Not a markov chain

2020-08-02

Let sequence (X_n) be defined by $X_n = Q_n + Q_{n-1}$

The definition of a Markov chain states that a sequence (X_n) is a MC if the following holds:

$$P(X_{n+1}|X_n, \ldots X_0) = P(X_{n+1}|X_n)$$

- for all n
- for every possible x value

That is the probability of what happens in the next step only depends on the previous step. Or put another way, knowing $X_{n-1}, \ldots, X_0$ does not change the probability of $P(X_{n+1}|X_n)$.

So to show that the sequence is NOT a markov chain, we will construct a situation where knowing $(X_{n-1}, \ldots, X_n)$ does change the probability.

Consider the sequence: * all 8's: $(X_{n-1} = 8, \ldots, X_0 = 8)$

Then fix $X_n = 6$. This means that $Q_n = 2, Q_{n-1} = 4$ (because we know $X_{n-1}, \ldots, X_n$ we can fully determine Q_n).

Then $P(X_{n+1}|X_n = 6, X_{n-1} = 8, \ldots, X_0 = 8)$ has distribution:

$$P(X_{n+1} = 4|\ldots) = P(Q_{n+1} = 2) = 1/4$$
$$P(X_{n+1} = 5|\ldots) = P(Q_{n+1} = 3) = 1/2$$
$$P(X_{n+1} = 6|\ldots) = P(Q_{n+1} = 4) = 1/4$$
$$P(X_{n+1} = 7|\ldots) = P(Q_{n+1} = 5) = 0$$

But if $(X_{n-1} = 4, \ldots, X_0 = 4)$ and then we fix $X_n = 6$ then we know $Q_n = 4, Q_{n-1} = 2$

$P(X_{n+1}|X_n = 6, X_{n-1} = 4, \ldots, X_0 = 4)$ has distribution

$$P(X_{n+1} = 4|\ldots) = P(Q_{n+1} = 0) = 0$$
$$P(X_{n+1} = 5|\ldots) = P(Q_{n+1} = 1) = 0$$
$$P(X_{n+1} = 6|\ldots) = P(Q_{n+1} = 2) = 1/4$$
$$P(X_{n+1} = 7|\ldots) = P(Q_{n+1} = 3) = 1/2$$
$$P(X_{n+1} = 8|\ldots) = P(Q_{n+1} = 3) = 1/4$$

That is we've changed the distribution given that we have knowledge other than X_n. The process is NOT memoryless and thus not a Markov chain.

Formally a Markov chain must have

$$P(X_{n+1}|X_n, \dots X_0) = P(X_{n+1}|X_n)$$

which also means more specifically

$$P(X_{n+1}|X_n = 6) =$$

$$P(X_{n+1}|X_n = 6, X_{n-1} = 8, \dots, X_0 = 8) = P(X_{n+1}|X_n = 6, X_{n-1} = 4, \dots, X_0 = 4)$$

but we've shown that the last equality does not hold.