Public Key Cryptography

Lecture 12

Elliptic Curve Cryptography

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Elliptic Curves - Motivation

- **Problem:**
  Asymmetric schemes like RSA and ElGamal require exponentiations in integer rings and fields with parameters of more than 1000 bits.
  High computational effort on CPUs with 32-bit or 64-bit arithmetic.
  Large parameter sizes critical for storage on small and embedded ones.
- **Motivation:**
  Smaller field sizes providing equivalent security are desirable.
- **Solution:**
  Elliptic Curve Cryptography uses a group of points (instead of integers) for cryptographic schemes with coefficient sizes of 160-256 bits, reducing significantly the computational effort.

Elliptic Curves

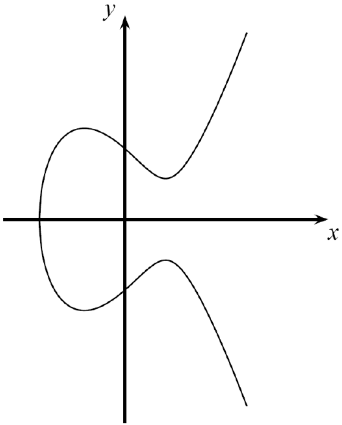
- Elliptic curves are polynomials that define points based on the (simplified) Weierstrass equation:

$$y^2 = x^3 + ax + b$$

  for parameters a, b that specify the exact shape of the curve.
- Elliptic curves cannot just be defined over the real numbers $\mathbb{R}$, but over many other types of finite fields.

Elliptic Curves (cont.)

On the real numbers and with parameters $a, b \in \mathbb{R}$, an elliptic curve looks like this:



Example: $y^2 = x^3 - 3x + 3$ over R

Elliptic Curves in Cryptography

In cryptography we are interested in elliptic curves modulo a prime number $p > 3$ (more generally, elliptic curves over finite fields):

Definition

An elliptic curve over $\mathbb{Z}_p$ *is the set of all pairs* $(x, y) \in \mathbb{Z}_p \times \mathbb{Z}_p$ *which fulfill*

$$y^2 = x^3 + ax + b \bmod p$$

together with an imaginary point at infinity θ*, where* $a, b \in \mathbb{Z}_p$ *and* $4a^3 + 27b^2 \neq 0 \bmod p$.

The last condition on a and b ensures that the elliptic curve is non-singular, that is, the plot has no self-intersections or vertices.

Computations on Elliptic Curves

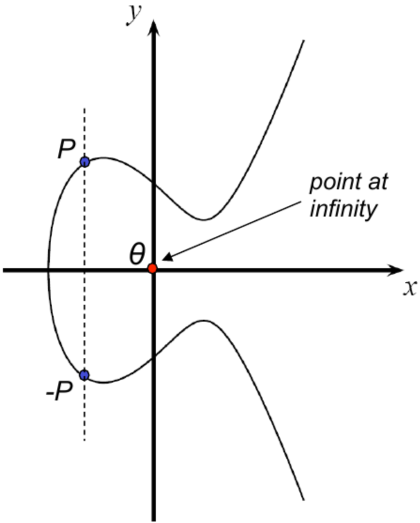
Some special considerations are required to convert elliptic curves into a group of points:

- In any group, a special element is required to allow for the identity operation, i.e., given $P \in E$:

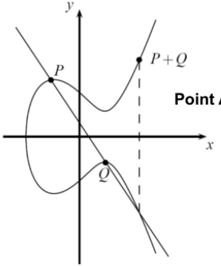
  $$P + \theta = P = \theta + P.$$

  This identity point (which is not on the curve) is additionally added to the group definition.
  This (infinite) identity point is denoted by θ.
- Elliptic curves are symmetric along the x-axis.
  Up to two solutions y and $-y$ exist for each quadratic residue x of the elliptic curve.
  For each point $P = (x, y)$, the inverse or negative point is defined as $-P = (x, -y)$.

Computations on Elliptic Curves (cont.)



- Generating a *group of points* on elliptic curves based on point addition operation *P+Q = R, i.e.,* $(x_P,y_P)+(x_Q,y_Q) = (x_R,y_R)$
- Geometric Interpretation of point addition operation
  - *Draw straight line through P and Q; if P=Q use tangent line instead*
  - *Mirror third intersection point of drawn line with the elliptic curve along the x-axis*

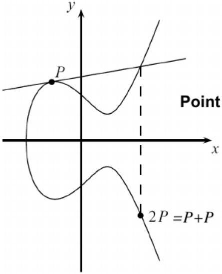


- Elliptic Curve Point Addition and Doubling Formulas

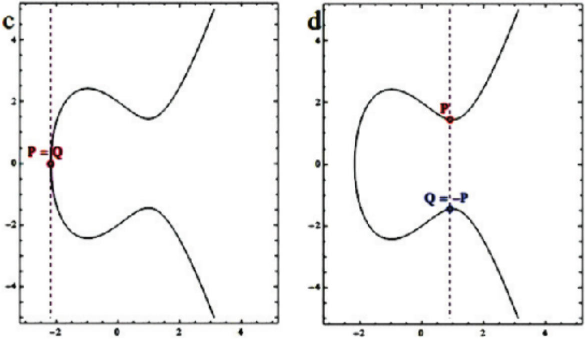
$$x_3 = s^2 - x_1 - x_2 \bmod p \quad \text{and} \quad y_3 = s(x_1 - x_3) - y_1 \bmod p$$

where

$$s = \begin{cases} \dfrac{y_2 - y_1}{x_2 - x_1} \bmod p & \text{; if P} \neq \text{Q (point addition)} \\ \dfrac{3x_1^2 + a}{2y_1} \bmod p & \text{; if P = Q (point doubling)} \end{cases}$$



Computations on Elliptic Curves (cont.)



Computations on Elliptic Curves - Example

Consider the elliptic curve

$$E : y^2 = x^3 + 2x + 2 \mod 17$$

and the point $P = (5, 1)$.
Let us compute $2P = P + P = (x_1, y_1) + (x_2, y_2)$. We have:

$$2P = P + P = (5, 1) + (5, 1) = (x_3, y_3).$$

Compute:

$$s = \frac{3x_1^2 + a}{2y_1} = (2 \cdot 1)^{-1}(3 \cdot 5^2 + 2) = 2^{-1} \cdot 9 = 9 \cdot 9 = 13 \mod 17.$$

$$x_3 = s^2 - x_1 - x_2 = 13^2 - 5 - 5 = 159 = 6 \mod 17.$$

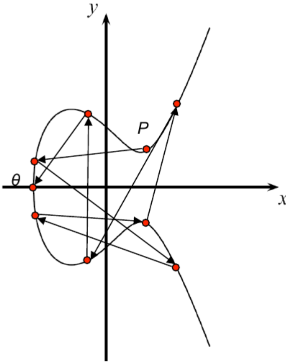
$$y_3 = s(x_1 - x_3) - y_1 = 13 \cdot (5 - 6) - 1 = -14 = 3 \mod 17.$$

Finally, we have $2P = (5, 1) + (5, 1) = (6, 3)$.

Computations on Elliptic Curves (cont.)

The points on an elliptic curve and the point at infinity θ form cyclic subgroups

$2P = (5,1)+(5,1) = (6,3)$	$11P = (13,10)$
$3P = 2P+P = (10,6)$	$12P = (0,11)$
$4P = (3,1)$	$13P = (16,4)$
$5P = (9,16)$	$14P = (9,1)$
$6P = (16,13)$	$15P = (3,16)$
$7P = (0,6)$	$16P = (10,11)$
$8P = (13,7)$	$17P = (6,14)$
$9P = (7,6)$	$18P = (5,16)$
$10P = (7,11)$	$19P = \theta$



This elliptic curve has order $\#E = |E| = 19$ since it contains 19 points in its cyclic group.

Number of Points on an Elliptic Curve

- How many points can be on an arbitrary elliptic curve?
  Consider the previous example $E: \; y^2 = x^3 + 2x + 2 \bmod 17$ has 19 points.
  However, determining the point count on elliptic curves in general is hard.
- But Hasse's theorem bounds the number of points to a restricted interval:

Theorem

Given an elliptic curve modulo p, the number of points on the curve is denoted by $\#E$ and is bounded by

$$p + 1 - 2\sqrt{p} \leq \#E \leq p + 1 + 2\sqrt{p}.$$

- Interpretation: the number of points is "close" to the prime p.
- Example: To generate a curve with about 2^{160} points, a prime with a length of about 160 bits is required.

We first recall Pollard's $p-1$ algorithm:

Let $n \geq 2$ be a composite integer to be factored.
Step 1: Set $a = 2$ (or any convenient value).
Step 2: Loop $d = 2, 3, 4, \ldots$ up to a specified bound.
  Step 3: Replace a with a^d mod n.
  Step 4: Compute $g = gcd(a-1, n)$.
  Step 5: If $1 < g < n$, then STOP and return g.
  Step 6: If $g = n$, go to **Step 1** and choose a new value a.
  Step 7: Increment d and loop again at **Step 2**.

Lenstra's Elliptic Curve Algorithm

Let $n \geq 2$ be a composite integer to be factored.
Step 1: Choose random integers a, x_1, y_1 modulo n.
Step 2: Set $P = (x_1, y_1)$ and $b = y_1^2 - x_1^3 - ax_1 \bmod n$.
Step 3: Consider the elliptic curve $E: \ y^2 = x^3 + ax + b$.
Step 4: Loop $d = 2, 3, 4, \ldots$ up to a specified bound d_{max}.

Step 5: Compute $q = dP \bmod n$.
Step 6: If the computation in **Step 5** fails, then we have a divisor $g > 1$ of n.
Step 7: If $g < n$, then STOP and return g.
Step 8: If $g = n$, then go to **Step 1** to choose a new curve and point.
Step 9: Increment d and, if $d \leq d_{max}$, then loop again at **Step 4**.

Step 10: Go to **Step 1** to choose a new curve and point.

Lenstra's Elliptic Curve Algorithm - Example

Example. Let us factor $n = 455839$. We choose the elliptic curve

$$y^2 = x^3 + 5x - 5$$

and its point $P = (1, 1)$. We try to compute $(10!)P$.

The slope of the tangent line at some point $A = (x, y)$ is $s = (3x^2 + 5) \cdot (2y)^{-1} \pmod{n}$. Using s we can compute $2A$. If the value of s is of the form $a \cdot b^{-1}$ where $b > 1$ and $gcd(a, b) = 1$, we have to find the inverse of b modulo n. If it does not exist, $gcd(n, b)$ is a non-trivial factor of n.

First we compute $2P$. We have $s(P) = s(1, 1) = 4$, so the coordinates of $2P = (x', y') \pmod{n}$ are:

$$x' = s^2 - 2x = 14,$$
$$y' = s(x - x') - y = 4 \cdot (1 - 14) - 1 = -53.$$

Note that 2P is on the curve: $(-53)^2 = 2809 = 14^3 + 5 \cdot 14 - 5$.

Lenstra's Elliptic Curve Algorithm - Example (cont.)

Then we compute $3(2P)$. We have

$$s(2P) = s(14, -53) = -593 \cdot 106^{-1} \pmod{n}.$$

Then $gcd(455839, 106) = 1$, and we obtain $106^{-1} = 81707$ mod 455839, and

$$s(2P) = -593 \cdot 106^{-1} = -133317 \pmod{455839}.$$

We can further compute $2(2P) = 4P = (259851, 116255)$. This is a point on the curve: $y^2 = 54514 = x^3 + 5x - 5 \pmod{455839}$. Next we can compute $3(2P) = 4P + 2P$.

Continuing the computations, we get to $8!P$, which requires the modular inverse of 599 modulo 455839. The Euclidean algorithm gives that 455839 is divisible by 599, and we have found a factorization $455839 = 599 \cdot 761$.

Elliptic Curve Discrete Logarithm Problem

- Cryptosystems rely on the hardness of the Elliptic Curve Discrete Logarithm Problem (ECDLP)

> ***Definition: Elliptic Curve Discrete Logarithm Problem (ECDLP)***
>
> *Given a primitive element P and another element T on an elliptic curve E. The ECDL problem is finding the integer d, where $1 \le d \le \#E$ such that*
>
> $$\underbrace{P + P + \ldots + P}_{d\ \text{times}} = dP = T.$$

- Cryptosystems are based on the idea that *d* is large and kept secret and attackers cannot compute it easily
- If *d* is known, an efficient method to compute the point multiplication *dP* is required to create a reasonable cryptosystem
  - Known Square-and-Multiply Method can be adapted to Elliptic Curves
  - The method for efficient point multiplication on elliptic curves: Double-and-Add Algorithm

Double-and-Add Algorithm for Point Multiplication

- **Double-and-Add Algorithm**

Input: Elliptic curve E, an elliptic curve point P and a scalar d with bits d_i

Output: $T = d\,P$

Initialization:

$T = P$

Algorithm:

FOR $i = t-1$ DOWNTO 0

$T = T + T \bmod n$

IF $d_i = 1$

$T = T + P \bmod n$

RETURN (T)

Example: $26P = (11010_2)P = (d_4d_3d_2d_1d_0)_2\,P$.

Step		
#0	$P = \mathbf{1}_2P$	inital setting
#1a	$P+P = 2P = \mathbf{10}_2P$	DOUBLE (bit d_3)
#1b	$2P+P = 3P = 10^2\,P+1_2P = \mathbf{11}_2P$	ADD (bit d_3=1)
#2a	$3P+3P = 6P = 2(11_2P) = \mathbf{110}_2P$	DOUBLE (bit d_2)
#2b		no ADD ($d_2 = 0$)
#3a	$6P+6P = 12P = 2(110_2P) = \mathbf{1100}_2P$	DOUBLE (bit d_1)
#3b	$12P+P = 13P = 1100_2P+1_2\,P = \mathbf{1101}_2P$	ADD (bit d_1=1)
#4a	$13P+13P = 26P = 2(1101_2P) = \mathbf{11010}_2P$	DOUBLE (bit d_0)
#4b		no ADD ($d_0 = 0$)

Theorem

Given a point P of an elliptic curve over $\mathbb{Z}_p$, the coordinates of dP can be computed in $O(\log d \log^3 p)$ bit operations.

Elliptic Curve Diffie-Hellman Protocol (ECDH)

- Given a prime p, a suitable elliptic curve E and a point $P=(x_P,y_P)$
- The Elliptic Curve Diffie-Hellman Key Exchange is defined by the following protocol:

Alice		**Bob**
Choose $k_{PrA} = a \in \{2, 3, \ldots, \#E-1\}$ Compute $k_{PubA} = A = aP = (x_A, y_A)$		Choose $k_{PrB} = b \in \{2, 3, \ldots, \#E-1\}$ Compute $k_{PubB} = B = bP = (x_B, y_B)$
	$\xrightarrow{A}$	
	$\xleftarrow{B}$	
Compute $aB = T_{ab}$		Compute $bA = T_{ab}$

- Joint secret between Alice and Bob: $T_{AB} = (x_{AB}, y_{AB})$
- Proof for correctness:
  - Alice computes aB=a(bP)=abP
  - Bob computes bA=b(aP)=abP since group is associative
- One of the coordinates of the point T_{AB} (usually the x-coordinate) can be used as session key (often after applying a hash function)

Elliptic Curve Diffie-Hellman Protocol (ECDH) (cont.)

- The ECDH is often used to derive session keys for (symmetric) encryption
- One of the coordinates of the point T_{AB} (usually the x-coordinate) is taken as session key

Alice		**Bob**	
Choose $k_{PrA} = a \in \{2, 3, \ldots, \#E-1\}$ Compute $k_{PubA} = A = aP = (x_A, y_A)$	$\xrightarrow{A}$	Choose $k_{PrB} = b \in \{2, 3, \ldots, \#E-1\}$ Compute $k_{PubB} = B = bP = (x_B, y_B)$	**ECDH**
	$\xleftarrow{B}$		
Compute $aB = T_{ab} = (x_T, y_T)$		Compute $bA = T_{ab} = (x_T, y_T)$	
Define key $k_{AES} = x_T$		Define key $k_{AES} = x_T$	**Symmetric encryption/decryption**
Given a message m: Encrypt $c = AES_{kAES}(m)$	$\xrightarrow{c}$	Received ciphertext c: Decrypt $m = AES^{-1}_{kAES}(c)$	

- In some cases, a hash function (see next chapters) is used to derive the session key

ECDH - Security Aspects

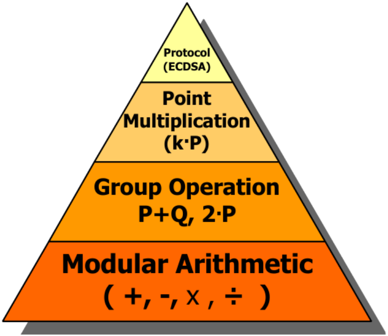
- Why are parameters signficantly smaller for elliptic curves (160-256 bit) than for RSA (1024-3076 bit)?
  - Attacks on groups of elliptic curves are weaker than available factoring algorithms or integer DL attacks
  - Best known attacks on elliptic curves (chosen according to cryptographic criterions) are the Baby-Step Giant-Step and Pollard-Rho method
  - Complexity of these methods: on average, roughly $\sqrt{p}$ steps are required before the ECDLP can be successfully solved

- Implications to practical parameter sizes for elliptic curves:
  - An elliptic curve using a prime p with 160 bit (and roughly 2^{160} points) provides a security of 2^{80} steps that required by an attacker (on average)
  - An elliptic curve using a prime p with 256 bit (roughly 2^{256} points) provides a security of 2^{128} steps on average

Implementations in Hardware and Software

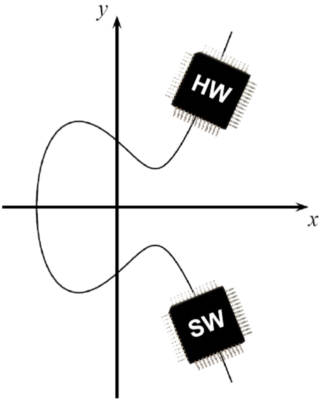
- Elliptic curve computations usually regarded as consisting of four layers:
  - Basic modular arithmetic operations are computationally most expensive
  - Group operation implements point doubling and point addition
  - Point multiplication can be implemented using the Double-and-Add method
  - Upper layer protocols like ECDH and ECDSA

- Most efforts should go in optimizations of the modular arithmetic operations, such as
  - Modular addition and subtraction
  - Modular multiplication
  - Modular inversion



Implementations in Hardware and Software (cont.)

- Software implementations
  - Optimized 256-bit ECC implementation on 3GHz 64-bit CPU requires about *2 ms* per point multiplication
  - Less powerful microprocessors (e.g, on SmartCards or cell phones) even take significantly longer (>*10 ms*)
- Hardware implementations
  - High-performance implementations with 256-bit special primes can compute a point multiplication in a few hundred microseconds on reconfigurable hardware
  - Dedicated chips for ECC can compute a point multiplication even in a few ten microseconds



Elliptic Curve Digital Signature Algorithm (ECDSA)

The ECDSA standard is defined for elliptic curves over $\mathbb{Z}_p$ and Galois fields $GF(2^m)$, the former being often preferred in practice.

Key Generation for ECDSA

1. Use an elliptic curve E with
   - modulus p
   - coefficients a and b
   - a point A which generates a cyclic group of prime order q
2. Choose a random integer d with $0 < d < q$.
3. Compute $B = dA$.

The keys are now:

$k_{pub} = (p, a, b, q, A, B)$

$k_{pr} = (d)$

Similar to DSA, the cyclic group has an order q which should have a size of at least 160 bit or more for higher security levels.

ECDSA - Signature and Verification

ECDSA Signature Generation

1. Choose an integer as random ephemeral key k_E with $0 < k_E < q$.
2. Compute $R = k_E A$.
3. Let $r = x_R$.
4. Compute $s \equiv (h(x) + d \cdot r)\, {k_E}^{-1} \bmod q$.

ECDSA Signature Verification

1. Compute auxiliary value $w \equiv s^{-1} \bmod q$.
2. Compute auxiliary value $u_1 \equiv w \cdot h(x) \bmod q$.
3. Compute auxiliary value $u_2 \equiv w \cdot r \bmod q$.
4. Compute $P = u_1 A + u_2 B$.
5. The verification $ver_{k_{pub}}(x, (r, s))$ follows from:

$$x_P \begin{cases} \equiv r \bmod q \implies \text{valid signature} \\ \not\equiv r \bmod q \implies \text{invalid signature} \end{cases}$$

ECDSA - Proof of Verification

Proof. Consider the hash value $h(x)$ of the message x. We show that (r, s) verifies the condition $r = x_P \mod q$. We have:

$$s = (h(x) + dr)k_E^{-1} \mod q,$$

which is equivalent to:

$$k_E = s^{-1}h(x) + ds^{-1}r \mod q.$$

and further to:

$$k_E = u_1 + du_2 \mod q.$$

Since the point A generates a cyclic group of order q, we can multiply both sides of the equation with A and we get:

$$k_E A = (u_1 + du_2)A = u_1 A + du_2 A = u_1 A + u_2 B.$$

But this is the condition that we check in the verification process by comparing the x-coordinates of $P = u_1 A + u_2 B$ and $R = k_E A$.

ECDSA - Example

Note that finding an elliptic curve with good cryptographic properties is a nontrivial task.
Bob wants to send a message to Alice, signed with ECDSA. He chooses the elliptic curve:

$$E : y^2 = x^3 + 2x + 2 \mod 17.$$

Because all points of the curve form a cyclic group of prime order 19, there are no subgroups and hence in this case $q = \#E = 19$.

Bob - key generation:

- Choose E with $p = 17$, $a = 2$, $b = 2$, and $A = (5, 1)$ with $q = 19$.
- Choose $d = 7$.
- Compute $B = dA = 7 \cdot (5, 1) = (0, 6)$.
- Send $(p, a, b, q, A, B) = (17, 2, 2, 19, (5, 1), (0, 6))$ to Alice.

ECDSA - Example (cont.)

Bob - signature:

- Compute the hash of a message x, say $h(x) = 26$.
- Choose ephemeral key $k_E = 10$
- Compute $R = 10 \cdot (5, 1) = (7, 11)$, so $r = x_R = 7$.
- Compute $s = (26 + 7 \cdot 7) \cdot 2 = 17 \mod 19$.
- Send $(x, (r, s)) = (x, (7, 17))$ to Alice.

Alice - verification:

- Compute $w = 17^{-1} = 9 \mod 19$.
- Compute $u_1 = 9 \cdot 26 = 6 \mod 19$.
- Compute $u_2 = 9 \cdot 7 = 6 \mod 19$.
- Compute $P = 6 \cdot (5, 1) + 6 \cdot (0, 6) = (7, 11)$.
- Check $x_P = r \mod 19$, so it is a valid signature.

Computational aspects of ECDSA

- **Key generation**: Finding an elliptic curve with good cryptographic properties is a nontrivial task. In practice, standardized curves such as the ones proposed by NIST are often used.
- **Signature**: The point multiplication, which is in most cases by the far the most arithmetic intensive operation, can be precomputed by choosing the ephemeral key ahead of time.
- **Verification**: The main computational load occurs during the evaluation of $P = u_1 A + u_2 B$. This can be accomplished by two separate point multiplications or by specialized methods for simultaneous exponentiations.

Conclusions

- Elliptic Curve Cryptography (ECC) is based on the discrete logarithm problem. It requires, for instance, arithmetic modulo a prime.
- ECC can be used for key exchange, for digital signatures and for encryption.
- ECC provides the same level of security as RSA or discrete logarithm systems over $\mathbb{Z}_p$ with considerably shorter operands (approximately 160-256 bit vs. 1024-3072 bit), which results in shorter ciphertexts and signatures.
- In many cases ECC has performance advantages over other public-key algorithms.
- ECC is slowly gaining popularity in applications, compared to other public-key schemes, i.e., many new applications, especially on embedded platforms, make use of elliptic curve cryptography.

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