* $R^2_{adj} < R^2$

$$R^2_{adj} - R^2 = \left(1 - \frac{MS_E}{MS_T}\right) - \left(1 - \frac{SS_E}{SS_T}\right)$$

$$= \frac{SS_E}{SS_T} - \frac{SS_E/(n-p-1)}{SS_T/(n-1)}$$

$$= \frac{SS_E}{SS_T}\left[1 - \frac{n-1}{n-p-1}\right]$$

$$= -\frac{p}{n-p-1} \times \frac{SS_E}{SS_T} < 0$$

$$\Rightarrow \underline{R^2_{adj} < R^2}$$

* Standardized Regression - Unit Normal Scaling

We know, $y_i = \hat{b}_0 + \hat{b}_1 x_{i1} + \hat{b}_2 x_{i2} + \cdots + \hat{b}_p x_{ip} + \epsilon_i$

$$\Rightarrow y_i = \bar{y} - \hat{b}_1\bar{x}_1 - \hat{b}_2\bar{x}_2 - \cdots - \hat{b}_p\bar{x}_p + \hat{b}_1 x_{i1} + \hat{b}_2 x_{i2} + \cdots + \hat{b}_p x_{ip} + \epsilon_i$$

$$\Rightarrow y_i - \bar{y} = \hat{b}_1(x_{i1} - \bar{x}_1) + \hat{b}_2(x_{i2} - \bar{x}_2) + \cdots + \hat{b}_p(x_{ip} - \bar{x}_p) + \epsilon_i$$

$$= \hat{b}_1 s_1\left[\frac{x_{i1} - \bar{x}_1}{s_1}\right] + \hat{b}_2 s_2\left[\frac{x_{i2} - \bar{x}_2}{s_2}\right] + \cdots + \hat{b}_p s_p\left[\frac{x_{ip} - \bar{x}_p}{s_p}\right] + \epsilon_i$$

$$\Rightarrow \frac{y_i - \bar{y}}{s_y} = \hat{b}_1\frac{s_1}{s_y} z_{i1} + \hat{b}_2\frac{s_2}{s_y} z_{i2} + \cdots + \hat{b}_p\frac{s_p}{s_y} z_{ip} + \frac{\epsilon_i}{s_y}$$

$$\Rightarrow \underline{y_i^* = \gamma_1 z_{i1} + \gamma_2 z_{i2} + \cdots + \gamma_p z_{ip} + \epsilon_i'},$$

where $\gamma_i = \hat{b}_i \times \frac{s_i}{s_y}$, $s_j = \sqrt{\frac{\sum(x_{ij} - \bar{x}_j)^2}{n-1}}$ & $s_y = \sqrt{\frac{\sum(y_i - \bar{y})^2}{n-1}}$

* Standardized Regression - Unit Length Scaling

As above

$$y_i - \bar{y} = \hat{b}_1(x_{i1} - \bar{x}_1) + \hat{b}_2(x_{i2} - \bar{x}_2) + \cdots + \hat{b}_p(x_{ip} - \bar{x}_p) + \epsilon_i$$

$$= \hat{b}_1\sqrt{SS_1}\left[\frac{x_{i1} - \bar{x}_1}{\sqrt{SS_1}}\right] + \hat{b}_2\sqrt{SS_2}\left[\frac{x_{i2} - \bar{x}_2}{\sqrt{SS_2}}\right] + \hat{b}_p\sqrt{SS_p}\left[\frac{x_{ip} - \bar{x}_p}{\sqrt{SS_p}}\right] + \epsilon_i$$

$$\Rightarrow \frac{y_i - \bar{y}}{\sqrt{SS_y}} = \hat{b}_1\sqrt{\frac{SS_1}{SS_y}} W_{i1} + \hat{b}_2\sqrt{\frac{SS_2}{SS_y}} W_{i2} + \cdots + \hat{b}_p\sqrt{\frac{SS_p}{SS_y}} W_{ip} + \frac{\epsilon_i}{\sqrt{SS_y}}$$

$$\Rightarrow \underline{y_i^\circ = \delta_1 W_{i1} + \delta_2 W_{i2} + \cdots + \delta_p W_{ip} + \epsilon_i'},$$ where

$\delta_i = \hat{b}_i\sqrt{\frac{SS_i}{SS_y}}$, $SS_j = \sum(x_{ij} - \bar{x}_j)^2$ & $SS_y = \sum(y_i - \bar{y})^2$

* Unit Length scaling $- W^TW$

$$W = \begin{bmatrix} w_{11} & w_{12} & \cdots & w_{1p} \\ w_{21} & w_{22} & \cdots & w_{2p} \\ \vdots & \vdots & & \vdots \\ w_{n1} & w_{n2} & \cdots & w_{np} \end{bmatrix}_{n\times p}$$

$$w_{ij} = \frac{x_{ij} - \bar{x}_j}{\sqrt{SS_j}} \quad \left.\right\} \begin{array}{l} i = 1, 2, \ldots, n \\ j = 1, 2, \ldots, p \end{array}$$

with $SS_j = \sum_{i=1}^{n} (x_{ij} - \bar{x}_j)^2$

$$\Rightarrow S_j = \frac{SS_j}{n-1}$$

$$\Rightarrow \sqrt{S_j} = \sqrt{\frac{SS_j}{n-1}}$$

$$W^TW = \begin{bmatrix} \sum w_{i1}^2 & \sum w_{i1}w_{i2} & \cdots & \sum w_{i1}w_{ip} \\ \sum w_{i2}w_{i1} & \sum w_{i2}^2 & \cdots & \sum w_{i2}w_{ip} \\ \vdots & \vdots & & \vdots \\ \sum w_{ip}w_{i1} & \sum w_{ip}w_{i2} & \cdots & \sum w_{ip}^2 \end{bmatrix}$$

Now $\sum_{i=1}^{n} w_{ij}^2 = 1$

$$\sum w_{i1}w_{i2} = \sum_{i=1}^{n} \left\{ \frac{x_{i1} - \bar{x}_1}{\sqrt{SS_1}} \right\} \left\{ \frac{x_{i2} - \bar{x}_2}{\sqrt{SS_2}} \right\}$$

$$= \frac{\sum (x_{i1} - \bar{x}_1)(x_{i2} - \bar{x}_2)}{\sqrt{SS_1 \times SS_2}} = r_{12}$$

So, $$W^TW = \begin{bmatrix} 1 & r_{12} & \cdots & r_{1p} \\ r_{12} & 1 & \cdots & r_{2p} \\ \vdots & \vdots & & \vdots \\ r_{1p} & r_{2p} & \cdots & 1 \end{bmatrix}$$

Again, $$W^Ty^0 = \begin{bmatrix} \sum w_{i1} y_i^0 \\ \sum w_{i2} y_i^0 \\ \vdots \\ \sum w_{ip} y_i^0 \end{bmatrix} = \begin{bmatrix} r_{1y} \\ r_{2y} \\ \vdots \\ r_{py} \end{bmatrix}$$

$\because \sum w_{ij} y_i^0$

$$= \frac{\sum (x_{ij} - \bar{x}_j)(y_i - \bar{y})}{\sqrt{SS_j \times SS_y}}$$

$= r_{jy}$, $j = 1, 2, \ldots, p$

— x —

* $Z^TZ = (n-1)W^TW$

$$z_{ij} = \frac{(x_{ij} - \bar{x}_j)}{\sqrt{\frac{\sum (x_{ij} - \bar{x}_j)^2}{n-1}}} = (n-1)^{1/2} w_{ij}$$

$\Rightarrow Z = \sqrt{n-1}\, W$

$\therefore Z^TZ = (n-1)W^TW$

* $\hat{\gamma}$ & $\hat{\delta}$ are identical

$$\hat{\gamma} = (Z^TZ)^{-1} Z^T y^* \quad \cdots\cdots (1)$$

$$y_i^* = \frac{y_i - \bar{y}}{s_y} = \frac{y_i - \bar{y}}{\sqrt{\frac{SS_y}{n-1}}} = \sqrt{n-1}\, y_i^0 \Rightarrow y^* = \sqrt{n-1}\, y^0$$

$$\therefore \hat{\gamma} = \frac{1}{(n-1)} (W^TW)^{-1} \{\sqrt{n-1}\, W^T\} \{\sqrt{n-1}\, y^0\}$$

$$= (W^TW)^{-1} W^T y^0$$

$$= \hat{\delta}$$