



In the name of GOD.

Sharif University of Technology

## Stochastic Process

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Homework 2

point process, poisson process, power spectrum

Deadline : 1404/08/25

- Find the PSD for  $X(t)$  if:

$$R(\tau) = \begin{cases} 1 - |\tau|, & |\tau| \leq 1 \\ 0, & \text{elsewhere} \end{cases}$$

**Solution:**

The PSD of the process is given by  $S(\omega) = \int_{-\infty}^{\infty} R(\tau)e^{-i\omega\tau}d\tau$

$$\begin{aligned} &= \int_{-1}^1 (1 - |\tau|)e^{-i\omega\tau}d\tau \\ &= \int_{-1}^1 (1 - |\tau|)(\cos \omega\tau - i \sin \omega\tau)d\tau \\ &= \int_{-1}^1 (1 - |\tau|) \cos \omega\tau d\tau - i \int_{-1}^1 (1 - |\tau|) \sin \omega\tau d\tau \\ &= \int_{-1}^1 (1 - |\tau|) \cos \omega\tau d\tau - i(0) \\ &= 2 \int_0^1 (1 - |\tau|) \cos \omega\tau d\tau \\ &= 2 \left[ (1 - \tau) \left( \frac{\sin \omega\tau}{\omega} \right) - (-1) \left( -\frac{\cos \omega\tau}{\omega^2} \right) \right]_0^1 \\ &= 2 \left[ \left( 0 - \frac{\cos \omega}{\omega^2} \right) - \left( 0 - \frac{1}{\omega^2} \right) \right] \\ &= 2 \left[ \frac{1 - \cos \omega}{\omega^2} \right] \end{aligned}$$

- Let  $X(t)$  be a white Gaussian noise with  $S_X(f) = \frac{N_0}{2}$ . Assume that  $X(t)$  is input to an LTI system with

$$h(t) = e^{-t}u(t).$$

Let  $Y(t)$  be the output.

- Find  $S_Y(f)$ .
- Find  $R_Y(\tau)$ .
- Find  $E[Y(t)^2]$ .

**Solution:**

First, note that

$$\begin{aligned} H(f) &= \mathcal{F}\{h(t)\} \\ &= \frac{1}{1 + j2\pi f}. \end{aligned}$$

a. To find  $S_Y(f)$ , we can write

$$\begin{aligned} S_Y(f) &= S_X(f)|H(f)|^2 \\ &= \frac{N_0/2}{1 + (2\pi f)^2}. \end{aligned}$$

b. To find  $R_Y(\tau)$ , we can write

$$\begin{aligned} R_Y(\tau) &= \mathcal{F}^{-1}\{S_Y(f)\} \\ &= \frac{N_0}{4}e^{-|\tau|}. \end{aligned}$$

c. We have

$$\begin{aligned} E[Y(t)^2] &= R_Y(0) \\ &= \frac{N_0}{4}. \end{aligned}$$

3. Let  $\{N(t), t \in [0, \infty)\}$  be a Poisson process with rate  $\lambda$  and  $X_1$  be its first arrival time. Show that given  $N(t) = 1$ , then  $X_1$  is uniformly distributed in  $(0, t]$ . That is show that

$$P(X_1 \leq x | N(t) = 1) = \frac{x}{t}, \quad \text{for } 0 \leq x \leq t.$$

**Solution:**

For  $0 \leq x \leq t$ , we can write

$$P(X_1 \leq x | N(t) = 1) = \frac{P(X_1 \leq x, N(t) = 1)}{P(N(t) = 1)}.$$

We know that

$$P(N(t) = 1) = \lambda t e^{-\lambda t},$$

and

$$\begin{aligned} P(X_1 \leq x, N(t) = 1) &= P(\text{one arrival in } (0, x] \text{ and no arrivals in } (x, t]) \\ &= [\lambda x e^{-\lambda x}] \cdot [e^{-\lambda(t-x)}] \\ &= \lambda x e^{-\lambda t}. \end{aligned}$$

Thus,

$$P(X_1 \leq x | N(t) = 1) = \frac{x}{t}, \quad \text{for } 0 \leq x \leq t.$$

4. Let  $\{N(t), t \in [0, \infty)\}$  be a Poisson process with rate  $\lambda$ . Find its covariance function

$$C_N(t_1, t_2) = \text{Cov}(N(t_1), N(t_2)), \quad \text{for } t_1, t_2 \in [0, \infty)$$

**Solution:**

Let's assume  $t_1 \geq t_2 \geq 0$ . Then, by the **independent increment property** of the Poisson process, the two random variables  $N(t_1) - N(t_2)$  and  $N(t_2)$  are independent. We can write

$$\begin{aligned} C_N(t_1, t_2) &= \text{Cov}(N(t_1), N(t_2)) \\ &= \text{Cov}(N(t_1) - N(t_2) + N(t_2), N(t_2)) \\ &= \text{Cov}(N(t_1) - N(t_2), N(t_2)) + \text{Cov}(N(t_2), N(t_2)) \\ &= \text{Cov}(N(t_2), N(t_2)) \\ &= \text{Var}(N(t_2)) \\ &= \lambda t_2, \quad \text{since } N(t_2) \sim \text{Poisson}(\lambda t_2). \end{aligned}$$

Similarly, if  $t_2 \geq t_1 \geq 0$ , we conclude

$$C_N(t_1, t_2) = \lambda t_1.$$

Therefore, we can write

$$C_N(t_1, t_2) = \lambda \min(t_1, t_2), \quad \text{for } t_1, t_2 \in [0, \infty).$$

5. Arrivals of customers into a store follow a **Poisson process** with rate  $\lambda = 20$  arrivals per hour. Suppose that the probability that a customer buys something is  $p = 0.30$ .

- (a) Find the expected number of sales made during an eight-hour business day.
- (b) Find the probability that 10 or more sales are made in one hour.
- (c) Find the expected time of the first sale of the day. If the store opens at 8 a.m.

**Solution:**

Let  $N_1(t)$  be the number of arrivals who buy something. Let  $N_2(t)$  be the number of arrivals who do not buy something.  $N_1$  and  $N_2$  are two independent **Poisson processes** with rate for  $N_1$  is  $\lambda_1 = \lambda p = (20)(0.3) = 6$ . The rate for  $N_2$  is  $\lambda_2 = \lambda(1 - p) = (20)(0.7) = 14$ .

- (a)  $E[X_1](80) = (8)(6) = 48$ .
- (b)  $P(N_1 \geq 10) = 1 - \sum_{j=0}^9 P(N_1 = j) = 1 - \sum_{j=0}^9 \frac{e^{-6} 6^j}{j!}$ .
- (c)  $E[T_{1,1}] = \frac{1}{\lambda_1} = \frac{1}{6}$  hours or 10 minutes. The expected time of the first sale is 8 : 10.

6. Ali finds coins on his way to work at a Poisson rate of 0.5 coins/minute. The denominations are randomly distributed:

- 60% of the coins are worth 1 each
- 20% of the coins are worth 5 each
- 20% of the coins are worth 10 each.

- (a) Calculate the probability that in the first ten minutes of his walk he finds at least 2 coins worth 10 each, and in the first twenty minutes finds at least 3 coins worth 10 each.
- (b) Calculate the variance of the value of the coins Ali finds during his one-hour walk to work.

**Solution:**

- (a) **Solution:** Let  $X$  be the number of coins worth 10 each Ali finds in the first 10 minutes.  $X$  has a Poisson distribution with mean  $(0.5)(10)(0.2) = 1$ . Let  $Y$  be the number of coins worth 10 each Ali finds between time= 10 minutes and time= 20 minutes.  $Y$  has a Poisson distribution with mean  $(0.5)(10)(0.2) = 1$ . We need to find

$$\begin{aligned} P(X \geq 2, X + Y \geq 3) &= P(X = 2, Y \geq 1) + P(X \geq 3) \\ &= \frac{e^{-1}}{2}(1 - e^{-1}) + 1 - e^{-1} - e^{-1} - \frac{e^{-1}}{2} = 0.1965. \end{aligned}$$

- (b) **Solution:** Let  $N_1(t)$  be the number of coins of value 1 which Ali finds until time  $t$  minutes. Let  $N_2(t)$  be the number of coins of value 5 which Ali finds until time  $t$  minutes. Let  $N_3(t)$  be the number of coins of value 10 which Ali finds until time  $t$  minutes.  $N_1, N_2$  are two independent Poisson processes with respective rates

$$\lambda_1 = (0.5)(0.6) = 0.30, \quad \lambda_2 = (0.5)(0.2) = 0.10, \quad \lambda_3 = (0.5)(0.2) = 0.10.$$

The total value of the coins Lucky Ali finds is  $Y = N_1(60) + 5N_2(60) + 10N_3(60)$ . The variance of  $Y$  is

$$\begin{aligned} \text{Var}(Y) &= \text{Var}(N_1(60)) + 5^2\text{Var}(N_2(60)) + 10^2\text{Var}(N_3(60)) \\ &= (60)(0.3) + 5^2(60)(.1) + 10^2(60)(.1) = 768 \end{aligned}$$

7. Two independent Poisson processes have respective rates 1 and 2. Two players start with fortunes  $a$  and  $b$ . The game evolves as follows:

- Each time an event occurs in the first Poisson process, player 2 pays one unit to player 1.
- Each time an event occurs in the second Poisson process, player 1 pays one unit to player 2.

The game ends when one player's fortune reaches zero; that player loses. Find the probability that player 1 wins.

**Solution:**

Since the two processes are independent, we can assume a single Poisson process with rate 3, where each event favors player 1 with probability one-third and favors player 2 with probability two-thirds. Therefore, the problem is equivalent to Example 3-15 in Papoulis's book. (The rest of the solution can be followed from the textbook.)