# Last Passage Percolation on Complete Graphs

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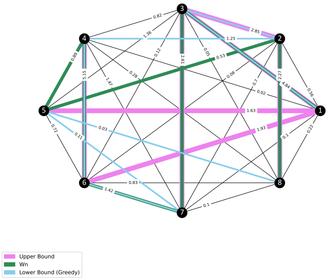
August 4, 2022

# Problem Description - Last Passage Percolation

- First Passage Percolation - Fluid Flow in Porous Media
- Complete graph $G_n$, each edge $e$ has i.i.d weight $X_e$
- Label vertices V $= \{1,2,...,n\}$, consider self-avoiding paths from $1 \to n$
- $W_n =$ highest weight path among all paths
- How does $W_n$ grow and vary as $n \to \infty$?



# Extreme Value Theory

- Given i.i.d. random variables $X_1, \ldots, X_n$ distributed as $X$, define

$$M_n^{(i)} := \text{ the ith maximum} \qquad M_n := M_n^{(1)}$$

- Goal: identify deterministic $b_n$ which grows approximately like $M_n$
- $b_n$ strictly increasing, but speed of $b_n$ growth dependent upon distribution $X$
  - Heavy-tailed $\implies$ greater probability of high values $\implies$ fast growth
  - Light-tailed $\implies$ lower probability of high values $\implies$ slow growth
- Would describe LPP well if each path was independent, but this isn't the case

# Upper Bound

- Path Length $< n$
- Consider $n$ largest edges in entire graph $M_{n^2}^{(1)}, M_{n^2}^{(2)}, ..., M_{n^2}^{(n)}$

This has weight:

$$W_n \leq \sum_{i=1}^{n} M_{n^2}^{(i)} \leq \sum_{i=1}^{n} b_{n^2}^{(i)} \approx \begin{cases} O(nb_n) & \text{Light Tail} \\ O(nb_n g(n)) & \text{Heavy Tail} \end{cases}$$

where $g(n) \to \infty$ as $n \to \infty$.

# Lower Bounds

- **Single Edge:**
  - There must exist an edge through the largest edge
- **Greedy Approach:**
  - Start at node 1 and choose the largest edge $M_n$ to any unvisited node (besides node $n$)
  - Repeat until all nodes visited. When $i$ nodes besides $n$ left, the next edge has weight $M_i$
  - Connect last node to node $n$
- $k_n$ **largest edges:**
  - If $k_n$ edges all disjoint, path can be drawn through them
  - For all sublinear $k_n$ ($\frac{k_n}{n} \to 0$), $k_n$ edges will be disjoint **w.h.p.**
  - **W.h.p.**, $W_n$ greater than sum of $k_n$ largest edge weights

$$M_{n^2} \leq W_n$$

$$\sum_{i=1}^{n-1} M_i \leq W_n$$

$$\sum_{i=1}^{o(n)} M_{n^2}^{(i)} \leq W_n$$

# Lower Bounds(cont.)

# General Theorems

**Bounded Distributions**:

$$\mu := \inf\{x : \mathbb{P}(X \geq x) = 0\}$$

$$\frac{W_n}{n} \xrightarrow[n\to\infty]{} \mu$$

**Unbounded Distributions**:
For sufficiently light tailed(slowly varying, Hegde Condition)

$$\frac{W_n}{nb_n} \xrightarrow[n\to\infty]{} 1$$

# Results

$$\lim_{n\to\infty} \frac{W_n}{n} \approx h(n)$$

| Distribution | Exp $e^{-x}$ | Rayleigh $e^{-x^2}$ | Power $x^{-\alpha}$ $(\alpha < 1)$ | Power $x^{-\alpha}$ $(\alpha = 1)$ | Power $x^{-\alpha}$ $(\alpha > 1)$ | Bounded $X \leq M < \infty$ |
|---|---|---|---|---|---|---|
| $h(n)$ | $\log(n)$ | $\sqrt{2\log(n)}$ | $n^{\frac{1}{\alpha}}$ | $n\log(n)$ | $n^{\frac{2}{\alpha}-1}$ | $M$ |
| Method used | Greedy | Greedy | Single Edge | Single Edge | Single Edge | Greedy |

Thank you!