CS 411: Artificial Intelligence

Integer Programming

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[Adapted from slides by Ariel Procaccia]

Agenda

- Announcements
- Discussion of Integer Programming
- Questions on Programming 2

Announcements

- Programming Assignment 2 due Sunday 23:59
  - Please install ASAP!
  - Sudoku problems are a bit harder
  - Autograder on Gradescope coming soon
- Week 6 content released
  - We'll discuss Quiz 1 on Tuesday

Integer Programming

- An integer programming (IP) problem:
  - $a_{ij} \in \mathbb{R}$ for $i \in [k] = \{1, \ldots, k\}, j \in [\ell]$
  - $b_i \in \mathbb{R}$ for $i \in [k]$
  - Variables x_j for $j \in [\ell]$
- The (feasibility) problem is:

$$\begin{aligned} &\text{find } x_1 \ldots, x_\ell \\ &\text{s.t.} \quad \forall i \in [k],\ \textstyle\sum_{j=1}^{\ell} a_{ij} x_j \leq b_i \\ &\qquad \forall j \in [\ell],\ x_j \in \mathbb{Z} \end{aligned}$$

Example: Sudoku

- For each $i, j, k \in [9]$, binary variable x_k^{ij} s.t. $x_k^{ij} = 1$ iff we put k in entry (i, j)
- For t $= 1, \ldots, 27$, S_t is a row, column, or 3×3 square

$$\begin{aligned} &\text{find } x_1^{11}, \ldots, x_9^{99} \\ &\text{s.t. } \forall t \in [27], \forall k \in [9], \sum_{(i,j) \in S_t} x_k^{ij} = 1 \\ &\qquad \forall i, j \in [9], \sum_{k \in [9]} x_k^{ij} = 1 \\ &\qquad \forall i, j, k \in [9], x_k^{ij} \in \{0,1\} \end{aligned}$$

Example: Fair Division

- Players $N = \{1, \dots, n\}$ and items $M = \{1, \dots, m\}$
- Player i has value v_{ij} for item j
- Partition items to bundles $A_1, \dots, A_n$
- $A_1, \dots, A_n$ satisfies Maximin Share iff $\forall i, \sum_{j \in A_i} v_{ij} \geq MMS_i$

1	$30	$50	$2	$5	$5	$3	$5
2	$2	$10	$5	$20	$20	$3	$40

Example: Fair Division

- Variables: $x_{ij} \in \{0,1\}$, $x_{ij} = 1$ iff $j \in A_i$
- AllocateMaximinShare as an IP:

$$\begin{aligned}
&\text{find } x_{11}, \ldots, x_{nm} \\
&\text{s.t.} \quad \forall i \in N, \ \textstyle\sum_{j \in M} v_{ij} x_{ij} \geq MMS_i \\
&\qquad \forall j \in M, \ \textstyle\sum_{i \in N} x_{ij} = 1 \\
&\qquad \forall i \in N, j \in M, x_{ij} \in \{0,1\}
\end{aligned}$$

Example: Fair Division

- Variables: $x_{i'j} \in \{0,1\}$, $x_{ij} = 1$ iff $j \in A_i$, $d \in Reals$
- CALCULATEMAXIMINSHARE(I) as an MIP:

$$\begin{aligned} &\max d \\ &\text{s.t.} \quad \forall i' \in N \quad \textstyle\sum_{j \in M} v_{ij} x_{i'j} \geq d \\ &\qquad \forall j \in M, \ \textstyle\sum_{i \in N} x_{ij} = 1 \\ &\qquad \forall i' \in N, j \in M, x_{i'j} \in \{0,1\} \end{aligned}$$

Other IPs in EconCS*

Dodgson's voting rule

Stackelberg security games

Questions on Programming 2?

Clarification on Q4: the solution should be interpreted as a tour (return to your start city)

- Useful Python fact:
  - L[-1] is the last element of L