 5x^7 + 13x^6 - 25x^5 + 44x^4 - 75x^3 + 117x^2 - 135x + 81$;
- $\rightsquigarrow$ isogeny class of an simple ordinary abelian varieties over $\mathbb{F}_3$ of dimension 4;
- Let F be a root of $h(x)$ and put $R := \mathbb{Z}[F, 3/F] \subset \mathbb{Q}(F)$;
- 8 over-orders of R: two of them are not Gorenstein;
- $\#\mathrm{ICM}(R) = 18 \rightsquigarrow 18$ isom. classes of AV in the isogeny class;
- 5 are not invertible in their multiplicator ring;
- 8 classes admit principal polarizations;
- 10 isomorphism classes of princ. polarized AV.

Concretely:

$$
\begin{aligned}
I_1 =& 2645633792595191\mathbb{Z} \oplus (F + 836920075614551)\mathbb{Z} \oplus (F^2 + 1474295643839839)\mathbb{Z} \oplus \\
&\oplus (F^3 + 1372829830503387)\mathbb{Z} \oplus (F^4 + 1072904687510)\mathbb{Z} \oplus \\
&\oplus \frac{1}{3}(F^5 + F^4 + F^3 + 2F^2 + 2F + 6704806986143610)\mathbb{Z} \oplus \\
&\oplus \frac{1}{9}(F^6 + F^5 + F^4 + 8F^3 + 2F^2 + 2991665243621169)\mathbb{Z} \oplus \\
&\oplus \frac{1}{27}(F^7 + F^6 + F^5 + 17F^4 + 20F^3 + 9F^2 + 68015312518722201)\mathbb{Z}
\end{aligned}
$$

principal polarizations:

$$
\begin{aligned}
x_{1,1} =& \frac{1}{27}(-121922F^7 + 588604F^6 - 1422437F^5 + \\
& + 1464239F^4 + 1196576F^3 - 7570722F^2 + 15316479F - 12821193) \\
x_{1,2} =& \frac{1}{27}(3015467F^7 - 17689816F^6 + 35965592F^5 - \\
& - 64660346F^4 + 121230619F^3 - 191117052F^2 + 315021546F - 300025458)
\end{aligned}
$$

$$\mathrm{End}(I_1) = R$$

$$\#\mathrm{Aut}(I_1, x_{1,1}) = \#\mathrm{Aut}(I_1, x_{1,2}) = 2$$

$$
\begin{aligned}
I_7 =& 2\mathbb{Z} \oplus (F+1)\mathbb{Z} \oplus (F^2+1)\mathbb{Z} \oplus (F^3+1)\mathbb{Z} \oplus (F^4+1)\mathbb{Z} \oplus \frac{1}{3}(F^5+F^4+F^3+2F^2+2F+3)\mathbb{Z} \oplus \\
&\oplus \frac{1}{36}(F^6+F^5+10F^4+26F^3+2F^2+27F+45)\mathbb{Z} \oplus \\
&\oplus \frac{1}{216}(F^7+4F^6+49F^5+200F^4+116F^3+105F^2+198F+351)\mathbb{Z}
\end{aligned}
$$

principal polarization:

$$x_{7,1} = \frac{1}{54}(20F^7 - 43F^6 + 155F^5 - 308F^4 + 580F^3 - 1116F^2 + 2205F - 1809)$$

$$
\begin{aligned}
\mathrm{End}(I_7) = & \mathbb{Z} \oplus F\mathbb{Z} \oplus F^2\mathbb{Z} \oplus F^3\mathbb{Z} \oplus F^4\mathbb{Z} \oplus \frac{1}{3}(F^5+F^4+F^3+2F^2+2F)\mathbb{Z} \oplus \\
&\oplus \frac{1}{18}(F^6+F^5+10F^4+8F^3+2F^2+9F+9)\mathbb{Z} \oplus \\
&\oplus \frac{1}{108}(F^7+4F^6+13F^5+56F^4+80F^3+33F^2+18F+27)\mathbb{Z}
\end{aligned}
$$

$\#\mathrm{Aut}(I_7, x_{7,1}) = 2$

I_1 is invertible in R, but I_7 is not invertible in $\mathrm{End}(I_7)$.

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Final remarks

- Using Centeleghe-Stix '15 we can compute the isomorphism classes in $\mathscr{C}_h$ over $\mathbb{F}_p$ where h is square-free and $h(\sqrt{p}) \neq 0$ much larger subcategory ... but no polarizations in this case.

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- we can also deal with the case $\mathscr{C}_{h^d}$ (with h square-free) when $\mathbb{Z}[F, q/F]$ is Bass.

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- base field extensions (ordinary case).
- period matrices (ordinary case) of the canonical lift.

Thank you!