Autocorrelation <Page 58>

$$\epsilon_t = \rho\epsilon_{t-1} + a_t$$

$$= \rho(\rho\epsilon_{t-2} + a_{t-1}) + a_t$$

$$= \rho^2\epsilon_{t-2} + \rho a_{t-1} + a_t$$

$$= \rho^2(\rho\epsilon_{t-3} + a_{t-2}) + \rho a_{t-1} + a_t$$

$$= \rho^3\epsilon_{t-3} + \rho^2 a_{t-2} + \rho a_{t-1} + a_t$$

$$= \rho^4\epsilon_{t-4} + \rho^3 a_{t-3} + \rho^2 a_{t-2} + \rho a_{t-1} + a_t$$

$$\vdots$$

$$= \rho^N\epsilon_{t-N} + \rho^{N-1} a_{t-(N-1)} + \cdots + \rho^1 a_{t-1} + a_t$$

$$= \sum_{u=0}^{N-1} \rho^u a_{t-u} + \rho^N \epsilon_{t-N}$$

Now, as $N \to \infty$, $\rho^N \to 0$ as $-1 < \rho < 1$.

So, $$\epsilon_t = \sum_{u=0}^{\infty} \rho^u a_{t-u}$$

* $E(\epsilon_t) = 0$

$$E(\epsilon_t) = \sum_{u=0}^{\infty} \rho^u E(a_{t-u})$$

$$= 0 \quad [\because E(a_{t-u}) = 0]$$

* $Var(\epsilon_t) = \sigma_a^2 \left(\frac{1}{1-\rho^2}\right)$

$$Var(\epsilon_t) = \sum_{u=0}^{\infty} \rho^{2u} Var(a_{t-u})$$

$$= \sigma_a^2 \sum_{u=0}^{\infty} \rho^{2u}$$

$$= \sigma_a^2 [1 + \rho^2 + \rho^4 + \rho^6 + \cdots]$$

$$= \sigma_a^2 \left(\frac{1}{1-\rho^2}\right)$$

* $Cov(\epsilon_t, \epsilon_{t+u}) = \rho^u Var(\epsilon_t)$

$$Cov(\epsilon_t, \epsilon_{t+1}) = E(\epsilon_t \epsilon_{t+1}) - E(\epsilon_t) E(\epsilon_{t+1})$$

$$= E[\epsilon_t(\rho \epsilon_t + a_{t+1})]$$

$$= E(\rho \epsilon_t^2 + \epsilon_t a_{t+1})$$

$$= E(\rho \epsilon_t^2) + a_{t+1} E(\epsilon_t)$$

$$= \rho E(\epsilon_t^2) = \rho\, var(\epsilon_t)$$

..... Contd.

$$\cancel{Cov(\epsilon_t, \epsilon_{t+2}) = cov(\epsilon_t(\rho\epsilon_{t+1} + a_{t+2}))}$$
$$\cancel{= Cov(\rho\epsilon_t\epsilon_{t+1} + a_{t+2}\epsilon_t)}$$
$$\cancel{= Cov(\rho\epsilon_t(\rho\epsilon_t + a_{t+1}) + a_{t+2}\epsilon_t)}$$
$$\cancel{= C}$$

$$Cov(\epsilon_t, \epsilon_{t+2}) = E(\epsilon_t\epsilon_{t+2})$$

Now, $\epsilon_t\epsilon_{t+2} = \epsilon_t(\rho\epsilon_{t+1} + a_{t+2})$

$$= \rho\epsilon_t(\rho\epsilon_t + a_{t+1}) + \epsilon_t a_{t+2}$$
$$= \rho^2\epsilon_t^2 + \rho\epsilon_t a_{t+1} + \epsilon_t a_{t+2}$$

$$E(\epsilon_t\epsilon_{t+2}) = \rho^2 E(\epsilon_t^2) \quad [\because E(\epsilon_t) = 0]$$
$$= \rho^2 var(\epsilon_t)$$

Similarly,

$$Cov(\epsilon_t, \epsilon_{t+u}) = \rho^u Var(\epsilon_t)$$

Durbin-Watson Statistic (Page 59):

$$d = \frac{\sum_{t=2}^{n} (e_t - e_{t-1})^2}{\sum_{t=1}^{n} e_t^2}$$

$$= \frac{\sum e_t^2 + \sum e_{t-1}^2 - 2\sum e_t e_{t-1}}{\sum e_t^2}$$

$$= 1 + 1 - 2\,\frac{\sum e_t e_{t-1}}{\sum e_t^2} \quad \langle \text{for large } n \rangle$$

$$= 2 - 2\,\frac{Cov(e_t\, e_{t-1})}{Var(e_t)}$$

$$= 2 - 2\,\frac{\rho\, var(e_{t-1})}{Var(e_t)}$$

$$= 2 - 2\rho$$

$$= 2(1-\rho)$$

- $cov(e_t\, e_{t-1})$

$$= \frac{1}{n}\sum [e_t - E(e_t)] \times [e_{t-1} - E(e_{t-1})]$$

$$= \frac{1}{n}\sum e_t e_{t-1}$$

- $Var(e_t)$

$$= \frac{1}{n}\sum [e_t - E(e_t)]^2$$

$$= \frac{1}{n}\sum e_t^2$$

So, under H_0 $d \simeq 2$,
otherwise $d < 2$