4. Decoherence

I recall that during one walk Einstein suddenly stopped, turned to me and asked whether I really believed that the moon exists only when I look at it. The rest of the walk was devoted to a discussion of what a physicist should mean by the term 'to exist'.
– Abraham Pais.

4.1 Problems with Quantum Mechanics

4.1.1 Copenhagen Interpretation

The pioneering physicists, Niels Bohr, Werner Heisenberg and Wolfgang Pauli, often met in Copenhagen to discuss difficulties with the (then) new quantum theory, whence the name *Copenhagen Interpretation*. We give a survey of some of the difficulties faced, and the interpretations, ideas and experiments in the subsequent decades.

One of the first difficulties was in the translation of the mathematical symbols into physical, commonsense concepts. In classical physics, there is no such difficulty, e.g. position and velocity expressed mathematically elevate these concepts by adding a level of precision afforded by mathematics. In quantum theory, the first difficulty was the issue of complex numbers in the wavefunction, which made the the association with intuitive everyday experience becomes more distant. This was resolved by Max Born's probabilistic interpretation, in which the absolute value-squared of the wavefunction $|\psi(x)|^2$ took on the meaning of probability density.

However, the Born's probabilistic interpretation of the wavefunction became a source of considerable debate between Einstein and Bohr. Determinism is an intrinsic part of classical physics, which describes the dynamics of a system usually by differential equations. Randomness in classical physics is an appearance originating from ignorance or lack of complete knowledge of the system and its interactions. Determinism is also part of the mathematical formalism of quantum theory, when calculating the dynamics of the wavefunction – the Schrödinger equation determines the wavefunction's time-dependence. Yet, even if everything entering into the quantum theory is known, one cannot predict with certainty where the quantum particle described by the wavefunction will be detected, and has to resort to a probabilistic interpretation. In part 6 of his lecture series " The Character of Physical Law", Richard Feynman remarked,

> *A philosopher once said, "It is necessary for the very existence of science that the same conditions always produce the same results." Well, they don't!*

Another problem of interpretation is how to take into account two contradictory accounts of physical reality. On the microscopic quantum level, the uncertainty relations, e.g. between position and momentum, lead to a breakdown of classical concepts, like "particle" or "trajectory", which were cornerstones of classical physics, the physics of macroscopic, everyday experience. At the microscopic level, single electrons projected onto a double slit apparatus and detected on a screen behind, produces a build up of an interference pattern that can only be described as wave-like. Yet each spot on the screen – the position where an electron was detected – shows up its particle nature. This dual nature – being wave-like when traversing the slits and the space to the screen, and particle-like when detected on the screen – would be contradictory according to the macroscopic world of experience. The Copenhagen trio recognized that quantum phenomena must involve a radically new manner of thinking and understanding. In 1927, Bohr suggested that the concept of *complementarity* was the essential logical character, fundamental to modes of thinking about quantum phenomena.

The complementary principle states that mutually exclusive modes of language can be applied to the description of a quantum object, although not simultaneously. We may speak of an electron by using the language of waves when it crosses the double slits in an interference apparatus, and we may speak of the same electron as a particle when it is detected, but we cannot use the two modes of speaking at the same time. When using the language of particles, we may speak of the position or of the momentum at some instant of time, but not simultaneously.

In 1920, Bohr also suggested the *correspondence principle*, which states that the behaviour of systems described by the old quantum theory reproduces classical physics in the so-called classical or correspondence limit, e.g. $h \to 0$. The term codifies the idea that a new theory should reproduce under some conditions the results of older well-established theories in those domains where the old theories work.

Finally, the validity of the rules of quantum mechanics is not in doubt; they have been checked and validated by countless experiments to date. Yet, the rule of wavefunction collapse after a measurement introduced much difficulty in interpretation. Quantum mechanics relies on explicit dynamics governed by the Schrödinger equation, which is a linear equation that describes a deterministic time-evolution that is unitary. Measurement, from this standpoint, is a special type of interaction between the system and the measuring device that, if one were to apply quantum theory consistently, should also be described by the same Schrödinger equation. Yet this is contradicted by the wavefunction collapse rule, proposed by Werner Heisenberg, which is an indeterministic, nonlinear and non-unitary evolution. At best, one could say that having two distinct rules for dynamics and measurement is inconsistent. At worst, it may be argued that a complete theory has yet to be found. The wavefunction collapse issue has come to be known as "**the measurement problem**".

4.1.2 Interpretation after Copenhagen

The post-1925 new quantum theory came in two different formulations – Heisenberg's matrix formulation of quantum mechanics, and Schrödinger approach. Once the Schrödinger equation was given a probabilistic interpretation, Ehrenfest showed that Newton's laws hold on average: the quantum statistical expectation value of the position and momentum obey Newton's laws.

In 1930, Dirac proposed a framework for quantum mechanics in his book, "*Quantum Mechanics*". A direct connection with mathematics was not made then, and came only later. Dirac's mathematics could therefore not be considered as rigorous.

In 1932, John von Neumann published his work "*Mathematical Foundations of Quantum Mechanics*" which was to put the new quantum theory on a rigorous foundation with the mathematical theory of Hilbert spaces, which is the formulation we learn today in undergraduate classes. Another of von Neumann's important contributions was a theory of quantum measurements, in which both the quantum system and the measurement apparatus are described quantum-mechanically. We shall elaborate on this model later in the chapter. Von Neumann model of measurement has a great effect on interpretation. In contrast with the correspondence principle, it indicated that a quantum property could be transferred to the seemingly classical world of macroscopic objects. However, its difficulty was that it implied that superpositions and interferences can be transferred to a macroscopic level, and this could not be removed. Since we do not observe superpositions at the macroscopic level, this posed a real issue.

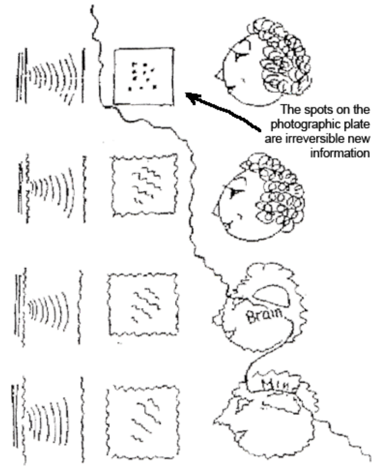


Figure 4.1: **Shifty split.** Illustrated by John Bell.

Von Neumann reasoned this away by requiring a "cut" (*schnitt* in German) between the quantum system and quantum apparatus, and the observer, as the boundary between quantum mechanics in which the superpositions persist, and the classical world in which we observe only one outcome. He said it did not matter where this cut was placed, because the mathematics would produce the same experimental results. Von Neumann thought that, since all material objects obey quantum mechanics at the fundamental level, even the chemical changes at the position of the screen hit by the double-slit electron, the subsequent photon that scattered off it into the eye of the observer, and the electrochemical signals induced in the human nervous system, should all be part of a quantum mechanical chain (the so-called *von Neumann chain*) capable of a description by quantum superposition, and that the final information of the measurement outcome – the wavefunction collapse – must be due to the consciousness of the observer.

No less a great physicist like Eugene Wigner also thought that consciousness was required for wavefunction collapse. He attempted to reason that quantum mechanics required consciousness with a thought experiment published n a 1961 article "Remarks on the Mind-Body Question". This has come to be known as the "*Wigner's friend*" paradox. He noted that most physicists in the then recent past had been materialists who insisted that "mind" or "soul" were illusory, and that nature is fundamentally deterministic, and argued that quantum physics has changed this situation, by writing,

> *All that quantum mechanics purports to provide are probability connections between subsequent impressions (also called "apperceptions") of the consciousness, and even though the dividing line between the observer, whose consciousness is being affected, and the observed physical object can be shifted towards the one or the other to a considerable degree, it cannot be eliminated.*

Von Neumann's schnitt, also called the Heisenberg cut, led John Bell later to sketch a cartoon titled "The shifty split" (see Fig. 4.1).

In the 1940s, David Bohm began questioning whether a reformulation of quantum mechanics could be possible, and that the seemingly probabilistic behavior of quantum systems stems from underlying, deterministic (hidden variable) mechanisms. In 1952, Bohm proposed that particles are indeed particles at all times, not just when they are observed, say by a detector. Their behaviour is determined by a force that Bohm called the "pilot wave." Any effort to observe a particle alters its behaviour by disturbing the pilot wave. Bohm thus gave the uncertainty principle a purely physical rather than metaphysical meaning. His theory was a non-local hidden variable theory, called the de Broglie-Bohm pilot wave theory or simply Bohmian mechanics.[1] Although Bohmian mechanics makes the same predictions as quantum mehanics, its nonlocal

[1] David Bohm, *A Suggested Interpretation of the Quantum Theory in Terms of "Hidden Variables", I*, Physical Review, (1952), 85, pp. 166–179. David Bohm, *A Suggested Interpretation of the Quantum Theory in Terms of "Hidden Variables", II*, Physical Review, (1952), 85, pp. 180–193.

nature makes it incompatible with special relativity (a local theory). Today, Bohmian mechanics is not widely accepted in the mainstream of the physics community, nor is it taught as part of the physics curriculum, perhaps due in no small part to Bohm's communist affiliations during the MacCarthyism era, a period in the 1940s and 50s, of intense anti-Communist suspicion in the United States.

Finally, in an earlier chapter, we saw how Bell's inequalities and experimental Bell tests resolved this issue, which showed that the correlations of Bell's inequalities cannot be described by local, hidden variable theories. That is to say, quantum indeterminism is fundamental and not due to ignorance or hidden variables.

4.1.3 Present state of interpretation: Decoherence

We complete this historical survey by mentioning the framework of decoherence, which is the subject of this chapter. In in 1970, H. Dieter Zeh introduced the concept of decoherence[2], motivated by the problem of the lack of macroscopic superpositions. Decoherence can be viewed as the loss of information from an open quantum system into the environment. Taken as a whole, the system-environment composite is a larger, closed quantum system and obeys unitary dynamics. However, when viewed in isolation, the system's dynamics are non-unitary. This framework was later developed further by Joos[3], and Wojciech Zurek.[4] Today, it is widely used to explain many issues related to the measurement problem, such as the so-called quantum-classical transition. We shall motivate the study of decoherence with the Schrödinger's cat thought experiment next, and leave out alternative approaches like objective collapse theories (e.g. Ghirardi–Rimini–Weber model), Consistent Histories, and Quantum Bayesianism (or QBism).

4.2 Principle of Superposition and Schrödinger's Cat

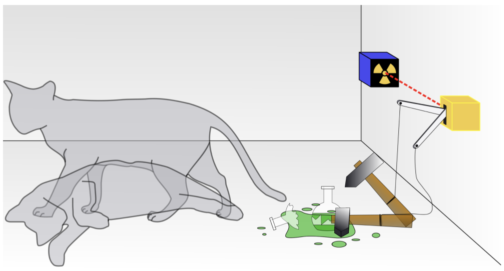


Figure 4.2: **Schrödinger's cat.** Picture from Wikipedia.

The difficulty reconciling quantum superpositions was pointed out by Erwin Schrödinger in 1935, with his famous Schrödinger's Cat thought experiment. The Schrödinger's cat thought experiment refers to the persistence of the superposition of cat alive and cat dead, according to the standard interpretation of quantum mechanics, until the box is open and its contents are directly observed, leading to the postulate of wavefunction collapse. This raises several questions:

1. How can we account for the observer's direct experience of a definite state of the cat being either alive or dead, but never in a superposition?
2. Is there a conflict between the experimentally confirmed role of the superposition principle and the observation of single definite outcomes in measurement?
3. What is the role of the observer, and why should the fate of the cat be left to the intervention of an external observer?

[2]H. Dieter Zeh, "*On the Interpretation of Measurement in Quantum Theory*", Foundations of Physics, vol. 1, pp. 69–76, (1970).

[3]E. Joos and H. D. Zeh, "*The emergence of classical properties through interaction with the environment*", Zeitschrift für Physik B, 59(2), pp. 223–243 (1985)

[4]See, e.g. W. H. Zurek, *Pointer Basis of Quantum Apparatus: Into what Mixture does the Wave Packet Collapse?*, Physical Review D, 24, pp. 1516–1525 (1981); W. H. Zurek, *Environment-Induced Superselection Rules*, Physical Review D, 26, pp. 1862–1880, (1982); W. H. Zurek, *Decoherence and the transition from quantum to classical*", Physics Today, 44, pp. 36–44 (1991); W. H. Zurek, *Decoherence, einselection, and the quantum origins of the classical*, Reviews of Modern Physics. 75 (2003).

The questions above are related what is called **"the measurement problem"** in quantum theory. We follow Schlosshauer[5], and break down the issue into specific problems, below.

Measurement problems

1. **The problem of the non-observability of interference.** Why is it so difficult to observe quantum interference effects, especially on the macroscopic scale?
2. **The problem of the preferred basis.** What singles out preferred physical quantities in nature? For instance, why are physical systems usually observed to be states of definite positions, rather than in its superpositions, which is simply related to the former by a change of basis?
3. **The problem of outcomes.** Why do measurements have outcomes at all, and what selects a particular outcome among the different possibilities described by the quantum probability distribution?

Using the framework of decoherence, we will discuss the how the framework attempts to resolve the first two problems. The third problem is related to the collapse postulate – the most essential question in quantum theory – and is still unresolved today.

4.3 The Double-Slit Experiment

Let us revisit the famous double-slit experiment. This experiment was first performed by Thomas Young in 1801, as a demonstration of the wave behaviour of visible light. With the development of quantum theory following the blackbody radiation, photoelectric effect and Compton scattering, we now know that light possesses a wave-particle dual nature. In 1927, Davisson and Germer demonstrated that electrons show the same interference behaviour through a double slit, showing that matter particles too, possess a wave-particle duality.

In 1976, a group Italian physicists, Merli, Missiroli and Pozzi, performed the double slit experiment with single electrons. A similar experiment was also performed by researchers at Hitachi in 1989 and documented online with video clips showing that while single bright spots recorded on the screen appear randomly, demonstrating the detection of electrons one by one as particles, over time, a pattern reminiscent of wave interference builds up, as seen in Fig. 4.3. This is despite having at any one instant, at most one electron in the experimental set up.

Richard Feynman called the double-slit experiment,

> "*a phenomenon which is impossible ... to explain in any classical way, and which has in it the heart of quantum mechanics. In reality, it contains the only mystery [of quantum mechanics].*"

Quantum mechanically, the electron is described by a superposition of the components $|\psi_1\rangle, |\psi_2\rangle$ corresponding to passage through slit 1 and 2 respectively. This is given by

$$|\Psi\rangle = \frac{1}{\sqrt{2}}\left(|\psi_1\rangle + |\psi_2\rangle\right). \tag{4.1}$$

However, this wavefunction $\Psi(x) = \langle x|\Psi\rangle$ is delocalized spatially, and the only measurement we have available is the position of the electron behind each slit. The electron probability density as a function of spatial position, is given by the square of the sum of probability amplitudes through each slit (1 and 2). This is given by

$$\varrho(x) = \frac{1}{2}|\psi_1(x) + \psi_2(x)|^2 = \frac{1}{2}|\psi_1(x)|^2 + \frac{1}{2}|\psi_2(x)|^2 + \mathrm{Re}\{\psi_1(x)\psi_2^\star(x)\}, \tag{4.2}$$

where the last term is responsible for the characteristic interference pattern (Fig. 4.4 right), and $\psi_1(x) = \langle x|\psi_1\rangle$ and $\psi_2(x) = \langle x|\psi_2\rangle$. Without the last term, $\frac{1}{2}|\psi_1(x)|^2 + \frac{1}{2}|\psi_2(x)|^2$ would give rise to a "classical" distribution (Fig. 4.4 center). This is because quantum states represent probability amplitudes rather than probabilities, which implies that a superposition state leading

[5] Schlosshauer, Decoherence and the quantum-to-classical transition. Springer (2008). Second printing.

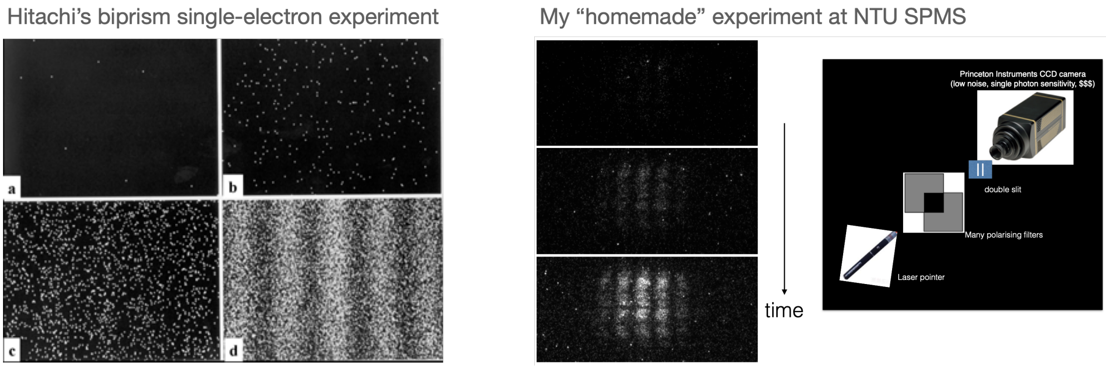


Figure 4.3: **Double-slit experiment.** Left: Single electron events build up to from an interference pattern in the double-slit experiments. From `https://www.hitachi.com/rd/research/materials/quantum/doubleslit/index.html`. Right: My “homemade” experiment at NTU SPMS, albeit not with single photons but with a heavily attenuated laser pointer.

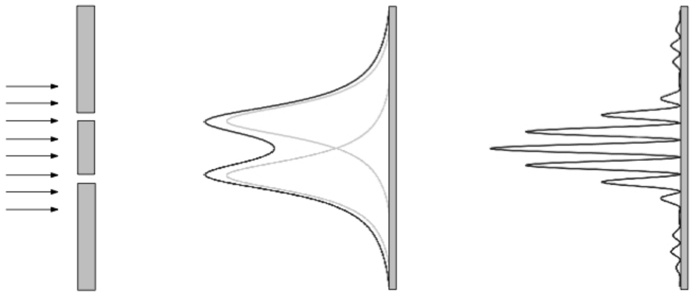


Figure 4.4: **Double-slit experiment schematic.** Center: The resulting “classical” density pattern obtained on a distant detection screen. This pattern corresponds to a simple addition of the contributions from each individual slit. Right: The interference pattern obtained in the quantum setting (with the envelope modulated by diffraction). From: Schlosshauer.

to interference terms in the probability distribution Eq. (4.2). Therefore, within the standard formalism[6] of quantum mechanics, the individual electron cannot be described by either one of the wave functions describing the passage through a particular slit, but only by a superposition of these wave functions.

4.4 Which-Path Information and Von Neumann Measurement

There are two important observations which are confirmed by single particle double-slit experiments, namely,

1. when no measurement of *which path* (slit 1 or 2) was taken by the particle, the interference pattern of Fig. 4.4(right) is observed, whereas
2. when a detector is placed at one of the slits to measure *which path* was taken by the particle, the “classical” distribution pattern of Fig. 4.4(center) is observed.

The explanation of case 2 in standard quantum mechanics comes from the collapse postulate. Here, the argument goes as follows: the state of the particle is given by the wavefunction $\Psi(x)$, Eq. (4.1), which is a superposition of the components $\psi_1(x), \psi_2(x)$. It is spread out in space encompassing the two slits. If a measurement is made on the position of the particle at one of the slits, then the measurement has *collapsed* the wavefunction $\Psi(x)$ into one of the two components $\psi_1(x)$ or $\psi_2(x)$. As such, there is no longer any interference to be observed.

[6] There exist alternative formalisms which take a different view, notably the de Broglie-Bohm pilot wave theory. See: https://www.quantamagazine.org/pilot-wave-theory-gains-experimental-support-20160516/ and https://www.science.org/doi/10.1126/sciadv.1501466.

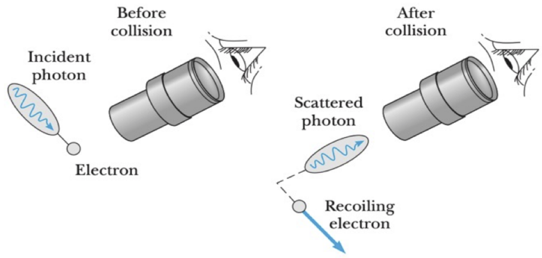


Figure 4.5: **Heisenberg's microscope.**

In Bohr's terms, there are *complementary* aspects between *which-path* information (the particle aspect) and interference (the wave aspect). We can only observe either the particle or the wave aspect, depending on the experimental setup, but not both *at the same time.* This "wave-particle duality" has been a cornerstone of quantum theory and the subject of philosophical discussions.

In addition, it is often argued, e.g. in Heisenberg's microscope Fig. 4.5, that a measurement of the position of a particle disturbs its. Since such a measurement involves observing photons that scatter off it, inevitably the particle recoils from the momenta of the photons. In other words, obtaining information about one aspect of a quantum system implies an inevitable disturbance of the complementary aspect of the system. Similarly, it is often claimed that obtaining which-path information implies an inevitable disturbance of the system. We shall discuss this next.

4.4.1 Von Neumann Measurement

According to the collapse postulate, the wavefunction collapse is a discontinuous and irreversible process.[7] What if we now perform an imprecise measurement of which path the particle has travelled? To this end, we pursue a quantum mechanical account of the measurement process, or sometimes called pre-measurement process, according to von Neumann, and the entanglement between the quantum system and the measuring apparatus. This will be in contrast to the classical account of measurement in the Heisenberg microscope example described in the preceding paragraph.

Von Neumann's goal was to formulate a theory of measurement in completely quantum-mechanical terms. To do so, not only is the system treated as a quantum object, but also the measurement apparatus as well. The von Neumann measurement scheme also allows us to understand how entanglement arises, and to regard decoherence as a consequence of a von Neumann–type measurement interaction between the system and its environment.

The von Neumann measurement scheme is formulated as follows.

[7] Wikipedia (Wave function collapse): *The concept of wavefunction collapse was introduced by Werner Heisenberg in his 1927 paper on the uncertainty principle, "Über den anschaulichen Inhalt der quantentheoretischen Kinematik und Mechanik", and incorporated into the mathematical formulation of quantum mechanics by John von Neumann, in his 1932 treatise Mathematische Grundlagen der Quantenmechanik. Heisenberg did not try to specify exactly what the collapse of the wavefunction meant. However, he emphasized that it should not be understood as a physical process. Niels Bohr also repeatedly cautioned that we must give up a "pictorial representation", and perhaps also interpreted collapse as a formal, not physical, process.*

The von Neumann measurement scheme

Pre-measurement (unitary evolution):

- **The quantum system.** The quantum system $\mathcal{S}$ is described by a Hilbert space $\mathcal{H}_{\mathcal{S}}$ with basis vectors $\{|s_i\rangle\}$. It is typically microscopic.
- **The measurement apparatus.** The measurement apparatus $\mathcal{A}$ can be microscopic or macroscopic. We typically consider an apparatus as macroscopic since it is associated with a large number of degrees of freedom and we expect such a system to show behaviour corresponding to that of classical physics. It is, however, treated quantum-mechanically, represented by basis vectors $\{|a_i\rangle\}$ in a Hilbert space $\mathcal{H}_{\mathcal{A}}$. We note that whether the system or apparatus are considered microscopic or macroscopic does not have any bearing on the measurement argument.
- **Pointer states.** The purpose of the apparatus is to measure the state of the system. We think of the apparatus as having some kind of pointer that moves to position "i", represented by state $|a_i\rangle$, if the system is measured to be in state $|s_i\rangle$.
- **Pre-measurement evolution.** The apparatus starts in some "ready" state $|a_r\rangle$, and the dynamical measurement interaction between the system and the apparatus is as such:
$$|s_i\rangle\,|a_r\rangle \longrightarrow |s_i\rangle\,|a_i\rangle \quad \text{for all } i, \tag{4.3}$$
where the initial and final system-apparatus states $|s_i\rangle\,|a_r\rangle$ and $|s_i\rangle\,|a_i\rangle$ reside in the joint Hilbert space $\mathcal{H}_{\mathcal{SA}} = \mathcal{H}_{\mathcal{S}} \otimes \mathcal{H}_{\mathcal{A}}$.
- **Quantum non-demolition measurement.** In writing Eq. (4.3), we tacitly assumed that the measurement interaction does not change the state of the system. Such a scheme is called ideal, or quantum non-demolition measurements.

Measurement (collapse):

- **Collapse postulate.** The *pre-measurement* scheme described in the preceding steps are unitary, continuous time-evolution, which is deterministic. Von Neumann, following Heisenberg, now postulated a second type of time-evolution: the probabilistic, non-unitary, non-local, discontinuous change brought about by observation and measurement. This was given as the fourth postulate of quantum mechanics in the earlier chapters.

Now, consider the system in a superposition $|\psi\rangle = \sum_i c_i\,|s_i\rangle$. Then the ideal von Neumann quantum measurement scheme gives us the pre-measurement evolution,

$$|\psi\rangle\,|a_r\rangle = \left(\sum_i c_i\,|s_i\rangle\right)|a_r\rangle \longrightarrow |\Psi\rangle = \sum_i c_i\,|s_i\rangle\,|a_i\rangle\,. \tag{4.4}$$

Measurement thus entangles the system and apparatus and we can no longer attribute an individual state vector to the system or the apparatus. Entanglement has been created *dynamically*, through the measurement interaction. The superposition initially present only in the system has been *amplified* to the level of the (typically macroscopic) apparatus, in the sense that the final superposition involves both the system and the apparatus.

Let us now apply this pre-measurement scheme to the double-slit experiment. If we prepared the state of the particle to be in $|\psi_{1/2}\rangle$ by covering slit 2/1, then the pre-measurement evolution scheme will yield

$$|\psi_1\rangle\,|\text{“ready”}\rangle \longrightarrow |\psi_1\rangle\,|1\rangle\,, \tag{4.5}$$
$$|\psi_2\rangle\,|\text{“ready”}\rangle \longrightarrow |\psi_2\rangle\,|2\rangle\,, \tag{4.6}$$

where the detector states $|1\rangle$ and $|2\rangle$ act as "pointers" for the states $|\psi_1\rangle$ and $|\psi_2\rangle$ of the system. If both slits are open, then the particle is described by the state $|\Psi\rangle$ in Eq. (4.1), and the pre-measurement evolution will be

$$|\Psi\rangle\,|\text{“ready”}\rangle = \frac{1}{\sqrt{2}}\left(|\psi_1\rangle + |\psi_2\rangle\right)|\text{“ready”}\rangle \longrightarrow \frac{1}{\sqrt{2}}\,|\psi_1\rangle\,|1\rangle + \frac{1}{\sqrt{2}}\,|\psi_2\rangle\,|2\rangle\,. \tag{4.7}$$

The quantity of interest is the reduced density operator of the particle, given by

$$\hat{\rho}_{\text{particle}} = \frac{1}{2}\left(|\psi_1\rangle\langle\psi_1| + |\psi_2\rangle\langle\psi_2| + |\psi_1\rangle\langle\psi_2|\langle 2|1\rangle + |\psi_2\rangle\langle\psi_1|\langle 1|2\rangle\right). \tag{4.8}$$

The particle probability density at the detection screen is

$$\begin{aligned} \varrho(x) =& \mathrm{Tr}(|x\rangle\langle x|\,\hat{\rho}_{\text{particle}}) \\ =& \langle x|\hat{\rho}_{\text{particle}}|x\rangle \\ =& \frac{1}{2}|\psi_1(x)|^2 + \frac{1}{2}|\psi_2(x)|^2 + \mathrm{Re}\{\psi_1(x)\psi_2^\star(x)\langle 2|1\rangle\}, \end{aligned} \tag{4.9}$$

which looks very similar to Eq. (4.2), except for an extra detector overlap term $\langle 2|1\rangle$. The visibility of the interference pattern is quantified by the overlap $\langle 2|1\rangle$. In the case of perfect distinguishability of detector states, $\langle 2|1\rangle = 0$ and the interference term disappears. Conversely, if the detector is unable to resolve the path of the particle at all, then $\langle 2|1\rangle = 1$ and we recover the interference.

It is here that we can now quantify the intermediate situation, where the detector may imprecisely resolve the path of the particle, $|\langle 2|1\rangle| < 1$, obtaining partial, incomplete which-path information. In such a case, we can see that we can still have interference, but the pattern will decay compared to when there was no which-path information.

Therefore, it is possible to simultaneously observe an interference pattern and obtain some information about the path of the particle through the slits, provided this information remains incomplete. The degree to which an interference pattern can be observed is simply determined by the available which-path information encoded in some system entangled with the object of interest, and this amount can be changed without necessarily influencing the spatial wave function of the object itself, in contrast to the disturbance explanation of Heisenberg's microscope.

Thus, complementarity can be regarded as a consequence of quantum entanglement.

4.5 Environmental Monitoring

A quantum system is not perfectly isolated; there is always some interaction with its environment. Consider the double slit experiment: the particle passing through the double slit may interact with the enormous number of air molecules or ambient photons present. Even if the experiment were to be conducted in a vacuum or a dark room, there will still be thermal photons from the cosmic microwave background, an artifact left over from the Big Bang. In other word, we have to consider the inevitable fact that quantum systems are *open*, and interacting with its environment which typically has a huge number of degrees of freedom.

Suppose the environment of, say N molecules, in the double slit experiment is given by its initial quantum state $|E_0\rangle$. When the molecules collide and interact with the particle, each molecule will fly off in a different way with some other velocity, giving rise to a different quantum state from its initial state. Let us denote the post-scattering states corresponding to the particle's passage through slits 1 and 2 respectively, by $|E_1\rangle$ and $|E_2\rangle$, such that after the environment interacts with the particle, the pure state of the system-environment composite evolves as

$$\frac{1}{\sqrt{2}}\left(|\psi_1\rangle + |\psi_2\rangle\right)|E_0\rangle \longrightarrow \frac{1}{\sqrt{2}}|\psi_1\rangle|E_1\rangle + \frac{1}{\sqrt{2}}|\psi_2\rangle|E_2\rangle. \tag{4.10}$$

However, when we only have access to the system, we obtain the reduced density matrix of the particle,

$$\hat{\rho}_{\text{particle}} = \frac{1}{2}|\psi_1(x)|^2 + \frac{1}{2}|\psi_2(x)|^2 + \mathrm{Re}\{\psi_1(x)\psi_2^\star(x)\langle E_2|E_1\rangle\} \tag{4.11}$$

The connection to the which-path argument with detectors in the preceding section is clear. If these two states $|E_1\rangle$ and $|E_2\rangle$ are distinguishable in principle, i.e. the information contained in these two states is sufficient to discriminate between the paths through either slit, then the overlap $\langle E_1|E_2\rangle$ is small, and interference will be weak.

The crucial point is that these environmental which-path, or more accurately, which-state detectors are present everywhere in nature. Every object interacts with its environment, which in turn will obtain information about certain physical properties of the system. Only measurements that include both system and environment could possibly reveal the persistence coherence (which is never lost according to this viewpoint) between the components of the system-environment entangled state. However, it is impossible in practice to include so many environmental degrees of freedom that have interacted with the system, in our observation. Thus, interference remains at the global level but have become unobservable at the local level. Even though some environmental degrees of freedom will be measured, e.g. by directly observing a scattered photon, the point is that there remains an incomparably larger number of environmental degrees of freedom not observed directly.

Environment as a Which-State Monitor

The environment is thus an ubiquitous "measuring device" which continuously performs effective measurements (in the von Neumann sense) on the system. Importantly, the environmental monitoring process does not require a human observer of any sort. The fact that the which-state information encoded in the environment could *in principle* be read out is sufficient for the interference pattern to disappear. Environmental monitoring and the resulting decoherence thus resolves the first of the measurement problems we stated at the start of this chapter: *the problem of the non-observability of interference.*

4.5.1 Decoherence and Local Damping of Interference

The discussion in Sec. 4.5 resolves qualitatively, the problem non-observability of interference, and is a formal representation of the decoherence process. A realistic modeling of the system-environment interactions in various experimental situations is required to understand quantitatively, how fast this decoherence or decay of coherence or interference terms happen. Typically, the dynamics of the overlap is found to follow an exponential decay with a characteristic decoherence time τ_d depending on the strength of the system-environment interaction,

$$\langle E_i(t)|E_j(t)\rangle \propto e^{-t/\tau_d}, \tag{4.12}$$

for $i \neq j$. Here, $t = 0$ corresponds to the time at which the interaction is "switched on"; at earlier times, the environment and system are assumed to be completely uncorrelated. We shall make this more quantitative with specific models with explicit forms of $|E_i(t)\rangle$, e.g. an explicit model of scattering-induced decoherence gives[8]

$$\langle E_i(t)|E_j(t)\rangle \propto e^{-\Lambda|x-x'|^2 t}, \tag{4.13}$$

where $|x - x'|$ is the difference between the centre-of-mass positions x and x' of the system, i.e. it is exponentially harder to find a particle in a positional superposition the further the two locations are from each other.

4.6 The Preferred Basis Problem

The preferred basis problem may be explained by a simple example. Suppose the von Neumann measurement scheme takes the system-apparatus state from such an initial superposition of states to a final state as follows,

$$|\psi\rangle\,|a_r\rangle = \left(\sum_i c_i\,|s_i\rangle\right)|a_r\rangle \longrightarrow |\Psi\rangle = \sum_i c_i\,|s_i\rangle\,|a_i\rangle\,. \tag{4.14}$$

Note that the RHS may be written in any choice of basis, including some primed basis,

$$|\Psi\rangle = \sum_i c_i\,|s_i\rangle\,|a_i\rangle = \sum_i c_i\,|s_i'\rangle\,|a_i'\rangle\,. \tag{4.15}$$

[8]This was first presented by E. Joos and H. D. Zeh in 1984, in their paper titled "*The Emergence of Classical Properties Through Interaction with the Environment*".

However, the freedom in the choice of basis is constrained by two requirements. Firstly, the states of the apparatus $|a_i\rangle$ should be mutually orthogonal (which ensures classically distinct outcomes of measurement such that possible system states $|s_i\rangle$ can be reliably distinguished). Secondly, for an arbitrary choice of apparatus states $|a_i\rangle$, the relative states $|s_i\rangle$ of the system may fail to be mutually orthogonal. As such, we may therefore require the apparatus states to be mutually orthogonal $\langle a_i|a_j\rangle = \delta_{ij}$, which means that the **Schmidt** decomposition below is unique,

$$|\Psi\rangle = \sum_i c_i |s_i\rangle |a_i\rangle , \tag{4.16}$$

with $c_i \in \mathbb{R}$ and $\sum_i c_i^2 = 1$, provided all coefficients c_i are different from one another.

When this condition on the coefficients does not hold, Eq. (4.16) is not unique, as can be seen from this next example. Given the system spin-apparatus interaction,

$$|0_z\rangle_S |\text{“ready”}\rangle_A \longrightarrow |0_z\rangle_S |0_z\rangle_A , \tag{4.17}$$
$$|1_z\rangle_S |\text{“ready”}\rangle_A \longrightarrow |1_z\rangle_S |1_z\rangle_A , \tag{4.18}$$
$$\tag{4.19}$$

when the system starts in the superposition of spin states, $(|0_z\rangle_S + |1_z\rangle_S)/\sqrt{2}$, the final system-apparatus state will be

$$|\Psi\rangle = \frac{1}{\sqrt{2}} (|0_z\rangle_S |0_z\rangle_A + |1_z\rangle_S |1_z\rangle_A) . \tag{4.20}$$

Looking at Eq. (4.20), the answer to the question “*what observable has been measured by the apparatus A?*” might seem obvious: it must be σ_z, of course, i.e. spin in the z-direction. But, we saw that Eq. (4.20) can be transformed and written in the x-basis, to become

$$|\Psi\rangle = \frac{1}{\sqrt{2}} (|0_x\rangle_S |0_x\rangle_A + |1_x\rangle_S |1_x\rangle_A) . \tag{4.21}$$

What can we now deduce from Eq. (4.21)? Apparently, the answer to the same question might now be that the apparatus is measuring spin in the x-direction.

We now encounter two problems. Firstly, the preceding line of argument implies that the apparatus has formed a one-to-one correlation with both the z-spin and x-spin states of the system. In the von Neumann measurement scheme, the apparatus would have been said to have measured non-commuting observables of the system simultaneously.

Secondly, such a situation of simultaneous measurement of a set of non-commuting observables is not only forbidden by quantum mechanics, but also contradicts our experience that measurement devices seemed to be designed to measure only very particular quantities. For example, a Stern-Gerlach apparatus with the magnetic field aligned in the z-direction, is designed to distinguish only the $|0_z\rangle_S$ and $|1_z\rangle_S$ spin states, and not those in the x-direction. Performing a measurement in the x or any other direction requires a physical rotation of the device or magnetic field, and would correspond to a different setup.

The conclusion to be drawn is that the formalism of quantum mechanics, with the von Neumann measurement scheme applied to the isolated system-apparatus combination, does not automatically specify the observable that has been measured. Yet this is hard to reconcile with our experience in the laboratory that measuring devices are designed to measure highly specific quantities!

We can generalize this problem by asking why (especially macroscopic) objects are usually found in a very small set of eigenstates, most prominently in position eigenstates, whereas the linearity of the Hilbert space of the quantum mechanical formalism would, in principle, allow for arbitrary superposition of positions (and other eigenbases).

The Preferred Basis Problem

The existence of such a “*preferred basis*” is thus not explained by the final system-apparatus state arrived at through a von Neumann measurement scheme.

4.7 Environment-Induced Superselection (*einselection*)

The inclusion of interactions between the quantum system-quantum apparatus (SA) combination and an environment (E) suggests a solution to the preferred basis problem.

Einselection, short for "environment-induced superselection", is a term given by Zurek. Due to the interaction with the environment, the vast majority of states in the Hilbert space of an open quantum system become highly unstable due to the entangling interaction with the environment, which in effect monitors *selected* observables of the system.

We may now ask: *What singles out the particular superpositions as those which interference is suppressed through the interaction with the environment?*

Since the system and apparatus are both quantum mechanical, let us consider just the quantum system interacting with the environment (dropping the apparatus) for simplicity. The following argument can be generalized to include the quantum apparatus.

Consider the example of the double slit experiment again, where the environment monitors the state of the particle localized at slit 1 or 2,

$$|\psi_1\rangle\,|E_0\rangle \rightarrow |\psi_1\rangle\,|E_1\rangle\,, \tag{4.22}$$
$$|\psi_2\rangle\,|E_0\rangle \rightarrow |\psi_2\rangle\,|E_2\rangle\,. \tag{4.23}$$

We motivate this form of the system-environment interaction by referring to the fact that the system states correspond to sufficiently distinct physical states (e.g. different spatial positions), and that this difference is resolved by the environment continuously monitoring the system, which may be written as the assumptions $\langle E_1|E_2\rangle \rightarrow 0$ and $\langle\psi_1|\psi_2\rangle = 0$. For the conjugate states

$$|\psi_\pm\rangle = \frac{1}{\sqrt{2}}\left(|\psi_1\rangle \pm |\psi_2\rangle\right), \tag{4.24}$$

corresponding to nonclassical delocalized states, i.e. superpositions of localized states, the system-environment interaction rules in Eqs. (4.22)-(4.23) lead to

$$|\psi_+\rangle\,|E_0\rangle \rightarrow \frac{1}{\sqrt{2}}\left(|\psi_1\rangle\,|E_1\rangle + |\psi_2\rangle\,|E_2\rangle\right) = \frac{1}{\sqrt{2}}\left(|\psi_1'\rangle\,|E_1'\rangle + |\psi_2'\rangle\,|E_2'\rangle\right), \tag{4.25}$$
$$|\psi_-\rangle\,|E_0\rangle \rightarrow \frac{1}{\sqrt{2}}\left(|\psi_1\rangle\,|E_1\rangle - |\psi_2\rangle\,|E_2\rangle\right) = \frac{1}{\sqrt{2}}\left(|\psi_1'\rangle\,|E_1'\rangle - |\psi_2'\rangle\,|E_2'\rangle\right), \tag{4.26}$$

where the RHS is a non-separable Bell-type state that can be written in infinitely many different basis states, as denoted by the primed states.

As a result of the effective environmental monitoring of which-state information, a system state prepared in any one of the superpositions of Eqs. (4.25)-(4.26), lead to the suppression of interference between the components $|\psi_1\rangle$ and $|\psi_2\rangle$ (as well as the alternative primed states).

However, the environment states $|E_1\rangle$ and $|E_2\rangle$ are entangled with the system states and measurements performed on the environment in the $\{|E_1\rangle\,, |E_2\rangle\}$ or the primed bases will reveal no information whatsoever about which of the two states $|\psi_+\rangle$ and $|\psi_-\rangle$ has been prepared initially. This is because measurements on the environment will only produce outcomes $|E_1\rangle$ and $|E_2\rangle$ with equal probability. Therefore, a superposition of the conjugate states, say

$$|\phi\rangle = \frac{1}{\sqrt{2}}\left(|\psi_+\rangle + |\psi_-\rangle\right), \tag{4.27}$$

(which is essentially state $|\phi\rangle = |\psi_1\rangle$) would give rise to a system-environment state,

$$|\phi\rangle\,|E_0\rangle \longrightarrow |\phi\rangle\,|E_1\rangle\,, \tag{4.28}$$

an un-entangled state. Thus, such a state has no environmental overlap terms to decay and cause decoherence of any system interference terms.

In short, according to the rules of the system-environment interaction of Eqs. (4.22)-(4.23), not all superpositions of states are equal when it comes to decoherence. Each spatially localized state, $|\psi_1\rangle$ and $|\psi_2\rangle$, results in no system-environment entanglement and hence no decoherence, whereas spatially non-localized states, $|\psi_+\rangle$ and $|\psi_-\rangle$ result in system-environment entanglement and hence decoherence. We call the former, *environment-superselected states* or *preferred states*. They also take the name *pointer states* when referred to quantum apparatus states. We will take these terms as synonymous.

Therefore, it is the *particular form of the interaction* between a system and its environment that determines whether the environment is sensitive or not, to the difference between system states. If no information is encoded in the environment as a result of this interaction that would allow us to distinguish these system states by looking at the environment, then the environment cannot bring about any suppression of interference between the system states. In this way the environment induces effective superselection rules. Einselection thus precludes the stable existence of pure superpositions of pointer states.

4.7.1 Pointer States

Some states of the system are more prone to decoherence than others, and the sensitivity of a particular state is determined by the structure of the system-environment interaction. Those states of the system that are the least sensitive, or most robust, to the interaction with the environment, in the sense that they become *least entangled* with the environment, are called **preferred states** or **pointer states**. They are thus most immune to decoherence.

This terminology is motivated by the idea that, because of their robustness, the environment-superselected states correspond to the physical quantities that are most easily "read-off" at the level of the system, analogous to the reading-off of a pointer on the dial of a measurement apparatus.

Let us consider the case of a system interacting with an environment without the explicit presence of an apparatus. In general, the Hamiltonian corresponding to such a situation can be written as the bare Hamiltonians of the system and environment, and the interaction term between the system and the environment,

$$\hat{H} = \hat{H}_S + \hat{H}_E + \hat{H}_{\text{int}}. \tag{4.29}$$

4.7.2 Pointer States in the Quantum-Measurement Limit

In the *quantum measurement limit* where the interaction between the system and its environment is strong enough to completely dominate the dynamics, we can approximate the total Hamiltonian as

$$\hat{H} \approx \hat{H}_{\text{int}}. \tag{4.30}$$

We assume that the initial system-environment state is unentangled, $|s_i\rangle |E_0\rangle$, as usual. Our aim is to find the set of robust system states $|s_i\rangle$ that do not get entangled under this evolution, i.e. we require that the evolution under $\hat{H}_{\text{int}}$ give

$$e^{-i\hat{H}_{\text{int}}t/\hbar} |s_i\rangle |E_0\rangle = \lambda_i |s_i\rangle e^{-i\hat{H}_{\text{int}}t/\hbar} |E_0\rangle \equiv \lambda_i |s_i\rangle |E_i(t)\rangle , \tag{4.31}$$

which implies that in the quantum measurement limit, the pointer states of the system $|s_i\rangle$ will be the eigenstates of the interaction Hamiltonian $\hat{H}_{\text{int}}$. Equivalently, the pointer observables are simply linear combinations of the pointer state projectors,

$$\hat{O}_S = \sum_i o_i |s_i\rangle \langle s_i| . \tag{4.32}$$

Since $|s_i\rangle$ are eigenstates of $\hat{H}_{\text{int}}$, it means that all $|s_i\rangle \langle s_i|$ commute with $\hat{H}_{\text{int}}$. This is the commutativity criterion:

Commutativity criterion

$$[\hat{O}_S, \hat{H}_{\text{int}}] = 0. \tag{4.33}$$

■ **Example 4.1** If the system-environment interaction Hamiltonian is of the form

$$\hat{H}_{\text{int}} = \hat{x} \otimes \hat{E}, \tag{4.34}$$

where $\hat{x}$ is the system position operator, (i) what would be the pointer states of the system, and (ii) how would a robust system-environment composite state evolve, in the quantum measurement limit? ■

4.7.3 Pointer States in the Decoherence Limit and Intermediate Regime

The opposite limit, called the decoherence limit, is when the self-Hamiltonian $\hat{H}_S$ dominates the evolution of the system. In such a scenario, the preferred states will be the eigenstates of $\hat{H}_S$, i.e. the system energy eigenstates.

The intermediate regime is when the self-Hamiltonian $\hat{H}_S$ and the interaction Hamiltonian $\hat{H}_{\text{int}}$ are roughly equal in strength. The resulting preferred states will represent a compromise between the first two limits.

4.8 The Problem of Outcomes

The very definition of *outcome*, in accordance with experience, tells us that every measurement results in a *definite value* of the measured quantity. A superposition of states that is obtained from the von Neumann pre-measurement scheme, is fundamentally distinct from a classical ensemble. Without an additional physical process (e.g. a collapse mechanism) or a suitable interpretation of a superposition of states (e.g. Born's probabilistic interpretation), we are unable to account for the definite pointer states (i.e. definite outcomes) that are observed as a result of a measurement.

The problem of outcomes may be broken down into two distinct aspects. First is the question of why we do not perceive the pointer of the apparatus in a superposition of different pointer states $|a_i\rangle$ at the conclusion of the measurement. It is also inconceivable what is would mean to perceive such a superposition, since we have no such experience. Second is the question of what selects a specific outcome.

The standard interpretation of quantum mechanics prescribes that a system has a *definite value* of a physical quantity if and only if it is in an eigenstate of the observable corresponding to that physical quantity. This is called the eigenvalue-eigenstate link. Within this interpretation of quantum mechanics, the decoherence framework cannot solve the problem of outcomes. This is because the interference between macroscopically different pointer states is preserved in the larger system-apparatus-environment composite, i.e. superposition of states persists in a larger entangled system.

In addition, in the decoherence framework, the statistics of any local measurement performed on the system only is captured by the system's reduced density matrix. And these statistics are the same if the system is in a proper mixture (ensemble) of pure states, or if the system density matrix is an improper mixture computed from tracing out the environment. That is to say, in the decoherence framework, the conceptual difference in the nature of the reduced density matrix (proper or improper mixtures) does not lead to any operational difference. Furthermore, the trace operation is nonunitary and the terms in the trace, Eq. (3.7), was interpreted as the possible outcomes weighted by the Born probabilities, i.e. **the decoherence framework already presumes that measurements have outcomes and the Born rule holds**. In order words, taking the partial trace amounts to the statistical version of the collapse postulate.

By working within the framework of the collapse postulate, decoherence does not tell you what happens to the system when you measure it; it merely gives you probabilities for what you observe. Decoherence is powerful in that it explains why we do not normally observe quantum interference effects for macroscopic objects. However, it does not explain how it happens that a system when measured, ends up in one, and only one, possible measurement outcome.

Therefore, the decoherence framework, which presupposes the collapse postulate (even if implicitly), cannot be used to derive the existence of outcomes from its formal structure. Decoherence thus **does not** solve the most challenging one of the three measurement problems – the problem of outcomes – although it manages to explain the appearance of classicality of quantum systems while working within standard quantum mechanics.

4.9 What Decoherence is Not

4.9.1 Decoherence versus Dissipation

Decoherence is a consequence of system-environment entanglement, and is a purely quantum-mechanical effect. Dissipation, on the other hand, is the loss of energy from the system into the environment (or another system) to approach thermal equilibrium, and is a classical effect. The characteristic time for which a system interacting with an environment to approach thermal equilibrium is called the relaxation time. This time is inversely related to the system-environment interaction strength.

Decoherence and dissipation should be understood as distinct effects. Decoherence may, but does not have to, be accompanied by dissipation, whereas the presence of dissipation implies the occurrence of decoherence.

Even if dissipation is absent, there is still decoherence in general. For example, when the environment obtains which-state information, it can do so without absorbing energy from the system. This can be seen when the environment monitors the energy eigenstates of the system, e.g.

$$\hat{H}_S = \hbar\omega_0\hat{\sigma}_z, \qquad \hat{H}_{\text{int}} = \hbar g\hat{\sigma}_z \otimes \hat{E}. \tag{4.35}$$

If dissipation and decoherence are both present, then they are usually quite easily distinguished because of their very different timescales. The decoherence timescale is typically many orders of magnitude shorter than the relaxation timescale.

In his article in Physics Today[9], Zurek considered a macroscopic object of mass $m = 1$ g at room temperature ($T = 300$ K). The thermal de Broglie wavelength of the object is $\lambda_{dB} = \hbar/\sqrt{2mk_BT} \approx 10^{-23}$ m. He considered the object being in a superposition of two spatial locations a macroscopic distance of $\Delta x = 1$ cm apart. Then the ratio of relaxation to decoherence timescales is estimated to be $\tau_r/\tau_d \sim (\Delta x/\lambda_{dB})^2 \approx 10^{40}$!

For macroscopic objects, the dissipative influence of the environment is usually completely negligible with respect to the dynamics of the system on any timescale relevant to the decoherence induced by this environment. For example, photons scattering off a bowling ball will hardly affect the ball's motion in any way (no dissipation), while they will lead to virtually instantaneous decoherence of a superposition state involving macroscopically distinguishable positions of the ball.

4.9.2 Decoherence versus Classical Noise

Decoherence is a distinctly quantum-mechanical effect with no classical analogue. However, especially in quantum computation literature, the observation that processes that lead to the disappearance of off-diagonal elements (in some basis) in the density matrix of the system have been referred to as "decoherence", even if those processes are classical in nature and not related to tracing out the environment degrees of freedom. It is therefore important to remember that the density-matrix description is only a formal tool that, somewhat misleadingly and deceptively, hides the crucial conceptual differences between the processes whose effects may have a similar representation in the density-matrix formalism. The source of confusion stems from the erroneous use of the density-matrix formalism to describe physical ensembles – a collection of N physical systems in which each individual system is described by a pure state – even though operationally, the statistics from ensemble averages are identical to those of mixed states. See Sec. 3.4.2.

The most common example of such confusion is in the description of classical fluctuations in parameters that govern the evolution of qubits in quantum computation. For example a qubit initialized into the state $|\psi(0)\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$ would gain random relative phases as it evolves under a Hamiltonian that contains random fluctations,

$$\hat{H} = \hbar\left(\frac{\omega}{2}\hat{\sigma}_z + \delta\omega(t)\hat{\sigma}_z\right), \tag{4.36}$$

where $\delta\omega(t)$ is a time-dependent random variable. Such noise – the addition of random fluctuations to the Hamiltonian of the system – does not create system-environment entanglement and is unitary; it can be completely undone (at least in principle) by local operations. Alternatively,

[9] W. H. Zurek, *Decoherence and the transition from quantum to classical*", Physics Today, 44, pp. 36–44 (1991)

the state preparation may be such that the relative phase is randomized, an example of which you will work out in the Discussion Set.

A confusion of terms, from Nielsen & Chuang

An unfortunate confusion of terms has arisen with the word 'decoherence'. Historically, it has been used to refer just to a phase damping process, particularly by Zurek. Zurek and other researchers recognized that phase damping has a unique role in the transition from quantum to classical physics; for certain environmental couplings, it occurs on a time scale which is much faster than any amplitude damping process, and can therefore be much more important in determining the loss of quantum coherence. The major point of these studies has been this emergence of classicality due to environmental interactions. However, by and large, the usage of decoherence in quantum computation and quantum information is to refer to any noise process in quantum processing.

4.10 A Model Decoherence System

We illustrate the basic features and dynamics of environment-induced decoherence and superselection in the context of a model first introduced and studied by Zurek in 1982, a spin 1/2 system in a spin environment or bath. Such a system is now known as the central spin model.

4.10.1 Central Spin Model

In this model, a two-level system or spin 1/2 particle (central spin) linearly and pairwise coupled to an environment of N other two-level systems, or spin 1/2 particles. We will write the basis of the central spin as $|0\rangle$ and $|1\rangle$, and the environment spins as $|\uparrow\rangle_i$ and $|\downarrow\rangle_i$. is described by the full Hamiltonian,

$$\begin{aligned} \hat{H} =& \hat{H}_S + \hat{H}_E + \hat{H}_{\text{int}}, \\ \text{where } \hat{H}_S = \frac{1}{2}\hbar\omega\hat{\sigma}_z, \quad \hat{H}_E =& \sum_i \frac{1}{2}\hbar\omega_i\hat{\sigma}_z^{(i)}, \quad \hat{H}_{\text{int}} = \frac{1}{2}\hat{\sigma}_z \otimes \left(\sum_i \hbar g_i \hat{\sigma}_z^{(i)} \bigotimes_{i'\neq i} \hat{\mathbb{I}}^{(i)}\right), \end{aligned} \tag{4.37}$$

where $\hat{\sigma}_z \equiv |0\rangle\langle 0| - |1\rangle\langle 1|$ and $\hat{\sigma}_z^{(i)} \equiv |\uparrow\rangle\langle\uparrow| - |\downarrow\rangle\langle\downarrow|$. From this point, we drop the identity operator for notational simplicity. In the quantum measurement limit, $g_i \gg \omega_i, \omega$, and the interaction Hamiltonian dominates,

$$\hat{H} \approx \hat{H}_{\text{int}} = \frac{1}{2}\hat{\sigma}_z \otimes \left(\sum_i \hbar g_i \hat{\sigma}_z^{(i)}\right) \equiv \frac{1}{2}\hat{\sigma}_z \otimes \hat{E}. \tag{4.38}$$

Here, we have defined $\hat{E} \equiv \sum_i \hbar g_i \hat{\sigma}_z^{(i)}$. Since Eq. (4.38) commutes with $\hat{H}_S$, by the commutativity criterion (Eq. (4.33)), the energy eigenstates of $\hat{H}_S$ are simultaneous eigenstates of $\hat{H}_{\text{int}}$, and are robust pointer states in the quantum measurement limit.

Therefore, energy is a conserved quantity or constant of the system; there is no exchange of energy between the system and the environment (i.e. no dissipation). Also, populations of the system in the $|0\rangle$ and $|1\rangle$ states, given by operators $\frac{1}{2}(\mathbb{I} \pm \hat{\sigma}_z)$ are conserved quantities. Furthermore, the interaction Hamiltonian, Eq. (4.38), is diagonal in the environment basis $\{|\uparrow\rangle_i, |\downarrow\rangle_i\}$, meaning that energy eigenstates of the environment, which we write as $|n\rangle$, are also simultaneous eigenstates of the interaction Hamiltonian, with eigenenergies,

$$\epsilon_n \equiv \hbar\Omega_n \equiv \hbar\sum_{i=1}^{N}(-1)^{n_i}\hbar g_i, \tag{4.39}$$

where $n_i = 1$ if the i-th environment spin is down, and $n_i = 0$ if the environment spin is up. We can then write the eigenstates of the interaction Hamiltonian as $|0\rangle|n\rangle, |1\rangle|n\rangle$. A general system-environment state can be written as

$$|\Psi\rangle = \sum_{n=0}^{2^N-1} \left(c_n |0\rangle |n\rangle + d_n |1\rangle |n\rangle\right). \tag{4.40}$$

Assuming that the system and environment are initially un-entangled,

$$|\Psi(0)\rangle = (a |0\rangle + b |1\rangle) \sum_{n=0}^{2^N-1} c_n |n\rangle, \tag{4.41}$$

the time evolution is given by

$$\begin{aligned}
|\Psi(t)\rangle =& e^{-i\hat{H}_{\text{int}}t/\hbar} |\Psi(0)\rangle \\
=& e^{-\frac{i}{2}\hat{\sigma}_z \otimes \hat{E}t/\hbar} \left(a |0\rangle \sum_{n=0}^{2^N-1} c_n |n\rangle + b |1\rangle \sum_{n=0}^{2^N-1} c_n |n\rangle \right) \\
=& a |0\rangle \sum_{n=0}^{2^N-1} c_n e^{-\frac{i}{2}\hat{E}t/\hbar} |n\rangle + b |1\rangle \sum_{n=0}^{2^N-1} c_n e^{\frac{i}{2}\hat{E}t/\hbar} |n\rangle \\
=& a |0\rangle \underbrace{\sum_{n=0}^{2^N-1} c_n e^{-\frac{i}{2}\Omega_n t} |n\rangle}_{\equiv |E_0(t)\rangle} + b |1\rangle \underbrace{\sum_{n=0}^{2^N-1} c_n e^{\frac{i}{2}\Omega_n t/} |n\rangle}_{\equiv |E_1(t)\rangle} \\
\equiv& a |0\rangle |E_0(t)\rangle + b |1\rangle |E_1(t)\rangle.
\end{aligned} \tag{4.42}$$

Recall the earlier discussion about the environment as a which-state monitor, the more distinguishable the states $|E_0(t)\rangle, |E_1(t)\rangle$ are (or the smaller the overlap $\langle E_1(t)|E_0(t)\rangle$), the more information about the system state gets encoded in the environment, and the greater the damping of the interference or coherence between system states $|0\rangle$ and $|1\rangle$. Therefore, defining the overlap as the coherence factor $r(t)$,

$$r(t) \equiv \langle E_1(t)|E_0(t)\rangle = \sum_{n=0}^{2^N-1} |c_n|^2 e^{-i\Omega_n t}, \tag{4.43}$$

with $|c_n|^2 \leq 1$, and $\sum_{n=0}^{2^N-1} |c_n|^2 = 1$. The geometrical interpretation of Eq. (4.43) is the sum of 2^N vectors, of length $|c_n|^2$ rotating with time at a rate Ω_n, in the complex plane. The reduced density matrix of the system is

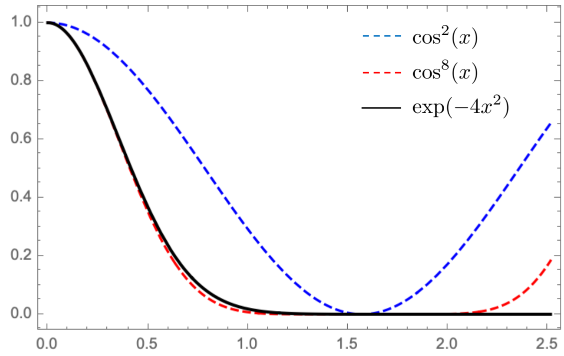
$$\begin{aligned}
\hat{\rho}_S(t) =& \text{Tr}_E \hat{\rho}(t) \\
=& \text{Tr}_E |\Psi(t)\rangle \langle \Psi(t)| \\
=& |a|^2 |0\rangle \langle 0| + ab^\star r(t) |0\rangle \langle 1| + a^\star b r^\star(t) |1\rangle \langle 0| + |b|^2 |1\rangle \langle 1|.
\end{aligned} \tag{4.44}$$

The derivation of the coherence factor was first shown by Cucchietti, Paz and Zurek in 2005.[10] For any fixed time t, this corresponds to a two-dimensional random-walk problem. Each of the 2^N steps of the random walk has length $|c_n|^2$ and follows the direction given by the phase $\Omega_n t$. Since $\sum_{n=0}^{2^N-1} |c_n|^2 = 1$, the average step length

$$\langle |c_n|^2 \rangle = \frac{1}{2^N}. \tag{4.45}$$

Assuming that the couplings g_i are sufficiently concentrated near their average value $\langle g_i \rangle$ so that their standard deviation $\langle (g_i - \langle g_i \rangle)^2 \rangle$ exists and is finite, and that N is large, then the

[10] F. M. Cucchietti, J. P. Paz, and W. H. Zurek, Phys. Rev. A 72, 052113 (2005).



distribution of Ω_n is in general Gaussian. In this case, the coherence factor follows a Gaussian decay with time,

$$|r(t)| \approx \frac{1}{2^N} e^{-\Gamma^2 t^2}, \tag{4.46}$$

where the precise value of Γ depends on the initial state of the environment and the distribution of the couplings g_i.

However, it must be stressed that we arrived at the coherence factor $r(t)$ from a unitary evolution of the system-environment composite, as a result of which $r(t)$ is a sum of periodic functions. This means that $r(t)$ is also periodic, and will eventually "revive" to 1. If all $g_i = g$, then $r(t) = \cos^N(gt)$ and a revival will happen at $t = \pi/g$. Note also that if the environment started in one of the eigenstates $|n\rangle$, then there will be no decay and decoherence will not occur.

In realistic cases, such highly ordered initial states and symmetrical system-environment couplings are unlikely to be relevant. In such cases, the characteristic recurrence time τ_{rec} is typically extremely long and of the of the order $\tau_{\text{rec}} \propto N!$ (the exclamation mark is the factorial symbol). For macroscopic environments of realistic but finite sizes, τ_{rec} can exceed the age of the universe.

Therefore, decoherence of a system is practically irreversible because the recurrence timescale is huge in physically realistic situations, and because we are unable to control and observe the many degrees of freedom of the environment.

4.11 Historical aside

The first paper[11] on what is now known as "decoherence" was written by H. Dieter Zeh at the University of Heidelberg and published in 1970. In the paper, Zeh pointed out that realistic quantum systems are never closed and interact with their environments. Hence, if the Schrödinger equation is assumed to be universally valid, such systems will typically be found in states that are quantum-correlated with the environment, leading to the inability to describe the dynamics of the system itself by the Schrödinger equation.

Later in 1981, Wojciech Zurek (a postdoc of John Wheeler's in 1981 and a Tolman Fellow at Caltech in 1982) in his paper "Pointer basis of quantum apparatus: Into what mixture does the wave packet collapse?"[12] pointed out the importance of the so-called "preferred-basis problem". Zurek developed the concept of environment-induced superselection (einselection) that is a cornerstone of the decoherence programme, and a precise framework for determining the environment-superselected preferred states (the so-called "pointer states").

Zeh faced great difficulty in getting his ideas noticed, and called the period his first contact with the measurement problem in the late 1960s and the early 1980s when the decoherence programme gained greater acceptance, the "dark ages". That first paper in 1970 was turned down by several journals. One referee wrote, "*the paper is completely senseless. It is clear that the author has not fully understood the problem and the previous contributions in this field.*"

[11] H. D. Zeh, On the interpretation of measurement in quantum theory, Foundations of Physics, vol 1, pp 69–76 (1970). https://link.springer.com/article/10.1007/BF00708656.

[12] W. H. Zurek, Pointer basis of quantum apparatus: Into what mixture does the wave packet collapse?, Phys. Rev. D 24, 1516–1525 (1981). https://journals.aps.org/prd/abstract/10.1103/PhysRevD.24.1516.

In 1976, historian of science David Edge had asked Zeh, ”*Do you feel that physicists who hold unorthodox views about QM have any great difficulties in carrying out their work?*'. Zeh's answer: "*exceptional difficulties.*"

It is perhaps a very human acceptance of the orthodox and mainstream ideas, or an avoidance of the difficulties of confronting foundational issues, that led to the challenges faced by Zeh. It is for reasons like these that those who hold social constructivist views of science perceive scientific knowledge to be social in nature; there is social influence even with technical judgments.

David Mermin summarized the attitude towards such issues thus, "*If I were forced to sum up in one sentence what the Copenhagen interpretation says to me, it would be 'Shut up and calculate!'*" Zeh expressed his resentment at the professional obstacles he faced while tackling the subject to John Archibald Wheeler in 1986,

> *I have always felt bitter about the way how Bohr's authority together with Pauli's sarcasm killed any discussion about the fundamental problems of the quantum. . . . I expect that the Copenhagen interpretation will some time be called the greatest sophism in the history of science, but I would consider it a terrible injustice if – when some day a solution should be found – some people claim that 'this is of course what Bohr always meant', only because he was sufficiently vague.*

4.12 References

1. **Decoherence:** Maximilian Schlosshauer, Decoherence and the Quantum-to-Classical Transition. Springer (2007). Chapter 2.
2. **Historical Survey of Interpretation:** Roland Omnes, Understanding Quantum Mechanics. Parts 1 and 2.
3. **History of Decoherence:** Olival Freire Junior, The Quantum Dissidents. Springer (2015). Chapter 8.3.

4.13 Discussion Set

Problem 4.1 — Decoherence-free subspaces. We note that it is always possible to write an arbitrary interaction Hamiltonian in the form of a diagonal composition of unitary, but not necessarily Hermitian system $\hat{S}_\alpha$ and environment $\hat{E}_\alpha$ operators,

$$\hat{H}_{\text{int}} = \sum_\alpha \hat{S}_\alpha \otimes \hat{E}_\alpha. \tag{4.47}$$

If $\hat{S}_\alpha$ are Hermitian, such a term describes the environment simultaneously monitoring different observables $\hat{S}_\alpha$ on the system. The pointer states of the system are then simultaneous eigenstates $|s_i\rangle$ of all the $\hat{S}_\alpha$ operators.

$$\hat{S}_\alpha |s_i\rangle = \lambda_i^{(\alpha)} |s_i\rangle, \quad \forall \ i, \alpha. \tag{4.48}$$

Given that the system starts in the state $|\psi\rangle = \sum_i c_i |s_i\rangle$, derive the resulting system-environment evolution $U(t)|\psi\rangle |E_0\rangle$ for two cases:

1. Case 1: If all eigenvalues are equal, i.e $\lambda_i^{(\alpha)} = \lambda^{(\alpha)}$
2. Case 2: If not all eigenvalues are equal, i.e $\lambda_i^{(\alpha)} \neq \lambda_j^{(\alpha)}$ $(i \neq j)$

Discussion points:

1. What do your results imply, in terms of environmental monitoring?
2. Which case is preferred, if a quantum system is to be free from decoherence? Why?

Reference: D. A. Lidar, I. L. Chuang, K. B. Whaley, Decoherence-free subspaces for quantum computation, Phys. Rev. Lett. 81, 2594–2597 (1998).

Problem 4.2 — Quantum measurement limit. Consider an atom with spin 1/2 in a magnetic field. Its Hamiltonian is

$$\hat{H}_{\text{atom}} = \gamma \hat{S}_z. \tag{4.49}$$

It is projected into a Stern-Gerlach measurement apparatus, where the interaction Hamiltonian for measurement is

$$\hat{H}_{\text{int}} = g(t)\hat{P}_d \otimes \hat{S}_z. \tag{4.50}$$

Here, the momentum P_d is is conjugate to the coordinate Q_d of the measuring device, and Q_d would correspond to the transverse deflection of the atom. We identify Q_d with the position of a pointer on the Stern-Gerlach measuring device. Also, $g(t)$ is non-zero only within the time interval $0 \le t \le T$ and satisfies $\int_0^T g(t)dt = g_0$. We assume that the measurement is impulsive, $T \to 0$, so that we regard the interaction Hamiltonian as the total Hamiltonian during measurement.

(i) Write down the Heisenberg equation of motion for the operator $\hat{Q}_d$ during measurement, and show that the change in the position operator of the device after the impulse, $\hat{Q}_d(T) - \hat{Q}_d(0)$, is dependent on the spin operator $\hat{S}_z$.

(ii) The initial state of the atom-device composite system is $|\Psi(0)\rangle = |\uparrow\rangle_{\text{atom}} \otimes |0\rangle_{\text{device}}$, where $|0\rangle_{\text{device}}$ is the state of the measuring device sharply peaked around $Q_d = 0$. Show that the action of

$$\hat{U}(T,0) = \exp\left\{-\frac{i}{\hbar}\int_0^T \hat{H}_{\text{int}}(t)dt\right\} \tag{4.51}$$

on this state is to evolve it to $|\uparrow\rangle_{\text{atom}} \otimes |\frac{\hbar g_0}{2}\rangle_{\text{device}}$. (In other words, the pointer or needle of the meter is displaced by $\frac{\hbar g_0}{2}$.)

(iii) If, instead, the initial atomic state is a superposition, so that the initial state of the composite system is $|\Psi(0)\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle_{\text{atom}} + |\downarrow\rangle_{\text{atom}}) \otimes |0\rangle_{\text{device}}$, show that we must obtain a superposition of pointer states at the end of the measurement.

Hint: The momentum operator in position representation is $\hat{p} = -i\hbar\frac{d}{dx}$, so that $e^{-ix_0\hat{p}/\hbar} = e^{-x_0\frac{d}{dx}}$, and by Taylor expanding, we get $e^{-ix_0\hat{p}/\hbar}\psi(x) = \psi(x - x_0)$. That is, the momentum operator is the generator of translation.

Problem 4.3 — **Classical fluctuations (ensemble dephasing).** Write the density matrix for this single qubit pure state, $|\psi\rangle = (|0\rangle + e^{i\phi_0}|1\rangle)/\sqrt{2}$.

Now consider a physical ensemble of qubits prepared in this state. Because of imprecise preparation, each qubit experiences a slightly different state preparation such that the relative phase has a probability distribution $p(\phi)$.

(i) Find $\langle\hat{\rho}\rangle_\phi$, the ensemble average of the density matrix over the phase, for a Gaussian distribution $p(\phi) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left\{-\frac{(\phi-\phi_0)^2}{2\sigma^2}\right\}$, where σ being the standard deviation of the initial phase.

(ii) What do the off-diagonal terms imply about the interference between $|0\rangle$ and $|1\rangle$ states? Is this decoherence?

Hint:

$$\text{Given } x \sim \mathcal{N}(\mu, \sigma^2), \qquad \begin{cases} \mathbb{E}(\cos(ax)) = \cos(a\mu)\; e^{-a^2\sigma^2/2}, \\ \mathbb{E}(\sin(ax)) = \sin(a\mu)\; e^{-a^2\sigma^2/2}. \end{cases} \tag{4.52}$$