Uncertainty Quantification of Low-Dimensional Models

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Outline

Topic

Motivation

- Low-dimensional modeling is a useful method for examining high-dimensional data
  - Proper Orthogonal Decomposition (POD)
  - Dynamic Mode Decomposition (DMD)
  - Identify inexpensive models
  - Identify dominant system dynamics
- Uncertainty quantification is a useful method for investigating statistical variations
  - Polynomial Chaos Expansions (PCE)
  - Efficient algorithms for exploring uncertain parameter space
  - Accurate surrogate model
- **Can we investigate statistical variations in low-dimensional models efficiently/accurately using uncertainty quantification tools?**

Applications

- POD Galerkin Models of Fluid Flows
  - Computationally-inexpensive model
    - Collect data from simulations/experiments
    - Calculate POD of output data
    - Project governing flow equations onto POD modes
  - Performance of models is sensitive to parametric uncertainty/variations
- DMD Analysis of Fluid Flows
  - Determine linear modes/frequencies describing flow behavior
  - Spatial modes and frequencies can change with parametric uncertainty
- Examples of parametric variations/uncertainties
  - Physical parameters (eg. Reynolds number)
  - Boundary conditions

Topic

Background

- Proper Orthogonal Decomposition (POD)
  - Data compression, dominant spatial features
  - Modes are eigenvectors of the dataset covariance matrix
  - Modes describe dataset better than any other linear basis
- Dynamic Mode Decomposition (DMD)
  - Describe dataset as linear dynamical system
  - Spatial modes + (frequencies, growth/decay rates)
- Polynomial Chaos Expansions (PCE)
  - Method for quantifying parametric uncertainty efficiently
  - Spectral method in probability space
  - Expand output in terms of basis polynomial functions of random variables

Methodology

Quantify Uncertain Input

- Identify source of uncertainty
  - Physical parameters (eg. Reynolds number)
  - Boundary conditions
- Write a probabilistic description of uncertainty (ie. PDF)

Explore Uncertain Parameter Space

- Utilize efficient sampling of probability space
  - Quadrature nodes corresponding to a spectral basis
- Collect simulation data using discrete points in probability space
  - Immersed boundary projection method (IBPM) code

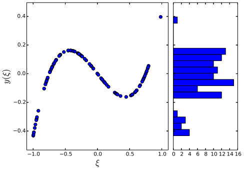
Quantify Uncertain Output

- Quantify uncertainty in outputs
  - POD modes
  - DMD modes
  - DMD eigenvalues

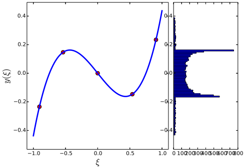
Polynomial Chaos Expansions (PCE)

- We have an uncertain parameter $\xi = \mathcal{U}[-1,1]$ which maps to output $y(\xi)$
- What is the distribution of $y(\xi)$?

$$y(\xi) = a\xi^3 + b\xi^2 + c\xi + d$$



Monte Carlo



Polynomial Chaos

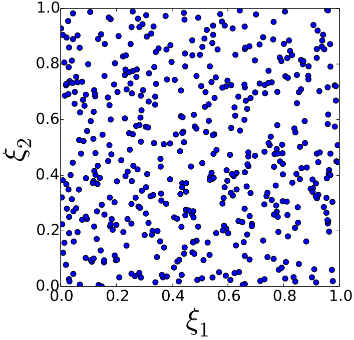
$$y \approx \delta(\xi - \xi_k)$$

$$y \approx \sum_i^Q c_i \psi_i(\xi)$$

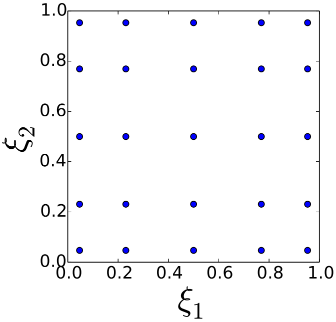
- Draw random samples
- Function exists at discrete points

- Use Q quadrature points
- $(Q-1)$ order polynomial fit

Polynomial Chaos Expansions (PCE)



Monte Carlo Sampling



Quadrature Sampling Grid

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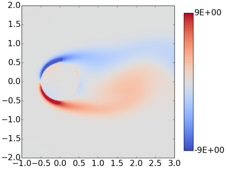
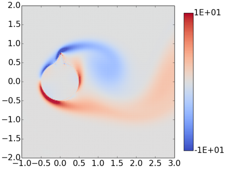
Setup

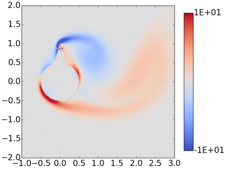
Small Spike

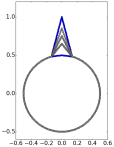


Medium Spike



Large Spike



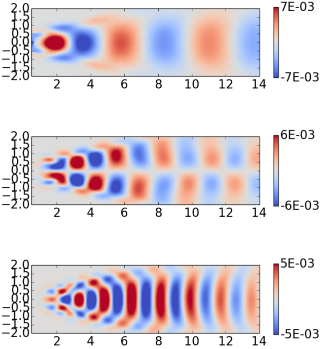


- Assume spike height is uniformly distributed between limits shown
- Re = 100
- Output = wake POD modes, DMD eigenvalues

Range of Flow Behavior

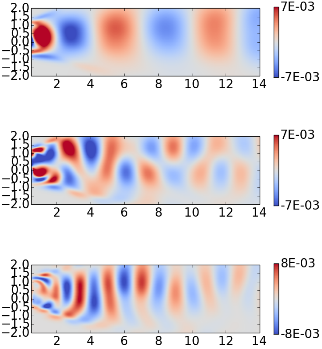
Cylinder, Re $= 100$

POD Modes

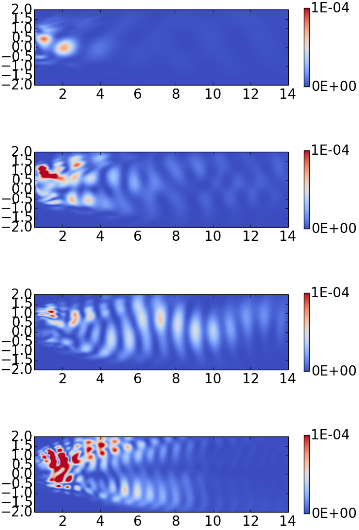


Perturbed Cylinder, Re $= 100$

POD Modes



Statistical Variance in Modes



Projection Error

- Choose the $Q-1$ points halfway between Q quadrature nodes
- Calculate true modes and interpolated modes at $Q-1$ points
- Compare error between true modes and interpolated modes vs. true modes and mean modes

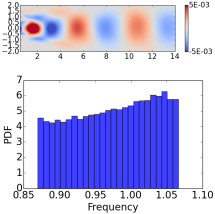
$$N(Y) \equiv max(||Y(\xi_k) - \Phi(\xi_k)||_2) \quad , \quad k = 1 \ldots Q-1$$

MODE	$N(y_P)$	$N(\bar{y})$	$N(\bar{y})/N(y_P)$
1	4e-3	3e-1	75
2	3e-2	9e-1	30
3	2e-1	1.2	6
4	7e-1	1.5	2
5	2e-1	1.8	9

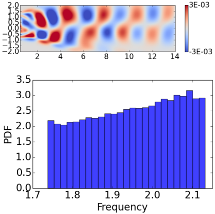
- PCE model captures range of symmetrical to asymmetrical modes

DMD Eigenvalues

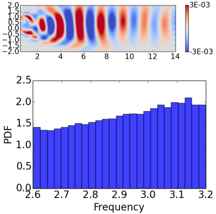
Low Frequency



Medium Frequency



High Frequency



Conclusions

- Uncertainty quantification techniques provide a fast, efficient, and accurate methodology for quantifying how low-dimensional models change with parametric uncertainty
  - POD modes
  - DMD modes/eigenvalues