risk-neutral measure flavored FX, IR

pq

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1 The Basics

Theorem 1.1 (Ito's Lemma)
Let X(s) be a stochastic process, ie:

$$dX(s) = a(s)ds + b(s)dW_s^X$$

, Then the stochastic chain rule will have an extra second order term that the common smooth calculus chain rule does not have, due to non-zero quadratic variation of Brownian motion, ie:

$$df(X(s)) = f'(X(s))dX(s) + \frac{1}{2}f''(X(s))(dX(s))^2$$

To continue the derivation you square X(s)'s SDE and plug into the $(dX(s))^2$ part, with the known short hand that $ds^2 = 0, dsdW = 0, (dW_s)^2 = ds$.

Sanity check: The full result is,

$$df(X(s)) = [f'(X(s))a(d) + \frac{1}{2}f''(X(s))b^2(s)]ds + f'(X(s))b(s)dW_s^X$$

2D version :

$$d\ f(X,t) = \partial_X\ f(X,t)dX + \partial_t\ f(X,t)dt + \frac{1}{2}\partial_X^2\ f(X,t)(dX)^2$$

- Ito intergal I := $\int_s^T f(d)\ dW_u$
- Thm: Ito integral is martingale: $E[\int_s^T f(d)\ dW_u] = 0$
- Ito isometry: for any integrand,

$$Var(I^2) = E[I^2] - 0 = E[\int_0^T (f(u))^2 \boxed{du}]$$

- if $X \sim N(a,b)$, then $Y = e^X$ is lognormal, and has $E[Y] = e^{a+\frac{1}{2}b}, Var[Y] = e^{2a+b}(e^b - 1)$

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2.1 Duration

- Duration = modified duration = mathematical absolute sensitivity to yield change

$$= \frac{-1}{P}\frac{dP}{dy} = \sum_{i}^{M} -i \cdot \frac{CF_i}{(1+y)^{i+1}}\frac{1}{P}$$

- maclaulay duration = weighted average time of receiving fractions of price

$$= \sum_{i}^{M} i \cdot \frac{CF_i}{(1+y)^i}\frac{1}{P} = -(1+y)\cdot duration = \sum_i i \cdot \frac{PV(CF_i)}{P}$$

- if yield held constant, for any given maturity duration falls as coupon increases. The intuition behind this fact is that higher-coupon bonds have a greater fraction of their value paid earlier. The higher the coupon, the larger the weights on the duration terms of early years relative to those of later years. Hence, higher-coupon bonds are effectively shorter-term bonds and therefore have lower durations.

2.2 curves

- spot curve = zero curve = the curve used to discount each CF: get from each actual zero bonds, or bootstrapped from existing coupon bonds.

- par curve: IRR of par coupon bonds: bootstrapped from fictional coupon bonds. for each T, suppose existence of a par coupon bond with coupon rate c_T, whose each CF is discounted with the zero curve; set its price to 100, and solve for c_T. Such points (T, c_T) is then the par curve.

3 FX

3.1 notation

Intuition notation:

Notation 3.1 ($\frac{YYY}{XXX}$)
If 1x of XXX == Sx of YYY, ⇔ Sx YYY per XXX ⇔ XXX is the base currency, S has unit $\frac{YYY}{XXX}$

Market notation: note that this convention is counterintuitive.

Notation 3.2 (XXXYYY == XXX/YYY = aaa/bbb)
(note that these slashes here does not signify mathematic fraction!!) def: XXX is called base currency; YYY is called the term currency. def: "aaa/bbb" denotes "bid/quote" Aaa == the bid price == I get aaa YYY to sell per XXX; the amount I get from the market maker bbb == the offer/ask price == pay bbb YYY to buy per XXX; the amount I pay the market maker

the naming "bid/ask" is named from the dealer/market maker's pespective, and not from the market participant's perspective; From your (the client's) perspective, it's the opposite: You sell at the dealer's bid; You buy at the dealer's ask.

Quick sanity check: If XXX appreciates (worth more), then the math notation value $\frac{XXX}{YYY}$ == how many X per 1 Y will decrease; the market convention XXXYYY rate will increase, the market convention YYYXXX will decrease.

3.2 FX forward of YYYXXX fx rate

Definition 3.3
Underlying S = YYYXXX rate, (XXX per YYY) Strike: K XXX per YYY YYY buying party: gets 1 YYY, give K unit of XXX. gets (S(T) - K) amt in XXX.
Two party agrees today to exchange K unit of XXX for 1 unit of YYY at time T. What's the value of this contract for the YYY buyer at time $t \leq T$?

Replicate argument: For the YYY buyer, cashflow at T == pay K unit XXX, get 1 YYY. To replicate this cashflow, at time 0, sell bond $K * e^{-r_x * T}$ unit of XXX + lend $e^{-r_y * T}$ unit of YYY. (Hold negative YYY, and hold positive amt of XXX)

*** So at time t, this arrangement net out to be $V(t) == e^{-r_y*(T-t)} * X(t) - K * e^{-r_x*(T-t)}$value in XXX currency, (*** YYY convert to XXX value instantaneously), where X(t) is the instaneous spot FX rate in XXX per YYY.

** And this replicating portfolio should be the same value as the forward contract value at all times, especially at initiation which == zero at any initiation time 0.

So solve to have

$$\boxed{K = e^{-r_y*(T-0)} * X(0)/e^{-r_x*(T-0)} := F(0,T)}$$

, the forward price in amount XXX for the YYY buyer.
The value of the forward contract at any time s is, for YYY buyer, in amt XXX:

$$\boxed{V(s) = X(s) * P_y(s,T) - K * P_x(s,T)}$$

value of forward contract for XXX buyer, in amt XXX: is the exact opposite.

$$\boxed{K * P_x(s,T) - X(s) * P_y(s,T)}$$

This gives the interest rate parity formula: For a fx forward of rate YYYXXX, initiated at time t, the forward strike price is:

$$\boxed{F(t,T) = X(t)P_y(t,T)/P_x(t,T)}$$

Convention 3.4
FX trade is settled on T, and delivered on T+2. So in fact, the spot FX rate shown on market whenever settled at any time t, S(t), is actually the forward FX rate, $F(t,t+\delta)$ which then is $== X(t) * e^{-r_y*\delta}/e^{-r_x*\delta} = X(t) * e^{(r_x-r_y)*\delta}$.
So

$$F(t,T) = X(t)*e^{(r_x-r_y)(T-t)} = S(t)e^{(r_x-r_y)(T-t-\delta)} \sim S(t)+S(t)*(r_x-r_y)(T-t-\delta)$$

3.3 FX pricing SDEs

- let X(s) be FX rate in YYYXXX, let $Y(s) = \frac{1}{X(s)}$ be FX rate in XXXYYY, derive SDE for both.
- $\frac{1}{X}dX = (r_x - r_y)ds + \sigma(s)dW^X$
- $\frac{1}{Y}dY = [(r_y - r_x) + \sigma^2(s)]ds - \sigma(s)dW^X = (r_y - r_x)ds - \sigma(s)dW^Y$,
- where the relation of W^Y and W^X is:

$$dW^Y + \sigma(s)ds = dW^X$$

$$\frac{dX(t)}{X(t)} = \begin{cases} (r_d - r_f)dt + \sigma dW^d(t) & \text{, in } \mathbb{Q}^d \\ (r_d - r_f + \sigma^2)dt + \sigma dW^f(t) & \text{, in } \mathbb{Q}^f \end{cases}$$

$$\frac{dY(t)}{Y(t)} = \begin{cases} (r_f - r_d)dt - \sigma dW^f(t) & \text{, in } \mathbb{Q}^f \\ (r_f - r_d + \sigma^2)dt - \sigma dW^d(t) & \text{, in } \mathbb{Q}^d \end{cases}$$

Figure 1: Summary of rate process's mutual relation

- derive SDE of the forward price $F(s,T)$ for a fixed const T from SDE of X:
- $\frac{1}{F(s,T)}dF(s,T) = \sigma(s)dW^X$

3.4 FX option

- Def: call option position on Y == put option position on X == buy 1 unit YYY, give $K = \frac{M}{N}$ unit XXX
- payoff of this call option on YYY at maturity T is:
$$V_X(T) = P_X(T, T+\delta) \cdot (S(T) - K)^+ \text{in XXX}$$
(Because the actual delivery date is $T + \delta$)
- (notation reminder: $P_X(s, T)$ is the discount factor in currency XXX)
- Put-Call parity: (of put on YYY and call on YYY)
$$Call^{onY}(s) - Put^{onY}(s) = P_X(s, T+\delta) \cdot (S(s) - K)$$

- $Call^{onYYY}$ explicit formula, in amt of XXX:
$$V_X(t) = P_X(t, T+\delta) \cdot [F(t, T+\delta) \cdot \Phi(d_+) - K \cdot \Phi(d_-)]$$
Or thru plug in formula of $F(t, T+\delta)$,
$$V_X(t) = X(t) \cdot P_Y(t, T+\delta) \cdot \Phi(d_+) - K \cdot \Phi(d_-) \cdot P_X(t, T+\delta)$$
- $Call^{onXXX} == Put^{onYYY}$ explicit formula, in amt XXX:
$$V_X(t) = P_X(t, T+\delta) \cdot [-F(t, T+\delta) \cdot \Phi(-d_+) + K \cdot \Phi(-d_-)]$$
- where $d_\pm = \frac{ln\frac{F(t,T+\delta)}{K} \pm \frac{\sigma^2}{2}(T-t)}{\sigma\sqrt{T-t}}$

FX option date convention:

- Expiry date:
- In FX options, you typically decide to exercise the option at the exercise date T; however, the actual exchange of currencies (settlement) occurs on the delivery date Td (often two business days after T). So the actual rate exchanged are the rates of Td; therefore S(T) == F(T, Td)
- trade date:
- if t is the trade date of the option ie. The day to enter in the contract and pay premium for the option position, the actual premium payment will also occur on $t + \delta$. So the actual value of the call on YYY price on actual payment day $t + \delta$ would be
$$p_x^{callonYYY} = call^{onYYY} / P_X(t, t+\delta)$$
(time value on $t + \delta$).
- $\boxed{\text{so the actual premium price you pay is } p_x^{callonYYY} \text{at time } t+\delta}$

3.5 Exotic FX option

Important identity:

- dirac delta function: $\delta_{dirac}(x) = \infty$ when $x = 0$, and $= 0$ elsewhere
- identity 1: $\boxed{\frac{\partial \mathbb{1}_{\{S(T)-x\}}}{\partial x} = \boxed{-}\delta_{dirac}(S(T) - x)}$
- identity 2: $\boxed{\frac{\partial^2 (S(T) - x)^+}{\partial x^2} = \delta_{dirac}(S(T) - x)}$

Theorem 3.5 (the Breeden-Litzenberger Recipe)
know implied vol surface
$\Rightarrow$ know $\sigma(T, K)$ for any T, K
$\Rightarrow$ can get implied vol for any T, K and plug into B-S option price to get $V_{call of T,K}(0)$
$\Rightarrow$ Can know $p(T, S(T) = x)$:= pdf of r.v. $S(T)$ under XXX risk neutral measure Q^X

Proof.

- def: $p(t, x)$:= pdf of S(T)$|\mathcal{F}_t$
- by def of pdf and cdf, have

$$\int_A p(t, x)dx = \mathbb{P}^{Q^X}(S(T) \in A|\mathcal{F}_t)$$

  and

$$p(t, x)dx = \mathbb{P}^{Q^X}(S(T) \in [x, x + dx] \,|\mathcal{F}_t) = \mathbb{E}^{Q^X}[\mathbb{1}_{\{S(T)\in[x,x+dx]\}} \,|\mathcal{F}_t]$$

- So $p(t, x) = \mathbb{E}^{Q^X}[\frac{\mathbb{1}_{\{S(T)\le x+dx\}} - \mathbb{1}_{\{S(T)\le x\}}}{dx} \,|\mathcal{F}_t] = \mathbb{E}^{Q^X}[\frac{d\mathbb{1}_{S(T)\le x}}{dx} \,|\mathcal{F}_t]$
- therefore $p(t, x) = \mathbb{E}^{Q^X}[\delta_{dirac}(S(T) - x) \,|\mathcal{F}_t]$
-
- also since $V_{call}(0) = \mathbb{E}^{Q^X}[\frac{(S(T)-K)^+}{\beta_X(T)}]$,
- then $\frac{\partial^2 V_{call}(0)}{\partial K^2} = \mathbb{E}^{Q^X}[\frac{\partial^2 (S(T)-K)^+}{\partial K^2}] \cdot P_X(0, T)$,
  $= \mathbb{E}^{Q^X}[\delta_{dirac}(S(T) - K)] \cdot P_X(0, T)$
-
- match this part with the previous part, can get

$$\boxed{P(T, K) = \beta_X(T) \cdot \frac{\partial^2 V_{call}(0)}{\partial K^2}}$$

$\square$

Utility: is that having gotten the $p(T, K)$, we can now get any time0 PV == can price any European type(:= option always achieves payoff on and only on maturity) option with any maturity-payoff func $V(T) = g(S(T))$:

$$V(0) = P_X(0,T)\,\mathbb{E}^{Q^X}[V(T) = g(S(T))] = P_X(0,T)\int_R g(x)\cdot p(T,x)\,dx$$

Example: $V_{digiCall}(t) = -\frac{\partial V_{call}(t)}{\partial K}$. Derivation: use Leibniz rule to derive K-derivative of regular call price; or use replicating portfolio argument: can easily show that $V_{digiCall}(T) = -[\frac{V_{call}(T,K+\epsilon)-V_{call}(T,K-\epsilon)}{2\epsilon}]$, so to replicate the digital call's cash flow at maturity, is to get a pair of call spreads.

digital

- Barrier digital option, named "American digi option" under this lecture:= it's still European type in the sense that payoff and closure occur and only occur on maturity; its "americaness" is in that the payoff-indicator-criteria is observed throughout life; so under this naming context the euorpean digi barrier option would mean the payoff-indicator-criteria is only observed at time T.

- To define the payouts of some common exotics, we need a bit of notation for the running minimum and maximum of the spot FX rate:

$$m(t) = \min(S(u), u \in [0,t])$$
$$M(t) = \max(S(u), u \in [0,t]).$$

- Domestic American digital calssification (payout is one unit of domestic currency):

Name	Payout at T
One-touch (up)	$1_{M(T)\geq K}$
One-touch (down)	$1_{m(T)\leq K}$
Double-touch	$1_{M(T)\geq K_u}$ or $1_{m(T)\leq K_l}$
No-touch (up)	$1_{M(T)<K}$
No-touch (down)	$1_{m(T)>K}$
Double no-touch	$1_{m(T)>K_l}\cdot 1_{M(T)<K_u}$

Figure 2: FX american digital

- Certain obvious parity results hold here. For instance, the sum the price of a one-touch and a no-touch must equal a zero-coupon bond

- Notice that all the digitals discussed so far pay out at time T. American digitals also exist that pay out "instantly" at the time the barrier is breached, but these are less common

- A common rule of thumb is this: an American one-touch is about twice as expensive as a European one. Intuition of this is if drift of the brownian motion is small or 0, or maturity is small, mathematically by reflection principle of Brownian motion, P(hitting time comes before T)= 2 P(cross barrier at T). So, on average the American one-touch pays out 1 about twice as often as the European one-touch

barrier

- def:= payoff also still at T, but payoff = regular option payoff $\cdot \mathbb{1}_{\text{barrier criteria}}$
- eg. European(:= criteria is only monitoring maturity) Up and Out barrier option: $V(T) = (S(T) - K)^{+} \cdot \mathbb{1}_{S(T) \leq K_*}$, where $K_* < K$
- American up and out barrier: $V(T) = (S(T) - K)^{+} \cdot \mathbb{1}_{M(T) \leq K_*}$

4 IR

- def: $P(t, T)$ is the time-t PV of a zero bond that pays P(T,T) = 1 at time T, where t is now == so by def it is deterministic and informed by $\mathcal{F}_t$.
- Simply compounded spot rate:= $L(t, T)$:= the flat rate over period [t, T], where t is now:

$$L(t, T) \cdot \tau(t, T) = \frac{1 - P(t, T)}{P(t, T)}$$

  := invest 1 dollar at time t, the IR amt that it will gain over this period [t, T]. this is saying: if i invest 1 dollar today, its value would be $\frac{1}{P(t,T)}$ at time T; so the absolute growth amt == $\frac{1}{P(t,T)} - 1$ == the IR growth amt.
- therefore $P(t, T) = \frac{1}{1+L(t,T)\cdot\tau(t,T)}$
- Simply compounded forward rate:= $F(t; T, S)$:= the flat rate applied to future period [T, S] that is determined at time t today.

$$F(t; T, S)\tau(S, T) = \frac{P(t, T) - P(t, S)}{P(t, S)}$$

  this formula is saying: if I today (time t) decide to invest 1 dollar at time T, its time-t-PV would be $P(t, T)$, its time-S-PV would be $\frac{P(t,T)}{P(t,S)}$. So the absolute IR growth across period [T,S]== $\frac{P(t,T)}{P(t,S)} - 1$
- Instantaneous simple Forward rate:=

$$lim_{S \to T^+} F(t; T, S) = \frac{-\partial \; ln \; P(t, T)}{\partial T}$$

change of measure This idea is analogous to the change of variables in a integral, where $\int \cdot\, dx = \int \cdot \boxed{\dfrac{dx}{dy}} dy$.

- let the "Jacobian" be $\frac{dQ^N}{dQ^U}$, have identity:

$$\Lambda_S = \boxed{\frac{dQ^N}{dQ^U}|_{\mathcal{F}_S} = \frac{U_0}{N_0} \cdot \frac{N_S}{U_S}}$$

- and the "change of measure" is done through:

$$\boxed{\frac{Z_t}{N_t} = \mathbb{E}^{Q^N}\left[\tfrac{Z_T}{N_T} \mid \mathcal{F}_t\right] = \frac{1}{\Lambda_t}\mathbb{E}^{Q^U}\left[\tfrac{Z_T}{N_T}\,\Lambda_T \mid \mathcal{F}_t\right]}$$

Theorem 4.1 (Roll forward identity)

$$H(t) = E^Q[\frac{B(t)}{B(T)}H(T)|\mathcal{F}_t] = E^Q[\frac{B(t)}{B(S)}H(T)\frac{1}{P(T,S)}|\mathcal{F}_t]$$

Proof.

- Tower property: $\boxed{E[\ E[X \mid \mathcal{F}_T]\ \mid \mathcal{F}_t], \text{ where } t < T}$.
- by B-measure martingale,

$$\frac{P(T,S)}{B(T)} = E[\frac{P(S,S)}{B(S)} \mid \mathcal{F}_T] = E[\frac{1}{B(S)} \mid \mathcal{F}_T]$$

- $\frac{H(t)}{B(t)} = E[\frac{H(T)}{B(T)} \cdot 1 \mid \mathcal{F}_t] = E[\frac{H(T)}{B(T)} \cdot \boxed{\dfrac{E[\frac{B(T)}{B(S)} \mid \mathcal{F}_T]}{P(T,S)}} \mid \mathcal{F}_t]$
- $= E[\frac{H(T)}{B(T)} \cdot \frac{\frac{B(T)}{B(S)}}{P(T,S)} \mid \mathcal{F}_t]$
- $\frac{H(t)}{B(t)} = E[H(T) \cdot \frac{1}{P(T,S)\cdot B(S)} \mid \mathcal{F}_t]$

$\square$

Utility: =price things with different payoff time under the same measure, the Terminal-forward-measure, where Terminal is the latest maturity of these assets