In the name of GOD.

Stochastic Processes

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Quiz 7 (25 minutes) — Estimation Theory 2 — Date: Dec. 23th

1. **Question 1.**

   Let $Y \in \{0,1\}$ satisfy

   $$\mathbb{P}(Y=0)=\alpha, \qquad \mathbb{P}(Y=1)=1-\alpha, \quad \alpha \in (0,1) \text{ known.}$$

   Assume μ is known and

   $$X \mid Y=0 \sim \mathcal{N}(0,\sigma^2), \qquad X \mid Y=1 \sim \mathcal{N}(\mu,\sigma^2).$$

   - Suppose (X_i, Y_i), $i=1,\ldots,n$, are i.i.d. Write the likelihood and derive the MLE of σ^2.

2. **Question 2.**

   Let $X_1,\ldots,X_n$ be i.i.d. Gamma(α,β) with shape $\alpha>0$ and scale $\beta>0$, having probability density function

   $$f(x \mid \alpha,\beta)=\frac{1}{\Gamma(\alpha)\beta^\alpha}\, x^{\alpha-1}e^{-x/\beta}, \qquad x>0.$$

   You are given

   $$\mathbb{E}[X]=\alpha\beta, \qquad \mathrm{Var}(X)=\alpha\beta^2.$$

   Define the sample raw moments

   $$m_1=\frac{1}{n}\sum_{i=1}^n X_i, \qquad m_2=\frac{1}{n}\sum_{i=1}^n X_i^2.$$

   Using the method of moments based on m_1, m_2, derive estimators $\hat{\alpha}$ and $\hat{\beta}$ in terms of m_1 and m_2.

3. **Question 3.**

   Suppose $X_1,\ldots,X_n$ are i.i.d. observations from a parametric family, and let $A(X)$ and $B(X)$ be two estimators of a parameter θ. Assume that

   $$\mathrm{Var}_\theta(A) < \mathrm{Var}_\theta(B) \quad \text{for all } \theta.$$

   Does this imply that $A(X)$ is preferred in the sense of UMVUE or that it attains the Cramér–Rao lower bound? Explain.