Asymptotic Behavior of Last Passage Percolation on Complete Graphs

Charles Beck, Stanley Jian, Catherine Lyu, Harrison Wang

Department of Mathematics
Columbia University

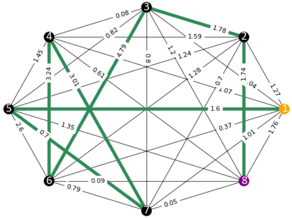
August 4, 2022

Problem Description - Last Passage Percolation

- Complete graph G_n, each edge e has i.i.d weight $X_e \geq 0$
- Label vertices V = {1,2,...,n}, consider self-avoiding paths γ from $1 \to n$
- Define the weight of path γ to be

$$W(\gamma) = \sum_{e \in \gamma} X_e, \quad W_n = \max_{\gamma} W(\gamma)$$

- How quickly does W_n grow as $n \to \infty$?[1]
- First & Last Passage Percolation in $\mathbb{Z}^2$ - Fluid Flow in Porous Media



General Theorems

Bounded Distributions:

$$\mu := \sup\{x : \mathbb{P}(X > x) > 0\}$$

$$\frac{W_n}{n} \xrightarrow[n\to\infty]{} \mu$$

Unbounded Distributions:
For sufficiently light tailed (slowly varying and sub-exponential)

$$\frac{W_n}{nb_n} \xrightarrow[n\to\infty]{} 1$$

For heavy-tailed distributions, the limit will not be a constant.

Extreme Value Theory

- Given i.i.d. random variables $X_1, \ldots, X_n$ distributed as X, define

$$M_n^{(i)} := \text{ the ith maximum} \qquad M_n := M_n^{(1)}$$

- Goal: identify deterministic b_n which grows approximately like M_n
- b_n strictly increasing, but speed of b_n growth dependent upon distribution X
  - Heavy-tailed $\implies$ b_n grows quickly, light-tailed $\implies$ b_n grows slowly
- Would describe LPP well if each path was independent, but this isn't the case

	Exp	Rayleigh	Power	Uniform
$\mathbb{P}(X \geq x)$	e^{-x}	e^{-x^2}	$x^{-\alpha}$	$1 - x$
b_n	$\log n$	$\sqrt{\log n}$	$n^{\frac{1}{\alpha}}$	$1 - \frac{1}{n}$

Extreme Value Theory

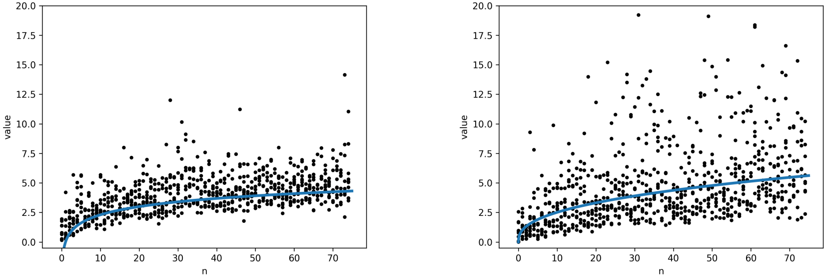


Figure: Simulated M_n and b_n for light-tailed (left) and heavy-tailed (right)

Results

$$\frac{W_n}{n} \approx h(n)$$

Distribution	Exp e^{-x}	Rayleigh e^{-x^2}	Power $x^{-\alpha}$ $(\alpha < 1)$	Power $x^{-\alpha}$ $(\alpha = 1)$	Power $x^{-\alpha}$ $(\alpha > 1)$	Bounded $X \leq \mu < \infty$
$h(n)$	$\log(n)$	$\sqrt{\log(n)}$	$n^{\frac{2}{\alpha}-1}$	$n\log(n)$	$n^{\frac{1}{\alpha}}$	μ

Note that $h(n)$ is like b_n for most distributions

Upper Bound Method

- Path Length $< n$
- Consider n largest edges in entire graph $M_{n^2}^{(1)}, M_{n^2}^{(2)}, ..., M_{n^2}^{(n)}$ $(n^2 \sim \binom{n}{2})$

This has weight:

$$W_n \leq \sum_{i=1}^{n} M_{n^2}^{(i)} \approx \sum_{i=1}^{n} b_{n^2}^{(i)} \approx n b_n (1 + o(1))$$

Lower Bound Methods

- **Single Edge:**
  - There must exist an edge through the largest edge

$$W_n \geq M_{n^2}$$

- **Greedy Approach:**
  - Start at node 1 and choose the largest edge M_{n-2} to any unvisited node (besides node n)
  - Repeat until all nodes visited. When i nodes besides n left, the next edge has weight M_i(independent).
  - Connect last node to node n

$$W_n \geq \sum_{i=1}^{n-2} M_i$$

- k_n **largest edges:**
  - If k_n edges all disjoint, path can be drawn through them
  - For all sublinear k_n ($\frac{k_n}{n} \to 0$), k_n largest edges will be disjoint **w.h.p.**

$$W_n \geq \sum_{i=1}^{k_n} M_{n^2}^{(i)}$$

Future Directions and Acknowledgements

- **Central Limit Theorem**
  - The next step is to study the second order behavior of W_n.
- **Acknowledgements**
  - We would like to thank Professor Milind Hegde, as well as Anushka Murthy for their guidance. We would also like to thank Professor George Dragomir and the Columbia University Math Department for organizing and supporting the program.
- **References**

[1] Feng Wang, Xian-Yuan Wu, and Rui Zhu. Last passage percolation on the complete graph. *Statistics Probability Letters*, 164:108798, 2020.

Thank you for giving us your time!

$$\frac{\bar{X} - \mathbb{E}[X]}{\sigma_X / \sqrt{n}} \xrightarrow{d} N(0,1)$$