



1. Find the PSD for $X(t)$ if:

$$R(\tau) = \begin{cases} 1 - |\tau|, & |\tau| \leq 1 \\ 0, & \text{elsewhere} \end{cases}$$

Solution:

The PSD of the process is given by $S(\omega) = \int_{-\infty}^{\infty} R(\tau) e^{-i\omega\tau} d\tau$

$$\begin{aligned} &= \int_{-1}^1 (1 - |\tau|) e^{-i\omega\tau} d\tau \\ &= \int_{-1}^1 (1 - |\tau|) (\cos \omega\tau - i \sin \omega\tau) d\tau \\ &= \int_{-1}^1 (1 - |\tau|) \cos \omega\tau d\tau - i \int_{-1}^1 (1 - |\tau|) \sin \omega\tau d\tau \\ &= \int_{-1}^1 (1 - |\tau|) \cos \omega\tau d\tau - i(0) \\ &= 2 \int_0^1 (1 - \tau) \cos \omega\tau d\tau \\ &= 2 \left[(1 - \tau) \left(\frac{\sin \omega\tau}{\omega} \right) - (-1) \left(-\frac{\cos \omega\tau}{\omega^2} \right) \right]_0^1 \\ &= 2 \left[\left(0 - \frac{\cos \omega}{\omega^2} \right) - \left(0 - \frac{1}{\omega^2} \right) \right] \\ &= 2 \left[\frac{1 - \cos \omega}{\omega^2} \right] \end{aligned}$$

2. Let $X(t)$ be a white Gaussian noise with $S_X(f) = \frac{N_0}{2}$. Assume that $X(t)$ is input to an LTI system with

$$h(t) = e^{-t} u(t).$$

Let $Y(t)$ be the output.

- (a) Find $S_Y(f)$.
- (b) Find $R_Y(\tau)$.
- (c) Find $E[Y(t)^2]$.

Solution:

First, note that

$$\begin{aligned} H(f) &= \mathcal{F}\{h(t)\} \\ &= \frac{1}{1 + j2\pi f}. \end{aligned}$$

a. To find $S_Y(f)$, we can write

$$\begin{aligned} S_Y(f) &= S_X(f)|H(f)|^2 \\ &= \frac{N_0/2}{1 + (2\pi f)^2}. \end{aligned}$$

b. To find $R_Y(\tau)$, we can write

$$\begin{aligned} R_Y(\tau) &= \mathcal{F}^{-1}\{S_Y(f)\} \\ &= \frac{N_0}{4}e^{-|\tau|}. \end{aligned}$$

c. We have

$$\begin{aligned} E[Y(t)^2] &= R_Y(0) \\ &= \frac{N_0}{4}. \end{aligned}$$

3. Let $\{N(t), t \in [0, \infty)\}$ be a Poisson process with rate λ and X_1 be its first arrival time. Show that given $N(t) = 1$, then X_1 is uniformly distributed in $(0, t]$. That is show that

$$P(X_1 \leq x | N(t) = 1) = \frac{x}{t}, \quad \text{for } 0 \leq x \leq t.$$

Solution:

For $0 \leq x \leq t$, we can write

$$P(X_1 \leq x | N(t) = 1) = \frac{P(X_1 \leq x, N(t) = 1)}{P(N(t) = 1)}.$$

We know that

$$P(N(t) = 1) = \lambda t e^{-\lambda t},$$

and

$$\begin{aligned} P(X_1 \leq x, N(t) = 1) &= P(\text{one arrival in } (0, x] \text{ and no arrivals in } (x, t]) \\ &= [\lambda x e^{-\lambda x}] \cdot [e^{-\lambda(t-x)}] \\ &= \lambda x e^{-\lambda t}. \end{aligned}$$

Thus,

$$P(X_1 \leq x | N(t) = 1) = \frac{x}{t}, \quad \text{for } 0 \leq x \leq t.$$

4. Let $\{N(t), t \in [0, \infty)\}$ be a Poisson process with rate λ . Find its covariance function

$$C_N(t_1, t_2) = \text{Cov}(N(t_1), N(t_2)), \quad \text{for } t_1, t_2 \in [0, \infty)$$

Solution:

Let's assume $t_1 \geq t_2 \geq 0$. Then, by the **independent increment property** of the Poisson process, the two random variables $N(t_1) - N(t_2)$ and $N(t_2)$ are independent. We can write

$$\begin{aligned} C_N(t_1, t_2) &= \text{Cov}(N(t_1), N(t_2)) \\ &= \text{Cov}(N(t_1) - N(t_2) + N(t_2), N(t_2)) \\ &= \text{Cov}(N(t_1) - N(t_2), N(t_2)) + \text{Cov}(N(t_2), N(t_2)) \\ &= \text{Cov}(N(t_2), N(t_2)) \\ &= \text{Var}(N(t_2)) \\ &= \lambda t_2, \quad \text{since } N(t_2) \sim \text{Poisson}(\lambda t_2). \end{aligned}$$

Similarly, if $t_2 \geq t_1 \geq 0$, we conclude

$$C_N(t_1, t_2) = \lambda t_1.$$

Therefore, we can write

$$C_N(t_1, t_2) = \lambda \min(t_1, t_2), \quad \text{for } t_1, t_2 \in [0, \infty).$$

5. Arrivals of customers into a store follow a **Poisson process** with rate $\lambda = 20$ arrivals per hour. Suppose that the probability that a customer buys something is $p = 0.30$.
- Find the expected number of sales made during an eight-hour business day.
 - Find the probability that 10 or more sales are made in one hour.
 - Find the expected time of the first sale of the day. If the store opens at 8 a.m.

Solution:

Let $N_1(t)$ be the number of arrivals who buy something. Let $N_2(t)$ be the number of arrivals who do not buy something. N_1 and N_2 are two independent **Poisson processes** with rate for N_1 is $\lambda_1 = \lambda p = (20)(0.3) = 6$. The rate for N_2 is $\lambda_2 = \lambda(1 - p) = (20)(0.7) = 14$.

- $E[X_1](80) = (8)(6) = 48$.
- $P(N_1 \geq 10) = 1 - \sum_{j=0}^9 P(N_1 = j) = 1 - \sum_{j=0}^9 \frac{e^{-6} 6^j}{j!}$.
- $E[T_{1,1}] = \frac{1}{\lambda_1} = \frac{1}{6}$ hours or 10 minutes. The expected time of the first sale is 8 : 10.

6. Ali finds coins on his way to work at a Poisson rate of 0.5 coins/minute. The denominations are randomly distributed:
- 60% of the coins are worth 1 each
 - 20% of the coins are worth 5 each
 - 20% of the coins are worth 10 each.

- (a) Calculate the probability that in the first ten minutes of his walk he finds at least 2 coins worth 10 each, and in the first twenty minutes finds at least 3 coins worth 10 each.
- (b) Calculate the variance of the value of the coins Ali finds during his one-hour walk to work.

Solution:

- (a) **Solution:** Let X be the number of coins worth 10 each Ali finds in the first 10 minutes. X has a Poisson distribution with mean $(0.5)(10)(0.2) = 1$. Let Y be the number of coins worth 10 each Ali finds between time = 10 minutes and time = 20 minutes. Y has a Poisson distribution with mean $(0.5)(10)(0.2) = 1$. We need to find

$$\begin{aligned} P(X \geq 2, X + Y \geq 3) &= P(X = 2, Y \geq 1) + P(X \geq 3) \\ &= \frac{e^{-1}}{2}(1 - e^{-1}) + 1 - e^{-1} - e^{-1} - \frac{e^{-1}}{2} = 0.1965. \end{aligned}$$

- (b) **Solution:** Let $N_1(t)$ be the number of coins of value 1 which Ali finds until time t minutes. Let $N_2(t)$ be the number of coins of value 5 which Ali finds until time t minutes. Let $N_3(t)$ be the number of coins of value 10 which Ali finds until time t minutes. N_1, N_2 are two independent Poisson processes with respective rates

$$\lambda_1 = (0.5)(0.6) = 0.30, \quad \lambda_2 = (0.5)(0.2) = 0.10, \quad \lambda_3 = (0.5)(0.2) = 0.10.$$

The total value of the coins Lucky Ali finds is $Y = N_1(60) + 5N_2(60) + 10N_3(60)$. The variance of Y is

$$\begin{aligned} \text{Var}(Y) &= \text{Var}(N_1(60)) + 5^2 \text{Var}(N_2(60)) + 10^2 \text{Var}(N_3(60)) \\ &= (60)(0.3) + 5^2(60)(.1) + 10^2(60)(.1) = 768 \end{aligned}$$

7. Two independent Poisson processes have respective rates 1 and 2. Two players start with fortunes a and b . The game evolves as follows:

- Each time an event occurs in the first Poisson process, player 2 pays one unit to player 1.
- Each time an event occurs in the second Poisson process, player 1 pays one unit to player 2.

The game ends when one player's fortune reaches zero; that player loses. Find the probability that player 1 wins.

Solution:

Since the two processes are independent, we can assume a single Poisson process with rate 3, where each event favors player 1 with probability one-third and favors player 2 with probability two-thirds. Therefore, the problem is equivalent to Example 3-15 in Papoulis's book. (The rest of the solution can be followed from the textbook.)