Network Optimization

Part I Integer programming Modeling techniques
Wolsey, Integer Programming, Chapter 1

Mixed Integer Linear Program

Data

A: m by n rational matrix G: m by n rational matrix

c: n-dimensional rational vector h: p-dimensional rational vector

b: m-dimensional rational vector

Variables

x: n-dimensional vector of variables y: p-dimensional vector of integer variables

$$\max c'x + h'y$$

$$\boxed{Ax + Gy \leq b} \longrightarrow \textbf{Linear constraints}$$

$$x \geq 0$$

$$y \geq 0,\ \text{integer}$$

Integer Linear Program

Data

A: m by n rational matrix
c: n-dimensional rational vector
b: m-dimensional rational vector

Variables

x: n-dimensional vector of integer variables

$$\max c'x$$
$$Ax \leq b$$
$$x \geq 0, \text{ integer}$$

Integer Linear Program

Solution
An assignment of values to variables

Feasible Solution
An assignment of values to variables such that all the constraints are satisfied

Objective function value
Value of a solution obtained by evaluating the objective function at a given point

Optimal solution
(max) is one whose objective function value is greater than or equal to that of all other feasible solutions.

Binary Integer Program

Data
A: m by n rational matrix
c: n-dimensional rational vector
b: m-dimensional rational vector

Variables
x: n-dimensional vector of $\{0,1\}$ variables

$$\max c'x$$
$$Ax \leq b$$
$$x \in \{0,1\}^n$$

Combinatorial Optimization Problem

Given

A finite set $N=\{1, \ldots, n\}$

A vector c of weights $\{c_1, c_2, \ldots, c_n\}$ for each $j \in N$

A collection *F* of feasible subsets of N

Find

A minimum (maximum) weight feasible subset

$$\min_{S \subseteq N} \{\sum_{j \in S} c_j : S \in \mathcal{F}\} \quad (\max_{S \subseteq N} \{\sum_{j \in S} c_j : S \in \mathcal{F}\})$$

S can be represented by an incidence vector $x = (x_1, x_2, \ldots, x_n)$

where $x_j = \begin{cases} 1 & \text{if } j \in S, \\ 0 & \text{otherwise} \end{cases}$

An example: the Knapsack Problem

Given

A set of items $N = \{1, \ldots, n\}$

Each item has associated a weight a_j and a profit c_j

A capacity b

Find

A subset of items such that the sum of weights does not exceed b and the sum of profits is maximized

Variables definition

$$x_j = \begin{cases} 1 & \text{if item } j \text{ is selected,} \\ 0 & \text{otherwise} \end{cases}$$

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An example: the Knapsack Problem

Constraints definition

$$\sum_{j=1}^{n} a_j x_j \leq b$$

$$x \in \{0,1\}^n$$

Objective function

$$\max \sum_{j=1}^{n} c_j x_j$$

An example: the Knapsack Problem

$$\max\ 10x_1 + 14x_2 + 12x_3 + 8x_4$$
$$\text{s.t.}$$
$$2x_1 + 3x_2 + 4x_3 + x_4 \le 5$$
$$x \in \{0,1\}^4$$

	a	**c**
1	2	10
2	3	14
3	4	12
4	1	8

Capacity: $b = 5$

$$\mathcal{F} = \{\{\emptyset\},\{1\},\{2\},\{3\},\{4\},\{1,2\},\{1,4\},\{2,4\},\{3,4\}\}$$

The family *F* can be represented by a characteristic vector x

An example: the Knapsack Problem

Contraints

$$2x_1 + 3x_2 + 4x_3 + x_4 \leq 5$$
$$x \in \{0, 1\}^4$$

represent the set of feasible solutions

$$x \in \mathcal{F} \rightarrow \left\{ \begin{pmatrix}0\\0\\0\\0\end{pmatrix}, \begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\0\\1\end{pmatrix}, \begin{pmatrix}0\\0\\0\\0\end{pmatrix}, \begin{pmatrix}1\\1\\0\\0\end{pmatrix}, \begin{pmatrix}1\\0\\0\\1\end{pmatrix}, \begin{pmatrix}0\\1\\0\\1\end{pmatrix}, \begin{pmatrix}0\\0\\1\\1\end{pmatrix} \right\}$$

Set Covering, Set Packing, Set Partitioning

Given

Two finite sets $M=\{1, \ldots, m\}$ and $N=\{1, \ldots, n\}$
A collection of n subsets of *M*: $\{M_1, M_2, \ldots, M_n\}$
A weight c_j for each Mj

A subset $F \subset N$ is a **cover** of M if

$$\cup_{j \in F} M_j = M$$

A subset $F \subset N$ is a **packing** of M if

$$M_h \cap M_k = \emptyset \ \ \forall h, k, h \neq k$$

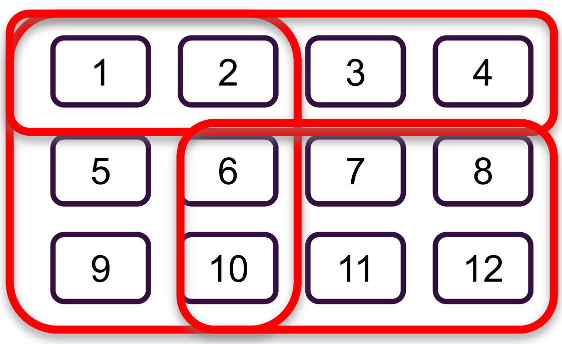
A subset $F \subset N$ is a **partition** of M if it is both a cover and a packing of M

Set Covering, Set Packing, Set Partitioning

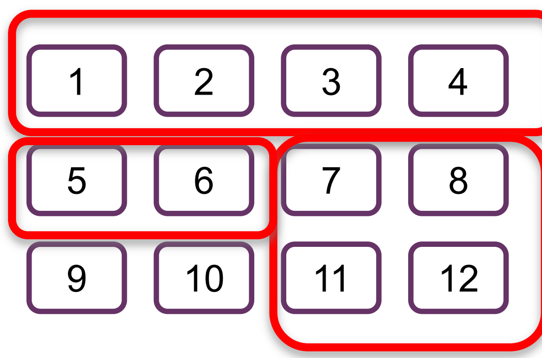
$M = \{1,2,3,4,5,6,7,8,9,10,11,12\}$

$M_1 = \{1,2,3,4\}$
$M_2 = \{1,2,5,6,9,10\}$
$M_3 = \{7,8,11,12\}$
$M_4 = \{5,6\}$

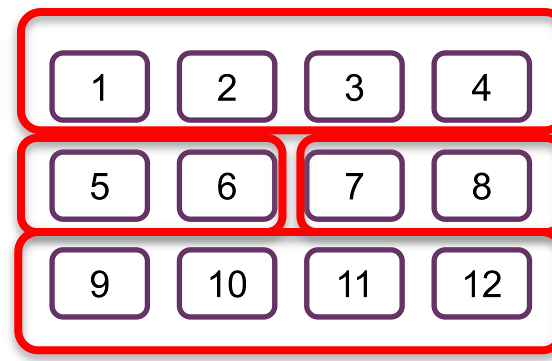
$M_5 = \{6,7,8,10,11,12\}$
$M_6 = \{9,10,11,12\}$
$M_7 = \{7,8\}$



M_1, M_2, M_5 is a **cover**



M_1, M_3, M_4 is a **packing**



M_1, M_4, M_6, M_7 is a **partition**

Set Covering, Set Packing, Set Partitioning

Define

A $m \times n$ incidence matrix A of the family $\{M_j | j \in N\}$

$$a_{ij} = \begin{cases} 1 \text{ if } i \in M_j \\ 0 \text{ otherwise} \end{cases}$$

A decision variable x_j, $j=1, \ldots, n$

$$x_j = \begin{cases} 1 \text{ if } j \in F \\ 0 \text{ otherwise} \end{cases}$$

Set Covering, Set Packing, Set Partitioning

$$\begin{aligned} \min c'x \\ \text{s.t.} \quad Ax \geq 1 \\ x \in \{0,1\}^n \end{aligned}$$

Set covering

$$\begin{aligned} \max c'x \\ \text{s.t.} \quad Ax \leq 1 \\ x \in \{0,1\}^n \end{aligned}$$

Set packing

$$\begin{aligned} \min(\text{ or } \max)\ c'x \\ \text{s.t.} \quad Ax = 1 \\ x \in \{0,1\}^n \end{aligned}$$

Set partitioning

Modeling logical conditions by binary variables

Let x,y,z be binary variables

Negation

$$z = \neg x \rightarrow z = 1 - x$$

AND

x	y	$z = x \wedge y$
0	0	0
0	1	0
1	0	0
1	1	1

$$\rightarrow \begin{cases} z - x \leq 0 \\ z - y \leq 0 \\ x + y - z \leq 1 \end{cases}$$

Modeling logical conditions by binary variables

OR

x	y	$z = x \vee y$
0	0	0
0	1	1
1	0	1
1	1	1

$$\rightarrow \begin{cases} z - x \geq 0 \\ z - y \geq 0 \\ x + y - z \geq 0 \end{cases}$$

Exclusive OR

x	y	$z = x \oplus y$
0	0	0
0	1	1
1	0	1
1	1	0

$$\rightarrow \begin{cases} x - y + z \geq 0 \\ -x + y + z \geq 0 \\ x + y - z \geq 0 \\ z + x + y \leq 2 \end{cases}$$

Modeling dependent decisions

Binary variables can be used to model dependency between two choices.

Suppose x and y are binary variables such that:

$$x = \begin{cases} 1 & \text{if project } x \text{ is selected,} \\ 0 & \text{otherwise} \end{cases} \qquad y = \begin{cases} 1 & \text{if project } y \text{ is selected,} \\ 0 & \text{otherwise} \end{cases}$$

Suppose that project x can be selected **only if** project y has already been selected.

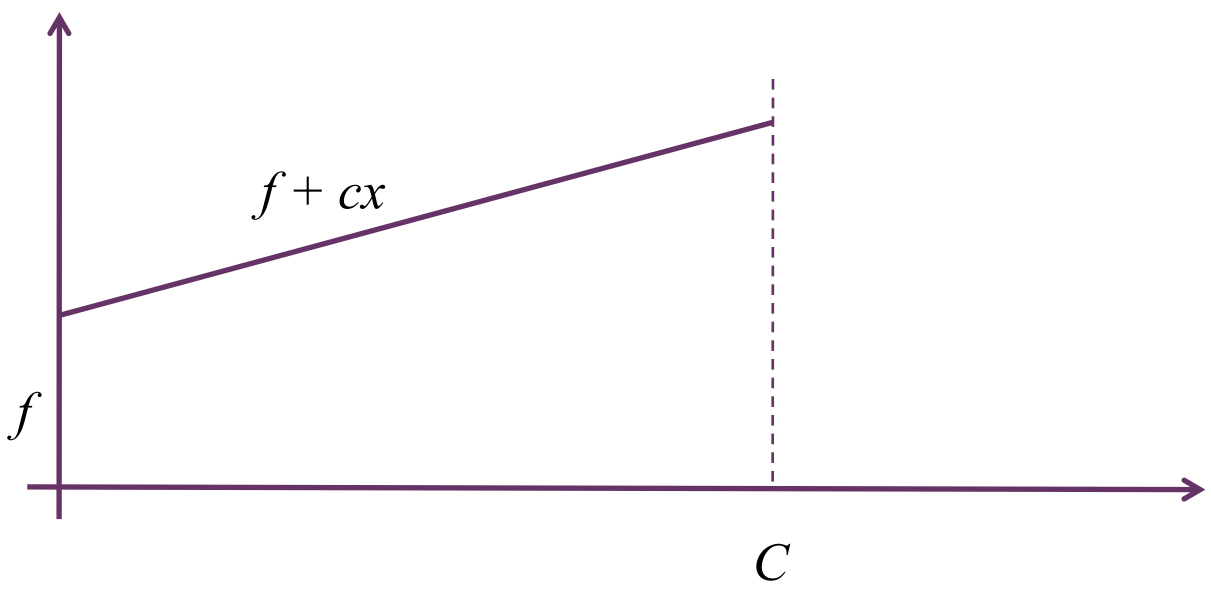
This can be expressed by the (linear) constraint

$$x - y \leq 0$$

Modeling fixed cost

Consider the following nonlinear objective function (f, c >0)

$$\min h(x) = \begin{cases} f + cx \text{ if } 0 < x \leq C \\ 0 \text{ if } x = 0 \end{cases}$$



Modeling fixed cost

Define an additional binary variable

$$y = \begin{cases} 1 \text{ if } x > 0 \\ 0 \text{ if } x = 0 \end{cases} \qquad (*)$$

Replace $$h(x) \rightarrow f + cx$$

Add the constraints

$$x - Cy \leq 0$$
$$y \in \{0, 1\}$$

Observation: $x = 0$ and $y = 1$ is a feasible solution in contrast to (*), but an optimal solution will always have y=0 if x=0 (minization problem)

Uncapacitated Facility Location

Given

A set $N=\{1, \ldots, n\}$ of potential depots (facilities) and a set $M=\{1, \ldots, m\}$ of clients

A cost f_j of opening facitity j

A cost c_{ij} associated with serving customer i from facility j

Variables definition

$$y_j = \begin{cases} 1 \text{ if facility } j \text{ is open,} \\ 0 \text{ otherwise} \end{cases}$$

$$x_{ij} = \begin{cases} 1 \text{ if client } i \text{ is served from facility } j, \\ 0 \text{ otherwise} \end{cases}$$

Uncapacitated Facility Location

Constraints definition

Demand of client i must be satisfied

$$\sum_{j=1}^{n} x_{ij} = 1 \text{ for } i = 1, \ldots, m$$

A client can be served from facility j only if facility j is open

$$x_{ij} \leq y_j \quad \forall i, j$$

Objective function

$$\min \sum_{j=1}^{n} f_j y_j + \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij}$$

Uncapacitated Facility Location

Formulation

$$\min \sum_{j=1}^{n} f_j y_j + \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij}$$

$$\sum_{j=1}^{n} x_{ij} = 1 \text{ for } i = 1, \ldots, m$$

$$x_{ij} \leq y_j \quad \forall i, j$$

$$x_{ij} \in \{0, 1\}$$

$$y_j \in \{0, 1\}$$

Modeling restricted set of values

x only takes values in a given set $\{a_1, a_2, \ldots, a_m\}$.

Model

Introduce m binary variables y_j, $j = 1, \ldots, m$ and the following constraints:

$$x = \sum_{j=1}^{m} a_j y_j,$$

$$\sum_{j=1}^{m} y_j = 1$$

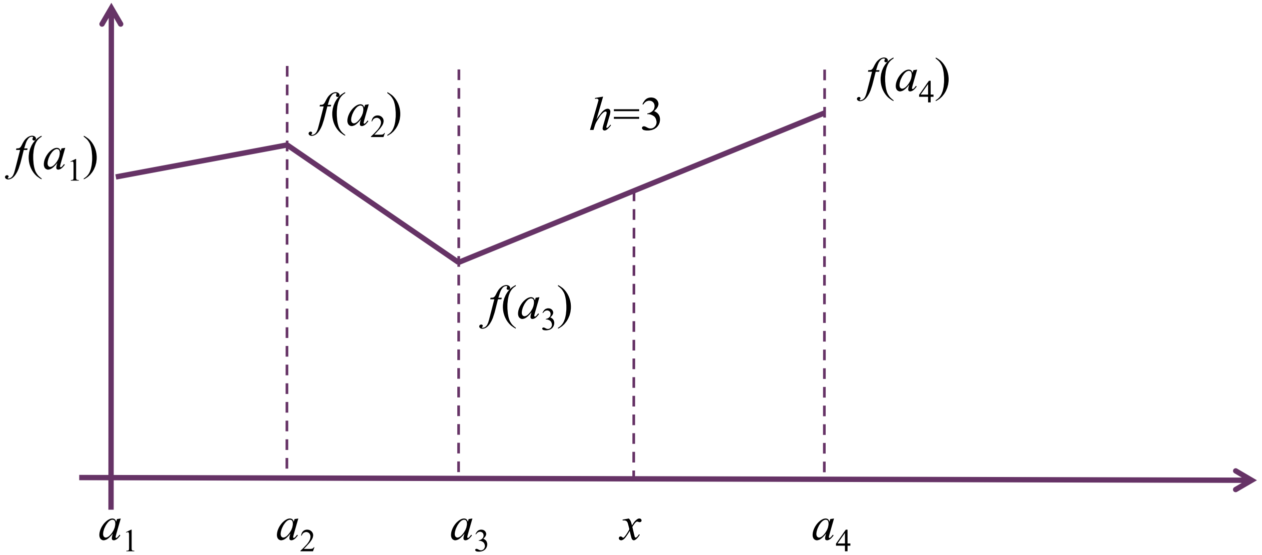
$$y_j \in \{0, 1\}$$

Piecewise linear cost function

A **piecewise linear** cost function is specified by m ordered pairs $(a_i, f(a_i))$

At a given point x in the h-th interval $[a_h, a_{h+1}]$ the function can be evaluated by:

$$
\begin{aligned}
f(x) = \lambda_h f(a_h) + \lambda_{h+1} f(a_{h+1}) \\
\lambda_h + \lambda_{h+1} = 1 \\
\lambda_h, \lambda_{h+1} \geq 0
\end{aligned}
$$



Piecewise linear cost function

Define **binary variables**

$$y_h = \begin{cases} 1 \text{ if } x \text{ is in the interval } [a_h, a_{h+1}] \\ 0 \text{ otherwise} \end{cases} \qquad \text{for } h = 1, \ldots, m-1$$

Define **continous variables**

$$\lambda_h \geq 0 \text{ for } h = 1, \ldots, m$$

Objective function

$$\min \sum_{i=1}^{m} \lambda_i f(a_i)$$

Piecewise linear cost function

Constraints

$$
\begin{aligned}
\sum_{i=1}^{m} \lambda_i &= 1, \\
\lambda_1 &\le y_1, \\
\lambda_i &\le y_{i-1} + y_i, && i \in [2, \ldots, m-1], \\
\lambda_m &\le y_{m-1}, \\
\sum_{i=1}^{m-1} y_i &= 1,
\end{aligned}
$$

This set of constraints implies that there are only two "active" λ-s corresponding to interval in which x lies, i.e.,

$$y_j = 1 \text{ implies } \lambda_i = 0 \ \ \forall i \neq j, j+1.$$

Piecewise linear cost function

Formulation

$$
\begin{aligned}
\min \sum_{i=1}^{m} \lambda_i f(a_i) \\
\sum_{i=1}^{m} \lambda_i &= 1, \\
\lambda_1 &\leq y_1, \\
\lambda_i &\leq y_{i-1} + y_i, \qquad\qquad i \in [2, \ldots, m-1], \\
\lambda_m &\leq y_{m-1}, \\
\sum_{i=1}^{m-1} y_i &= 1, \\
\lambda_h &\geq 0 \text{ for } h = 1, \ldots, m \\
y_i &= \{0, 1\} \text{ for } i = 1, ..., m-1
\end{aligned}
$$

Fixed Charge Network Flow Problem

Given

A directed graph $G=(N,A)$ with a source node s

A demand vector

$$b \in \mathbb{Z}^{|N|} \text{ such that: } b_s > 0, b_i \leq 0 \; \forall i \in N, i \neq \{s\} \text{ and } \sum_{i \in N} b_i = 0$$

A capacity vector $u \in \mathbb{Z}^{|A|}$

A cost vector $c \in \mathbb{Z}^{|A|}$ representing the arcs activation cost

Find

A feasible flow vector (see definition of feasible flow in Module I)

$$0 \leq x \leq u$$

that minimizes the cost of activated arcs

Fixed Charge Network Flow Problem

Variables

Flow variables

$$0 \leq x_{ij} \leq u_{ij} \quad \forall (i,j) \in A$$

Activation variables

$$y_{ij} = \begin{cases} 1 \text{ if arc } (i,j) \text{ is activated,} \\ 0 \text{ otherwise.} \end{cases}$$

Objective function

$$\min \sum_{(i,j) \in A} c_{ij} y_{ij}$$

Fixed Charge Network Flow Problem

Constraints

Flow balance constraints

$$\sum_{(i,j)\in\delta^+(i)} x_{ij} - \sum_{(j,i)\in\delta^-(i)} x_{ji} = b_i \quad \forall i \in N$$

Activation constraints

$$0 \leq x_{ij} \leq u_{ij} y_{ij}$$

Fixed Charge Network Flow Problem

Formulation

$$
\begin{array}{rcll}
\min \displaystyle\sum_{(i,j)\in A} c_{ij} y_{ij} & & & \\
\text{s.t.} & & & \\
\displaystyle\sum_{(i,j)\in\delta^+(i)} x_{ij} - \sum_{(j,i)\in\delta^-(i)} x_{ji} & = & b_i & \forall i \in N \\
x_{ij} - u_{ij} y_{ij} & \leq & 0 & \forall (i,j) \in A \\
x & \geq & 0 & \\
y & \in & \{0,1\}^{|A|} &
\end{array}
$$

Disjunctive constraints

Given two constraints

$$a'x \geq b$$

$$e'x \geq f$$

$$\text{with } a, b, e, f \geq 0 \text{ and } x \geq 0$$

Suppose we want to model $a'x \geq b \vee e'x \geq f$
(i.e., at least one of the two constraints is satisfied)

Introduce a binary variable y and impose

$$a'x \geq by$$

$$e'x \geq f(1-y)$$

$$y \in \{0, 1\}$$

Disjunctive constraints

In general, given,

$$a_i' x \geq b_i \quad i = \{1, \ldots, m\}$$

if you want that at least k out of m constraints are satisfied, you can impose

$$a_i' x \geq b_i y_i \quad i = \{1, \ldots, m\}$$

$$\sum_{i=1}^{m} y_i \geq k$$

$$y_i \in \{0, 1\}$$

Disjunctive constraints

Single machine scheduling problem: n jobs, with processing time p_j. Let $t_j > 0$ be the start time of job j.

Consider the pair of jobs h and k. One has:

$$t_k \geq t_h + p_h \text{ if } h \text{ precedes } k$$

$$t_h \geq t_k + p_k \text{ if } k \text{ precedes } h$$

Introduce the binary variable

$$y_{hk} = \begin{cases} 1 \text{ if } h \text{ precedes } k, \\ 0 \text{ otherwise.} \end{cases} \Rightarrow \begin{aligned} t_k &\geq t_h + p_h - M(1 - y_{hk}) \\ t_h &\geq t_k + p_k - M y_{hk} \end{aligned}$$

where M is a constant large enough. The reasoning can be easily extended to the whole set of jobs.

Minimum Spanning Tree

Given

A symmetric graph G=(V,E) and a cost $c_e \geq 0$ for each edge in E

Find

A minimum cost spanning tree T

Notation

$E(S)$: set of edges induced by $S \subset V$

Cutset $\delta(S) = \{\{i,j\} \in E | i \in S, j \notin S\}$

Variables

$$x_e = \begin{cases} 1 \text{ if edge } e \text{ is selected,} \\ 0 \text{ otherwise.} \end{cases}$$

Constraints

T is spanning

$$\sum_{e \in E} x_e = |V| - 1$$

Minimum Spanning Tree

Connectivity
Can be imposed by one of the sets of constraints

Subtour inequalities

$$\sum_{e \in E(S)} x_e \leq |S| - 1$$

$$\forall S \subset V, 2 < |S| \leq |V| - 1$$

Cutset inequalities

$$\sum_{e \in \delta(S)} x_e \geq 1$$

$$\forall S \subset V, S \neq \emptyset, V$$

Observation
Both sets contains an exponential number (in $|V|$) of constraints

Minimum Spanning Tree

Formulation 1

$$\begin{aligned}
&\min \sum_{e\in E} c_e x_e \\
&\text{s.t.} \sum_{e\in E} x_e = |V|-1, \\
&\sum_{e\in E(S)} x_e \le |S|-1 \qquad \forall S\subset V, 2<|S|\le |V|-1, \\
&x\in\{0,1\}^{|E|}
\end{aligned}$$

Formulation 2

$$\begin{aligned}
&\min \sum_{e\in E} c_e x_e \\
&\text{s.t.} \sum_{e\in E} x_e = |V|-1, \\
&\sum_{e\in \delta(S)} x_e \ge 1 \qquad \forall S\subset V, S\neq\emptyset, V \\
&x\in\{0,1\}^{|E|}
\end{aligned}$$

Traveling Salesman Problem

Given
A symmetric graph $G=(V,E)$ and a cost $c_e \geq 0$ for each edge in E
Find
A Hamiltonian tour (a cycle that visits all nodes) of minimum cost

Decision variable

$$x_e = \begin{cases} 1 \text{ if edge } e \text{ is selected,} \\ 0 \text{ otherwise.} \end{cases}$$

Property of a Hamiltonian tour

$$\sum_{e \in \delta(\{i\})} x_e = 2$$

Traveling Salesman Problem

Cut property

$$\sum_{e \in \delta(S)} x_e \geq 2 \;\; \forall S \subset V, S \neq \emptyset, V$$

Cutset formulation

$$\begin{aligned}
&\min \sum_{e \in E} c_e x_e \\
\text{s.t.} \quad &\sum_{e \in \delta(\{i\})} x_e = 2, \\
&\sum_{e \in \delta(S)} x_e \geq 2 \qquad \forall S \subset V, S \neq \emptyset, V \\
&x \in \{0,1\}^{|E|}
\end{aligned}$$

Traveling Salesman Problem

Subtour inequalities

$$\sum_{e \in \delta(S)} x_e \leq |S| - 1 \quad \forall S \subset V, S \neq \emptyset, V$$

Subtour formulation

$$\begin{aligned}
&\min \sum_{e \in E} c_e x_e \\
&\text{s.t.} \sum_{e \in \delta(\{i\})} x_e = 2, \\
&\sum_{e \in \delta(S)} x_e \leq |S| - 1 \qquad \forall S \subset V, S \neq \emptyset, V \\
&x \in \{0,1\}^{|E|}
\end{aligned}$$