MIP → QUBO

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1 MIP

We consider the PhD Thesis of Thomas Ponsignon, who formulated the Master Planning in Semiconductor Manufacturing as a MIP:

Objective (3.1):

$$f(x,u,I,B,s^{sr}) = \sum_{p=1}^{p_{max}} \sum_{t=1}^{t_{max}} (rev_{pt} s_{pt}^{sr} - hc_{pt} I_{pt} - udc_{pt} B_{pt} - \sum_{m=1}^{m_{max}} mc_{pmt} x_{pmt} - \sum_{m=1}^{m_{max}} lc_{pmt} u_{pmt})$$

Constraints:

(3.2)

$$I_{pt} + s_{pt}^{sr} + s_{pt}^{fo} = I_{pt-1} + \sum_{m=1}^{m_{max}} x_{pmt} + \sum_{m=1}^{m_{max}} x_{pmt}^{i}$$

$$\forall p \in [p_{max}], t \in [t_{max}]$$

(3.3)

$$s_{pt}^{fo} + B_{pt} = d_{pt}^{fo} + B_{pt-1}$$

$$\forall p \in [p_{max}], t \in [t_{max}]$$

(3.4)

$$s_{pt}^{sr} \leq d_{pt}^{sr}$$

$$\forall p \in [p_{max}], t \in [t_{max}]$$

(3.5)

$$C_{mbt}^{min} \leq \sum_{p=1}^{p_{max}} \sum_{k=0}^{min(k_{max}, t_{max}-t)} cc_{bk}^{pm} (x_{p,m,t+k} + x_{p,m,t+k}^{i}) \leq C_{mbt}^{max}$$

$$\forall m \in [m_{max}], p \in [p_{max}], t \in [t_{max}]$$

(3.6)

$$x_{pmt} \leq \delta u_{pmt}$$

$$\forall p \in [p_{max}], m \in [m_{max}], t \in [t_{max}]$$

(3.7)

$$u_{pmt} \in \{0, 1\}$$

$$\forall p \in [p_{max}], m \in [m_{max}], t \in [t_{max}]$$

(3.8) All variables are non-negative.

2 QUBO

Let's first tackle the objective function (3.1):

2.1 Objective Function

Each term has as left side a fixed value and as right side a decision variable, e.g. $rev_{pt}s^{sr}_{pt}$. Note that all decision variables are discrete, only the fixed ones could be real-valued. For this reason, we will employ the log-trick for the decision variables, representing any value in $[K]$, for a fixed K, with $log_2(K)$ qubits, e.g. we know from (3.4) that $s^{sr}_{pt} \leq d^{sr}_{pt}$, thus we use in this case $K := d^{sr}_{pt}$, removing also the constraint.

For the other decision variables, we must have a context, such that we can extract the max. possible value.

In the end, we write:

$$k = \left\lfloor log_2(d^{sr}_{pt}) \right\rfloor$$
$$s^{sr}_{pt} = \sum_{j=0}^{k-1} \sigma^{sr}_{pt}(j) 2^j + (d^{sr}_{pt} + 1 - 2^k)\sigma^{sr}_{pt}(k),$$

where $\sigma^{sr}_{pt}(j)$ are the $k+1$ qubits which will represent s^{sr}_{pt}.

2.2 Constraints

(3.2)

$$I_{pt} + s^{sr}_{pt} + s^{fo}_{pt} = I_{pt-1} + \sum_{m=1}^{m_{max}} x_{pmt} + \sum_{m=1}^{m_{max}} x^{i}_{pmt}$$

$$\forall p \in [p_{max}], t \in [t_{max}]$$

is an equality constraint, thus is can be rewrited as it is into a QUBO formulation, whereby behind each variable we have the encoding from the log-trick. Thus, we have:

$$(I_{pt} + s^{sr}_{pt} + s^{fo}_{pt} - I_{pt-1} - \sum_{m=1}^{m_{max}} x_{pmt} - \sum_{m=1}^{m_{max}} x^{i}_{pmt})^2$$

$$\forall p \in [p_{max}], t \in [t_{max}]$$

which remains quadratic.

(3.3)

$$s^{fo}_{pt} + B_{pt} = d^{fo}_{pt} + B_{pt-1}$$

$$\forall p \in [p_{max}], t \in [t_{max}]$$

This constraint can be handled in the same manner as the previous one.

(3.4)

$$s^{sr}_{pt} \leq d^{sr}_{pt}$$

$$\forall p \in [p_{max}], t \in [t_{max}]$$

Remember that we already covered this one with the log-trick.

(3.5)

$$C^{min}_{mbt} \leq \sum_{p=1}^{p_{max}} \sum_{k=0}^{min(k_{max}, t_{max}-t)} cc^{pm}_{bk}(x_{p,m,t+k} + x^{i}_{p,m,t+k}) \leq C^{max}_{mbt}$$

$$\forall m \in [m_{max}], p \in [p_{max}], t \in [t_{max}]$$

We can split it in 2 trivial constraints:

$$C^{min}_{mbt} \leq \sum_{p=1}^{p_{max}} \sum_{k=0}^{min(k_{max}, t_{max}-t)} cc^{pm}_{bk}(x_{p,m,t+k} + x^{i}_{p,m,t+k})$$

$$\sum_{p=1}^{p_{max}} \sum_{k=0}^{min(k_{max}, t_{max}-t)} cc^{pm}_{bk}(x_{p,m,t+k} + x^{i}_{p,m,t+k}) \leq C^{max}_{mbt}$$

$$\forall m \in [m_{max}], p \in [p_{max}], t \in [t_{max}]$$

Such inequalities constraints are handled with slack variables, such that we obtain only equality constraints, thus:

$$C^{min}_{mbt} + l^{min}_{mbt} = \sum_{p=1}^{p_{max}} \sum_{k=0}^{min(k_{max}, t_{max}-t)} cc^{pm}_{bk}(x_{p,m,t+k} + x^{i}_{p,m,t+k})$$

$$\sum_{p=1}^{p_{max}} \sum_{k=0}^{min(k_{max}, t_{max}-t)} cc^{pm}_{bk}(x_{p,m,t+k} + x^{i}_{p,m,t+k}) + r^{max}_{mbt} = C^{max}_{mbt}$$

$$\forall m \in [m_{max}], p \in [p_{max}], t \in [t_{max}],$$

where l^{min}_{mbt} and r^{min}_{mbt} are the aforementioned slack variables. Again, they also have the log-trick behind them, in order to represent non-negative integers. The bound for the log-trick can be, for example, C^{max}_{mbt}, for each m, b and t.

As a quick optimization, we observe that cc^{pm}_{bk} and $x^{i}_{p,m,t+k}$ are constants in the sum, so we can move the extra factor before the sum. The formulation remains quadratic as well.

(3.6)

$$x_{pmt} \leq \delta u_{pmt}$$

$$\forall p \in [p_{max}], m \in [m_{max}], t \in [t_{max}]$$

Thomas wrote an explanation for this: "It fixes the binary variable to 1 whenever there is a positive production. On the other hand, $u_{pmt} = 0$ leads to $x_{pmt} = 0$.

Minor trick to check if the number is positive: $S := \sum_{j=0} \chi_{pmt}(j) = 0$, where $\chi_{pmt}(j)$ are the qubits encoding x_{pmt}. In this way, we avoid the OR-gate, which is not quadratic for a number of qubits > 2.

We distinguish 4 cases:
(i) $s = 0, u = 0 \rightarrow$ good $= 0$
(ii) $s > 0, u = 0 \rightarrow$ bad $!= 0$
(iii) $s = 0, u = 1 \rightarrow$ good $= 0$
(iv) $s > 0, u = 1 \rightarrow$ good $= 0$

How to represent this with a succint formula? $S(1 - u_{pmt})$, with $S := \sum_{j=0} \chi_{pmt}(j)$.