2. Entanglement

Bohr was inconsistent, unclear, willfully obscure and right. Einstein was consistent, clear, down-to-earth and wrong.
– John S. Bell

I told Wheeler that I had had a number of conversations with Bell about quantum theory. "He's a wonderful fellow," Wheeler noted. "Did he say to you," Wheeler asked, laughing, "'I'd rather be clear and wrong, than foggy and right'?" I told Wheeler that Bell had not used exactly those words, but that it certainly sounded like him. I also told Wheeler that from the time that Bell began to study the quantum theory, he had conceptual problems with it, and that I had asked Bell if, at that time, he thought that the theory might simply be wrong – to which Bell had answered, "I hesitated to think it might be wrong, but I knew that it was rotten."
– Jeremy Bernstein.

2.1 Composite Systems

We have been studying two-level systems so far, and will now start studying higher-dimensional systems. In the context of quantum computing, n qubits live in an $N = 2^n$-dimensional Hilbert space. In quantum sensing applications, e.g. NV centres, a central electron spin may interact with a large bath of nuclei spins in the solid state lattice. The many-body physics of interacting two-level systems require a description in higher dimensional Hilbert space.

In this section, we will therefore describe the mathematical procedure of constructing the larger Hilbert space.

2.1.1 Tensor product of Hilbert spaces

Consider 2 two-level systems, in Hilbert spaces $\mathcal{H}_A$ and $\mathcal{H}_B$. They are spanned by vectors $\{|0\rangle_A, |1\rangle_A\}$ and $\{|0\rangle_B, |1\rangle_B\}$ respectively. For each space, operators can be written in terms of the identity and Pauli matrices. Operators of one space act on the vector of its space, but does nothing to vectors of the other space. For example,

$$\begin{aligned}\sigma^x_A \left|0\right\rangle_A &= \left|1\right\rangle_A ,\\ \sigma^x_A \left|0\right\rangle_B &= \left|0\right\rangle_B .\end{aligned}$$

We define the joint Hilbert space by the tensor product of the component spaces, denoted with the $\otimes$ symbol: $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$. The joint Hilbert space has dimensions $d_{AB} = d_A \times d_B$, which is $d_{AB} = 4$ in the case of two qubits. Note that the order of joining the spaces will be important in keeping track of the ordering of basis states that span the joint space, when it comes to the matrix representation of the operators in the joint Hilbert space. If one has an operator A acting in $\mathcal{H}_A$ and B acting in $\mathcal{H}_B$, a joint operator composed of these two operations is constructed by the Kronecker product[1], given by

$$A \otimes B = \begin{pmatrix} A_{11} \times B & A_{12} \times B \\ A_{21} \times B & A_{22} \times B \end{pmatrix} = \begin{pmatrix} A_{11}B_{11} & A_{11}B_{12} & A_{12}B_{11} & A_{12}B_{12} \\ A_{11}B_{21} & A_{11}B_{22} & A_{12}B_{21} & A_{12}B_{22} \\ A_{21}B_{11} & A_{21}B_{12} & A_{22}B_{11} & A_{22}B_{12} \\ A_{21}B_{21} & A_{21}B_{22} & A_{22}B_{21} & A_{22}B_{22} \end{pmatrix}. \tag{2.1}$$

In the same way, the basis vectors spanning the joint Hilbert space are constructed by the Kronecker product, e.g. for two qubit systems,

$$\begin{aligned}
\left|0\right\rangle_A \otimes \left|0\right\rangle_B &= \begin{pmatrix}1\\0\end{pmatrix} \otimes \begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix} 1 \times \begin{matrix}1\\0\end{matrix} \\ 0 \times \begin{matrix}1\\0\end{matrix} \end{pmatrix} = \begin{pmatrix}1\\0\\0\\0\end{pmatrix}\\
\left|0\right\rangle_A \otimes \left|1\right\rangle_B &= \begin{pmatrix}1\\0\end{pmatrix} \otimes \begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix} 1 \times \begin{matrix}0\\1\end{matrix} \\ 0 \times \begin{matrix}0\\1\end{matrix} \end{pmatrix} = \begin{pmatrix}0\\1\\0\\0\end{pmatrix}\\
\left|1\right\rangle_A \otimes \left|0\right\rangle_B &= \begin{pmatrix}0\\1\end{pmatrix} \otimes \begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix} 0 \times \begin{matrix}1\\0\end{matrix} \\ 1 \times \begin{matrix}1\\0\end{matrix} \end{pmatrix} = \begin{pmatrix}0\\0\\1\\0\end{pmatrix}\\
\left|1\right\rangle_A \otimes \left|1\right\rangle_B &= \begin{pmatrix}0\\1\end{pmatrix} \otimes \begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix} 0 \times \begin{matrix}0\\1\end{matrix} \\ 1 \times \begin{matrix}0\\1\end{matrix} \end{pmatrix} = \begin{pmatrix}0\\0\\0\\1\end{pmatrix},
\end{aligned}$$

and by a linear extension of this formula for arbitrary states. The tensor product has linear properties:

$$(\left|a_1\right\rangle + \left|a_2\right\rangle) \otimes \left|b\right\rangle = \left|a_1\right\rangle \otimes \left|b\right\rangle + \left|a_2\right\rangle \otimes \left|b\right\rangle , \tag{2.2}$$

$$c(\left|a\right\rangle \otimes \left|b\right\rangle) = (c\left|a\right\rangle) \otimes \left|b\right\rangle = \left|a\right\rangle \otimes (c\left|b\right\rangle). \tag{2.3}$$

The joint Hilbert space $\mathcal{H}_{AB}$ is spanned by d_{AB}^2 operators. For example, in a two-qubit Hilbert space of dimension 4, there are $4^2 = 16$ combinations of operators from the two single-qubit spaces: $\{\mathbb{I}_A \otimes \mathbb{I}_B, \sigma^x_A \otimes \mathbb{I}_B, \ldots, \sigma^x_A \otimes \sigma^y_B, \ldots \sigma^z_A \otimes \sigma^z_B\}$. Therefore, any operator in the joint Hilbert space can be written as a linear combination of operators in the two spaces,

$$C = \sum_{i,j} c_{i,j} A_i \otimes B_j. \tag{2.4}$$

Tensor products of operators acting on states can be manipulated in a factorized manner, e.g.

$$(\sigma^z_A \otimes \sigma^x_B)\left(\left|0_A\right\rangle \otimes \left|1_B\right\rangle\right) = (\sigma^z_A \left|0\right\rangle_A) \otimes (\sigma^x_B \left|1\right\rangle_B). \tag{2.5}$$

[1]The term Kronecker product refers specifically to matrices; tensor product refers to linear maps between vector spaces. The Kronecker product of the two matrices represents the tensor product of the two linear maps.

The inner product of vectors in the joint space are

$$(\langle a_1| \otimes \langle b_1|) \otimes (|a_1\rangle \otimes |b_1\rangle) = \langle a_1|a_2\rangle \langle b_1|b_2\rangle . \tag{2.6}$$

We are not pedantic with notation. Often, simplicity of notation is preferred. For example, these notations are used interchangeably in the literature: $|0\rangle_A \otimes |1\rangle_B$, $|0\rangle_A |1\rangle_B$, $|0_A, 1_B\rangle$, $|01\rangle_{AB}$. For operators, often the identity is dropped: σ_A^x refers to the same operation as $\sigma_A^x \otimes \mathbb{I}_B$.

2.1.2 Local and non-local operations on two qubits

In the case of two-qubit Hilbert space, 'local' operations that act only on their component subspaces, e.g. $A \otimes B$, have no ability to generate entanglement. Only 'non-local' operations have the ability to generate entanglement, where non-local would refer to any operation that cannot be factored into the form $A \otimes B$. But this does not mean that all non-local operations generate entanglement. Non-factorizability is a necessary but insufficient condition for entanglement generation.

For example, the control-NOT (CNOT) gate operation (which flips the state of qubit B, conditioned on the state of qubit A being 1) is often used to generate entanglement between two qubits, given in the $\{|00\rangle_{AB}, |01\rangle_{AB}, |10\rangle_{AB}, |11\rangle_{AB}\}$ basis by

$$U_{\text{CNOT}} = \frac{1}{2} (\mathbb{I}_A \otimes \mathbb{I}_B + \sigma_A^z \otimes \mathbb{I}_B + \mathbb{I}_A \otimes \sigma_B^x + \sigma_A^z \otimes \sigma_B^x) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} . \tag{2.7}$$

On the other hand, the SWAP gate operation (which exchanges the states of qubits A and B) cannot be factored into the form $A \otimes B$, but it is not entangling. It is given by

$$U_{\text{SWAP}} = \frac{1}{2} (\mathbb{I}_A \otimes \mathbb{I}_B + \sigma_A^x \otimes \sigma_B^x + \sigma_A^y \otimes \sigma_B^y + \sigma_A^z \otimes \sigma_B^z) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} . \tag{2.8}$$

One way to look at the entanglement of a pure state is to express it as a Schmidt decomposition, the subject of the next section.

2.2 Schmidt Decomposition

An important characterization of the states of a *bipartite* composite quantum system is obtained with the help of the Schmidt decomposition theorem. This theorem asserts that for any given state $|\psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$, there exist orthonormal bases called the Schmidt bases $\{|\chi_i\rangle_A\} \in \mathcal{H}_A$ and $\{|\chi_i\rangle_B\} \in \mathcal{H}_B$ such that

$$|\psi\rangle = \sum_{i=1}^{d} \alpha_i |\chi_i\rangle_A \otimes |\chi_i\rangle_B , \tag{2.9}$$

where $d = \min(d_A, d_B)$. Here, α_i are called Schmidt coefficients, and the number of non-zero Schmidt coefficients is called the Schmidt number. This number is invariant with respect to unitary transformations U_A and U_B which act only in the respective spaces. For the same reason the Schmidt number does not depend on the particular Schmidt bases chosen and is thus uniquely defined for a given state $|\psi\rangle$. For a normalized state,

$$\langle\psi|\psi\rangle = \sum_i |\alpha_i|^2 = 1. \tag{2.10}$$

It must be noted that the Schmidt bases which allow a representation of the form Eq. (2.9) depend, in general, on the given state $|\psi\rangle$.

Some interesting properties of the Schmidt coefficients relate to entanglement:

1. If $|\psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$ **can** be written as a tensor product $|\phi\rangle_A \otimes |\phi\rangle_B$ of states of the subsystems, we call it a **product** state. This is an un-entangled state.

2. A state $|\psi\rangle$ is said to be **entangled** if it **cannot** be written as a tensor product. It follows from the Schmidt decomposition theorem that $|\psi\rangle$ is an entangled state if and only if the Schmidt number is larger than 1.
3. If the absolute values of all non-vanishing Schmidt coefficients for a given state $|\psi\rangle$ are **equal to each other**, the state is called **maximally entangled**.

2.2.1 Proof

We give the proof for the case where systems A and B have state spaces of the same dimension. Let $|j\rangle$ and $|k\rangle$ be any fixed orthonormal bases for systems A and B, respectively. Then $|\psi\rangle$ can be written as

$$|\psi\rangle = \sum_{j,k} a_{jk} |j\rangle |k\rangle , \tag{2.11}$$

for some matrix a of complex numbers a_{jk}. By the singular value decomposition (SVD), $a = udv$, where d is a diagonal matrix with non-negative elements, and u and v are unitary matrices. Then,

$$|\psi\rangle = \sum_{j,k} a_{jk} |j\rangle |k\rangle = \sum_{i,j,k} u_{ji} d_{ii} v_{jk} |j\rangle |k\rangle . \tag{2.12}$$

Defining $|\chi_i\rangle_A \equiv \sum_j u_{ji} |j\rangle$ and $|\chi_i\rangle_B \equiv \sum_k v_{ik} |k\rangle$, and $\alpha_i \equiv d_{ii}$, we see that this gives

$$|\psi\rangle = \sum_i \alpha_i |\chi_i\rangle_A |\chi_i\rangle_B . \tag{2.13}$$

Therefore, to perform the Schmidt decomposition, we need to know how to perform the SVD. It is not a difficult procedure, but we shall skip that since SVD not usually taught in Physics undergraduate courses.

Instead, we will make use of the alternative method, which is to find the eigen-decomposition of the reduced density matrices. However, density matrices is a topic for a later chapter, so we will defer working through examples of Schmidt decomposition until we get there.

Finally, in this course we will limit ourselves to sub-systems that have the same dimensionality. This implies that a will be a square matrix. However, if the sub-systems have different dimensionality, the a matrix will be rectangular.

2.3 Entanglement

The term "entanglement" ("Verschränkung" in German) was first coined by Schrödinger in 1935, who immediately emphasized the nonclassical implications of entanglement,

> *When two systems, of which we know the states by their respective representatives, enter into temporary physical interaction due to known forces between them, and when after a time of mutual influence the systems separate again, then they can no longer be described in the same way as before, viz. by endowing each of them with a representative of its own. I would not call that one but rather the characteristic trait of quantum mechanics, the one that enforces its entire departure from classical lines of thought.*

The peculiar features of entanglement have now been confirmed in many experiments. Entanglement is also at the heart of the "second quantum revolution" that is concerned with quantum technologies such as quantum computers and quantum cryptography.

Suppose a bipartite quantum system $\mathcal{S}$ comprises two subsystems, $\mathcal{S}_1$ and $\mathcal{S}_2$, i.e. $\mathcal{S} = \mathcal{S}_1 \otimes \mathcal{S}_2$. If the state of the system $|\Psi\rangle$ cannot be written as a tensor product of state vectors of these two subsystems, i.e. if there do not exist any state vectors $|\psi\rangle_1 \in \mathcal{S}_1$ and $|\phi\rangle_2 \in \mathcal{S}_2$ such that $|\Psi\rangle = |\psi\rangle_1 \otimes |\phi\rangle_2$, then we say that $\mathcal{S}_1$ and $\mathcal{S}_2$ are *entangled*. Another way to put it is to say that $\mathcal{S}$ is entangled with respect to subsystems $\mathcal{S}_1$ and $\mathcal{S}_2$.

As an example, consider two spin-1/2 particles, A and B, described by mutually orthogonal basis states $|\uparrow\rangle_{A/B}$ and $|\downarrow\rangle_{A/B}$, where the spin quantization axis implied by the arrows corresponds to some axis in real space. There are four possible *maximally entangled* states that can be written, called Bell states, given by

$$|\Phi^{\pm}\rangle = \frac{1}{\sqrt{2}} \left(|0\rangle_A |0\rangle_B \pm |1\rangle_A |1\rangle_B \right), \tag{2.14}$$

$$|\Psi^{\pm}\rangle = \frac{1}{\sqrt{2}} \left(|0\rangle_A |1\rangle_B \pm |1\rangle_A |0\rangle_B \right). \tag{2.15}$$

One can show that these Bell states cannot be factored into tensor products of the two spin-1/2 subsystems. This can be proven by contradiction.

2.4 Einstein-Bohr Debates

In a 1935 paper titled "Can Quantum-Mechanical Description of Physical Reality be Considered Complete?"[2], physicists Albert Einstein, Boris Podolsky and Nathan Rosen (EPR) argued argued that the description of physical reality provided by quantum mechanics was incomplete. In the paper, the authors proposed two rather reasonable criteria for a theory:

EPR criteria for theory

- **Completeness:** A complete theory is one in which "*every element of the physical reality must have a counterpart in the physical theory*".
- **Physical Reality:** For a definition of physical reality, EPR states: "*If, without in any way disturbing a system, we can predict with certainty (i.e., with probability equal to unity) the value of a physical quantity, then there exists an element of physical reality corresponding to this physical quantity*".

For example, in classical mechanics, we can write down the state of a system of N particles in terms of their positions and momenta $(\vec{x}_i, \vec{p}_i)$ for $i = 1 \ldots N$. This state vector is a complete description of the system in the sense that the results of any observation on the system can be calculated from it, and the dynamics of the system can be written in terms of it (equations of motion). Furthermore, each of its elements $(\vec{x}_i, \vec{p}_i)$ corresponds to a physical property in the world that exists *independently* of whether we write down the state or observe any property of the system.

In contrast, the quantum mechanical state, e.g. $|\psi\rangle = \cos(\theta/2)|0\rangle + e^{i\phi}\sin(\theta/2)|1\rangle$, has the following properties:

Quantum theory

1. **Statistical predictions.** In quantum mechanics, the outcomes of individual observations on the system cannot be predicted, only the statistics of possible outcomes of the observation. For example, statistics of measurements of the observable σ_z yields $+1$ or -1 with probabilities $|\cos(\theta/2)|^2$ and $|\sin(\theta/2)|^2$, but it is not possible to predict what any one particular outcome of measurement will be. This is in contrast with classical states where if the position of the particle is given by x, then the outcome of a measurement will be x, i.e. the measurement outcome is completely predicted by this state.
2. **State collapse.** When a measurement of σ_z yields the outcome $+1$, the state will have "collapsed" into $|0\rangle$, thus the post-measurement state is $|\psi\rangle = |0\rangle$. The parameters that determined the state prior to measurement, θ, ϕ are completely erased by the measurement. Unless, by accident, the quantum state is an eigenstate of the observable, in general the post-measurement state will not be the same as the prior state.

[2] Einstein, A.; Podolsky, B.; Rosen, N. (1935). "Can Quantum-Mechanical Description of Physical Reality Be Considered Complete?". Physical Review. 47 (10): 777–780. https://journals.aps.org/pr/abstract/10.1103/PhysRev.47.777

The second point then begets the question of whether the properties of the state given in quantum theory, θ, ϕ, are “real” by EPR’s criteria.

As per the mathematical formulation of quantum mechanics, the dynamical evolution of quantum states is deterministic whereas the measurement outcomes are non-deterministic. i.e. it generally does not predict the outcome of any measurement with certainty. Instead, it indicates what the probabilities of the outcomes are, with the indeterminism of observable quantities constrained by the uncertainty principle. Consequently, there has been much debate about how to think about the nature of the quantum mechanical state,[3] e.g. whether it directly corresponds to reality or it represents only knowledge or information about some aspect of reality.

Consider the case of classical coin tosses, to which is generally attributed a probability distribution of outcomes, heads or tails. In principle if all physical conditions are known precisely, e.g. initial position, orientation, linear and angular momenta, mass distribution of coin, air flow, table elasticity, etc., then classical mechanics is able to predict with certainty, the way the coin would land. However, if some of these information is missing, then the best we can do is ascribe a probability distribution for the coin toss. Analogous to the missing information described, “hidden” variables have been proposed to be underlying quantum properties that are the missing ingredients for a deterministic theory. That is to say, according to *hidden-variable* theories, measurement in quantum mechanics only appears to be probabilistic, although it is fundamentally deterministic. The most famous example of such a theory is the de Broglie-Bohm pilot wave theory or Bohmian mechanics[4].

In 1951, David Bohm proposed a variant of the EPR thought experiment in which the measurements have discrete ranges of possible outcomes, and it is this version that we shall examine. Suppose a pair of spin-1/2 particles are described by the singlet Bell state,

$$|\Psi^-\rangle = \frac{1}{\sqrt{2}} \left(|\uparrow\rangle_A |\downarrow\rangle_B - |\downarrow\rangle_A |\uparrow\rangle_B \right), \tag{2.16}$$

with the two particles traveling with equal momenta in opposite directions. For example, they could originate in the decay of an unstable particle of zero spin and zero momentum, in which case momentum conservation implies that the particles move in opposite directions and have a total spin of zero. An example is the rare decay of a η muon (mass 549 MeV/c^2) into a muon pair,

$$\eta \to \mu^+ + \mu^-. \tag{2.17}$$

Also, proton-proton scattering at low kinetic energies forces the interacting protons (each are of spin-1/2, and are fermions) to be in a state of zero orbital angular momentum and a spin singlet as given by Eq. (2.16), from the Pauli exclusion principle. The spin states of the scattered protons must be correlated as given in the singlet Bell state even after they are separated by a macroscopic distance.

Alice and Bob, measure the spin component of each particle along a certain axis when the particles are very far apart. As a first case, consider Alice and Bob both measuring in the z-axis, $\sigma_z \equiv |\uparrow\rangle\langle\uparrow| - |\downarrow\rangle\langle\downarrow|$. After they make the measurements, they pick up the telephone (they are far away from each other) to compare their results. Whenever Alice measures spin up $(+1)$, Bob will measure spin down (-1), and vice versa. This perfect anti-correlation may seem unimpressive; we can think of correlations that occur in our daily life. For instance, John Bell wrote about his colleague Bertlmann’s socks.[5]

> *Dr. Bertlmann likes to wear two socks of different colours. Which colour he will have on a given foot on a given day is quite unpredictable. But when you see that the first sock is pink (Fig. 1) you can already be sure that the second sock will not be pink. Observation of the first, and experience of Bertlmann, gives immediate information about the second. There is no accounting for tastes, but apart from that there is no mystery here. And is not the EPR business the same?*

[3] See, for example: M. Pusey, J. Barrett, and T. Rudolph, On the reality of the quantum state. Nature Phys 8, 475–478 (2012). https://www.nature.com/articles/nphys2309.

[4] See Bohm, D. (1952). “A Suggested Interpretation of the Quantum Theory in Terms of Hidden Variables, I”. Physical Review. 85 (2): 166–179. https://journals.aps.org/pr/abstract/10.1103/PhysRev.85.166.

[5] See J. S. Bell, “Speakable and Unspeakable in Quantum Mechanics”. Cambridge University Press, (2004) ISBN: 9780511815676. Also here: `https://cds.cern.ch/record/142461/files/198009299.pdf`

We shall see later that the correlations in quantum mechanics is very different from classical correlations like this.

In the usual classical picture of the world, EPR assumes the principles of locality and realism[6] (both of these hypothesis should be true at the same time).

EPR assumptions

- **Locality:** The principle of locality states that an object is influenced directly only by its immediate surroundings. An event at one point cannot cause a simultaneous result at another point. An event at point A cannot cause a result at point B in a time less than the time for the speed of light to travel between A and B. This is an alternative to the concept of instantaneous "action at a distance".
- **Realism:** Realism assumes the existence of objects, and properties of objects, even when they have not been measured. It allows us to speak "meaningfully" of the definiteness of the results of measurements that have not been performed. In the EPR discussions, "meaningfully" means the ability to treat these unmeasured results on an equal footing with measured results in statistical calculations.

That is to say, EPR or the local realist's interpretation of the experimental results is that the particles (like socks) possessed a definite, "real" property, spin-$\uparrow$ or spin-$\downarrow$ (pink or not-pink), independent of the observation. When the experimenter conducted the observation, he or she merely uncovered what the property was.

However, the standard Copenhagen interpretation[7] says that quantum mechanics is intrinsically indeterministic, with probabilities calculated using the Born rule, and the act of "observing" or "measuring" a quantum object is irreversible. Since no truth can be attributed to an object, except according to the results of its measurement, this implies that before measurement, it is meaningless to say what the spin of the particle is (contrary to the idea of realism). Some, like Niels Bohr, go further to assert that quantum particles do not have any definite properties before measurement. In this light, we can understand Bohr's declaration that

> *There is no quantum world. There is only an abstract quantum mechanical description. It is wrong to think that the task of physics is to find out how Nature* **is**. *Physics concerns what we can say about Nature.*

Pascual Jordan went further, by saying

> *... observations not only disturb what has to be measured, they produce it. In a measurement of position, for example, as performed with a gamma ray microscope, "the electron is forced to a decision". We compel it to assume a definite position; previously it was, in general, neither here nor there; it had not yet made its decision for a definite position ... If by another experiment the velocity of the electron is being measured, this means: the electron is compelled to decide itself for some exactly defined value of its velocity ... we ourselves produce the results of measurement.*

2.4.1 EPR Argument

Having said so much, let us now analyze the EPR argument with the singlet spin state.

1. Consider a system of two spin-1/2 particles which at some point in time t are located, respectively, in the spatially distant regions A and B and which are also in the entangled state $|\Psi^-\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle_A |\downarrow\rangle_B - |\downarrow\rangle_A |\uparrow\rangle_B)$.

2. At time t the particles in region A is measured with a Stern-Gerlach apparatus oriented in the z-direction. Suppose that the result of the measurement is that the particle is spin-up. According to the wavefunction collapse postulate, the result is that *immediately*

[6]See Wikipedia: Counterfactual Definiteness. `https://en.wikipedia.org/wiki/Counterfactual_definiteness`.

[7]See, for example: Omnes, Roland. (1999) "The Copenhagen Interpretation" in Understanding Quantum Mechanics. Princeton University Press. pp. 41–54. `https://www.jstor.org/stable/j.ctv173f2pm`

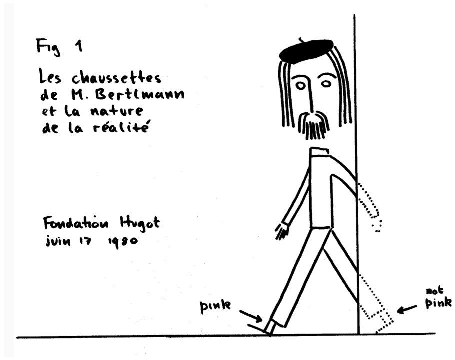

Figure 2.1: **Bertlmann's socks.** Fig. 1 of John Bell's essay "Bertlmann's socks and the nature of reality."

after measurement, the system goes from $|\Psi^-\rangle \longrightarrow |\uparrow\rangle_A |\downarrow\rangle_B$.

3. At this point, the observer in A who carried out the first measurement on particle 1, *without doing anything else that could disturb the system or the other particle* (realism assumption), can predict with certainty that particle 2 will be in a state of spin-down if it is measured in the z-direction. It follows that *particle 2 possesses an element of physical reality*: that of having a spin pointing down in the z-direction.

4. According to the assumption of locality, it cannot have been the action carried out in A which created this element of reality for particle 2. Therefore, we must conclude that the particle possessed the said property *before and independently of the measurement of particle 1.*

5. The observer in A could have decided to carry out a measurement at some other orientation, say some x-direction, obtaining a certain result, for example, that the particle is spin-up (or down) in that direction. In that case, he could have concluded that particle 2 turned out to be spin-down (or up) in the x-direction. Combining one of these alternatives with the conclusion reached in 4, it seems that particle 2, before the measurement took place, possessed with certainty, both the properties of spin-up in the z and x-directions. These properties are incompatible according to the formalism.

6. Since natural and obvious requirements have forced the conclusion that particle 2 simultaneously possesses incompatible properties, this means that, even if it is not possible to determine these properties simultaneously and with arbitrary precision, they are nevertheless possessed objectively by the system. But quantum mechanics denies this possibility and it is therefore an incomplete theory.

■ **Example 2.1** To see why an observation of σ_x will also give anti-correlations, express the Bell state $|\Psi^-\rangle$ in the x-basis, i.e. in the basis of $|\pm\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\rangle \pm |\downarrow\rangle\right)$.

■

2.4.2 Bohr's response to EPR

Five months later, Bohr published his response to EPR in the same journal, Physical Review, with the same title.[8] In his response, he attacked EPR's criterion of physical reality, saying

> *The statement of the criterion in question is ambiguous with regard to the expression "without disturbing the system in any way". Naturally, in this case no mechanical*

[8] N. Bohr, Phys. Rev. 48, 696 (1935). `https://journals.aps.org/pr/abstract/10.1103/PhysRev.48.696`

> *disturbance of the system under examination can take place in the crucial stage of the process of measurement. But even in this stage there arises the essential problem of an influence on the precise conditions which define the possible types of prediction which regard the subsequent behaviour of the system ... their arguments do not justify their conclusion that the quantum description turns out to be essentially incomplete ... This description can be characterized as a rational use of the possibilities of an unambiguous interpretation of the process of measurement compatible with the finite and uncontrollable interaction between the object and the instrument of measurement in the context of quantum theory.*

It is difficult to understand precisely what Bohr meant. Perhaps the "*influence*" he referred to comes from some disturbance arising from measurement. It is not clear if Bohr meant it to be the "spooky action at a distance", i.e. a non-local disturbance, that Einstein was against.

2.4.3 Measurement Statistics from Quantum Mechanics

Let us now look at an example of a possible set of measurements by Alice and Bob on each particle of the singlet Bell state of Eq. (2.16).

Suppose Alice measures her particle along the z-axis, while Bob measures his along some other axis, say b-axis: $\hat{b} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)^T$. This means that the Pauli matrix where the b-axis is the quantization axis is related to the original axes by $\sigma_b = \vec{\sigma}\cdot\hat{b} = \cos\theta\ \sigma_z + \sin\theta(\cos\phi\ \sigma_x + \sin\theta\sin\phi\ \sigma_y)$. Without loss of generality, we take $\phi = 0$ henceforth.

Therefore, the expectation value $\langle\sigma_z^A\sigma_b^B\rangle$ given the singlet Bell state $|\Psi^-\rangle$, will be

$$
\begin{aligned}
\langle\sigma_z^A\sigma_b^B\rangle =& \langle\Psi^-|\sigma_z^A\sigma_b^B|\Psi^-\rangle\\
=&\frac{1}{2}\left(\langle\uparrow_A\downarrow_B| - \langle\downarrow_A\uparrow_B|\right)\sigma_z^A\sigma_b^B\left(|\uparrow_A\downarrow_B\rangle - |\downarrow_A\uparrow_B\rangle\right)\\
=&\frac{1}{2}\left(\langle\uparrow_A\downarrow_B|\,\sigma_z^A\sigma_b^B\,|\uparrow_A\downarrow_B\rangle - \langle\uparrow_A\downarrow_B|\,\sigma_z^A\sigma_b^B\,|\downarrow_A\uparrow_B\rangle\right.\\
&\qquad\left. - \langle\downarrow_A\uparrow_B|\,\sigma_z^A\sigma_b^B\,|\uparrow_A\downarrow_B\rangle + \langle\downarrow_A\uparrow_B|\,\sigma_z^A\sigma_b^B\,|\downarrow_A\uparrow_B\rangle\right)\\
=&\frac{1}{2}\Bigg(\cancelto{+1}{\langle\uparrow_A|\sigma_z^A|\uparrow_A\rangle}\langle\downarrow_B|\sigma_b^B|\downarrow_B\rangle - \cancelto{0}{\langle\uparrow_A|\sigma_z^A|\downarrow_A\rangle}\langle\downarrow_B|\sigma_b^B|\uparrow_B\rangle\\
&\qquad - \cancelto{0}{\langle\downarrow_A|\sigma_z^A|\uparrow_A\rangle}\langle\uparrow_B|\sigma_b^B|\downarrow_B\rangle + \cancelto{-1}{\langle\downarrow_A|\sigma_z^A|\downarrow_A\rangle}\langle\uparrow_B|\sigma_b^B|\uparrow_B\rangle\Bigg)\\
=&\frac{1}{2}\left(\langle\downarrow_B|\sigma_b^B|\downarrow_B\rangle - \langle\uparrow_B|\sigma_b^B|\uparrow_B\rangle\right)\\
=& -\cos\theta.
\end{aligned}
\tag{2.18}
$$

More generally, Alice can choose to measure in two different axes $\hat{a}, \hat{a}'$, and Bob can choose to measure in another two different axes $\hat{b}, \hat{b}'$, but they don't tell each other which axis they choose. In any direction, the outcomes are either $+1$ or -1. After the measurement, they compare their results, written as a or a', and b or b'. We are interested in correlations of the measurement statistics, $\langle ab\rangle, \langle a'b\rangle, \langle ab'\rangle, \langle a'b'\rangle$. We know that these measurement statistics can be calculated from quantum mechanics as these respective expection values $\langle\sigma_a^A\sigma_b^B\rangle, \langle\sigma_{a'}^A\sigma_{b'}^B\rangle, \langle\sigma_a^A\sigma_{b'}^B\rangle, \langle\sigma_{a'}^A\sigma_b^B\rangle$.

■ **Example 2.2** Show by the same analysis, that quantum mechanics predicts

$$
\begin{cases}
\langle\sigma_a^A\sigma_b^B\rangle & \equiv \langle ab\rangle = -\cos\theta_{ab}\\
\langle\sigma_{a'}^A\sigma_{b'}^B\rangle & \equiv \langle a'b'\rangle = -\cos\theta_{a'b'}\\
\langle\sigma_a^A\sigma_{b'}^B\rangle & \equiv \langle ab'\rangle = -\cos\theta_{ab'}\\
\langle\sigma_{a'}^A\sigma_b^B\rangle & \equiv \langle a'b\rangle = -\cos\theta_{a'b}
\end{cases}
\tag{2.19}
$$

where θ_{ij} is the angle between axes i and j.

■

2.5 Bell's inequality (CHSH version)

The question of whether quantum mechanics can be "completed" by hidden variables remain unanswered until 1964. In that year, John Bell in his landmark paper "On the Einstein Podolsky Rosen Paradox",[9] showed that measurement outcomes which depend on hidden variables implied a mathematical constraint on how the outcomes on the measurements are correlated. This constraint became known as Bell's inequality. It allowed experiments to be performed that could falsify one theory or the other, thereby elevating the debate from philosophy to science.

We now consider a version of Bell's inequality due to four physicists, John Clauser, Michael Horne, Abner Shimony, and Richard Holt, who described it in a much-cited paper published in 1969.[10] This is called the CHSH version of Bell's inequality.

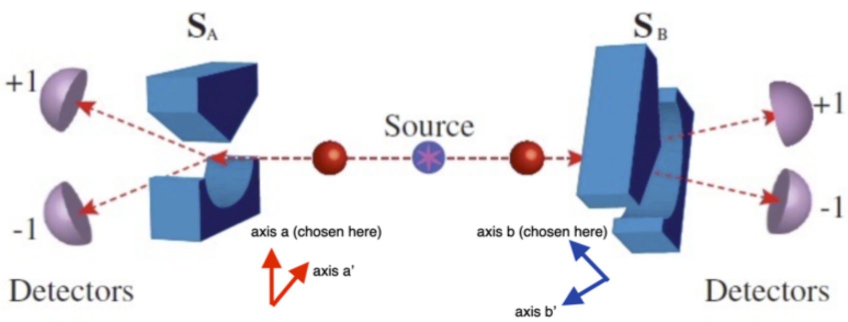

Figure 2.2: Schematic diagram of EPR experiment.

As shown schematically in Fig. 2.2, Alice can measure in either a or a' axes, and Bob can measure in either b or b' axes. Whichever the direction, the outcomes are either $+1$ or -1, which they record, i.e.

$$a, a' = \pm 1, \tag{2.20}$$
$$b, b' = \pm 1. \tag{2.21}$$

Now, a rather nuanced argument can also be made. Since the outcomes are binary,

$$\text{if } a' + a = 2 \text{ or } -2, \quad \text{then } a' - a = 0, \tag{2.22}$$
$$\text{if } a' + a = 0, \quad \text{then } a' - a = 2 \text{ or } -2. \tag{2.23}$$
$$\implies S \equiv ab + a'b' + a'b - ab' = (a' + a)b + (a' - a)b' = \pm 2. \tag{2.24}$$

One may ask, where is the nuance here? Recall the Stern-Gerlach experiment and the concept of incompatible observables, that *in quantum mechanics, it is not possible to measure* $\{a, a'\}$ *or* $\{b, b'\}$ *simultaneously.* Yet, in Eqs. (2.22, 2.23), by assigning outcomes ± 1 to the complementary observables at the same time, we have made an implicit assumption that properties exist, independent of the measurement (**realism**).

The expectation value of the quantity S in Eq. 2.24 is denoted by:

$$\langle S \rangle = \langle ab + a'b' + a'b - ab' \rangle = \langle ab \rangle + \langle a'b' \rangle + \langle a'b \rangle - \langle ab' \rangle \,. \tag{2.25}$$

We would have the prediction from quantum mechanics, putting in the angle dependences from Example 2.2 as

$$\langle S \rangle_{\mathrm{QM}} = -\left(\cos\theta_{ab} + \cos\theta_{a'b'} + \cos\theta_{a'b} - \cos\theta_{ab'}\right). \tag{2.26}$$

[9] See: J. S. Bell, Physics 1, 195-200 (1964) https://journals.aps.org/ppf/pdf/10.1103/PhysicsPhysiqueFizika.1.195.

[10] J.F. Clauser, M.A. Horne, A. Shimony, R.A. Holt (1969). "Proposed experiment to test local hidden-variable theories", Phys. Rev. Lett., 23 (15): 880–4. https://journals.aps.org/prl/abstract/10.1103/PhysRevLett.23.880.

With the locality assumption, we obtain from Eq. (2.25), this version

$$\langle S\rangle_{\mathrm{LHV}} = \langle ab + a'b' + a'b - ab'\rangle = \langle a' + a\rangle\langle b\rangle + \langle a' - a\rangle\langle b'\rangle\,, \tag{2.27}$$

where subscripts "LHV" stands for "local hidden variables". Here, the locality assumption is embedded in the factorizability condition[11] of the forms $\langle ab\rangle = \langle a\rangle\langle b\rangle$ and equivalently, $\langle (a+a')b\rangle = \langle a+a'\rangle\langle b\rangle$. What the factorization means is that the probability of getting outcomes a and b are independent of each other, i.e. what is measured at Alice's detector does not affect Bob's measurement and vice versa (**locality**).[12] Richard Feynman, in his Lectures on Physics, vol III[13], described this about EPR's locality argument,

> *[EPR] argue that "someone else making a measurement shouldn't be able to change the probability that I will find something." Our quantum mechanics says, however, that by making a measurement on photon number one, you can predict precisely what the polarization of photon number two is going to be when it is detected. This point was never accepted by Einstein, and he worried about it a great deal.*

A skeptic might suggest, what if some hidden signal informs Bob's experiment about Alice's detection basis, or vice versa, to create the correlations observed without actual entanglement? That possibility is called the locality loophole[14], which can be closed by arranging the experiment so that Alice and Bob are at places remote from one another, or that the decision to choose the measurement basis is made so late, that no light-speed signal with information about Alice's choice of basis can reach Bob until after his measurement is complete, and vice versa.

Finally, since average values must be within the bounds of the maximum and minimum possible values of the quantity, we have Bell's inequality, CHSH version:

$$-2 \leq \langle S\rangle_{\mathrm{LHV}} \leq 2. \tag{2.28}$$

As such, Bell's inequality is a constraint on the statistical occurrence of correlations in a Bell test, which is necessarily true if there exists underlying local, hidden variables (**local realism**). If experiments measure $\langle S\rangle$ outside of the bounds of Eq. (2.28), then it will falsify the hypothesis of local realism. For more than 50 years after the Einstein-Bohr debates, physicists had treated the arguments as belonging to the domain of philosophy. That John Bell was able to propose an experimentally falsifiable test that brought quantum philosophy into the domain of science, is a remarkable achievement.

■ **Example 2.3** Calculate $\langle S\rangle_{\mathrm{QM}}$ with the angle between detector axes θ as shown in Fig. 2.3. The EPR pair is the singlet state $|\Psi^-\rangle$. Sketch the result $\langle S\rangle_{\mathrm{QM}}$ against θ and indicate the regions where Bell's inequalities are violated, and the angles at which they are maximally violated. ■

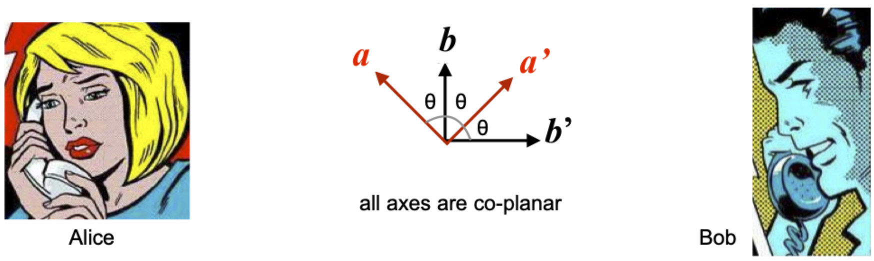

Figure 2.3: Axes of EPR experiment for Example 2.3.

[11] Bell refers to the factorizability condition as the condition that the correlations be locally explicable.

[12] A slightly more mathematical treatment is given in Bell's 1964 paper "On the Einstein Podolsky Rosen Paradox" for the interested student, but the key idea on factorizability is unchanged.

[13] `https://www.feynmanlectures.caltech.edu/III_18.html`

[14] For other loopholes, see e.g. John Preskill's notes (section 4.3.6 Experiments and loopholes) here: `http://theory.caltech.edu/~preskill/ph229/notes/chap4_01.pdf`.

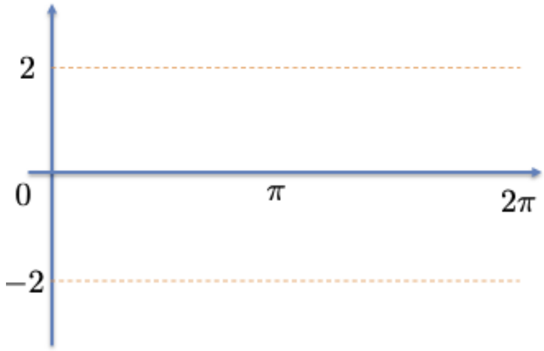

Figure 2.4: Answer to Example 2.3: (for the student to fill in)

2.6 Bell's Theorem

John S. Bell, in his paper "On the Einstein Podolsky Rosen Paradox" (1964), wrote the abstract:

> *The paradox of Einstein, Podolsky and Rosen was advanced as an argument that quantum mechanics could not be a complete theory but should be supplemented by additional variables. These additional variables were to restore to the theory causality and locality. In this note that idea will be formulated mathematically and shown to be incompatible with the statistical predictions of quantum mechanics. It is the requirement of locality, or more precisely that the result of a measurement on one system be unaffected by operations on a distant system with which it has interacted in the past, that creates the essential difficulty. There have been attempts to show that even without such a separability or locality requirement no "hidden variable" interpretation of quantum mechanics is possible. These attempts have been examined elsewhere and found wanting. Moreover, a hidden variable interpretation of elementary quantum theory has been explicitly constructed. That particular interpretation has indeed a grossly non-local structure. This is characteristic, according to the result to be proved here, of any such theory which reproduces exactly the quantum mechanical predictions.*

Bell's Theorem

No physical theory of local hidden variables can reproduce all of the predictions of quantum mechanics.

Bell's theorem has become central to both metaphysics and quantum information science. According to Bell[15], what his theorem implies is one of the following two conclusions:

1. Reality is irreducibly random, meaning that there are no hidden variables that "determine the results of individual measurements".
2. Reality is 'non-local', meaning that the "the setting of one measuring device can influence the reading of another instrument, however remote".

In fact, Bell wrote another paper in 1976 in which he distinguished between what he called "locality" and a new concept he termed "local causality". The interested student is persuaded to read the original paper[16], and a commentary[17] by Howard Wiseman.

2.7 Bell Tests

2.7.1 Loopholes

To date, predictions from quantum mechanics agree with Bell test experiments. Bell tests have been subject to intense scrutiny in order to identify flaws or *loopholes* in the reasoning or in the

[15] Bell, J.S. Physics 1,195 – 200(1964).
[16] Bell, J. S. Epistemol. Lett. 9, 11–24 (1976) https://cds.cern.ch/record/980036/files/197508125.pdf.
[17] https://www.nature.com/articles/510467a

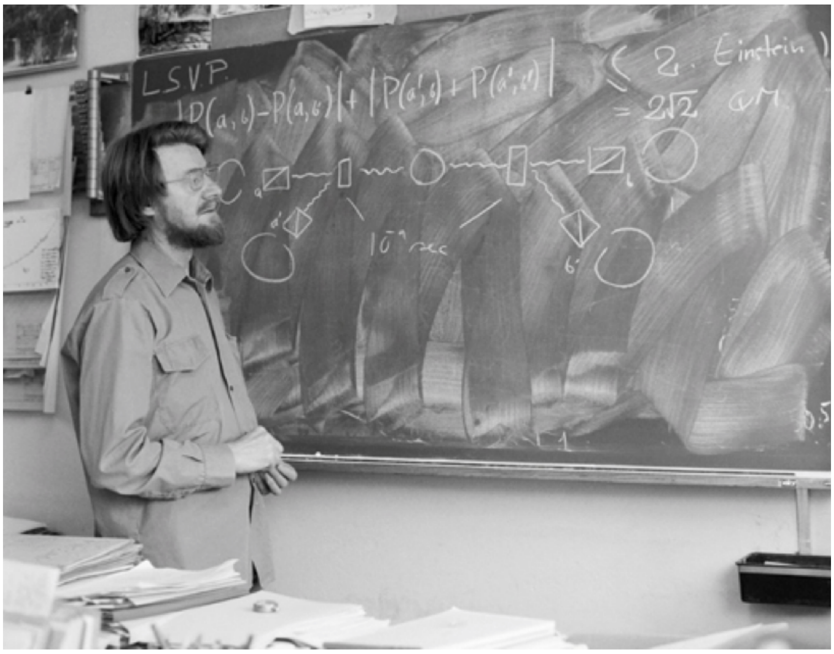

Figure 2.5: John Stewart Bell (1928 - 1990). Credit: CERN

experiments.

1. **Locality (or communication) loophole.**
   Detection events A and B must be separated by a distance too far for any hidden causal signal traveling at the speed of light to go from A to B thereby “informing” the other particle of the state of measurement of the first one.
2. **Detection (or unfair sampling) loophole.**
   If Alice and Bob do not perform sufficient experiments, or when instruments detect a subsample, e.g. due to detector inefficiency, then the statistics obtained may be from a biased sample. What if the correlations of the detected photons are unrepresentative: although they show a violation of a Bell inequality, if all photons were detected the Bell inequality would actually be respected? The assumption that this does not happen, i.e., that the small sample is actually representative of the whole is called the fair sampling assumption.
   To do away with this assumption it is necessary to detect a sufficiently large fraction of the photons. This is usually characterized in terms of the detection efficiency η, defined as the probability that a photodetector detects a photon that arrives at it. Garg and Mermin showed that when using a maximally entangled state and the CHSH inequality an efficiency of $\eta > 2\sqrt{2} - 2 \approx 0.83$ is required for a loophole-free violation.
3. **Memory loophole.**
   One of the basic assumptions in Bell tests is that different rounds of a test are *independent and identically distributed* (i.i.d.). In most experiments, measurements are repeatedly made at the same two locations. A local hidden variable theory could exploit the memory of past measurement settings and outcomes in order to increase the violation of a Bell inequality. Moreover, physical parameters might be varying in time. It has been shown that, provided each new pair of measurements is done with a new random pair of measurement settings, that neither memory nor time inhomogeneity have a serious effect on the experiment.[18]
4. **Free will or superdeterminism or skepticism loophole.**
   Results are predetermined by God or the Universe; we have no free will. This is the “ultimate” hidden variable. This loophole proposes that a particle detector’s settings may “conspire” with events in the shared causal past of the detectors themselves to determine

[18] Barrett et al., Quantum nonlocality, Bell inequalities, and the memory loophole, Phys. Rev. A 66, 042111 (2002). https://journals.aps.org/pra/abstract/10.1103/PhysRevA.66.042111

which properties of the particle to measure. This implies that the physicist running the experiment does not have complete free will in choosing each detector's setting. As Alain Aspect put it[19]

> *This is based on the idea that the choices of orientations we consider independent (because of relativistic causality) could in fact be correlated by an event in their common past. Since all events have a common past if we go back far enough in time – possibly to the big bang – any observed correlation could be justified by invoking such an explanation. Taken to its logical extreme, however, this argument implies that humans do not have free will, since two experimentalists, even separated by a great distance, could not be said to have independently chosen the settings of their measuring apparatuses. Upon being accused of metaphysics for his fundamental assumption that experimentalists have the liberty to freely choose their polarizer settings, Bell replied: "Disgrace indeed, to be caught in a metaphysical position! But it seems to me that in this matter I am just pursuing my profession of theoretical physics." I would like to humbly join Bell and claim that, in rejecting such an ad hoc explanation that might be invoked for any observed correlation, "I am just pursuing my profession of experimental physics."*

2.7.2 Timeline

1. **1982. First Bell Test – locality loophole closed.**
A. Aspect, J. Dalibard, and G. Roger, *Experimental Test of Bell's Inequalities Using Time-Varying Analyzers.* Phys. Rev. Lett. 49, 1804 (1982).
https://journals.aps.org/prl/abstract/10.1103/PhysRevLett.49.1804.
Locality loophole closed, but detector efficiency loophole not closed.
2. **2013. Two groups – Fair sampling loophole closed.**
Giustina, M. et al. *Bell violation using entangled photons without the fair-sampling assumption.* Nature 497,227 – 230 (2013) https://doi.org/10.1038%2Fnature12012.
Christensen, B. G. et al. *Detection-Loophole-Free Test of Quantum Nonlocality, and Applications.* Phys. Rev. Lett. 111, 130406 (2013)
https://journals.aps.org/prl/abstract/10.1103/PhysRevLett.111.130406.
Fair sampling loophole closed. But no space-like separation – locality loophole not closed.
3. **2015. Three groups – Both fair sampling and locality loopholes closed simultaneously.**
B. Hensen et al., *Loophole-free Bell Inequality Violation Using Electron Spins Separated by 1.3 Kilometres.* Nature 526, 682 (2015).
http://dx.doi.org/10.1038/nature15759
M. Giustina et al., *Significant-Loophole-Free Test of Bell's Theorem with Entangled Photons.* Phys. Rev. Lett. 115, 250401 (2015).
http://dx.doi.org/10.1103/PhysRevLett.115.250401
L. K. Shalm et al., *Strong Loophole-Free Test of Local Realism.* Phys. Rev. Lett. 115, 250402 (2015).
http://dx.doi.org/10.1103/PhysRevLett.115.250402

2.8 References

References

1. Sakurai, Chap. 3.9 Spin Correlation Measurements and Bell's Inequality. (This describes Bell's inequality not in the CHSH form.)
2. Preskill's Lecture Notes on Ph219/CS219: Quantum Information and Computation, Chap. 4. `http://theory.caltech.edu/~preskill/ph229/notes/chap4_01.pdf`

Further reading

1. Closing the Door on Einstein's and Bohr's Quantum Debate, by Alain Aspect.
https://physics.aps.org/articles/v8/123

[19] In further reading #1.

2. Three groups close the loopholes in tests of Bell's theorem, by Johanna L. Miller. https://physicstoday.scitation.org/doi/10.1063/PT.3.3039
3. Bell's theorem still reverberates, by Howard Wiseman. https://www.nature.com/articles/510467a
4. Nobel Prize: Quantum Entanglement Unveiled, by Michael Schirber. https://physics.aps.org/articles/v15/153

2.9 Discussion Set

Problem 2.1 Prove by contradiction, that the Bell states are entangled states, i.e. they cannot be written as tensor products of sub-system states.

Problem 2.2 We assumed an entangled pair of particles in the calculation of the CHSH quantity S. Is the CHSH inequality violated for a pair of particles that is not entangled, e.g. $|\Psi\rangle = |\uparrow\rangle_A |\uparrow\rangle_B$?

Problem 2.3 Check that the CHSH inequality can be violated by quantum mechanics, for another Bell state, the $|\Phi^+\rangle = \frac{1}{\sqrt{2}} (|\uparrow\rangle_A |\uparrow\rangle_B + |\downarrow\rangle_A |\downarrow\rangle_B)$.