ANOVA

Prerequisitos :

- $\epsilon_{ij} \sim N(0,\sigma^2) \quad \left[\begin{array}{l} i=1,2,\cdots,a \\ j=1,2,\cdots,n \end{array} \right.$
- $E(\epsilon_{ij}^2) = [E(\epsilon_{ij})]^2 + Var(\epsilon_{ij}) = \sigma^2$
- $E(\bar{\epsilon}_{i\cdot}^2) = E\left[\left(\frac{1}{n}\sum_j \epsilon_{ij}\right)^2\right] = \frac{1}{n^2}\sum_j E(\epsilon_{ij}^2) = \frac{1}{n^2} \times n\sigma^2 = \frac{\sigma^2}{n}$
- $E(\bar{\epsilon}_{\cdot\cdot}^2) = E\left[\left(\frac{1}{an}\sum_i\sum_j \epsilon_{ij}\right)^2\right] = \frac{1}{(an)^2} \times an\sigma^2 = \frac{\sigma^2}{an}$
- $E(\bar{\epsilon}_{i\cdot}) = E\left[\frac{1}{n}\sum_j \epsilon_{ij}\right] = \frac{1}{n}\sum_j E(\epsilon_{ij}) = 0$
- $Var(\bar{\epsilon}_{i\cdot}) = Var\left[\frac{1}{n}\sum_j \epsilon_{ij}\right] = \frac{1}{n^2}\sum Var(\epsilon_{ij}) = \frac{1}{n^2}\cdot n\sigma^2 = \frac{\sigma^2}{n}$
- $Var(\epsilon_{i\cdot}) = Var\left[\sum_j \epsilon_{ij}\right] = n\sigma^2$
- $y_{ij} = \mu_i + \epsilon_{ij} \quad \Rightarrow \bar{y}_{i\cdot} = \mu_i + \bar{\epsilon}_{i\cdot}$
- $E(\bar{y}_{i\cdot}) = \mu_i$
- $Var(\bar{y}_{i\cdot}) = Var(\bar{\epsilon}_{i\cdot}) = \frac{\sigma^2}{n}$

$$E(SS_T) = (a-1)\sigma^2 + n\sum \tau_i^2$$

$$y_{ij} = \mu + \tau_i + \epsilon_{ij} \qquad i = 1, 2, \cdots, a \ \& \ j = 1, 2, \cdots, n$$

$$SS_T = n\sum_i (\bar{y}_{i\cdot} - \bar{y}_{\cdot\cdot})^2$$

$$\bar{y}_{i\cdot} - \bar{y}_{\cdot\cdot} = (\mu + \tau_i + \bar{\epsilon}_{i\cdot}) - (\mu + \bar{\epsilon}_{\cdot\cdot})$$

$$= \tau_i + \bar{\epsilon}_{i\cdot} - \bar{\epsilon}_{\cdot\cdot}$$

$$SS_T = n\sum \tau_i^2 + n\sum \bar{\epsilon}_{i\cdot}^2 + an\bar{\epsilon}_{\cdot\cdot}^2 + 2n\sum \tau_i \bar{\epsilon}_{i\cdot} - 2n\bar{\epsilon}_{\cdot\cdot}\sum \tau_i - 2n\bar{\epsilon}_{\cdot\cdot}\sum \bar{\epsilon}_{i\cdot}$$

$$= n\sum \tau_i^2 + n\sum \bar{\epsilon}_{i\cdot}^2 + an\bar{\epsilon}_{\cdot\cdot}^2 + 2n\sum \tau_i \bar{\epsilon}_{i\cdot} - 2n\bar{\epsilon}_{\cdot\cdot}(a\bar{\epsilon}_{\cdot\cdot})$$

$$= n\sum \tau_i^2 + n\sum \bar{\epsilon}_{i\cdot}^2 - an\bar{\epsilon}_{\cdot\cdot}^2 + 2n\sum \tau_i \bar{\epsilon}_{i\cdot}$$

[τ_i and ϵ_{ij} are uncorrelated]

$$E(SS_T) = n\sum \tau_i^2 + n\sum E(\bar{\epsilon}_{i\cdot}^2) - an\,E(\bar{\epsilon}_{\cdot\cdot}^2)$$

$$= n\sum \tau_i^2 + n \times a\frac{\sigma^2}{n} - an \times \frac{\sigma^2}{an}$$

$$= n\sum \tau_i^2 + a\sigma^2 - \sigma^2$$

$$= \boxed{(a-1)\sigma^2 + n\sum \tau_i^2}$$

$$\left[\begin{aligned} &\sum \bar{\epsilon}_{i\cdot} \\ &= \frac{1}{n}\sum\sum \epsilon_{ij} \\ &= \frac{1}{n}\epsilon_{\cdot\cdot} \\ &= \frac{1}{n}(an\bar{\epsilon}_{\cdot\cdot}) \\ &= a\bar{\epsilon}_{\cdot\cdot} \end{aligned}\right.$$

$$E(SS_E) = a(n-1)\sigma^2$$

$$SS_E = \sum\sum (y_{ij} - \bar{y}_{i\cdot})^2 = \sum\sum (\epsilon_{ij} - \bar{\epsilon}_{i\cdot})^2$$

$$= \sum_i \sum_j \epsilon_{ij}^2 + n\sum_i \bar{\epsilon}_{i\cdot}^2 - 2\sum_i \sum_j \epsilon_{ij}\bar{\epsilon}_{i\cdot}$$

$$= \sum\sum \epsilon_{ij}^2 + n\sum \bar{\epsilon}_{i\cdot}^2 - 2n\sum \bar{\epsilon}_{i\cdot}^2 \qquad \left[\because \sum_j \epsilon_{ij} = \epsilon_{i\cdot} = n\bar{\epsilon}_{i\cdot}\right]$$

$$= \sum\sum \epsilon_{ij}^2 - n\sum \bar{\epsilon}_{i\cdot}^2$$

$$E(SS_T) = \sum\sum E(\epsilon_{ij}^2) - n\sum E(\bar{\epsilon}_{i\cdot}^2)$$

$$= an\sigma^2 - n\frac{a\sigma^2}{n}$$

$$= a(n-1)\sigma^2$$

⊛ Point estimate of μ_i in $y_{ij} = \mu_i + \epsilon_{ij}$:

$$y_{ij} = \mu_i + \epsilon_{ij} \quad i = 1, 2, \cdots, a \quad j = 1, 2, \cdots, n$$

$$L = \sum\sum \epsilon_{ij}^2 = \sum\sum (y_{ij} - \mu_i)^2$$

Intention is to find optimal parameter values by minimizing the **sum of squared residuals**, known as least-square methods.

$$\left.\frac{dL}{d\mu_i}\right|_{\mu_i = \hat{\mu}_i} = 0$$

$$\Rightarrow 2 \sum_j (y_{ij} - \hat{\mu}_i)(-1) = 0$$

$$\Rightarrow y_{i\cdot} = n\hat{\mu}_i$$

$$\Rightarrow \hat{\mu}_i = \bar{y}_{i\cdot}$$

Thus, residuals (e_{ij}) can be defined as

$$e_{ij} = y_{ij} - \bar{y}_{i\cdot}$$

Hence, fitted value of y_{ij} is

$$\hat{y}_{ij} = \bar{y}_{i\cdot}$$

* Simplified forms of SS :

* $TSS = \sum\sum (y_{ij} - \bar{y}_{\cdot\cdot})^2 = \sum\sum y_{ij}^2 + \sum\sum \bar{y}_{\cdot\cdot}^2 - 2\bar{y}_{\cdot\cdot} \sum\sum y_{ij}$

$= \sum\sum y_{ij}^2 + an\bar{y}_{\cdot\cdot}^2 - 2an\bar{y}_{\cdot\cdot}^2$ $[\sum\sum y_{ij} = an\bar{y}_{\cdot\cdot}]$

$= \sum\sum y_{ij}^2 - an\bar{y}_{\cdot\cdot}^2$

$= \sum\sum y_{ij}^2 - \dfrac{y_{\cdot\cdot}^2}{an} = \sum\sum y_{ij}^2 - CF$

⊛ $SS_T = n\sum (\bar{y}_{i\cdot} - \bar{y}_{\cdot\cdot})^2$

$= n\sum \bar{y}_{i\cdot}^2 + an\bar{y}_{\cdot\cdot}^2 - 2n\bar{y}_{\cdot\cdot} \sum \bar{y}_{i\cdot}$

$= n\sum \bar{y}_{i\cdot}^2 + an\bar{y}_{\cdot\cdot}^2 - 2an\bar{y}_{\cdot\cdot}^2$ $[\sum \bar{y}_{i\cdot} = a\bar{y}_{\cdot\cdot}]$

$= \frac{1}{n}\sum y_{i\cdot}^2 - an\bar{y}_{\cdot\cdot}^2 = \frac{1}{n}\sum y_{i\cdot}^2 - CF$

Random Effect Model (One-way Classification)

~~Ass~~ **Assumption** :

1. Treatments are 'random sample' of treatment levels in the experiment and hence treatment effect (τ_i) is a random variable.
2. Two random variables τ_i and ϵ_{ij} are uncorrelated
3. $\tau_i \sim N(0, \sigma_\tau^2)$ and $\epsilon_{ij} \sim N(0, \sigma^2)$

* $E(SS_T) = (a-1)(n\sigma_\tau^2 + \sigma^2)$

$$SS_T = n\sum_i (\bar{Y}_{i.} - \bar{Y}_{..})^2 \qquad \underline{Y_{ij} = \mu + \tau_i + \epsilon_{ij}}$$

$$= n\sum_i [(\mu + \tau_i + \bar{\epsilon}_{i.}) - (\mu + \bar{\tau}_. + \bar{\epsilon}_{..})]^2$$

$$= n[\sum \tau_i^2 + \sum \bar{\epsilon}_{i.}^2 + a\bar{\tau}_.^2 + a\bar{\epsilon}_{..}^2 + 2\sum \tau_i \bar{\epsilon}_{i.} - 2\bar{\tau}_. \sum \tau_i - 2\bar{\epsilon}_{..}\sum \tau_i - 2\bar{\tau}_. \sum \bar{\epsilon}_{i.} - 2\bar{\epsilon}_{..}\sum \bar{\epsilon}_{i.} + 2a\bar{\tau}_. \bar{\epsilon}_{..}]$$

$$\frac{1}{an}\sum\sum \tau_i = \frac{1}{an}\sum_j \tau_. = \frac{1}{a}\tau_. = \bar{\tau}_.$$

$$= n[\sum \tau_i^2 + \sum \bar{\epsilon}_{i.}^2 + a\bar{\tau}_.^2 + a\bar{\epsilon}_{..}^2 + 2\sum \tau_i \bar{\epsilon}_{i.} - 2a\bar{\tau}_.^2 - 2\bar{\epsilon}_{..}\tau_. - \cancel{2\bar{\tau}_.(a\bar{\epsilon}_{..})} - 2a\bar{\epsilon}_{..}^2 + \cancel{2a\bar{\tau}_.\bar{\epsilon}_{..}}]$$

$$= n\sum \tau_i^2 + n\sum \bar{\epsilon}_{i.}^2 - an\bar{\tau}_.^2 - an\bar{\epsilon}_{..}^2 + 2n\sum \tau_i \bar{\epsilon}_{i.} - 2n\bar{\epsilon}_{..}\tau_.$$

Threfore,

$$E(SS_T) = n\sum \sigma_\tau^2 + n\sum \frac{\sigma^2}{n} - an\frac{\sigma_\tau^2}{a} - an\frac{\sigma^2}{an}$$

$$= an\sigma_\tau^2 + a\sigma^2 - n\sigma_\tau^2 - \sigma^2$$

$$= \underline{(a-1)(n\sigma_\tau^2 + \sigma^2)}$$

<u>Note</u>: 1. $E(\tau_i^2) = [E(\tau_i)]^2 + Var(\tau_i) = \sigma_\tau^2$

2. $E(\bar{\tau}_.^2) = E[(\frac{1}{a}\sum \tau_i)^2] = \frac{1}{a^2}\sum E(\tau_i^2) = \frac{1}{a^2} \times a\sigma_\tau^2 = \frac{\sigma_\tau^2}{a}$

* $E(SS_E) = a(n-1)\sigma^2$ <similar to fixed-effect model>

Randomized Complete Block Design (RCBD)

$$E(SS_T) = (a-1)\sigma^2 + b\sum \tau_i^2$$

$$E(SS_B) = (b-1)\sigma^2 + a\sum \beta_j^2$$

Above two expressions can be proved following the method used in case one-way fixed effect model.

$$E(SS_E) = (a-1)(b-1)\sigma^2$$

$$SS_E = \sum_{i=1}^{a}\sum_{j=1}^{b}(y_{ij} - \bar{y}_{i.} - \bar{y}_{.j} + \bar{y}_{..})^2$$

$$= \sum\sum(\mu + \tau_i + \beta_j + \epsilon_{ij} - \mu - \tau_i - \bar{\epsilon}_{i.} - \mu - \beta_j - \bar{\epsilon}_{.j} + \mu + \bar{\epsilon}_{..})^2$$

$$= \sum\sum(\epsilon_{ij} - \bar{\epsilon}_{i.} - \bar{\epsilon}_{.j} + \bar{\epsilon}_{..})^2$$

* $\sum \bar{\epsilon}_{i.} = a\bar{\epsilon}_{..}$
* $\sum \bar{\epsilon}_{.j} = b\bar{\epsilon}_{..}$
* $\sum_j\sum_i \bar{\epsilon}_{i.}\bar{\epsilon}_{.j} = \sum_j (a\bar{\epsilon}_{..})\bar{\epsilon}_{.j} = a\bar{\epsilon}_{..}\sum\bar{\epsilon}_{.j} = ab\bar{\epsilon}_{..}^2$

$$= \sum\sum\epsilon_{ij}^2 + b\sum\bar{\epsilon}_{i.}^2 + a\sum\bar{\epsilon}_{.j}^2 + ab\bar{\epsilon}_{..}^2 - 2\sum\sum\epsilon_{ij}\bar{\epsilon}_{i.} - 2\sum\sum\epsilon_{ij}\bar{\epsilon}_{.j} + 2\bar{\epsilon}_{..}\sum\sum\epsilon_{ij} + 2\sum\sum\bar{\epsilon}_{i.}\bar{\epsilon}_{.j} - 2\bar{\epsilon}_{..}\, b\sum\bar{\epsilon}_{i.} - 2\bar{\epsilon}_{..}\, a\sum\bar{\epsilon}_{.j}$$

$$= \sum\sum\epsilon_{ij}^2 + b\sum\bar{\epsilon}_{i.}^2 + a\sum\bar{\epsilon}_{.j}^2 + ab\bar{\epsilon}_{..}^2 - 2b\sum\bar{\epsilon}_{i.}^2 - 2a\sum\bar{\epsilon}_{.j}^2 + \cancel{2ab\bar{\epsilon}_{..}^2} + \cancel{2\sum\sum\bar{\epsilon}_{i.}\bar{\epsilon}_{.j}} - \cancel{2ab\bar{\epsilon}_{..}^2} - \cancel{2ab\bar{\epsilon}_{..}^2}$$

$$= \sum\sum\epsilon_{ij}^2 - b\sum\bar{\epsilon}_{i.}^2 - a\sum\bar{\epsilon}_{.j}^2 + ab\bar{\epsilon}_{..}^2$$

$$E(SS_E) = \sum\sum E(\epsilon_{ij}^2) - b\sum E(\bar{\epsilon}_{i.}^2) - a\sum E(\bar{\epsilon}_{.j}^2) + ab\,E(\bar{\epsilon}_{..}^2)$$

$$= ab\sigma^2 - b\sum_{i=1}^{a}\frac{\sigma^2}{b} - a\sum_{j=1}^{b}\frac{\sigma^2}{a} + ab \times \frac{\sigma^2}{ab}$$

$$= ab\sigma^2 - b \times \frac{a\sigma^2}{b} - a \times \frac{b\sigma^2}{a} + \sigma^2$$

$$= ab\sigma^2 - a\sigma^2 - b\sigma^2 + \sigma^2$$

$$= (ab - a - b + 1)\sigma^2$$

$$= \underline{(a-1)(b-1)\sigma^2}$$

* $E(\bar{\epsilon}_{i.}^2) = E\left[\left(\frac{1}{b}\sum_j \epsilon_{ij}\right)^2\right] = \frac{1}{b^2}\sum_j E(\epsilon_{ij}^2) = \frac{1}{b^2}\cdot b\sigma^2 = \frac{\sigma^2}{b}$
* $E(\bar{\epsilon}_{.j}^2) = \sigma^2/a$ and
* $E(\bar{\epsilon}_{..}^2) = \frac{1}{(ab)^2}\sum\sum E(\epsilon_{ij}^2) = \frac{1}{(ab)^2} \times ab\sigma^2 = \frac{\sigma^2}{ab}$

Residuals of RCBD :

$$Y_{ij} = \mu + \tau_i + \beta_j + \epsilon_{ij} \quad \begin{cases} i = 1, 2, \cdots, a \\ j = 1, 2, \cdots, b \end{cases}$$

$$\sum_i \tau_i = 0$$
$$\sum_j \beta_j = 0$$

Estimation of μ, τ_i & β_j :

$$L = \sum\sum \epsilon_{ij}^2 = \sum\sum (y_{ij} - \mu - \tau_i - \beta_j)^2$$

* $\left.\frac{\partial L}{\partial \mu}\right|_{\mu = \hat{\mu}} = 0 \Rightarrow -2\sum\sum(y_{ij} - \hat{\mu} - \tau_i - \beta_j) = 0$

$$\Rightarrow Y_{..} - ab\hat{\mu} = 0$$
$$\Rightarrow \hat{\mu} = \bar{Y}_{..}$$

* $\left.\frac{\partial L}{\partial \tau_i}\right|_{\tau_i = \hat{\tau}_i} = 0 \Rightarrow -2\sum_j (y_{ij} - \hat{\mu} - \hat{\tau}_i - \beta_j) = 0$

$$\Rightarrow y_{i\cdot} - b\hat{\mu} - b\hat{\tau}_i = 0$$
$$\Rightarrow \hat{\tau}_i = \bar{Y}_{i\cdot} - \hat{\mu} = \bar{Y}_{i\cdot} - \bar{Y}_{..}$$

* Similarly, $\hat{\beta}_j = \bar{Y}_{\cdot j} - \bar{Y}_{..}$

So, $\hat{Y}_{ij} = \hat{\mu} + \hat{\tau}_i + \hat{\beta}_j = \bar{Y}_{i\cdot} + \bar{Y}_{\cdot j} - \bar{Y}_{..}$, and

residual $e_{ij} = Y_{ij} - \hat{Y}_{ij}$

$$= \underline{Y_{ij} - \bar{Y}_{i\cdot} - \bar{Y}_{\cdot j} + \bar{Y}_{..}}$$

Two-way Classification

$$SS_{AB} = \frac{1}{n}\sum\sum Y_{ij.}^2 - CF - SS_A - SS_B$$

$$SS_{AB} = n\sum\sum\left[\bar{Y}_{ij.} - \bar{Y}_{i..} - \bar{Y}_{.j.} + \bar{Y}_{...}\right]^2$$

$$= n\sum\sum\left[\bar{Y}_{ij.}^2 + \bar{Y}_{i..}^2 + \bar{Y}_{.j.}^2 + \bar{Y}_{...}^2 - 2\bar{Y}_{ij.}\bar{Y}_{i..} - 2\bar{Y}_{ij.}\bar{Y}_{.j.} + 2\bar{Y}_{ij.}\bar{Y}_{...} + 2\bar{Y}_{i..}\bar{Y}_{.j.} - 2\bar{Y}_{i..}\bar{Y}_{...} - 2\bar{Y}_{.j.}\bar{Y}_{...}\right]$$

$$= \frac{1}{n}\sum\sum Y_{ij.}^2 + bn\sum\bar{Y}_{i..}^2 + an\sum\bar{Y}_{.j.}^2 + abn\bar{Y}_{...}^2 - 2n\sum\sum\bar{Y}_{ij.}\bar{Y}_{i..} - 2n\sum\sum\bar{Y}_{ij.}\bar{Y}_{.j.} + 2n\sum\sum\bar{Y}_{ij.}\bar{Y}_{...} + 2n\sum\sum\bar{Y}_{i..}\bar{Y}_{.j.} - 2n\bar{Y}_{...}\sum\sum\bar{Y}_{i..} - 2n\bar{Y}_{...}\sum\sum\bar{Y}_{.j.} \quad (1)$$

* $2n\sum\sum\bar{Y}_{ij.}\bar{Y}_{i..} = 2bn\sum\bar{Y}_{i..}^2 \quad \left[\because \sum_j \bar{Y}_{ij.} = b\bar{Y}_{i..}\right]$

* $2n\sum\sum\bar{Y}_{ij.}\bar{Y}_{.j.} = 2an\sum\bar{Y}_{.j.}^2 \quad \left[\because \sum_i \bar{Y}_{ij.} = a\bar{Y}_{.j.}\right]$

* $2n\bar{Y}_{...}\sum\sum\bar{Y}_{ij.} = 2abn\bar{Y}_{...}^2 \quad \left[\because \sum\sum\bar{Y}_{ij.} = ab\bar{Y}_{...}\right]$

* $2n\sum\sum\bar{Y}_{i..}\bar{Y}_{.j.} = 2n\sum_j (a\bar{Y}_{...})\bar{Y}_{.j.} = 2abn\bar{Y}_{...}^2$

* $2n\bar{Y}_{...}\sum\sum\bar{Y}_{i..} = 2bn\bar{Y}_{...}\sum\bar{Y}_{i..} = 2abn\bar{Y}_{...}^2 \quad \left[\because \sum\bar{Y}_{i..} = a\bar{Y}_{...}\right]$

* $2n\bar{Y}_{...}\sum\sum\bar{Y}_{.j.} = 2an\bar{Y}_{...}\sum\bar{Y}_{.j.} = 2abn\bar{Y}_{...}^2 \quad \left[\because \sum\bar{Y}_{.j.} = b\bar{Y}_{...}\right]$

So, from (1),

$$SS_{AB} = \frac{1}{n}\sum\sum Y_{ij.}^2 - bn\sum\bar{Y}_{i..}^2 - an\sum\bar{Y}_{.j.}^2 + abn\bar{Y}_{...}^2$$

$$= \frac{1}{n}\sum\sum Y_{ij.}^2 - \frac{1}{bn}\sum Y_{i..}^2 - \frac{1}{an}\sum Y_{.j.}^2 + \frac{Y_{...}^2}{abn}$$

$$= \frac{1}{n}\sum\sum Y_{ij.}^2 - \frac{Y_{...}^2}{abn} - \left(\frac{1}{bn}\sum Y_{i..}^2 - \frac{Y_{...}^2}{abn}\right) - \left(\frac{1}{an}\sum Y_{.j.}^2 - \frac{Y_{...}^2}{abn}\right)$$

$$= \frac{1}{n}\sum\sum Y_{ij.}^2 - CF - SS_A - SS_B$$

Two-way Classification - Expected Mean Square

$$E(SS_A) = (a-1)\sigma^2 + bn\sum \tau_i^2$$

$$\sum_i \tau_i = \sum_j \beta_j = 0$$

$$\sum_i (\tau\beta)_{ij} = \sum_j (\tau\beta)_{ij} = 0$$

$$SS_A = bn\sum_i (\bar{y}_{i..} - \bar{y}_{...})^2$$

$$= bn\sum_i [(\mu + \tau_i + \bar{\epsilon}_{i..}) - (\mu + \bar{\epsilon}_{...})]^2$$

$$= bn\sum \tau_i^2 + bn\sum \bar{\epsilon}_{i..}^2 + abn\bar{\epsilon}_{...}^2 + 2bn\sum \tau_i\bar{\epsilon}_{i..} - \underbrace{\cancel{2bn\bar{\epsilon}_{...}\sum \tau_i}}_{0} - 2bn\bar{\epsilon}_{...}\sum \bar{\epsilon}_{i..}$$

$$= bn\sum \tau_i^2 + bn\sum \bar{\epsilon}_{i..}^2 + abn\bar{\epsilon}_{...}^2 + 2bn\sum \tau_i\bar{\epsilon}_{i..} - 2abn\bar{\epsilon}_{...}^2$$

$$= bn\sum \tau_i^2 + bn\sum \bar{\epsilon}_{i..}^2 - abn\bar{\epsilon}_{...}^2 + 2bn\sum \tau_i\bar{\epsilon}_{i..}$$

$$E(SS_A) = bn\sum \tau_i^2 + bn\sum \frac{\sigma^2}{bn} - abn \times \frac{\sigma^2}{abn}$$

$$= bn\sum \tau_i^2 + a\sigma^2 - \sigma^2 = \underline{(a-1)\sigma^2 + bn\sum \tau_i^2}$$

$$E(SS_{AB}) = (a-1)(b-1)\sigma^2 + n\sum_i\sum_j (\tau\beta)_{ij}^2$$

$$SS_{AB} = n\sum_i\sum_j (\bar{y}_{ij.} - \bar{y}_{i..} - \bar{y}_{.j.} + \bar{y}_{...})^2$$

$$= n\sum\sum [(\mu + \tau_i + \beta_j + (\tau\beta)_{ij} + \bar{\epsilon}_{ij.}) - (\mu + \tau_i + \bar{\epsilon}_{i..}) - (\mu + \beta_j + \bar{\epsilon}_{.j.}) + (\mu + \bar{\epsilon}_{...})]^2$$

$$= n\sum\sum [(\tau\beta)_{ij} + (\bar{\epsilon}_{ij.} - \bar{\epsilon}_{i..}) - (\bar{\epsilon}_{.j.} - \bar{\epsilon}_{...})]^2$$

$$= n\sum\sum (\tau\beta)_{ij}^2 + n\sum\sum (\bar{\epsilon}_{ij.} - \bar{\epsilon}_{i..})^2 + an\sum (\bar{\epsilon}_{.j.} - \bar{\epsilon}_{...})^2 - 2n\sum\sum (\bar{\epsilon}_{ij.} - \bar{\epsilon}_{i..})(\bar{\epsilon}_{.j.} - \bar{\epsilon}_{...}) + \text{terms involving } (\tau\beta)_{ij}$$

$$= n\sum\sum (\tau\beta)_{ij}^2 + n\sum\sum (\bar{\epsilon}_{ij.} - \bar{\epsilon}_{i..})^2 + an\sum (\bar{\epsilon}_{.j.} - \bar{\epsilon}_{...})^2 - 2an\sum_j (\bar{\epsilon}_{.j.} - \bar{\epsilon}_{...})^2 + \cdots$$

$$= n\sum\sum (\tau\beta)_{ij}^2 + n\sum\sum \bar{\epsilon}_{ij.}^2 - bn\sum \bar{\epsilon}_{i..}^2 - an\sum \bar{\epsilon}_{.j.}^2 + abn\bar{\epsilon}_{...}^2 + \cdots\cdots$$

$$E(SS_{AB}) = n\sum\sum (\tau\beta)_{ij}^2 + n\left(ab\frac{\sigma^2}{n}\right) - bn\left(a\frac{\sigma^2}{bn}\right) - an\left(b\frac{\sigma^2}{an}\right) + abn\left(\frac{\sigma^2}{abn}\right)$$

$$= n\sum\sum (\tau\beta)_{ij}^2 + ab\sigma^2 - a\sigma^2 - b\sigma^2 + \sigma^2$$

$$= \underline{n\sum\sum (\tau\beta)_{ij}^2 + (a-1)(b-1)\sigma^2}$$

Two-way Classification - Residual

$$\sum_i \tau_i = \sum_j \beta_j = 0$$
$$\sum_i (\tau\beta)_{ij} = \sum_j (\tau\beta)_{ij} = 0$$

$$y_{ijk} = \mu + \tau_i + \beta_j + (\tau\beta)_{ij} + \epsilon_{ijk}$$

$i = 1, 2, \cdots, a$
$j = 1, 2, \cdots, b$
$k = 1, 2, \cdots, n$

$$L = \sum\sum\sum \epsilon_{ijk}^2 = \sum\sum\sum (y_{ijk} - \mu - \tau_i - \beta_j - (\tau\beta)_{ij})^2$$

* $$\frac{\partial L}{\partial \mu}\Big|_{\mu=\hat{\mu}} = 0 \Rightarrow -2\sum\sum\sum (y_{ijk} - \hat{\mu} - \tau_i - \beta_j - (\tau\beta)_{ij}) = 0$$
$$\Rightarrow \hat{\mu} = \bar{y}_{...}$$

* $$\frac{\partial L}{\partial \tau_i}\Big|_{\tau_i=\hat{\tau}_i} = 0 \Rightarrow -2\sum_j\sum_k (y_{ijk} - \hat{\mu} - \hat{\tau}_i - \beta_j - (\tau\beta)_{ij}) = 0$$
$$\Rightarrow y_{i..} - bn\hat{\mu} - bn\hat{\tau}_i = 0$$
$$\Rightarrow \hat{\tau}_i = \bar{y}_{i..} - \bar{y}_{...}$$

* Similarly, $\hat{\beta}_j = \bar{y}_{.j.} - \bar{y}_{...}$

* $$\frac{\partial L}{\partial (\tau\beta)_{ij}}\Big|_{(\tau\beta)_{ij}=(\widehat{\tau\beta})_{ij}} = 0 \Rightarrow -2\sum_k (y_{ijk} - \hat{\mu} - \hat{\tau}_i - \hat{\beta}_j - (\widehat{\tau\beta})_{ij}) = 0$$
$$\Rightarrow y_{ij.} - n\hat{\mu} - n\hat{\tau}_i - n\hat{\beta}_j - n(\widehat{\tau\beta})_{ij} = 0$$
$$\Rightarrow (\widehat{\tau\beta})_{ij} = \bar{y}_{ij.} - \bar{y}_{...} - (\bar{y}_{i..} - \bar{y}_{...}) - (\bar{y}_{.j.} - \bar{y}_{...})$$
$$\Rightarrow (\widehat{\tau\beta})_{ij} = \bar{y}_{ij.} - \bar{y}_{i..} - \bar{y}_{.j.} + \bar{y}_{...}$$

So, $$\hat{y}_{ijk} = \bar{y}_{...} + (\bar{y}_{i..} - \bar{y}_{...}) + (\bar{y}_{.j.} - \bar{y}_{...}) + (\bar{y}_{ij.} - \bar{y}_{i..} - \bar{y}_{.j.} + \bar{y}_{...})$$
$$= \bar{y}_{ij.}$$

* Residual $= y_{ijk} - \hat{y}_{ijk} = y_{ijk} - \bar{y}_{ij.}$