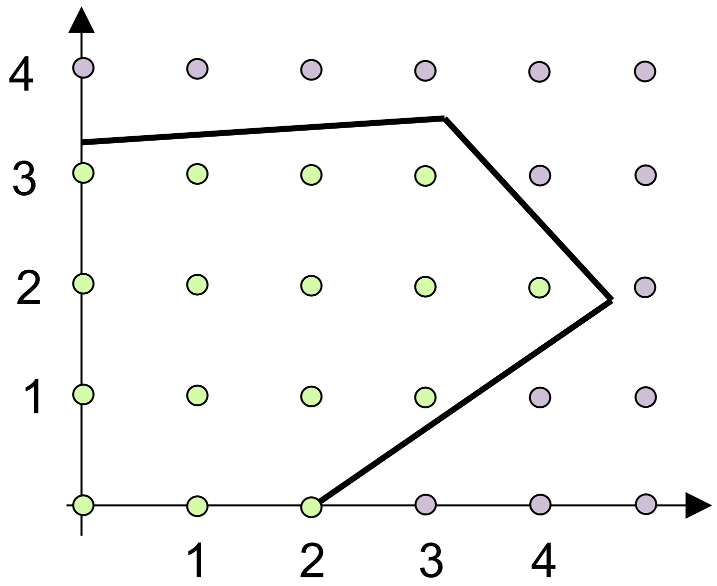
Network Optimization

Part II Branch-and-bound algorithm
Wolsey, Integer Programming Chapter 7

Integer Program: feasible region



$$z^* := \max c'x$$
$$x \in P$$
$$x \in \mathbb{Z}^n$$

$P = \{x \in \mathbb{R}^n : Ax \leq b, x \geq 0\}$

P bounded and not empty

$S = P \cap \mathbb{Z}^n$ set of feasible points

x^* **optimal solution** of value z^*

Linear Relaxation

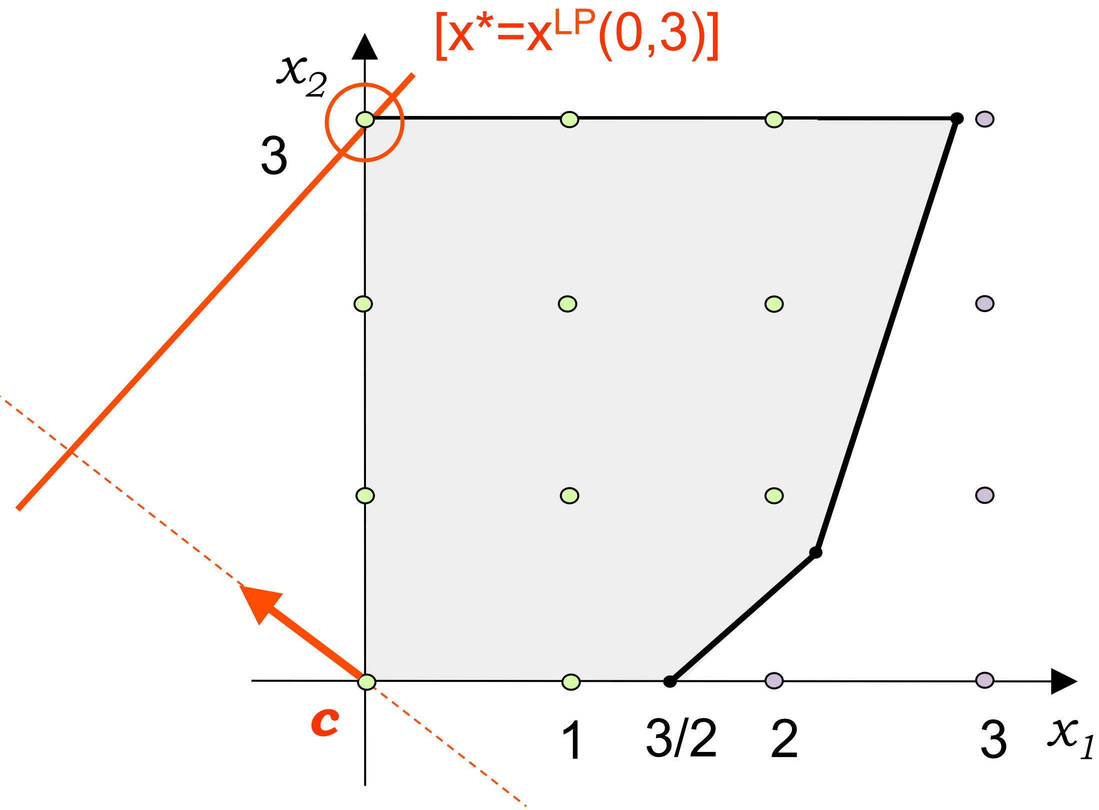
$$
\begin{aligned}
z^{\mathrm{LP}} := \max\, & c'x \\
Ax &\leq b \\
x &\geq 0
\end{aligned}
$$

If the optimal solution x^{LP} of the linear relaxation is **integral**, then x^{LP} is also **optimal** for the IP

$$
\begin{cases}
z^{\mathrm{LP}} = c'x^{\mathrm{LP}} \geq z^* \ \text{(Relaxation)} \\
x^{\mathrm{LP}} \in S, \ \text{i.e.} \ c'x^{\mathrm{LP}} \leq z^* \ \text{(Feasibility)}
\end{cases}
\qquad \Rightarrow c'x^{\mathrm{LP}} = z^*
$$

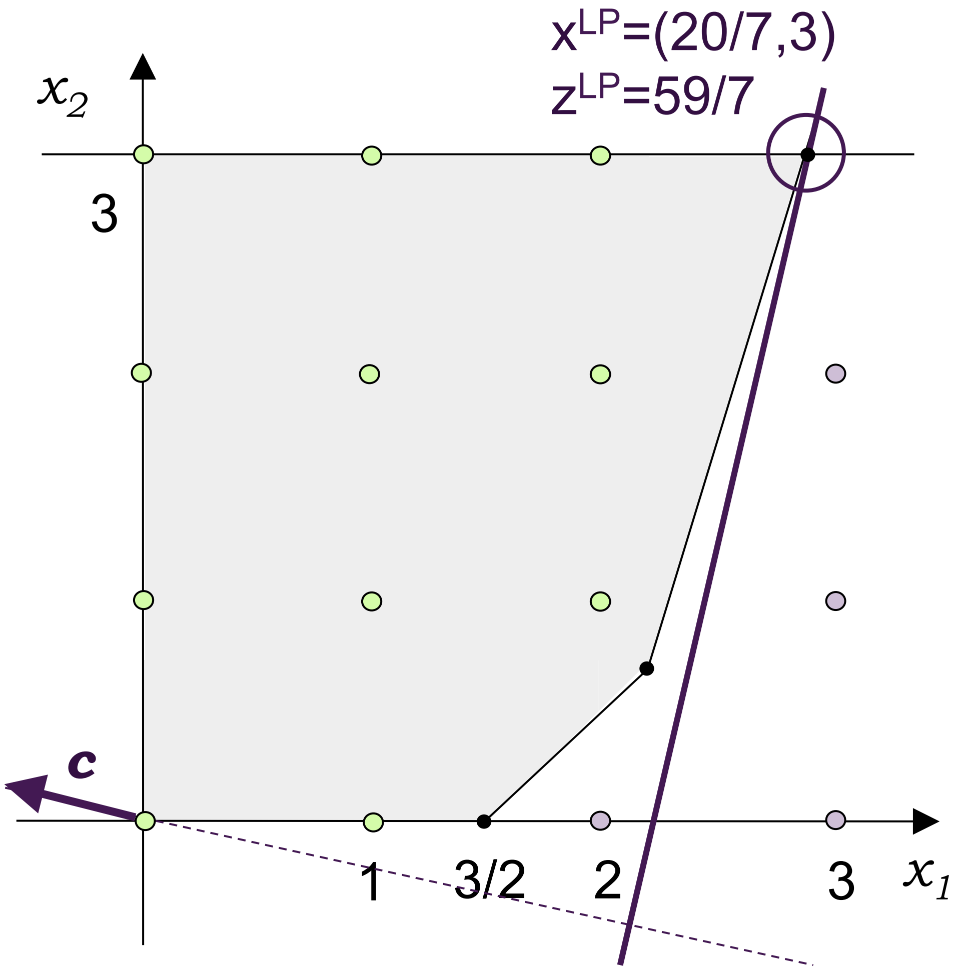
Example: x^{LP} is **integer**

$$\max -x_1 + x_2$$
$$7x_1 - 2x_2 \leq 14$$
$$x_2 \leq 3$$
$$2x_1 - 2x_2 \leq 3$$
$$x_1, x_2 \geq 0$$



Example: x^{LP} is **fractional**

$$\max 4x_1 - x_2$$
$$7x_1 - 2x_2 \leq 14$$
$$x_2 \leq 3$$
$$2x_1 - 2x_2 \leq 3$$
$$x_1, x_2 \geq 0$$



+ x^{LP} is **fractional**

$S \subset P$ implies that:

$$z^{\text{LP}} = \max_{x \in P} c'x \geq \max_{x \in S} c'x = z^*$$

z^{LP} is an **upper bound** on z^*

Note
The value z^{H} of **any integer feasible solution** x^{H} is a **lower bound** on z^*

x^{LP} is **fractional** (minimization case)

$S \subset P$ implies that:

$$z^{\mathrm{LP}} = \min_{x \in P} c'x \leq \min_{x \in S} c'x = z^*$$

z^{LP} is a **lower bound** on z^*

Note
The value z^{H} of **any integer feasible solution** x^{H} is an **upper bound** on z^*

x^{LP} is fractional: **branching**

Given the solution x^{LP}, **select** a **fractional component** x_h and **partition** the original problem into two supbroblems:

$$
\begin{aligned}
z_1^* &:= \max c'x \\
Ax &\leq b \\
x &\geq 0 \\
x_h &\leq \lfloor x_h^{\mathrm{LP}} \rfloor \\
x &\in \mathbb{Z}^n
\end{aligned}
$$

$$
\begin{aligned}
z_2^* &:= \max c'x \\
Ax &\leq b \\
x &\geq 0 \\
x_h &\geq \lceil x_h^{\mathrm{LP}} \rceil \\
x &\in \mathbb{Z}^n
\end{aligned}
$$

x^{LP} is fractional: **branching**

The branching defines two subproblems and two relaxations:

$$P_1 = \{x \in P : x_h \leq \lfloor x_h^{LP} \rfloor\}$$
$$S_1 = P_1 \cap \mathbb{Z}_+^n$$

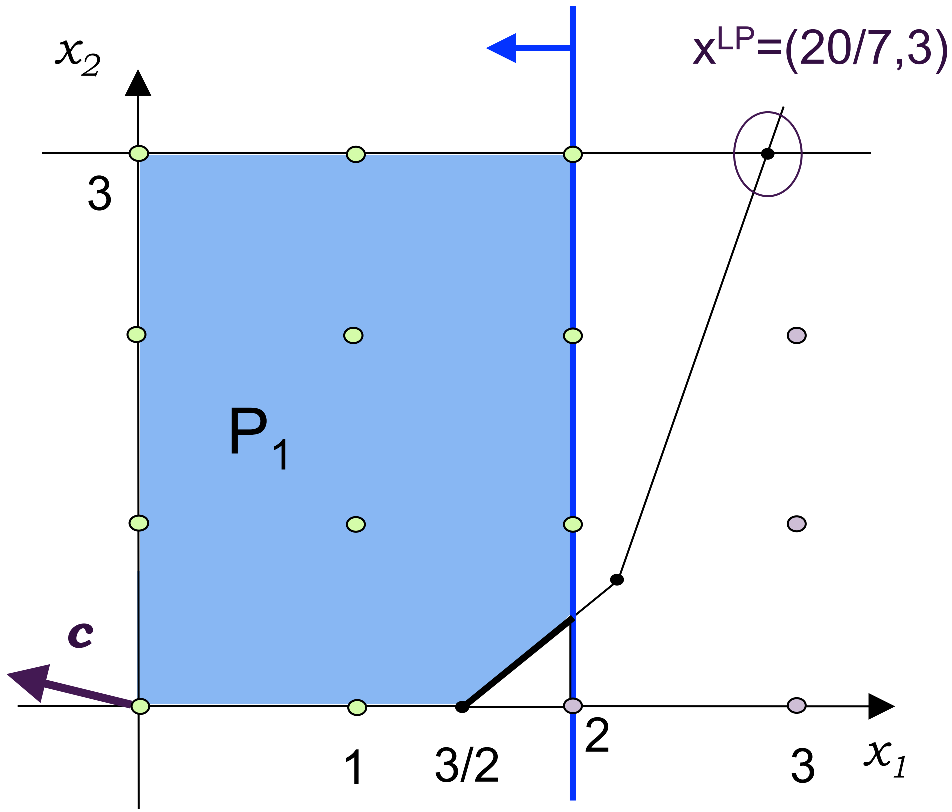
$$P_2 = \{x \in P : x_h \geq \lceil x_h^{LP} \rceil\}$$
$$S_2 = P_2 \cap \mathbb{Z}_+^n$$

Properties

$$x^{\mathrm{LP}} \notin P_1 \text{ and } x^{\mathrm{LP}} \notin P_2$$
$$S_1 \cup S_2 = S, S_1 \cap S_2 = \emptyset$$
$$z^* = \max\{z_1^*, z_2^*\}$$

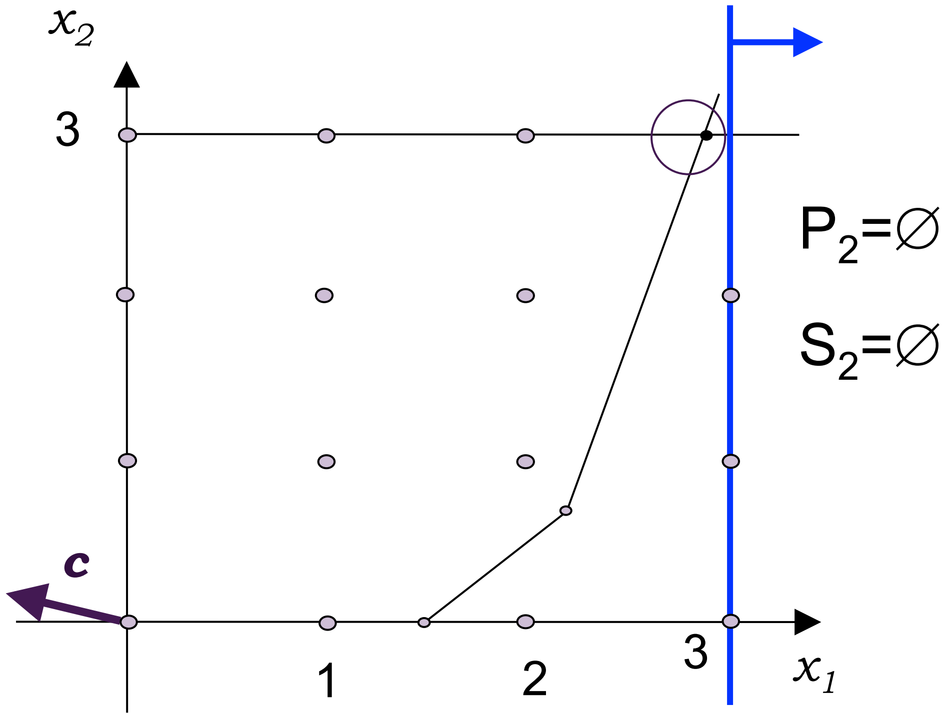
Example: **subproblem** S_1

$$\max 4x_1 - x_2$$
$$7x_1 - 2x_2 \leq 14$$
$$x_2 \leq 3$$
$$2x_1 - 2x_2 \leq 3$$
$$x_1 \leq 2$$
$$x_1, x_2 \geq 0$$



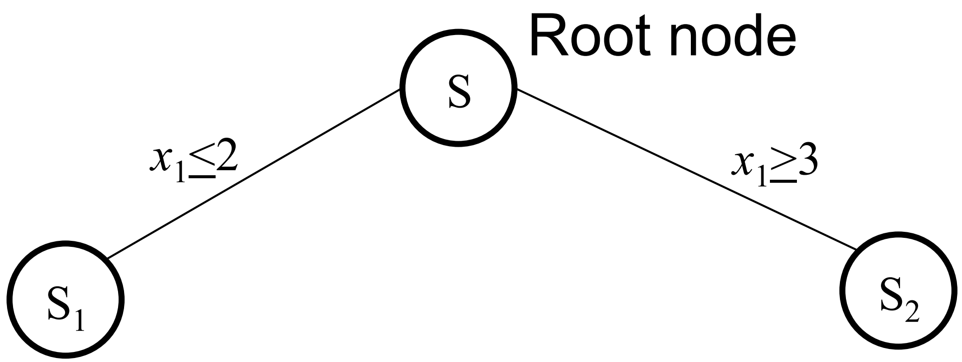
Example: **subproblem** S_2

$$\max 4x_1 - x_2$$
$$7x_1 - 2x_2 \leq 14$$
$$x_2 \leq 3$$
$$2x_1 - 2x_2 \leq 3$$
$$x_1 \geq 3$$
$$x_1, x_2 \geq 0$$



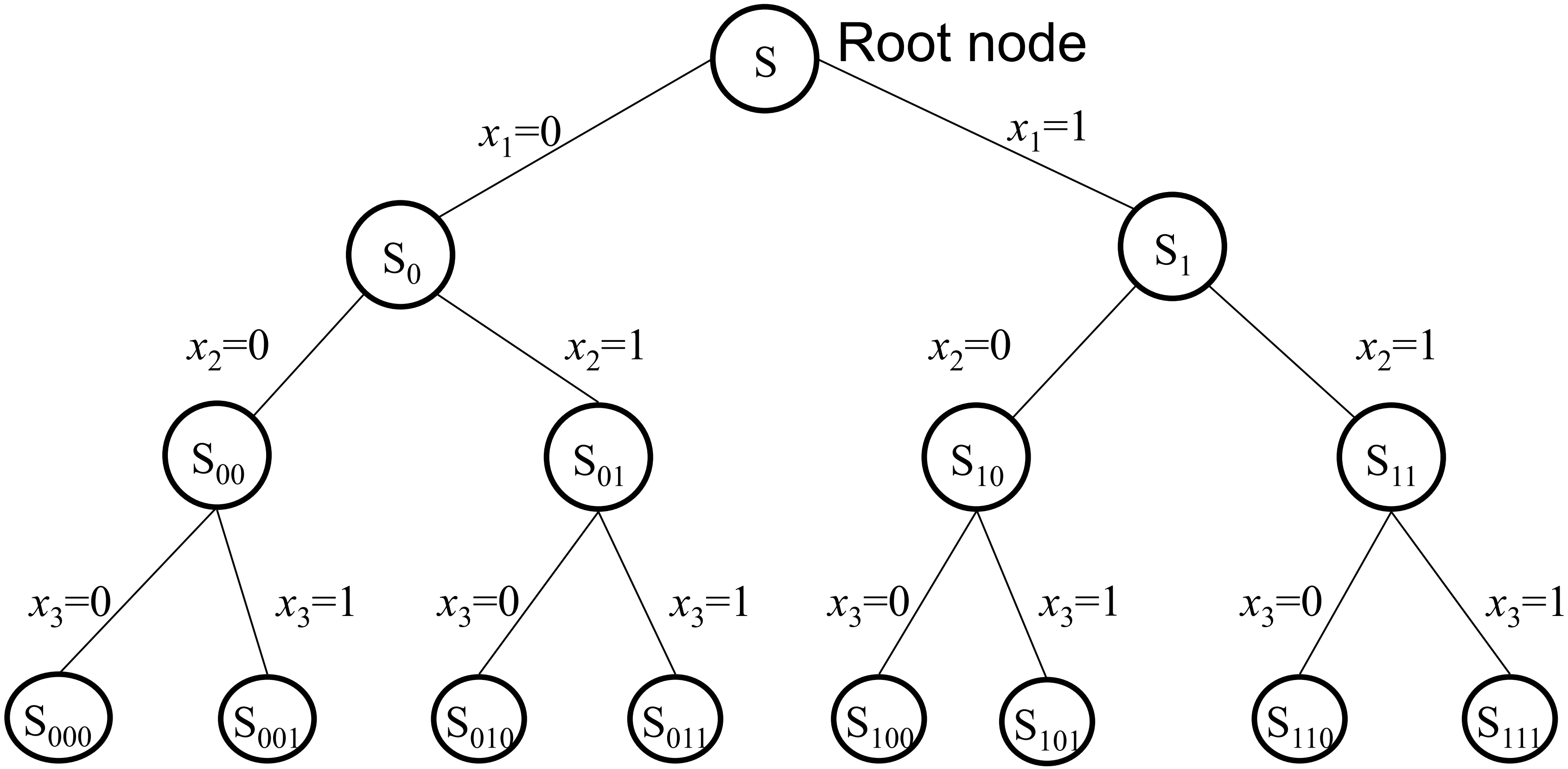
Enumeration tree

Subproblems can be represented by an **enumeration tree**



Enumeration tree

In the **binary case** $x \in \{0,1\}^n$ by applying recursively the partion scheme one can obtain 2^n leaves

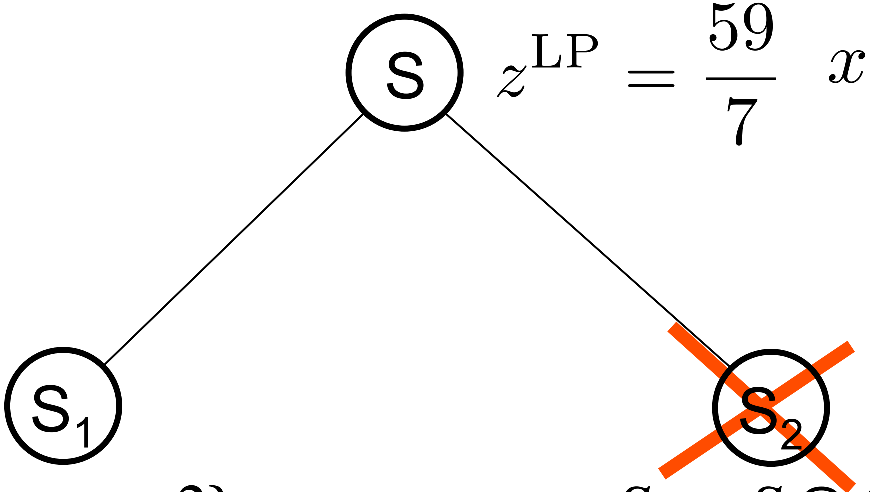


Combining the scheme with the linear relaxation: **implicit enumeration**

Consider subproblem S_2:

$$z^{\mathrm{H}} = -\infty$$

$$z^{\mathrm{LP}} = \frac{59}{7} \quad x^{\mathrm{LP}} = \left(\frac{20}{7}, 3\right)$$



$$S_1 = S \cap \{x : x_1 \leq 2\} \qquad S_2 = S \cap \{x : x_1 \geq 3\}$$

S_2 [P_2] is **empty:** one can avoid to explore the tree from S_2

The node is **pruned by infeasibility**

Combining the scheme with the linear relaxation: **implicit enumeration**

Consider subproblem S_1:

$$z_1^{\mathrm{LP}} = \max 4x_1 - x_2$$

$$7x_1 - 2x_2 \leq 14$$

$$x_2 \leq 3$$

$$2x_1 - 2x_2 \leq 3$$

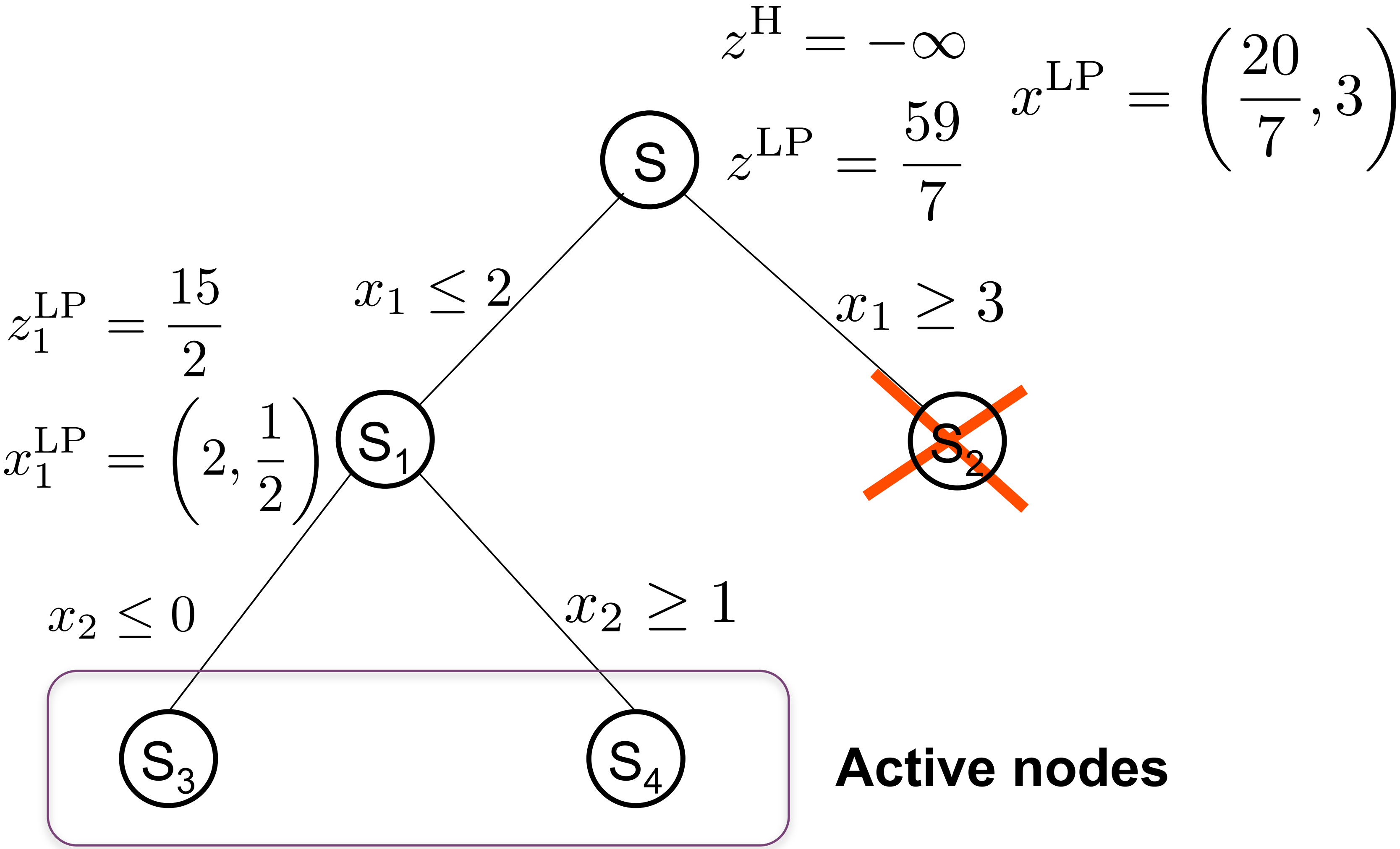
$$x_1 \leq 2$$

$$x_1, x_2 \geq 0$$

$$z_1^{\mathrm{LP}} = \frac{15}{2}$$

$$x_1^{\mathrm{LP}} = \left(2, \frac{1}{2}\right)$$

x_1^{LP} is fractional: **branching**



Subproblem S_4

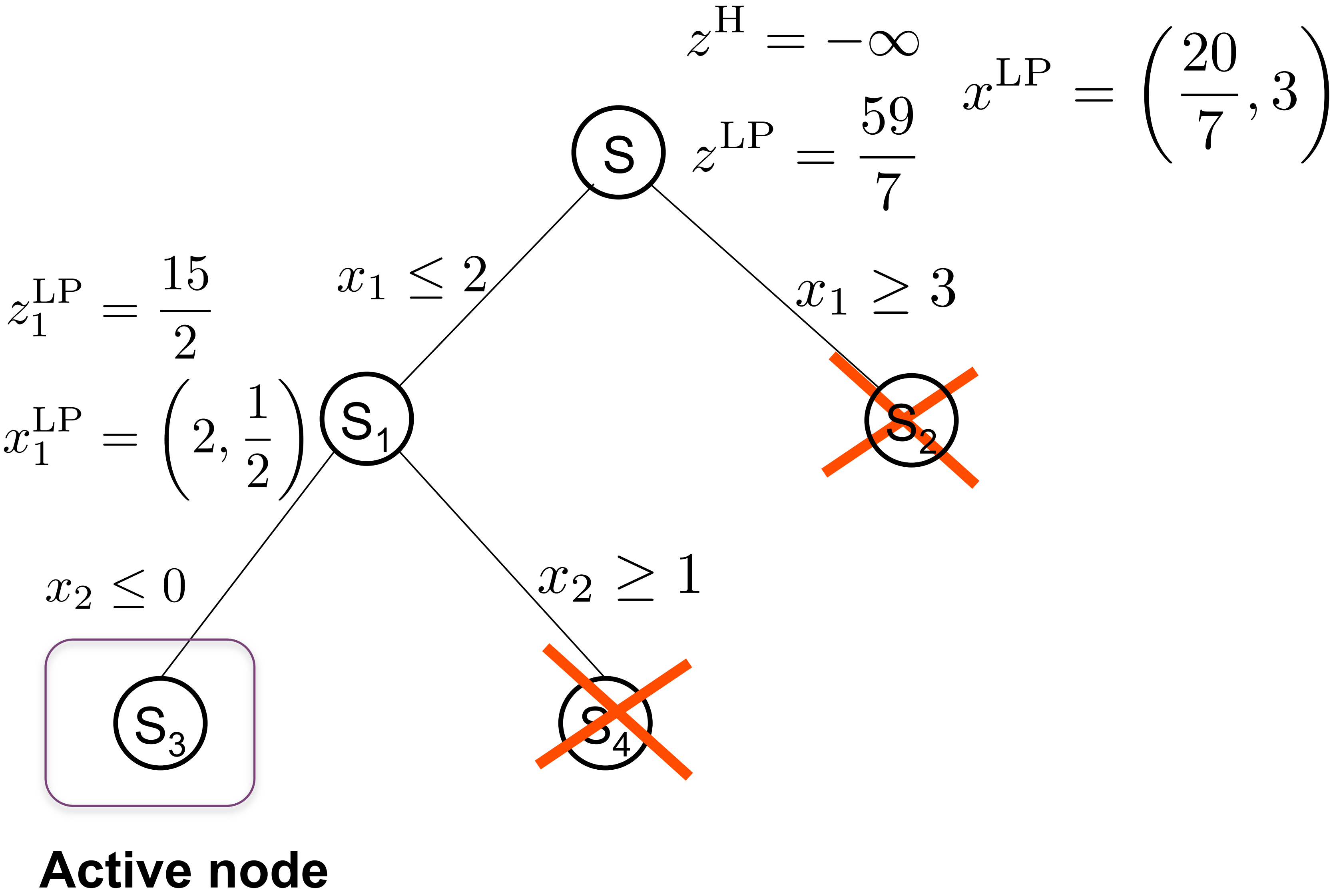
$$z_4^{\text{LP}} = \max 4x_1 - x_2$$
$$7x_1 - 2x_2 \leq 14$$
$$x_2 \leq 3$$
$$2x_1 - 2x_2 \leq 3$$
$$x_1 \leq 2$$
$$x_2 \geq 1$$
$$x_1, x_2 \geq 0$$

$$z_4^{\text{LP}} = 7$$
$$x_4^{\text{LP}} = (2, 1)$$

x_4^{LP} is **integer**

1. S_4 is **pruned by optimality**
2. z_4^{LP} is **better than the current incumbent** z^{H}, the incumbent is **updated** to 7

x_1^{LP} is fractional: **branching**



Subproblem S_3

$$
\begin{aligned}
z_3^{\mathrm{LP}} &= \max 4x_1 - x_2 \\
7x_1 - 2x_2 &\leq 14 \\
x_2 &\leq 3 \\
2x_1 - 2x_2 &\leq 3 \\
x_1 &\geq 2 \\
x_2 &\leq 0 \\
x_1, x_2 &\geq 0
\end{aligned}
$$

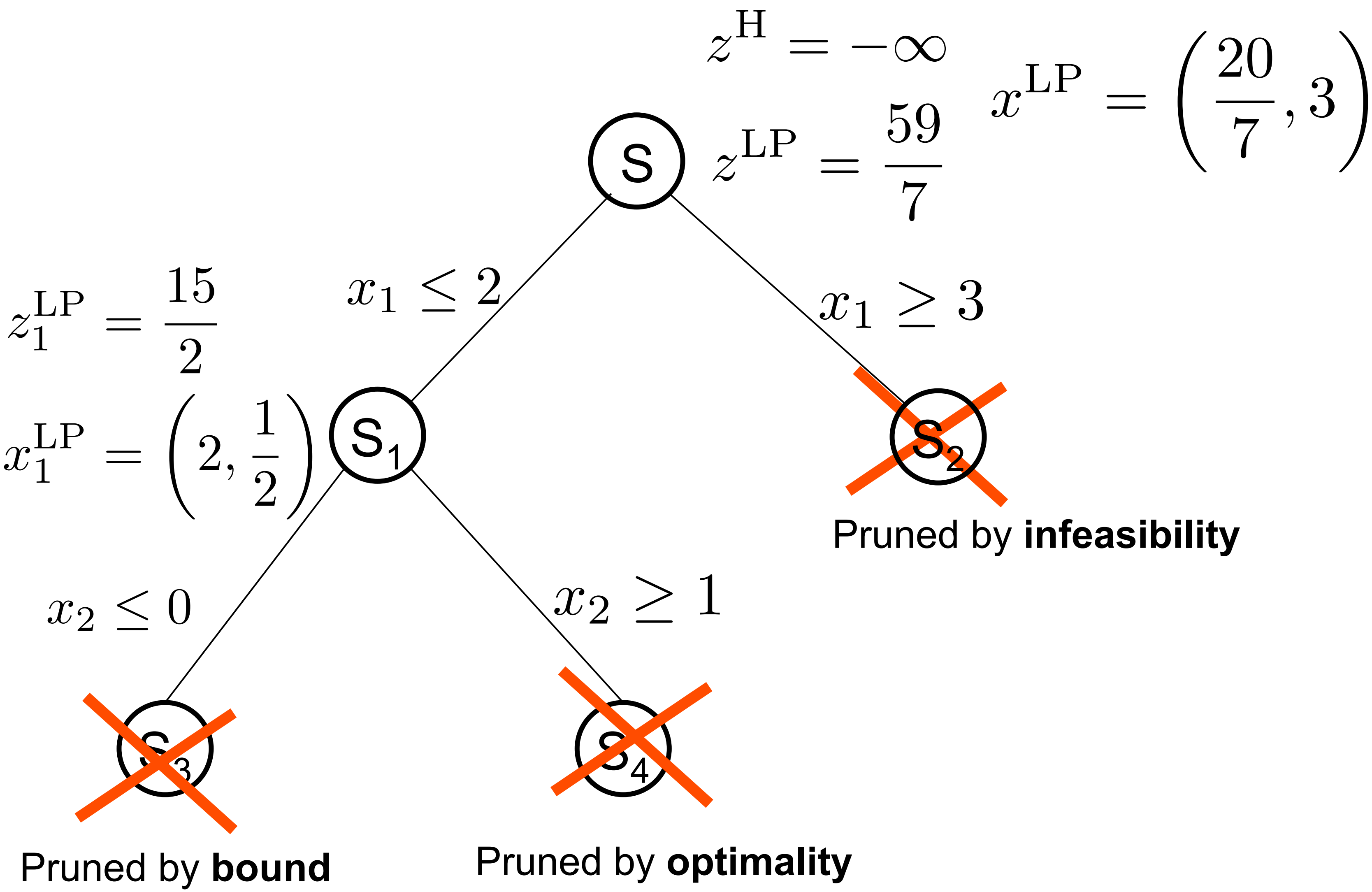
$$z_3^{\mathrm{LP}} = 6$$

$$x_3^{\mathrm{LP}} = \left(\frac{3}{2}, 0\right)$$

x_3^{LP} is **fractional** but $z_3^{LP} < z^H = 7$

S_3 can be **pruned by bound**

x_1^{LP} is fractional: **branching**



Pruning by infeasibility

Let S_t be the current subproblem and P_t be its linear relaxation

If P_t is **empty** then S_t is empty

Consequence

The node corresponding to S_t can be **pruned by infeasibility**

Pruning by optimality

If x_t^{LP} is **integral** then it is the **optimal** for the subproblem S_t.

Consequence

The node corresponding to S_t can be **pruned by optimality**.

Note

If x_t^{LP} is integral, then it is **feasible** also for S

If z_t^{LP} improves the current incumbent z^H, then the incumbent is **updated** z^H=z^{LP}

Pruning by bound

x_t^{LP} is **fractional** and $z^{LP}{}_t \leq z^H$

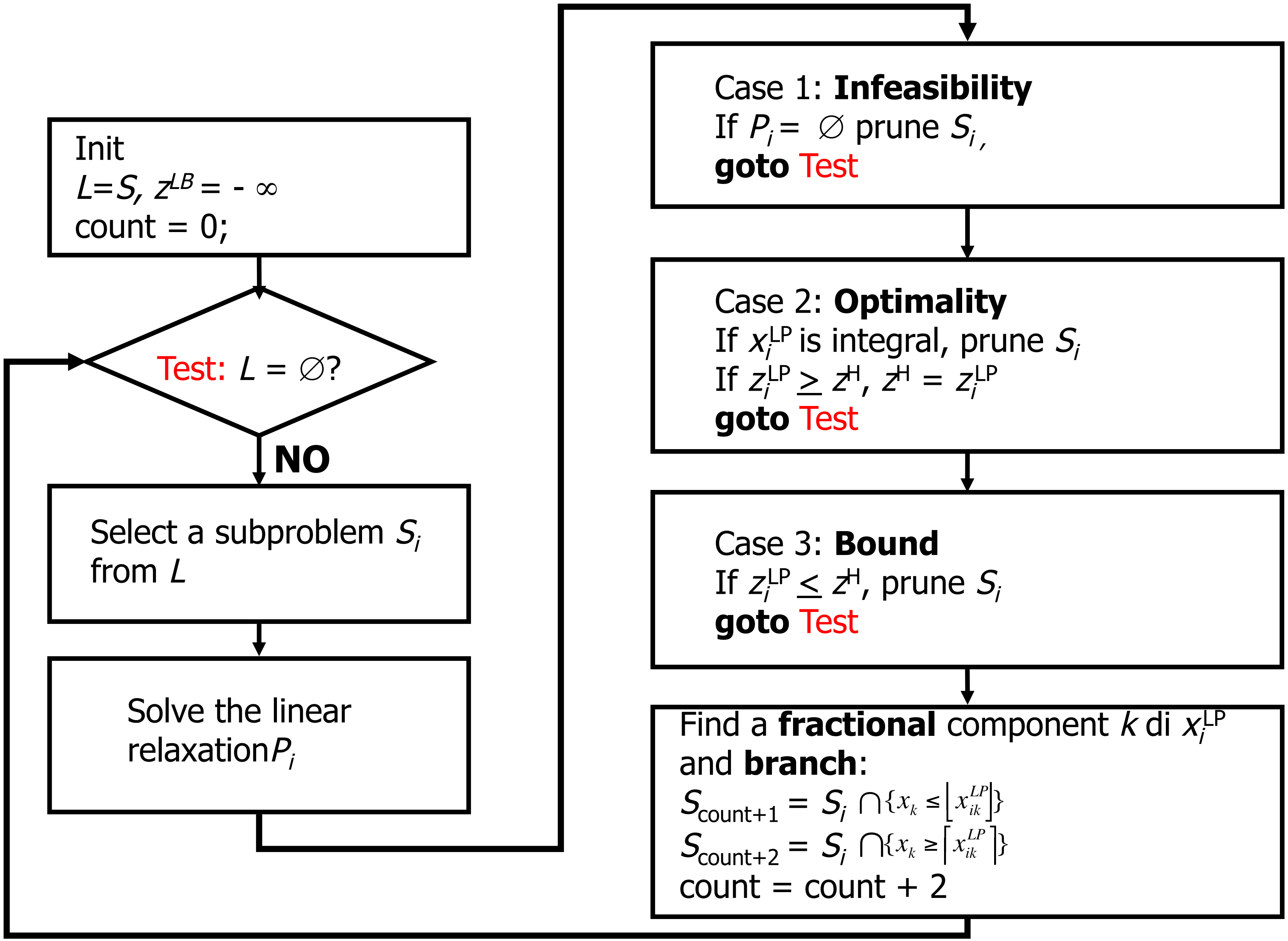
Consequence

S_t can be **pruned by bound**.

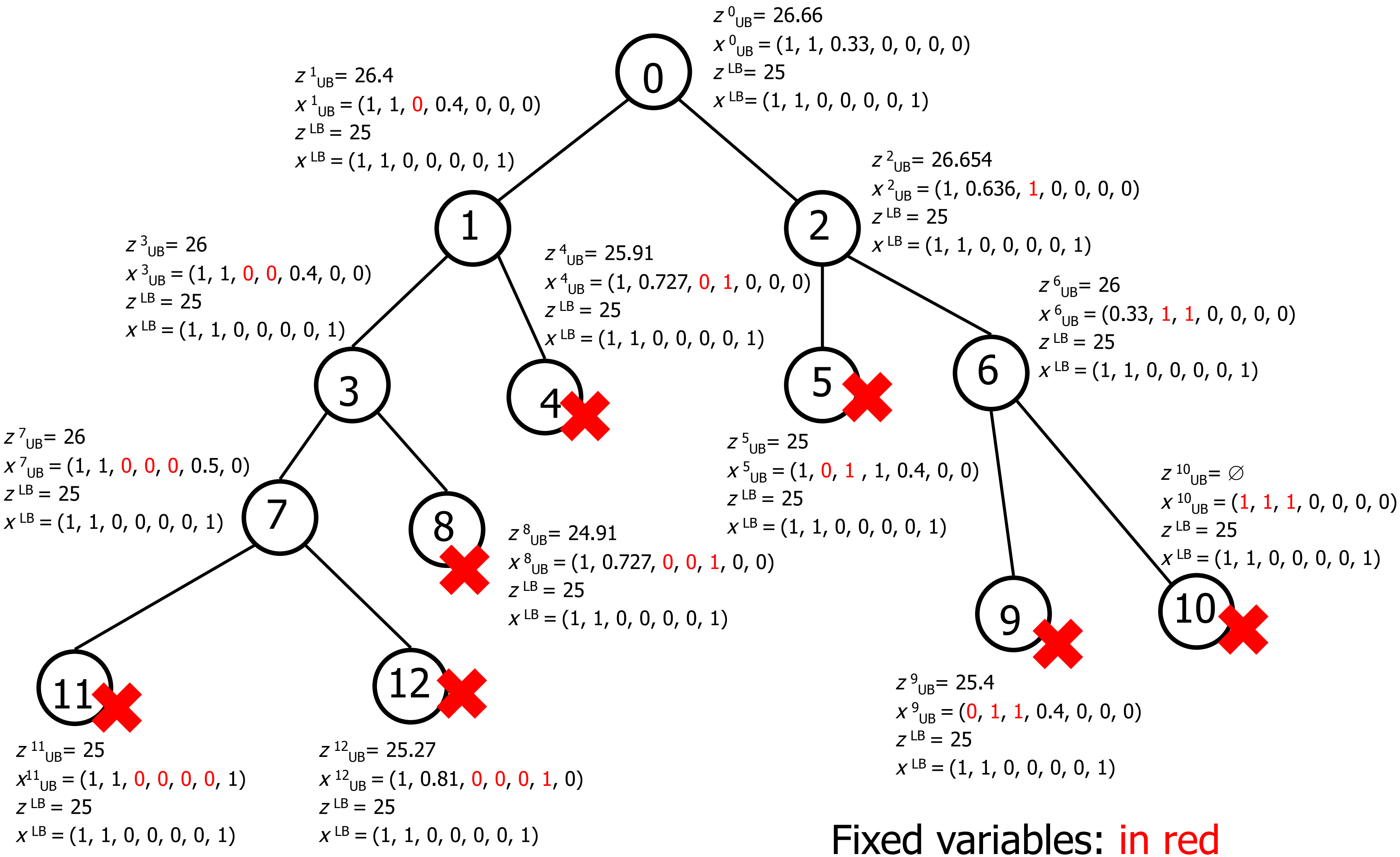
Branching

If x_t^{LP} is **fractional** and $z^{LP}{}_t > z^H$ then **branch**

LP-based Branch-and-Bound



{0,1} example



Fixed variables: in red