Uncertainty Quantification of Low-Dimensional Models

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Outline

Topic

Motivation

- Low-dimensional modeling is a useful method for examining high-dimensional data
  - Proper Orthogonal Decomposition (POD)
    - Data compression
    - Dominant spatial features
    - Inexpensive models
  - Dynamic Mode Decomposition (DMD)
    - Linear description of dynamical system
    - Spatial modes + frequencies
- Uncertainty quantification is a useful method for investigating statistical variations
  - Polynomial Chaos Expansions (PCE)
    - Assumes *a-priori* probabilistic knowledge of uncertainty
    - Efficient sampling algorithm
    - Accurate surrogate model
- **Can we investigate statistical variations in low-dimensional models efficiently/accurately?**

Applications

- POD Galerkin Models of Fluid Flows
  - Computationally-inexpensive model
    - Collect data from simulations/experiments
    - Calculate POD of output data
    - Project governing flow equations onto POD modes
- DMD Analysis of Fluid Flows
  - Determine linear modes/frequencies describing flow behavior
- Sensitive to parametric variations/uncertainties
  - Physical parameters (eg. Reynolds number)
  - Boundary conditions
  - Sensor noise

Topic

Methodology

- Identify source of uncertainty
  - Physical parameters (eg. Reynolds number)
  - Boundary conditions
  - Sensor noise
- Write a probabilistic description of uncertainty (ie. PDF)
- Utilize efficient sampling of probability space
  - Quadrature nodes corresponding to a spectral basis
- Collect simulation data using discrete points in probability space
  - Immersed boundary projection method (IBPM) code
- Quantify uncertainty in outputs
  - POD modes
  - DMD modes
  - DMD eigenvalues

Polynomial Chaos Expansions (PCE)

- **Polynomial Chaos Framework** [1]
  - Let $\mathbf{Z} = (Z_1 \ldots Z_d)$ be d random variables with PDF $\rho(\mathbf{Z})$ that parameterize ice
  - Let $\{\Phi_k\}$ denote the set of polynomials which are orthogonal w.r.t. $\rho(\mathbf{Z})$
  - Let $y(\mathbf{Z})$ denote the mapping from $\mathbf{Z}$ to an aerodynamic performance metric
- **Probabilistic Collocation Method:**
  - *Representation*

$$y(\mathbf{Z}) \approx \sum_{|i|=0}^{N} y_i \Phi_i(\mathbf{Z})$$

  - *Orthonormality*

$$\langle f, g \rangle = \int_{\Gamma} f(\mathbf{z}) g(\mathbf{z}) \rho(\mathbf{z}) d\mathbf{z}$$

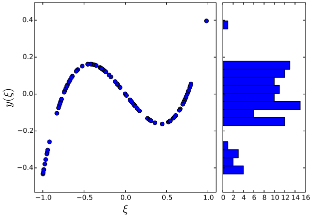
$$\langle \Phi_i, \Phi_j \rangle = \delta_{ij}$$

  - *Quadrature*

$$y_k = \langle y, \Phi_k \rangle \approx \sum_{i=0}^{Q} y(\mathbf{Z}^{(k)}) \Phi_k(\mathbf{Z}^{(k)}) w_k$$

[1] Xiu D. *Numerical Methods for Stochastic Computations: A Spectral Method Approach*. Princeton University Press, 2010.

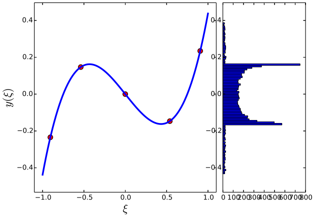
Polynomial Chaos Expansions (PCE)



Monte Carlo

$$y \approx \delta(\xi - \xi_k)$$

- Draw random samples from distribution
- Function exists at discrete points

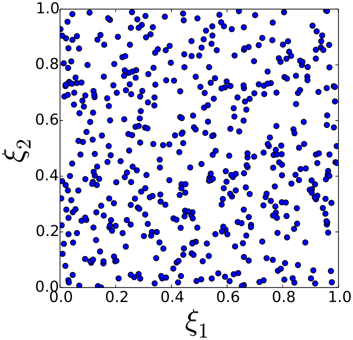


Polynomial Chaos

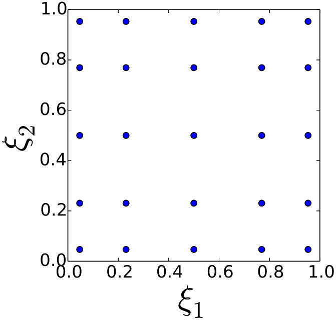
$$y \approx \sum_{i}^{Q} c_i \psi_i(\xi)$$

- Use Q quadrature points
- Construct $(Q-1)$ order polynomial fit

Polynomial Chaos Expansions (PCE)



Quadrature Sampling Grid

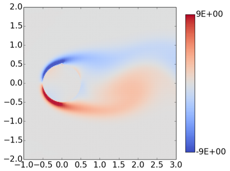


Monte Carlo Sampling

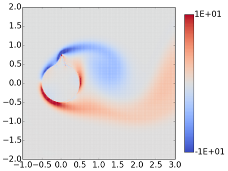
Topic

Setup

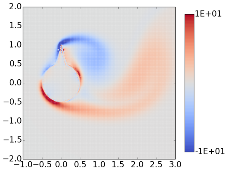
Small Spike

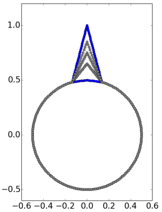


Medium Spike



Large Spike

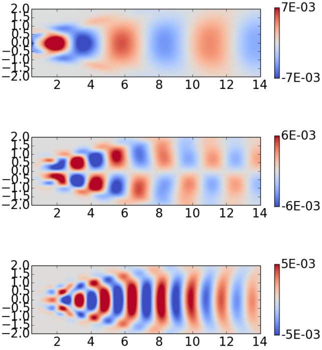




Range of Flow Behavior

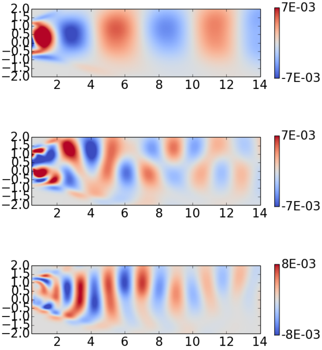
Cylinder, Re = 100

POD Modes

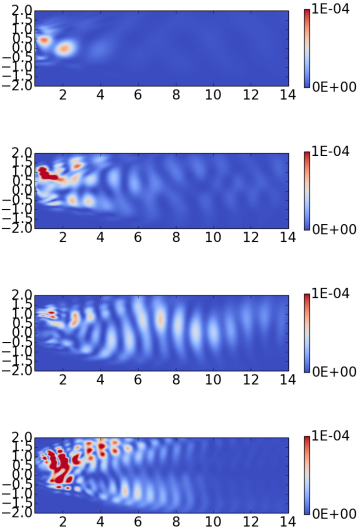


Perturbed Cylinder, Re = 100

POD Modes



Statistical Variance in Modes



Projection Error

- Choose the $Q-1$ points halfway between Q quadrature nodes
- Calculate true modes and interpolated modes at $Q-1$ points
- Compare error between true modes and interpolated modes vs. true modes and mean modes

$$N(Y) \equiv max(||Y(\xi_k) - \Phi(\xi_k)||_2) \quad , \quad k = 1 \ldots Q-1$$

MODE	$N(y_P)$	$N(\bar{y})$	$N(\bar{y})/N(y_P)$
1	4e-3	3e-1	75
2	3e-2	9e-1	30
3	2e-1	1.2	6
4	7e-1	1.5	2
5	2e-1	1.8	9

- PCE model captures range of symmetrical to asymmetrical modes

Topic