In the name of GOD.

Stochastic Process

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Homework 5 — Hypothesis Testing, Markov Chains, Hidden Markov Model — Deadline : 1403/10/01

1. [40] In automated food packaging machines, there is always a degree of error. We intend to purchase and evaluate the performance of several packaging machines for a food company.

   (a) There are two machines available for purchase at the same price. Based on their reported performance specifications, determine which machine has better accuracy. Provide a detailed explanation for your choice.

   - Machine Type A : Has 98% accuracy within a range of ±2%.
     (This means that if the packaging weight is set to 100 grams, 98% of the packages fall within the range of 98 to 102 grams.)
   - Machine Type B : Has 90% accuracy within a range of ±1%.
     (This means that if the packaging weight is set to 100 grams, 90% of the packages fall within the range of 99 to 101 grams.)

   (b) After purchasing the chosen machine, we set it up and configured the target packaging weight to 100 grams. We measured the weights of 10 samples packaged by the machine, which are as follows:

   $$106.69, 107.1, 106.74, 106.23, 106.99, 107.72, 106.29, 108.51, 106.59, 106.76$$

   It seems the machine is not packaging correctly. Calculate the P-Value to determine if the machine is functioning correctly. ($\alpha = 0.05$)

   (c) After reviewing the machine's user manual, we discovered that the calibration of the machine degrades over time or due to transportation, introducing bias into its operation. Therefore, it needs to be recalibrated after transportation or at regular monthly intervals. The calibration process is iterative, reducing some bias with each step. However, due to the time-consuming nature of calibration, we decided to repeat it only a few steps. After each calibration step, 10 samples were taken to check the calibration accuracy of the machine. The samples taken at the end of each step are shown in Table 1. Based on this data, determine how many calibration steps are required. ($\alpha = 0.05$)

   (d) After years of use, these machines become worn, leading to operational drift. However, due to financial constraints, the company does not plan to replace them. As a result, we must continue the calibration process as before. Based on Table 2, determine how many calibration steps are required. ($\alpha = 0.05$) Have we achieved reliable calibration in the end?

2. [10] In a complete graph with n vertices ($n > 1$), consider two distinct vertices u and v.

   (a) In a simple random walk, what is the expected number of steps to reach vertex v starting from vertex u?

Step 1	104.77	105.08	105.35	104.89	104.71	104.47	105.56	106.35	105.65	104.57
Step 2	103.98	102.67	103.63	102.91	103.01	103.95	104.36	103.56	103.53	103.72
Step 3	101.45	101.88	103.18	102.49	103.41	102.19	102.35	101.83	102.79	101.43
Step 4	102.57	101.56	101.66	102.07	102.19	101.86	100.81	101.71	100.74	101.87
Step 5	101.02	101.52	102.33	100.74	100.17	100.65	100.78	100.14	101.08	101.13
Step 6	101.01	100.04	101.58	100.75	100.44	101.04	100.31	100.66	100.39	100.49
Step 7	100.24	100.43	100.26	100.32	100.29	100.35	101.55	100.33	100.56	99.59
Step 8	100.19	101.07	101.18	100.32	100.19	101.11	99.16	101.03	99.53	99.73

Table 1: Sample data after each calibration step during setup

Step 1	91.29	92.64	97.74	95.75	95.71	92.11	98.63	92.13	91.68	91.73
Step 2	94.32	103.45	99.69	97.15	103.12	93.33	108.28	95.24	100.38	92.41
Step 3	99.68	96.11	98.93	103.91	103.2	101.01	95.28	103.29	98.58	94.98
Step 4	95.38	98.92	98.91	103.22	102.2	100.77	96.22	99.41	98.58	97.33
Step 5	107.45	102.52	89.22	98.19	96.73	90.65	101.74	104.21	106.00	98.61
Step 6	99.31	100.88	105.07	102.94	100.91	98.94	98.81	103.21	108.06	98.78

Table 2: Sample data from the calibration process of a worn machine

(b) If a new vertex v' is added to the graph and connected to v with a single edge, what is the expected number of steps to reach vertex v' starting from vertex u?

3. [10] A fair coin is flipped repeatedly until the sequence $\{Heads, Heads, Tails\}$ is observed. At this point, the experiment ends.

   (a) Model the problem using a Markov chain.

   (b) Calculate the expected number of flips until the experiment ends.

4. [20] A tourist alternates between traveling for k_n days and resting for k_m days. If $k_n \sim$ uniform$(1, n)$ and $k_m \sim$ uniform$(1, m)$:

   (a) Model the state of this tourist as a Markov chain.

   (b) Using the model obtained in the previous part, determine the long-term probability that the tourist is traveling on the i-th day of their trip, given that $n_0 \leq i$ $(n_0 \leq n)$.

5. [20] An HMM (Hidden Markov Model) with two Hidden States, H_0 and H_1, and two Visible States, V_0 and V_1, is given. The transition probability matrix is denoted by A, the observation probability matrix by B, and the initial probability vector by π.

$$A = \begin{bmatrix} p & 1-p \\ 1-p & p \end{bmatrix}, B = \begin{bmatrix} 0.8 & 0.2 \\ 0.2 & 0.8 \end{bmatrix}, \pi = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

If the following sequence is generated by this model, update the value of p by performing one step of the EM algorithm. (Assume the initial value of p is $p = \frac{1}{2}$.)

$$0, 1, 0, 1, 1, 0$$

(Hint: Compute the values of α, β, ξ, and γ.)

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
−3.6	.0002	.0002	.0001	.0001	.0001	.0001	.0001	.0001	.0001	.0001
−3.5	.0002	.0002	.0002	.0002	.0002	.0002	.0002	.0002	.0002	.0002
−3.4	.0003	.0003	.0003	.0003	.0003	.0003	.0003	.0003	.0003	.0002
−3.3	.0005	.0005	.0005	.0004	.0004	.0004	.0004	.0004	.0004	.0003
−3.2	.0007	.0007	.0006	.0006	.0006	.0006	.0006	.0005	.0005	.0005
−3.1	.0010	.0009	.0009	.0009	.0008	.0008	.0008	.0008	.0007	.0007
−3.0	.0013	.0013	.0013	.0012	.0012	.0011	.0011	.0011	.0010	.0010
−2.9	.0019	.0018	.0018	.0017	.0016	.0016	.0015	.0015	.0014	.0014
−2.8	.0026	.0025	.0024	.0023	.0023	.0022	.0021	.0021	.0020	.0019
−2.7	.0035	.0034	.0033	.0032	.0031	.0030	.0029	.0028	.0027	.0026
−2.6	.0047	.0045	.0044	.0043	.0041	.0040	.0039	.0038	.0037	.0036
−2.5	.0062	.0060	.0059	.0057	.0055	.0054	.0052	.0051	.0049	.0048
−2.4	.0082	.0080	.0078	.0075	.0073	.0071	.0069	.0068	.0066	.0064
−2.3	.0107	.0104	.0102	.0099	.0096	.0094	.0091	.0089	.0087	.0084
−2.2	.0139	.0136	.0132	.0129	.0125	.0122	.0119	.0116	.0113	.0110
−2.1	.0179	.0174	.0170	.0166	.0162	.0158	.0154	.0150	.0146	.0143
−2.0	.0228	.0222	.0217	.0212	.0207	.0202	.0197	.0192	.0188	.0183
−1.9	.0287	.0281	.0274	.0268	.0262	.0256	.0250	.0244	.0239	.0233
−1.8	.0359	.0351	.0344	.0336	.0329	.0322	.0314	.0307	.0301	.0294
−1.7	.0446	.0436	.0427	.0418	.0409	.0401	.0392	.0384	.0375	.0367
−1.6	.0548	.0537	.0526	.0516	.0505	.0495	.0485	.0475	.0465	.0455
−1.5	.0668	.0655	.0643	.0630	.0618	.0606	.0594	.0582	.0571	.0559
−1.4	.0808	.0793	.0778	.0764	.0749	.0735	.0721	.0708	.0694	.0681
−1.3	.0968	.0951	.0934	.0918	.0901	.0885	.0869	.0853	.0838	.0823
−1.2	.1151	.1131	.1112	.1093	.1075	.1056	.1038	.1020	.1003	.0985
−1.1	.1357	.1335	.1314	.1292	.1271	.1251	.1230	.1210	.1190	.1170
−1.0	.1587	.1562	.1539	.1515	.1492	.1469	.1446	.1423	.1401	.1379
−0.9	.1841	.1814	.1788	.1762	.1736	.1711	.1685	.1660	.1635	.1611
−0.8	.2119	.2090	.2061	.2033	.2005	.1977	.1949	.1922	.1894	.1867
−0.7	.2420	.2389	.2358	.2327	.2296	.2266	.2236	.2206	.2177	.2148
−0.6	.2743	.2709	.2676	.2643	.2611	.2578	.2546	.2514	.2483	.2451
−0.5	.3085	.3050	.3015	.2981	.2946	.2912	.2877	.2843	.2810	.2776
−0.4	.3446	.3409	.3372	.3336	.3300	.3264	.3228	.3192	.3156	.3121
−0.3	.3821	.3783	.3745	.3707	.3669	.3632	.3594	.3557	.3520	.3483
−0.2	.4207	.4168	.4129	.4090	.4052	.4013	.3974	.3936	.3897	.3859
−0.1	.4602	.4562	.4522	.4483	.4443	.4404	.4364	.4325	.4286	.4247
−0.0	.5000	.4960	.4920	.4880	.4840	.4801	.4761	.4721	.4681	.4641

Figure 1: z-table

df/p	0.40	0.25	0.10	0.05	0.025	0.01	0.005	0.0005
1	0.324920	1.000000	3.077684	6.313752	12.70620	31.82052	63.65674	636.6192
2	0.288675	0.816497	1.885618	2.919986	4.30265	6.96456	9.92484	31.5991
3	0.276671	0.764892	1.637744	2.353363	3.18245	4.54070	5.84091	12.9240
4	0.270722	0.740697	1.533206	2.131847	2.77645	3.74695	4.60409	8.6103
5	0.267181	0.726687	1.475884	2.015048	2.57058	3.36493	4.03214	6.8688
6	0.264835	0.717558	1.439756	1.943180	2.44691	3.14267	3.70743	5.9588
7	0.263167	0.711142	1.414924	1.894579	2.36462	2.99795	3.49948	5.4079
8	0.261921	0.706387	1.396815	1.859548	2.30600	2.89646	3.35539	5.0413
9	0.260955	0.702722	1.383029	1.833113	2.26216	2.82144	3.24984	4.7809
10	0.260185	0.699812	1.372184	1.812461	2.22814	2.76377	3.16927	4.5869
11	0.259556	0.697445	1.363430	1.795885	2.20099	2.71808	3.10581	4.4370
12	0.259033	0.695483	1.356217	1.782288	2.17881	2.68100	3.05454	43178
13	0.258591	0.693829	1.350171	1.770933	2.16037	2.65031	3.01228	4.2208
14	0.258213	0.692417	1.345030	1.761310	2.14479	2.62449	2.97684	4.1405
15	0.257885	0.691197	1.340606	1.753050	2.13145	2.60248	2.94671	4.0728
16	0.257599	0.690132	1.336757	1.745884	2.11991	2.58349	2.92078	4.0150
17	0.257347	0.689195	1.333379	1.739607	2.10982	2.56693	2.89823	3.9651
18	0.257123	0.688364	1.330391	1.734064	2.10092	2.55238	2.87844	3.9216
19	0.256923	0.687621	1.327728	1.729133	2.09302	2.53948	2.86093	3.8834
20	0.256743	0.686954	1.325341	1.724718	2.08596	2.52798	2.84534	3.8495
21	0.256580	0.686352	1.323188	1.720743	2.07961	2.51765	2.83136	3.8193
22	0.256432	0.685805	1.321237	1.717144	2.07387	2.50832	2.81876	3.7921
23	0.256297	0.685306	1.319460	1.713872	2.06866	2.49987	2.80734	3.7676
24	0.256173	0.684850	1.317836	1.710882	2.06390	2.49216	2.79694	3.7454
25	0.256060	0.684430	1.316345	1.708141	2.05954	2.48511	2.78744	3.7251
26	0.255955	0.684043	1.314972	1.705618	2.05553	2.47863	2.77871	3.7066
27	0.255858	0.683685	1.313703	1.703288	2.05183	2.47266	2.77068	3.6896
28	0.255768	0.683353	1.312527	1.701131	2.04841	2.46714	2.76326	3.6739
29	0.255684	0.683044	1.311434	1.699127	2.04523	2.46202	2.75639	3.6594
30	0.255605	0.682756	1.310415	1.697261	2.04227	2.45726	2.75000	3.6460

Figure 2: t-table