Moving back to finance (slowly)

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Example: Ornstein - Uhlenbeck (OU) process

1. Consider $dX_t = -\gamma(X_t - \mu)dt + \sigma dB_t$.

2. When $X_t > \mu$ the drift is negative towards μ. When $X_t < \mu$, the drift is positive towards μ.

3. σ: sensitivity to noise. Example of X_t: interest rates, commodity prices, etc.

4. Let $Y_t = e^{\gamma t}X_t$. ($f(t,x) = e^{\gamma t}x$)

5. Applying Itô's lemma on Y_t yields:

$$dY_t = e^{\gamma t}dX_t + \gamma e^{\gamma t}X_t dt = \gamma\mu e^{\gamma t}dt + e^{\gamma t}\sigma dB_t$$

OU Solution

$$\therefore Y_t - Y_0 = \gamma\mu \int_0^t e^{\gamma s} ds + \int_0^t e^{\gamma s} \sigma \, dB_s$$

$$\Rightarrow X_t = e^{-\gamma t} X_0 + (1 - e^{-\gamma t})\mu + e^{-\gamma t} \int_0^t e^{\gamma s} \sigma \, dB_s$$

X_t will have Gaussian distribution:

$$X_t \sim N(\tilde{\mu}, \tilde{\sigma}^2)$$

where

$$\tilde{\mu} = e^{-\gamma t} X_0 + (1 - e^{-\gamma t})\mu$$

$$\tilde{\sigma}^2 = \sigma^2 e^{-2\gamma t} E\left[\left(\int_0^t e^{\gamma s} dB_s\right)^2\right]$$

$$\tilde{\sigma}^2 = \frac{\sigma^2}{2\gamma}(1 - e^{-2\gamma t}) \quad \text{(By Itô's Isometry property)}$$

Black-Scholes Model

- The stock price process follows a stochastic differential equation for $0 \leq t \leq T$:

$$dS_t = \mu S_t dt + \sigma S_t dB_t$$

- Or more accurately

$$S_t = S_0 + \mu \int_0^t S_u du + \sigma \int_0^t S_u dB_u$$

- Both sides depend on $\{S_t\}$ as in an ordinary differential equation. Existence of solution is not obvious but can be shown in this case.

Black-Scholes Model

- Recall the stock price process follows stochastic differential equation

$$dS_t = \mu S_t dt + \sigma S_t dB_t$$

- Under SDE, $\{S_t\}_{0 \le t \le T}$ is a Markov process.

  - A process $(X_t : 0 \le t \le T)$ is Markov if, for any function f and for all $s < t$,
  $$\forall s < t \quad E[f(X_t)|\mathcal{F}_s] = g(X_s)$$
  for some function g.

  - The evolution of $(X_s : t \le s \le T)$ depends only on X_t and not on the values before that.

Back to Black Scholes

- Assume the existence of money market process $0 \leq t \leq T$:

$$d\beta_t = r\beta_t dt \quad \Rightarrow \quad \beta_t = e^{rt}$$

- Consider a European call option with payoff

$$h(S_T) = (S_T - K)^+.$$

- Let $c(t, S_t)$ denote the price process of the option at time t when the stock price equals S_t (it depends only on (t, S_t)).

- Applying Ito's Lemma to $c(t, S_t)$ we get

$$
\begin{aligned}
dc(t, S_t) &= c_t dt + c_s dS_t + \frac{1}{2} c_{ss} (dS_t)^2 \\
&= c_t dt + c_s dS_t + \frac{1}{2} c_{ss} \sigma^2 S_t^2 dt
\end{aligned}
$$

- Consider a self-financing replicating portfolio that matches $c(t, S_t)$ at $t = T$.

$$P_t = a_t S_t + b_t \beta_t$$

  - a_t : No. of stock units at time t.

  - b_t : No. of risk free security units at time t.

- Self-financing means that

$$dP_t = a_t dS_t + b_t d\beta_t. \tag{1}$$

Heuristically, $a_{t+\delta t}$ and $b_{t+\delta t}$ are decided at time t in a self financing manner. That is,

$$a_{t+\delta t} S_t + b_{t+\delta t} \beta_t = a_t S_t + b_t \beta_t$$

(i.e., $S_t da_t + \beta_t db_t = 0$).

- Then,

$$P_{t+\delta t} - P_t \approx a_{t+\delta t}(S_{t+\delta t} - S_t) + b_{t+\delta t}(\beta_{t+\delta t} - \beta_t).$$

Roughly, (1) holds.

Matching Coefficients

- We have

$$dc(t, S_t) = c_t dt + c_s(\mu S_t dt + \sigma S_t dB_t) + \frac{1}{2} c_{ss} \sigma^2 S_t^2 dt$$

and

$$dP_t = a_t dS_t + b_t d\beta_t = a_t \mu S_t dt + a_t \sigma S_t dB_t + b_t r \beta_t dt.$$

Matching dB_t co-efficient implies $a_t = c_s(t, S_t)$.

- Matching dt coefficient

$$c_t + c_s \mu S_t + \frac{1}{2} \sigma^2 S_t^2 c_{ss} = a_t \mu S_t + b_t r \beta_t$$

so that

$$b_t = \frac{c_t + \frac{1}{2} c_{ss} \sigma^2 S_t^2}{r \beta_t}$$

Black Scholes equation for options pricing

- Now

$$P_t = a_t S_t + b_t \beta_t = c(t, S_t) = c_s S_t + \frac{c_t + \frac{1}{2} c_{ss} \sigma^2 S_t^2}{r}$$

- It follows that

$$rc = rc_s S_t + c_t + \frac{1}{2} c_{ss} \sigma^2 S_t^2.$$

- This has to be true for all S_t, so that

$$rc = rc_s s + c_t + \frac{1}{2} c_{ss} \sigma^2 s^2.$$

- More explicitly, option price is a solution to the BS pde

$$rsc_s(t, s) + c_t(t, s) + \frac{1}{2} \sigma^2 s^2 c_{ss}(t, s) - rc_{s,s}(t, s) = 0$$

and $\qquad\qquad c(T, s) = (s - K)^+.$

1. Two big theorems in probability made for math finance:

   1. **Girsanov's Theorem**

   2. **Martingale representation theorem**

Change of measure

- Consider $(\Omega, \mathcal{F}, P)$. Random variable Z: $E_P Z = 1$, $Z > 0$ a.s.

- Can define new probability measure $\tilde{P}$: $\tilde{P}(A) = E_P[ZI(A)] \; \forall A \in \mathcal{F}$.

- Now

$$P(A) = 0 \implies \tilde{P}(A) = 0$$

so $\tilde{P}$ is abs. cont. w.r.t. P and $Z = \frac{d\tilde{P}}{dP}(\omega)$ is the Radon-Nikodym derivative of $\tilde{P}$ w.r.t. P.

- Since $\tilde{P}(A) = E_P[ZI(A)]$ by standard m/c, $E_{\tilde{P}} X = E_P[ZX]$ when X is simple

$$\implies E_{\tilde{P}}[X] = E_P[ZX]$$

$\forall X$ for which these expectations are well defined.

- Take $X = Y/Z$, legal since $Z > 0$ a.s. $\implies E_{\tilde{P}}\left[\frac{Y}{Z}\right] = E_P[Y]$

- Hence, $P(A) = E_{\tilde{P}}\left[\frac{1}{Z}I(A)\right]$.

- Thus, $P(A) = 0 \iff \tilde{P}(A) = 0$, so P & $\tilde{P}$ are **equivalent!** and

$$\frac{dP}{d\tilde{P}} = \frac{1}{Z}$$

- Recall that in discrete setting $Z(\omega) = \frac{\tilde{P}(\omega)}{P(\omega)}$, $1/Z(\omega) = \frac{P(\omega)}{\tilde{P}(\omega)}$

Martingale Property of Density

- Let $E_P Z_T = 1$, $Z_T \geq 0$, Z_T is $\mathcal{F}_T$ measurable.

- Set $Z_t = E_P(Z_T | \mathcal{F}_t)$. Recall that $\{Z_t\}$ is a martingale.

- Let Y be $\mathcal{F}_t$ measurable. Then for $Q(A) = E_P[Z_T I(A)]$

$$E_Q(Y) = E_P[Z_t Y]$$

- True since

$$E_Q[Y] = E_P[Z_T Y] = E_P[E_P(Z_T Y | \mathcal{F}_t)]$$

$$= E_P[Y E_P(Z_T | \mathcal{F}_t)] = E_P[Y Z_t]$$

Martingale Property of Density

- Important. Y is $\mathcal{F}_t$ mble, $s < t$:

$$E_Q(Y|\mathcal{F}_s) = \frac{E_P(YZ_t|\mathcal{F}_s)}{Z_s}$$

- That is, RHS integrates like Y under Q along all $A \in \mathcal{F}_s$.

$$E_Q\left[\frac{E_P(YZ_t|\mathcal{F}_s)}{Z_s}I(A)\right] = E_P\left[\frac{E_P(YZ_t|\mathcal{F}_s)}{Z_s}Z_tI(A)\right]$$

$$= E_P\left[\frac{E_P(YZ_t|\mathcal{F}_s)}{Z_s}Z_sI(A)\right] = E_P[YZ_tI(A)]$$

$$= E_Q[YI(A)] = \int_A YdQ$$

Exponential Martingale (The Density Process Z_t)

- Consider

$$Z_t = \exp\left[-\int_0^t \alpha_s dB_s - \int_0^t \frac{\alpha_s^2}{2}ds\right]$$

  $\{\alpha_s\}$ is an adapted process.

- Then $\{Z_t : 0 \leq t \leq T\}$ is a martingale (under mild technical conditions).

- To see this, let

$$X_t = -\int_0^t \alpha_s dB_s - \int_0^t \frac{\alpha_s^2}{2}ds$$

  or

$$dX_t = -\alpha_t dB_t - \frac{\alpha_t^2}{2}dt$$

- Applying Itô's lemma on e^{X_t} we get:

$$dZ_t = Z_t dX_t + \frac{1}{2} Z_t dX_t dX_t$$

$$= Z_t \left(-\alpha_t dB_t - \frac{\alpha_t^2}{2} dt \right) + \frac{1}{2} Z_t (\alpha_t^2 dt) = -\alpha_t Z_t dB_t$$

$$\implies Z_t = Z_0 - \int_0^t \alpha_s Z_s dB_s$$

is a martingale.

- $Z_0 = 1$ and $EZ_t = 1$. And $Z_t = E(Z_T | \mathcal{F}_t)$.

Girsanov Theorem

Let $(B_t : 0 \leq t \leq \tau)$ be a Brownian motion on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, P)$

$$Z_t = \exp\left[-\int_0^t \alpha_s dB_s - \frac{1}{2} \int_0^t \alpha_s^2 ds\right]$$

Let

$$\tilde{B}_t = B_t + \int_0^t \alpha_s ds$$

Girsanov's Theorem Under $\tilde{P}$ such that $\tilde{P}(A) = E_P[Z_T I(A)]$, $(\tilde{B}_t : 0 \leq t \leq T)$ is standard Brownian motion.

- Take $\alpha_s = \alpha > 0$. $\implies Z_t = e^{-\alpha B_t - \frac{1}{2}\alpha^2 t}$. This reweighing assigns small weight to large values of B_t and large weight to negative values of B_t.

- Then $B_t = \tilde{B}_t - \alpha t$ has a negative drift under the new measure.

Proof relies on showing that

1. $E_{\tilde{P}}(\tilde{B}_t | \mathcal{F}_s) = \tilde{B}_s \implies$ is a martingale.

2. It is continuous in t by definition.

3. $d\tilde{B}_t \cdot d\tilde{B}_t = (dB_t + \alpha dt)(dB_t + \alpha dt) = dt$

4. Lévy's theorem (to be discussed later) says that under above conditions $\tilde{B}(t)$ is a Brownian motion under $\tilde{P}$.

Proof of Girsanov

- Later

The Gaussian example again

- Consider rv $B_1, B_2, \ldots, B_T$ such that $B_1 = X_1 \sim N(0,1)$ and each increment $B_m - B_{m-1} = X_m \sim N(0,1)$ independent of $(B_1, \ldots, B_{m-1})$ under probability measure ¶. Let $X = (X_1, \ldots, X_T)$.

- Then, letting $x = (x_1, \ldots, x_T)$, $P(X \in A)$ equals

$$\int_{x \in A} f(x_1) \ldots f(x_T) dx_1 \ldots dx_T$$

where f is a pdf of $N(0,1)$.

- Let

$$M_T = \exp(-\sum_{i=1}^{T} \mu_i X_i - \sum_{i=1}^{T} \mu_i^2/2.)$$

Then, $M_T > 0$ everywhere and $E_P(M_T) = 1$

- Set Q such that $Q(A) = E_P(M_T I(A))$. Then, for

$$g_i(x) = e^{-\mu_i x - \mu_i^2/2} f(x)$$

we have

$$Q(A) = \int_{x \in A} g_1(x_1) \ldots g_T(x_T) dx_1 \ldots dx_T$$

Observe that g_i is a pdf of of $N(\mu_i, 1)$.

- Under Q, the density f for each X_i is replaced by the density g_i, so mean of each X_i reduces from 0 to $-\mu_i$.

- This can be extended to allow μ_i to be a function of $(B_1, \ldots, B_{i-1})$

Itô's Multi-variate Calculus

- Suppose that B_1 & B_2 are independent Brownian motions.

- Then the cross variation

$$\lim_{n \to \infty} \sum_{i=1}^{n} (B_1((i+1)t/n) - B_1(it/n))(B_2((i+1)t/n) - B_2(it/n)) = 0.$$

To see this, note that in distribution, the sum equals

$$\frac{T}{n} \sum_{i=1}^{n} U_i V_i$$

where $U_i \sim N(0,1)$ and $V_i \sim N(0,1)$ and each U_i is independent of each V_j.

- By law of large numbers

$$\frac{1}{n} \sum_{i=1}^{n} U_i V_i \to 0$$

Itô's Multi-variate Calculus

- We denote that the cross variation is zero by notation
  $dB_1(t) \cdot dB_2(t) = 0$.

- Let $f = f(t, X_t, Y_t)$ where both $(X_t : 0 \leq t \leq T)$ and
  $(Y_t : 0 \leq t \leq T)$ are Itô processes.

$$dX_t = a_1(t)dt + b_{11}(t)dB_1(t) + b_{12}(t)dB_2(t)$$

$$dY_t = a_2(t)dt + b_{21}(t)dB_1(t) + b_{22}(t)dB_2(t)$$

- B_1 & B_2 are independent Brownian motions.

- $a_1, a_2, b_{11}, b_{12}, b_{21}, b_{22}$ are adapted processes.

Then

$$df(t, X_t, Y_t) = f_t dt + f_x dX_t + f_y dY_t + \frac{1}{2} f_{xx} dX_t dX_t + \frac{1}{2} f_{yy} dY_t dY_t + f_{xy} dX_t dY_t$$

where

$$dX_t \cdot dX_t = [b_{11}^2(t) + b_{12}^2(t)]dt$$

$$dY_t \cdot dY_t = [b_{21}^2(t) + b_{22}^2(t)]dt$$

$$dX_t \cdot dY_t = [b_{11}(t)b_{21}(t) + b_{12}(t)b_{22}(t)]dt$$

Levy Theorem

Levy's Theorem: Given $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, P)$ and a continuous martingale M_t with $M_0 = 0$, then if the quadratic variation $(dM_t)^2 = dt$, then it is a Brownian motion.

- Consider two independent Brownian motions $(B_1(t) : 0 \leq t \leq T)$ and $(B_2(t) : 0 \leq t \leq T)$. For $\rho \in (0, 1)$, what is the distribution of

$$B(t) = \rho B_1(t) + \sqrt{1 - \rho^2} B_2(t).$$

- Observe that $B(0) = 0$, $B(t)$ is continuous in t, and it is a martingale. What is its quadratic variation?

$$dB(t)^2 = (\rho dB_1(t) + \sqrt{1 - \rho^2} dB_2(t))(\rho dB_1(t) + \sqrt{1 - \rho^2} dB_2(t)) = dt$$

Hence, it is a Brownian motion.

Proof of Girsanov Theorem

Let $(B_t : 0 \leq t \leq \tau)$ be a Brownian motion on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, P)$

$$Z_t = \exp\left[-\int_0^t \alpha_s dB_s - \frac{1}{2}\int_0^t \alpha_s^2 ds\right]$$

and

$$\tilde{B}_t = B_t + \int_0^t \alpha_s ds$$

Girsanov's Theorem Under $\tilde{P}$ such that $\tilde{P}(A) = E_P[Z_T I(A)]$, $(\tilde{B}_t : 0 \leq t \leq T)$ is standard Brownian motion.

Proof: Need to show that $E_{\tilde{P}}(\tilde{B}_t | \mathcal{F}_s) = \tilde{B}_s$, i.e., $\tilde{B}_t$ is a martingale under $\tilde{P}$.

- Since

$$E_{\tilde{P}}(Y|\mathcal{F}_s) = \frac{E_P(YZ_t|\mathcal{F}_s)}{Z_s}$$

  for $\mathcal{F}_t$ measurable Y, we need to show that

$$E_P(Z_t\tilde{B}_t|\mathcal{F}_s) = Z_s\tilde{B}_s$$

  that is, $Z_t\tilde{B}_t$ is a martingale under P.

- We had earlier shown that $dZ_t = -\alpha_t Z_t dB_t$.

- Now $d(\tilde{B}_t Z_t) = \tilde{B}_t dZ_t + Z_t d\tilde{B}_t + d\tilde{B}_t dZ_t$. This equals

$$-\tilde{B}_t \alpha_t Z_t dB_t + Z_t(dB_t - \alpha_t dt) + \alpha_t Z_t dt = (-\tilde{B}_t \alpha_t + 1)Z_t dB_t$$

  which is a martingale! QED.

Martingale Representation Theorem

- Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, P)$ be a prob. space where $\mathcal{F}_t = \sigma(B_s : 0 \leq s \leq t)$

- If $\{M_t\}$ is a martingale defined on this space, then there exists an adapted process $\{b_t\}$ such that

$$M_t = M_0 + \int_0^t b_s \, dB_s$$

or $dM_t = b_t \, dB_t$.

- Thus every stochastic integral is a martingale (when integrand is in $\mathcal{H}^2$), but under the basic filtration above, every martingale is also a stochastic integral.

Application of Girsanov to Two Security Model

- Consider 2 security model (stock S_t, money market M_t)

$$dS_t = \mu_t S_t dt + \sigma_t S_t dB_t$$

$$dM_t = r_t M_t dt$$

- Select $\tilde{P}$ so that under it

$$d\tilde{B}_t = dB_t + \alpha_t dt$$

is standard BM.

- Then

$$dS_t = \mu_t S_t dt + \sigma_t S_t (d\tilde{B}_t - \alpha_t dt)$$

- Choose

$$\alpha_t = \frac{\mu_t - r_t}{\sigma_t}$$

- Then

$$dS_t = r_t S_t dt + \sigma_t S_t d\tilde{B}_t$$

- Same rate of growth as risk free security.

- Two security model again

$$dS_t = \mu_t S_t dt + \sigma_t S_t dB_t$$

$$dR_t = -r_t R_t dt$$

or $R_t = e^{-\int_0^t r_s ds}$ (discount rate).

- Consider discounted process $R_t S_t$. By multi-variate Itô

$$d(R_t S_t) = R_t dS_t + S_t dR_t + dR_t dS_t$$

$$d(R_t S_t) = R_t(\mu_t S_t dt + \sigma_t S_t dB_t) + S_t(-r_t R_t dt) + 0$$
$$= S_t(R_t \mu_t - r_t R_t)dt + R_t \sigma_t S_t dB_t$$

Discounted Stock Price Martingale II

- Under $\tilde{P}$

$$= S_t(R_t\mu_t - r_tR_t)dt + R_t\sigma_tS_t(d\tilde{B}_t - \alpha_tdt)$$

- Choose

$$\alpha_t = \frac{\mu_t - r_t}{\sigma_t}$$

$$\implies d(R_tS_t) = R_t\sigma_tS_td\tilde{B}_t$$

a martingale.

- Here Z_T corresponding to $\tilde{P}$ is

$$Z_T = \exp\left[-\int_0^\tau \left(\frac{\mu_t - r_t}{\sigma_t}\right) dB_t - \frac{1}{2}\int_0^\tau \left(\frac{\mu_t - r_t}{\sigma_t}\right)^2 dt\right].$$

Replicating portfolio Let $\{X_t\}$ be a self financing portfolio process

$$dX_t = a_t dS_t + (X_t - a_t S_t) r_t dt$$

Then

$$
\begin{aligned}
d(R_t X_t) &= R_t dX_t + X_t dR_t + dR_t dX_t \\
&= R_t[a_t dS_t + (X_t - a_t S_t) r_t dt] + X_t(-r_t R_t dt) \\
&= R_t a_t dS_t - R_t a_t S_t r_t dt \\
&= a_t(R_t dS_t - R_t S_t r_t dt) \\
&= a_t d(R_t S_t) \\
&= a_t R_t \sigma_t S_t d\tilde{B}_t
\end{aligned}
$$

(since $d(R_t S_t) = R_t \sigma_t S_t d\tilde{B}_t$) so it is a martingale under $\tilde{P}$.

Let option price at time T be $V_T \in \mathcal{F}_T$ measurable (mble). Define,

$$V_t = \frac{1}{R_t} E_{\tilde{P}}(R_T V_T | \mathcal{F}_t)$$

$$\implies \{V_t R_t\} \text{ is a martingale under } \tilde{P}.$$

By martingale representation theorem,

$$R_t V_t = R_0 V_0 + \int_0^t \gamma_u d\tilde{B}_u$$

for some adapted $\{\gamma_u\}$

- Now $d(R_t V_t) = \gamma_t d\tilde{B}_t$ and $d(R_t X_t) = a_t R_t \sigma_t S_t d\tilde{B}_t$

- Choose

$$a_t = \frac{\gamma_t}{R_t \sigma_t S_t}$$

  and $X_0 = V_0$.

-

$$\implies R_t X_t = R_t V_t \quad \forall t$$

$$\implies X_t = V_t.$$

- So we have a replicating portfolio for the option V_T.

- Furthermore, option price

$$X_t = V_t = E_{\tilde{P}}\left[e^{-\int_t^T r_s ds} V_T | \mathcal{F}_t\right].$$

Black-Scholes Price Derivation

When $\mu_t = \mu$, $\sigma_t = \sigma$, $r_t = r$

$$dS_t = \mu S_t dt + \sigma S_t dB_t$$

$$S_T = S_t \exp\left[\left(\mu - \frac{\sigma^2}{2}\right)(T - t) + \sigma(B_T - B_t)\right]$$

Under $\tilde{P}$:

$$dS_t = r S_t dt + \sigma S_t d\tilde{B}_t$$

$$S_T = S_t \exp\left[\left(r - \frac{\sigma^2}{2}\right)(T - t) + \sigma(\tilde{B}_T - \tilde{B}_t)\right]$$

- $V_T = \max(S_T - K, 0)$

- Call price $C(t, S_t) = V_t$

$$= E_{\tilde{P}}\left[e^{-r(T-t)}\left[S_t e^{\left(r - \frac{\sigma^2}{2}\right)(T-t) + \sigma\sqrt{T-t}N(0,1)} - K \right]^{+} \right]$$

- This gives the **Black-Scholes price.**

Multi-variate Girsanov's Theorem

- Let $B_t = (B_1(t), \ldots, B_d(t))$ be d independent Brownian motions.

$$Z_t = \exp\left[-\int_0^t \alpha_t^T \, dB_t - \frac{1}{2}\int_0^t ||\alpha_t||^2 dt\right]$$

where $||\alpha_t||^2 = \sum_{i=1}^d \alpha_t^2(i)$

- $\alpha_t = (\alpha_t(1), \ldots, \alpha_t(d))$ is an adapted process

- $(Z_t : t \geq 0)$ is a martingale (check!). If $\tilde{P}(A) = E_P[Z_T I(A)]$, then

$$\tilde{B}_t = B_t + \int_0^t \alpha_s \, ds$$

is a d-dimensional Brownian motion under $\tilde{P}$. (i.e., all components of $\tilde{B}_t$ are independent under $\tilde{P}$).

Multi-dimensional Market Model

- Assume there are m assets in the market. Thus for $i = 1, \ldots, m$

$$dS_i(t) = a_i(t)S_i(t)dt + S_i(t)\sum_{j=1}^{d}\sigma_{ij}(t)dB_j(t)$$

- where $(B_1(t), \ldots, B_d(t))$ are d independent Brownian motions and $\sigma_{ij}(t)$ are adapted processes.

- Let $\sigma_i(t) = \sqrt{\sum_{j=1}^{d}\sigma_{ij}^2(t)}$ Define

$$dW_i(t) = \sum_{j=1}^{d}\frac{\sigma_{ij}(t)}{\sigma_i(t)}dB_j(t)$$

- Claim: Each $(W_i(t) : t \geq 0)$ is a Brownian motion.

- Since $\{B_j(t)\}$ are martingales $\forall j, \implies (W_i(t) : i = 1, \ldots, d)$ are also martingales.

- From integral representation, easy to see that each $W_i(t)$ is continuous in t.

-
$$dW_i(t) \cdot dW_i(t) = \sum_{j=1}^{d} \frac{\sigma_{ij}^2(t)}{\sigma_i^2(t)} dt = \frac{\sum_{j=1}^{d} \sigma_{ij}^2(t)}{\sigma_i^2(t)} dt = dt$$

- Thus, by Lévy's Theorem, each $(W_i(t) : 0 \leq t \leq T)$ is Brownian motion.

- So we have

$$dS_i(t) = a_i(t)S_i(t)dt + \sigma_i(t)S_i(t)dW_i(t)$$

- The processes $\{W_i(t)\}$ & $\{W_k(t)\}$ will now be correlated

$$E(W_i(t)W_k(t)) = E\left[\int_0^t \sum_{j=1}^d \frac{\sigma_{ij}(s)}{\sigma_i(s)} \frac{\sigma_{kj}(s)}{\sigma_k(s)} ds\right] = \rho_{ik}(t)$$

- To see this, use

$$d(W_i(t)W_k(t)) = W_i(t)dW_k(t) + W_k(t)dW_i(t) + dW_i(t) \cdot dW_k(t)$$

Multi-dimensional Market Model

- Let $dR_t = -r_t R_t dt$, so that $R_t = e^{-\int_0^t r_s ds}$

- Again,

$$d(R_t S_i(t)) = R_t dS_i(t) + S_i(t) dR(t) + dS_i(t) dR(t)$$

  and

$$dS_i(t) = a_i(t) S_i(t) dt + \sigma_i(t) S_i(t) dW_i(t)$$

- Thus,

$$d(R_t S_i(t)) = (a_i(t) - r_t) R_t S_i(t) dt + R_t S_i(t) \sum_{j=1}^{d} \sigma_{ij}(t) dB_j(t)$$

- Girsanov allows, $d\tilde{B}_j(t) = dB_j(t) + \beta_j(t)dt$.

- So existence of Equivalent Martingale Measure (EMM) boils down to finding $\{\beta_j(t)\}_{j=1,\ldots,d}$ so that $(R_t S_i(t))_{t\geq 0}$ is a martingale $\forall i$.

- We need
$$
\begin{pmatrix} a_1(t) - r_t \\ \vdots \\ a_m(t) - r_t \end{pmatrix} = \begin{pmatrix} \sum_{j=1}^d \sigma_{1j}(t)\beta_j(t) \\ \vdots \\ \sum_{j=1}^d \sigma_{mj}(t)\beta_j(t) \end{pmatrix}
$$
or $(\vec{a}_t - \vec{r}_t) = \tilde{\sigma}_t \vec{\beta}_t$

- Typically, $m = d$ and $\tilde{\sigma}_t$ is modellled to be invertible for each t and ω.

$$
\implies \vec{\beta}_t = \tilde{\sigma}_t^{-1}(\vec{a}_t - \vec{r}_t)
$$

- Then $d(R_t S_i(t)) = R_t S_i(t) \sum_{j=1}^d \sigma_{ij}(t) d\tilde{B}_j(t)$. So risk-neutral measure exists.

- On the other hand, if $(\vec{a}_t - \vec{r}_t) = \tilde{\sigma}_t \vec{\beta}_t$ does not have a solution $\implies$ **arbitrage**.

- Let $\tilde{S}_t$ be a diagonal matrix whose i th diagonal entry is $S_i(t)$.

- Check that $(\vec{a}_t - \vec{r}_t) = \tilde{\sigma}_t \vec{\beta}_t$ not having a solution corresponds to

$$\tilde{S}_t(\vec{a}_t - \vec{r}_t) = \tilde{S}_t \tilde{\sigma}_t \vec{\beta}_t$$

not having a solution.

- Since, then $\exists\, \vec{\theta}_t$ such that

$$\vec{\theta}_t^T \tilde{S}_t \tilde{\sigma}_t = 0$$

but

$$\vec{\theta}_t^T \tilde{S}_t (\vec{a}_t - \vec{r}_t) > 0.$$

(Depending on (ω, t), the above maybe true as $>$ or $<$ inequality. If it is always $<$, then set $\vec{\theta}_t = -\vec{\theta}_t$ to make it positive. Else, whenever $<$ holds we can keep $\vec{\theta}_t = 0$, so that $\vec{\theta}_t$ is non-negative and positive with positive measure).

- Recall that

$$d(R_t S_i(t)) = (a_i(t) - r_t)R_t S_i(t)dt + R_t S_i(t) \sum_{j=1}^{d} \sigma_{ij}(t)dB_j(t)$$

- Consider security $m+1$ as the risk free security, for a self financing portfolio (adjust investments to risk free security to make the portfolio self financing),

$$d(\sum_{i=1}^{m+1} \theta_t(i)R_t S_i(t)) = \sum_{i=1}^{m} \theta_t(i)d(R_t S_i(t)).$$

(since $dR_t S_{m+1}(t) = 0$)

- This equals,

$$= \sum_{i=1}^{m} \theta_t(i)(a_i(t) - r_t)R_t S_i(t)dt + \sum_{i=1}^{m} \theta_t(i)R_t S_i(t) \sum_{j=1}^{d} \sigma_{ij}(t)dB_j(t)$$

- This in turn equals $\sum_{i=1}^{m} \theta_t(i)(a_i(t) - r_t)R_t S_i(t)dt$ and leads to arbitrage, since, drift for the discounted portfolio

$$\vec{\theta}_t^T \tilde{S}_t(\vec{a}_t - \vec{r}_t) > 0.$$

Discounted Portfolio Process

- So the undiscounted portfolio earns more than the risk free rate at zero risk, leading to an **'arbitrage'**

- So risk neutral measure exists if

$$(\vec{a}_t - \vec{r}_t) = \sigma_t \vec{\beta}_t$$

has a solution for each t a.s. Then

$$d(R_t S_i(t)) = R_t S_i(t) \sum_{j=1}^{d} \sigma_{ij}(t) d\tilde{B}_j(t)$$

where $d\tilde{B}_j(t) = dB_j(t) + \vec{\beta}_j(t)dt$ is Brownian motion under $\tilde{P}(A) = E_P[Z_T I(A)]$

$$Z_T = \exp\left[-\int_0^T \vec{\beta}_t^T dB_t - \frac{1}{2} \int_0^T ||\vec{\beta}_t||^2 dt \right].$$

- Consider portfolio process

$$dX_t = \sum_{i=1}^{m} \alpha_i(t)dS_i(t) + r_t \left(X_t - \sum_{i=1}^{m} \alpha_i(t)S_i(t) \right) dt$$

$$\implies d(R_t X_t) = \sum_{i=1}^{m} \alpha_i(t)d(R_t S_i(t))$$

$$\implies d(R_t X_t) = \sum_{i=1}^{m} \sum_{j=1}^{d} \alpha_i(t)R_t S_i(t)\sigma_{ij}(t)d\tilde{B}_j(t)$$

- Hence under $\tilde{P}$, **discounted self financing portfolio process is a Martingale.**

- As before, we can see that under risk neutral measure there is no arbitrage:

- **Arbitrage** is a portfolio process $X(t)$ satisfying $X(0) = 0$ & for some T,
$$P(X(T) \geq 0) = 1, \quad P(X(T) > 0) > 0.$$

$$\implies P(R(T)X(T) \geq 0) = 1 \text{ and } P(R(T)X(T) > 0) > 0$$

$$\implies \tilde{E}[R(0)X(0)] = 0 < \tilde{E}[R(T)X(T)]$$

- Not possible if under $\tilde{P}$, $R(t)X(t)$ is a martingale.

Attainability and Market Completeness

- **When can an option be attained?** Suppose we want to price $V_T \in \mathcal{F}_T$ & risk neutral measure exists. Set

$$V_t = \tilde{E}\left[e^{-\int_t^T r_u du} V_T | \mathcal{F}_t\right]$$

$$\implies R_t V_t = \tilde{E}[R_T V_T | \mathcal{F}_t]$$

- So $\{R_t V_t\}$ is a martingale.

- Multi-dimensional martingale representation theorem is defined analogously to the single dimension one. By it, there exists an adapted $\{\gamma_j(t) : 1 \leq j \leq d\}$, such that

$$R_t V_t = V_0 + \sum_{j=1}^{d} \int_0^t \gamma_j(s) d\tilde{B}_j(s)$$

- It's a stochastic integral. Equivalently,

$$d(R_t V_t) = \sum_{j=1}^{d} \gamma_j(t) d\tilde{B}_j(t)$$

- Recall portfolio process

$$d(R_t X_t) = \sum_{j=1}^{d} \left(\sum_{i=1}^{m} \alpha_i(t) R_t S_i(t) \sigma_{ij}(t) \right) d\tilde{B}_j(t)$$

- To create a replicating portfolio, set $X_0 = V_0$ and

$$\gamma_j(t) = \sum_{i=1}^{m} \alpha_i(t) R_t S_i(t) \sigma_{ij}(t)$$

- **When does solution exist?** Again, suppose $m = d$.

$$R_t^{-1}\vec{\gamma}(t) = \sigma(t)^T(\vec{\alpha}_t \vec{S}_t)$$

- where

$$\vec{\alpha}_t \vec{S}_t = \begin{pmatrix} \alpha_1(t)S_1(t) \\ \vdots \\ \alpha_d(t)S_d(t) \end{pmatrix}$$

- So solution $\vec{\alpha}_t$ exists for all options if and only if $\sigma(t)^T$ is invertible.

$$\implies \vec{\alpha}_t \vec{S}_t = (\sigma(t)^T)^{-1} R_t^{-1} \vec{\gamma}_t$$

Then the market is **complete** and every option V_T can be attained & priced.

- Recall that EMM exists if and only if

$$(\vec{a}_t - \vec{\nu}_t) = \sigma_t \vec{\beta}_t$$

  has a solution.

- Now market is complete iff σ_t is invertible $\implies$ Risk neutral measure corresponding to

$$\vec{\beta}_t = \sigma_t^{-1}(\vec{a}_t - \vec{\nu}_t)$$

  is unique.

- **Thm: Consider a mkt. model that has a risk neutral prob. measure. The model is complete if and only if the risk neutral measure is unique.**

- In our $m = d$ setting, Iff σ_t is invertible for each t.