Stochastic Processes December 12, 2022
Sharif University of Technology CE 40-695
Hamid R. Rabiee

Time: 20 mins

Name: Std. Number:

Quiz 7 (Estimation)

Questions

1. Data $x[n] = A + w[n]$ for n = 0, 1, ..., N - 1 are observed. The unknown parameter A is assumed to have the prior PDF,

$$p(A) = \begin{cases} \lambda exp(-\lambda A), & A \geq 0 \\ 0 & A < 0 \end{cases}$$

where $\lambda > 0$, and $w[n]$ is white gaussian noise with variance σ^2 and is independent of A.

(a) Find the likelihood of observations (3).

(b) Find the MAP estimator of A (7).

Solution: Since the noise is Gaussian, its PDF is

$$p(w[n]) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{w^2[n]}{2\sigma^2}\right).$$

Consequently, the conditional probability of $x[n]$ given A is also a Gaussian process with mean A, i.e.,

$$p(x[n] \mid A) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x[n]-A)^2}{2\sigma^2}\right)$$

For simplicity, collect the observations into vector $\boldsymbol{x} = [x[0], x[1], \ldots, x[n-1]]^{\mathrm{T}}$. The likelihood of the i.i.d. observations is therefore the conditional PDF

$$p(\boldsymbol{x} \mid A) = \prod_{n=0}^{N-1} p(x[n] \mid A) = \frac{1}{\sqrt{2\pi\sigma^2}^N} \exp\left(-\frac{1}{2\sigma^2} \sum_{n=0}^{N-1} (x[n]-A)^2\right).$$

The MAP estimator of A maximizes the posterior, that is,

$$\hat{A}_{\mathrm{MAP}} = \arg\max_A p(A \mid \boldsymbol{x}) = \arg\max_A p(\boldsymbol{x} \mid A)p(A) = \arg\max_A (\log p(\boldsymbol{x} \mid A) + \log p(A)).$$