Aspects on Tensor Networks for Topological Orders

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Outline

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  - Topological orders and category theory
  - Tensor networks
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  - Strange correlators and partition functions of minimal CFTs
  - Construction of holographic tensor networks
  - Operator pushing
- Tensor network representations of Virasoro and Kac–Moody algebra
  - Review of 2d CFT
  - Basic construction
  - Examples: Ising, dimer and Fibonacci models

Motivation & background

Motivation: AdS/CFT correspondence

- Holographic principle
- Duality between a gravity theory in AdS$_{d+1}$ spacetime (bulk) and a CFT$_d$ (boundary)
- AdS/CFT dictionary:
  - $Z_{\mathrm{CFT}} = Z_{\mathrm{bulk}}$
- Ryu-Takayanagi formula:
  - $S_A = \mathrm{area}(\gamma_A)/4G^{(d+1)}$
  - Entanglement is geometry
- *p*-adic AdS/CFT and Einstein equation

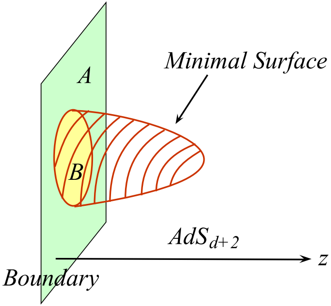


Image credit: Nishioka et al., *J. Phys. A: Math. Theor.* **42**: 504008 (2009)

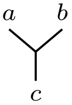
Topological orders

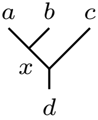
- Novel phases of matter beyond Landau's theory
  - Fractional quantum Hall effect
  - High temperature superconductivity
- Fundamental properties:
  - Ground state degeneracy
  - Non-abelian geometric phase
- Microscopic origin:
  - Long-range entanglement
  - Local unitary transformation
- Applications: fault-tolerant quantum computation
- Mathematical framework: modular tensor categories (fusion categories)

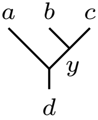
Tensor & fusion categories

- Basic of category theory:
  - Objects and morphisms
  - Functors, natural transformations, etc.
- Tensor product: $\otimes$
  - Associativity: $(a \otimes b) \otimes c = a \otimes (b \otimes c)$
  - Unit object: $\mathbb{1} \otimes a = a \otimes \mathbb{1} = a$
- Simple objects and their fusion: $a \otimes b = \bigoplus_c N_{ab}^c c$
  - Simple objects a, b: different types of anyon
  - Fusion: can't be distinguished at long distance
  - Fusion coefficients: $N_{ab}^c \in \mathbb{Z}^*$
  - Quantum dimension d_a: max eigenvalue of matrix $(N_a)_{bc} = N_{ab}^c$
- More structures: dual, braiding, ribbon, non-degeneracy, etc.

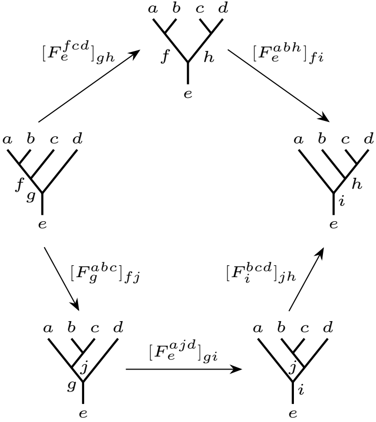
Fusion diagrams

- Basis in vector space $\mathrm{Hom}_{\mathcal{C}}(a \otimes b, c)$: 

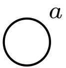
- F-move: 

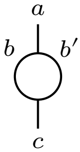
$= \sum_y [F_d^{abc}]_{xy}$ 

- Constraints: pentagon equations



- Bubble removal:



$= d_a,$


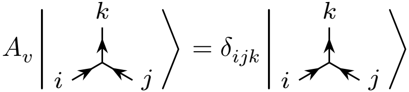
$= \delta_{ac} \sqrt{\frac{d_b d_{b'}}{d_a}}$


Examples of fusion categories

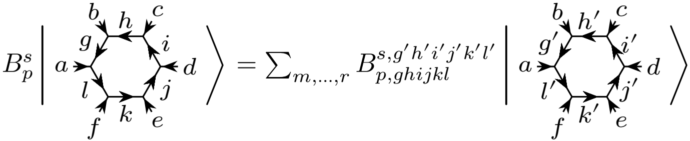
- Fibonacci
  - Anyon types: $\{\mathbb{1}, \tau\}$
  - Fusion rules: $\tau \otimes \tau = \mathbb{1} \oplus \tau$
  - Quantum dimensions: $d_{\mathbb{1}} = 1,\ d_\tau = \varphi$
  - F-symbols: $[F_\tau^{\tau\tau\tau}]_{ij} = \frac{1}{\varphi}\begin{pmatrix} 1 & \sqrt{\varphi} \\ \sqrt{\varphi} & -1 \end{pmatrix},\ i, j \in \{\mathbb{1}, \tau\}$
- Ising
  - Anyon types: $\{\mathbb{1}, \sigma, \psi\}$
  - Fusion rules: $\psi \otimes \psi = \mathbb{1},\ \sigma \otimes \sigma = \mathbb{1} \oplus \psi,\ \psi \otimes \sigma = \sigma$
  - Quantum dimensions: $d_{\mathbb{1}} = d_\psi = 1,\ d_\sigma = \sqrt{2}$
  - F-symbols: $[F_\sigma^{\psi\sigma\psi}]_{\sigma\sigma} = [F_\psi^{\sigma\psi\sigma}]_{\sigma\sigma} = -1,\ [F_\sigma^{\sigma\sigma\sigma}]_{ij} = -\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix},\ i, j \in \{\mathbb{1}, \psi\}$

String-net model

- Input data
  - Trivalent lattice (e.g. honeycomb)
  - Superselection sector (edge): simple objects
  - Branching rules (vertex): fusion rules
- Hamiltonian: $H = -\sum_v A_v - \sum_p B_p$
  - Electric charge:



  - Magnetic flux: $B_p = \sum_{s=0}^{N} \frac{d_s}{D^2} B_p^s\,, \quad D = \sqrt{\sum_{s=0}^{N} d_s^2}$



Tensor networks

- Tensor: a multi-dimensional array
- Contraction and decomposition (SVD)
- Why efficient?
  - Only keep the relevant (i.e. entanglement) degrees of freedom
  - Area-law: $S \sim \partial A$
- Algorithms:
  - MPS/MPO based: DMRG, TEBD, etc.
    - 2d generalization: PEPS/PEPO
  - Coarse-graining: TRG, TNR, HOTRG, etc.
  - MERA: holographic geometry

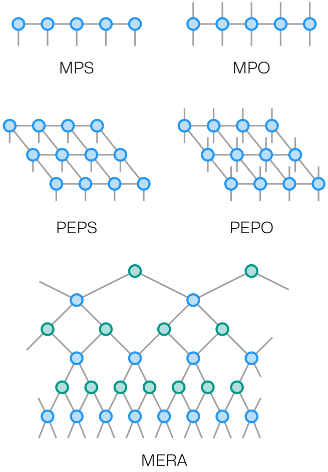
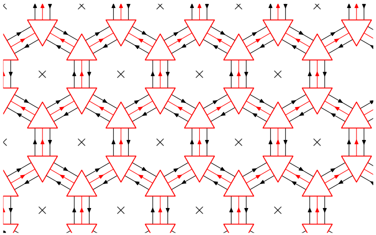


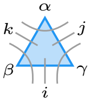
Image credit: Orús, *Nat. Rev. Phys.* **1**: 538–550 (2019) (with modification)

Strange correlators &
holographic tensor networks

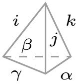
Tensor network representation of string-net model

- Construct ground state: is given by
  - Apply B_p on vacuum state $|\varnothing\rangle$
  - Weighted by quantum dimensions
  - Use F-moves to simplify
- PEPS structure:
  - Virtual indices: summed over
    (outside two legs: α, β, γ)
  - Physical indices: left uncontracted
    (inner one leg: i, j, k)
- Tetrahedral symmetry: A_4 group





$$= \frac{1}{D}(d_i d_j d_k)^{-\frac{1}{4}}(d_\alpha d_\beta d_\gamma)^{-\frac{1}{3}}$$



$$[F^{abc}_d]_{xy} = \sqrt{d_x d_y}\begin{bmatrix} a & b & x \\ c & d & y \end{bmatrix} = \frac{1}{\sqrt{d_a d_b d_c d_d}}$$

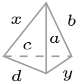
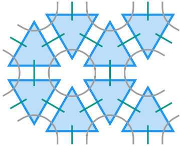
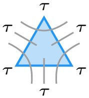


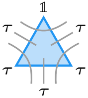
Image credit: Buerschaper et al., *Phys. Rev. B* **79**: 085119 (2009)

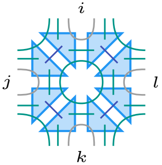
Strange correlators

- Original definition: $C(r, r') = \langle \Omega | \phi(r) \phi(r') | \Psi \rangle / \langle \Omega | \Psi \rangle$
  - $|\Psi\rangle$: a non-trivial short-range entangled (SPT) state
  - $|\Omega\rangle$: a direct product state
- In string-net model:
  - $|\Psi_{\mathrm{SN}}\rangle$: PEPS wave function for string-net ground state
  - $|\Omega\rangle$: some specific product state $|\omega\rangle^{\otimes N}$
  - Partition function: $Z = \langle \Omega | \Psi_{\mathrm{SN}} \rangle =$ 
    - Virtual indices (gray): summed over
    - Physical indices (green): fixed to some certain values (boundary conditions)

Examples

- Fibonacci
  - Boundary conditions: $|\omega\rangle = |\tau\rangle$
  - Building blocks: 

$= \varphi^{\frac{1}{4}} \left[F^{\tau\tau\tau}_{\tau}\right]_{\tau\tau} = -\varphi^{-\frac{3}{4}}$, 

$= \varphi^{\frac{7}{12}} \left[F^{\tau\tau\tau}_{\tau}\right]_{\tau\mathbb{1}} = \varphi^{\frac{1}{12}}$
- Ising
  - Boundary conditions: $|\omega(\beta)\rangle = \sqrt{2}\,(\cosh\beta\,|\mathbb{1}\rangle + \sinh\beta\,|\psi\rangle)$
  - Building blocks: $A_{ijkl} =$ 

where $i, j, k, l = \mathbb{1}$ or ψ, $| = \sigma$, $| = \omega$
  - Kramers–Wannier duality: shifted by 1/2 unit → $\beta_c = \frac{1}{2}\log(1+\sqrt{2})$

Holographic tensor networks in 2+1d

- $|\Psi\rangle$ is invariant under scaling transformation $\mathcal{H}_{\mathcal{C}}$
- Partition function is also invariant: $Z = \langle\Omega|\Psi\rangle = \langle\Omega|\exp(z\mathcal{H}_{\mathcal{C}})|\Psi\rangle$
- Eigenvalue problem: $\langle\Omega|\exp(z\mathcal{H}_{\mathcal{C}}) = \langle\Omega|\, FFF\cdots = \langle\Omega|$
    - Discrete Euclidean AdS space

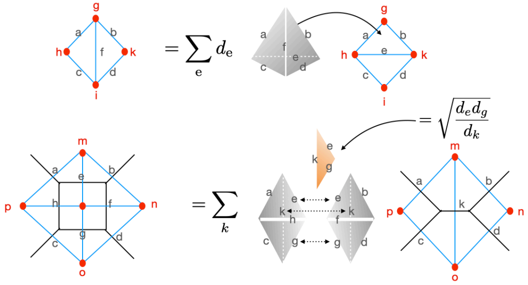


Image credit: Chen et al., arXiv:2210.12127 (2022)

Details of RG procedure

(a) PEPS tensor unit of $\langle \Omega |$: $T^{a_1 a_2 a_3}_{I_1 I_2 I_3}$

- $a_i \in \mathcal{C}$: physical indices
- I_i: virtual indices (trivial at first)

(b) Apply tetrahedra on surface to change its triangulation

(c) Use SVD to split coarse-grained tensor: $M^{acbd}_{ILJK} \to \tilde{T}^{akc}_{IHL}(k)\,\tilde{T}^{bdk}_{JHK}$

- H: new generated virtual index
- Bond dimension χ^2 truncated to χ

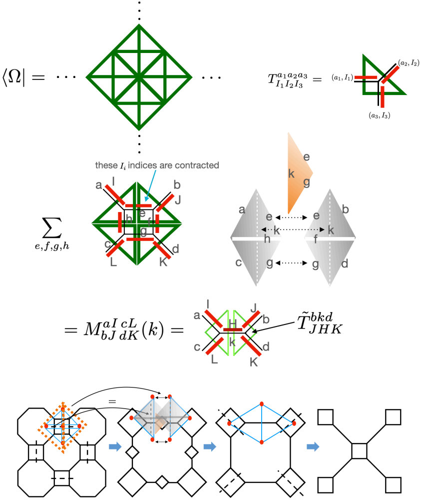


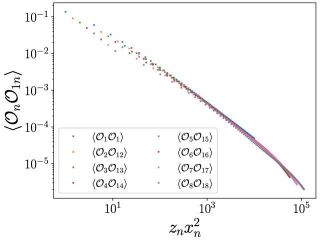
Image credit: Chen et al., arXiv:2210.12127 (2022)

Bulk-boundary propagators in 2d Ising model

- Correlation between bulk/boundary operators:

$$\langle \mathcal{O}_{n=1}(0,0)\, \mathcal{O}_n(x,y) \rangle$$
$$= \langle \Omega(T_{\Lambda_1}) \big| \sigma^z(0,0)\, U^{n-1}(\mathcal{C})\, \sigma^z(x,y)\, U^n(\mathcal{C}) \cdots \big| \Psi \rangle$$
$$= \langle \Omega(T_{\Lambda_1}) \big| \sigma^z(0,0)\, U^{n-1}(\mathcal{C})\, \sigma^z(x,y) \big| \Psi_{\Lambda_n} \rangle$$

- AdS/CFT prediction: $\langle \mathcal{O}_1\, \mathcal{O}_n \rangle \sim \left[\frac{z}{x^2+z^2}\right]^{\Delta}$
- $\langle \mathcal{O}_n \mathcal{O}_1 \rangle \sim z_n(x_n^2+1)$ plot indicates that the tensor network is holographic
  - $\mathcal{O}_{1n}$: $\mathcal{O}_1$ pushed to n-th layer
  - $z_n = (\sqrt{2})^{n-1}$, $x_n = x_1/z_n$



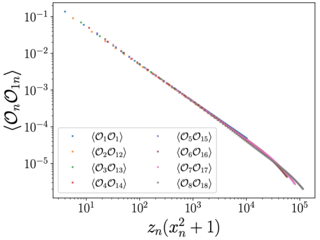
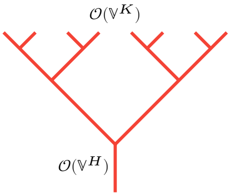


Image credit: Chen et al., arXiv:2210.12127 (2022)

Bulk operator reconstruction & operator pushing

- Generalized free fields (in the bulk)
  - Correlation functions satisfy Wick's theorem
  - Can be decomposed as a sum of *simple operators* (in the boundary)
- Operator pushing: $\mathcal{O}(\mathbb{V}^H) \cdot M = M \cdot \mathcal{O}(\mathbb{V}^K)$
  - Bulk operator: $\mathcal{O}(\mathbb{V}^H)$
    $= \mathbb{I}_1 \otimes \cdots \otimes \mathbb{I}_{i-1} \otimes X_i \otimes \mathbb{I}_{i+1} \otimes \cdots \otimes \mathbb{I}_H$
  - Boundary operator: $\mathcal{O}(\mathbb{V}^K)$
    $= \sum_{i=1}^{K} \alpha_i \left(\mathbb{I}_1 \otimes \cdots \otimes \mathbb{I}_{i-1} \otimes X_i \otimes \mathbb{I}_{i+1} \otimes \cdots \otimes \mathbb{I}_K\right)$



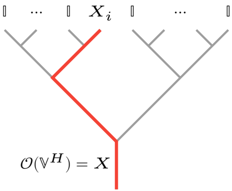
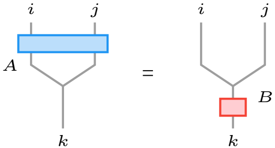


Image credit: Zeng et al., *Entropy* **25**: 1543 (2023)

Operator pushing in 1+1d

- Find boundary operator A for given bulk operator B, s.t.



$$\rightarrow \quad A_{(ij),(i'j')} M_k^{i'j'} = M_{k'}^{ij} B_{k'k}$$

- Generalized Pauli matrices: $\sigma_\mu := \sigma_{ns+t} = X^t Z^s$ where

$$X = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 1 & 0 & 0 & \cdots & 0 \end{pmatrix}, \; Z = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & \omega & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \omega^{n-2} & 0 \\ 0 & 0 & \cdots & 0 & \omega^{n-1} \end{pmatrix}$$

- Constraint equations: $A_{(ij),(i'j')} \delta_{G(i',j'),k} = \delta_{G(i,j),k'} B_{k'k} = (\sigma_\mu)_{G(i,j),k}$
  - One specific solution: $A^{(\mu)}_{(ij),(0j')} = (\sigma_\mu)_{G(i,j),j'}$
- For simple form A: $\tilde{A}_{ii'} \delta_{G(i',j),k} - \sum_\mu \alpha_\mu (\sigma_\mu)_{G(i,j),k} := \tilde{M}(\cdots) = 0$

Invitation: $\mathbb{Z}_2$

- Tensor unit: $M = \delta_{G(i,j),k} = \delta_{(i+j) \bmod 2,k} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 1 & 0 \end{pmatrix}$
- Null space: $\{v^{(p)}\} = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix} \right\} \implies A^* = \begin{pmatrix} \beta_{0,0} & \beta_{0,1} & -\beta_{0,1} & -\beta_{0,0} \\ \beta_{1,0} & \beta_{1,1} & -\beta_{1,1} & -\beta_{1,0} \\ \beta_{2,0} & \beta_{2,1} & -\beta_{2,1} & -\beta_{2,0} \\ \beta_{3,0} & \beta_{3,1} & -\beta_{3,1} & -\beta_{3,0} \end{pmatrix}$
- Full solutions: $\begin{cases} B = \mathbb{I} \implies A = A^* + \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \\ B = \sigma_x \implies A = A^* + \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \implies A = \mathbb{I} \otimes \sigma_x \text{ or } \sigma_x \otimes \mathbb{I} \\ B = \sigma_z \implies A = A^* + \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \\ B = -\mathrm{i}\sigma_y \implies A = A^* + \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \end{cases}$

Abelian example: $\mathbb{Z}_n$

- Fusion rules: $G(i,j) = (i+j) \bmod n$
- General solutions:
  - General part: $A^* = \begin{pmatrix} \beta_{0,0} v^{(0)} + \cdots + \beta_{0,n^2-n-1} v^{(n^2-n-1)} \\ \vdots \\ \beta_{n^2-1,0} v^{(0)} + \cdots + \beta_{n^2-1,n^2-n-1} v^{(n^2-n-1)} \end{pmatrix}$
  - Specific part: $A^{(\mu)}_{(ij),(0j')} = (\sigma_\mu)_{(i+j) \bmod n, j'}$
- Simple form solutions:
  - $\mathrm{size}(\tilde{M}) = n^3 \times 2n^2$, $\mathrm{rank}(\tilde{M}) = 2n^2 - n$
  - n solutions: $\tilde{A}^{(k)}_{ii'} = (\sigma_k)_{ii'}$, $\alpha^{(k)}_\mu = \delta_{k\mu}$
  - Generalized free fields $B = \sigma_k \rightarrow A = \sigma_k \otimes \mathbb{I}$ or $\mathbb{I} \otimes \sigma_k$
  - Tensor network of L layers: $A_L = \mathbb{I}^{\otimes L-l-1} \otimes \sigma_k \otimes \mathbb{I}^{\otimes l}$

Non-abelian example: S_3

- Group multiplication table:

	g_0	g_1	g_2	g_3	g_4	g_5
g_0	g_0	g_1	g_2	g_3	g_4	g_5
g_1	g_1	g_0	g_3	g_2	g_5	g_4
g_2	g_2	g_4	g_0	g_5	g_1	g_3
g_3	g_3	g_5	g_1	g_4	g_0	g_2
g_4	g_4	g_2	g_5	g_0	g_3	g_1
g_5	g_5	g_3	g_4	g_1	g_2	g_0

- Null space: spanned by $v^{(p)} = \left(\tilde{v}^{(p)},\ \hat{v}^{(p)}\right)^{\mathsf{T}}$
    - $\tilde{v}^{(1)} = (1,0,0,0,0,0)^{\mathsf{T}},\ \tilde{v}^{(2)} = (0,0,0,1,0,0)^{\mathsf{T}},\ \cdots;\quad \hat{v}_q^{(p)} = \delta_{pq}$
- Simple form solutions:
    - $\mathrm{size}(\tilde{M}) = 216 \times 72,\ \mathrm{rank}(\tilde{M}) = 66$ → has 6 solutions
    - $\tilde{A}^{(0)}_{L/R} = B^{(0)}_{L/R} = \begin{pmatrix} 1&0&0&0&0&0\\ 0&1&0&0&0&0\\ 0&0&1&0&0&0\\ 0&0&0&1&0&0\\ 0&0&0&0&1&0\\ 0&0&0&0&0&1 \end{pmatrix},\ \tilde{A}^{(1)}_{L/R} = B^{(1)}_{L/R} = \begin{pmatrix} 0&1&1&1&1&1\\ 1&0&1&1&1&1\\ 1&1&0&1&1&1\\ 1&1&1&0&1&1\\ 1&1&1&1&0&1\\ 1&1&1&1&1&0 \end{pmatrix},\ \cdots$
    - Possible $\mathbb{Z}_2$ subgroup structure

Operator pushing in 2+1d

- Tetrahedra:
  - Single: F-moves, change triangulation
  - Multiple: RG operators, boundary → bulk
- RG operator:
  - Map i, j, m, n (blue) → a (red)
  - Keep b, c, d, e unchanged
- Constraint equations:
  - $A_{(ijk),(i'j'k')} M_{(i'j'k'),I} = M_{(ijk),I'} B_{I'I}, \; B = \sigma_\mu$
  - $I = \Phi(b, c, a)$: face index, Φ: fusion rules
  - $M_{(ijk),I} = M_{(ijk),\Phi(b,c,a)} = \sqrt{d_j d_k d_b d_c}\, [F_c^{jkb}]_{ia}$

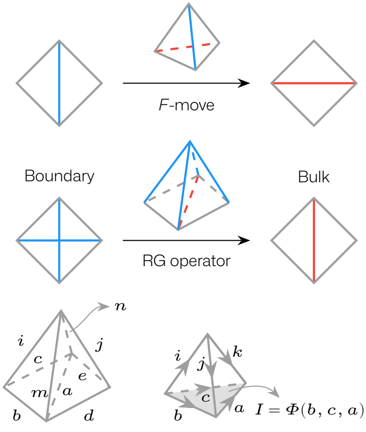


Image credit: Zeng et al., *Entropy* **25**: 1543 (2023)

Example: $\mathbb{Z}_n$

- Trivial $\mathbb{Z}_n$ Dijkgraaf-Witten models: $F = 1$
- Face index: $I = nb + c = n[(i+j) \bmod n] + [(i+k) \bmod n]$
- General solutions:
  - Null space: $v_q^{(p)} = \delta_{n[(-\lfloor p/n \rfloor - \lfloor p/n^2 \rfloor - 2) \bmod n] + [(-\lfloor p/n^2 \rfloor - 2) \bmod n], q} - \delta_{n^3 - p - 1, q}$
  - Specific part: $A^{(\mu)}_{(ijk),(0j'k')} = (\sigma_\mu)_{n[(i+j) \bmod n] + [(i+k) \bmod n], nj'+k'}$
- Simple form solutions:
  - $\mathrm{size}(\tilde{M}) = n^5 \times (n^4 + n^2)$, $\mathrm{rank}(\tilde{M}) = n^4 + n^2 - n$ → has n solutions
  - For small n (e.g. $\mathbb{Z}_2, \mathbb{Z}_3$): $B^{(i)} = \sigma_i \otimes \sigma_i$, $\tilde{A}^{(i)} = \sigma_i$, $A^{(i)} = \sigma_i \otimes \mathbb{1} \otimes \mathbb{1}$
  - Can be further iterated since B can be decomposed on edges

Example: Fibonacci

- Fusion rules: $\mathbb{1} \times \mathbb{1} = \mathbb{1},\ \mathbb{1} \times \tau = \tau \times \mathbb{1} = \tau,\ \tau \times \tau = \mathbb{1} + \tau$
- F-symbols: $[F_\tau^{\tau\tau\tau}]_{ij} = \begin{pmatrix} \varphi^{-1} & \varphi^{-1/2} \\ \varphi^{-1/2} & -\varphi^{-1} \end{pmatrix}$
- $M_{(ijk),I} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \varphi & 0 & 0 \\ 0 & 0 & 0 & \varphi & 0 \\ 0 & \varphi & 0 & 0 & \varphi^{3/2} \\ 0 & \varphi & 0 & 0 & 0 \\ 0 & 0 & 0 & \varphi & \varphi^{3/2} \\ 0 & 0 & \varphi & 0 & \varphi^{3/2} \\ \varphi & \varphi^{3/2} & \varphi^{3/2} & \varphi^{3/2} & -\varphi \end{pmatrix},\ \mathrm{rank}(M) = 5$
- Simple form solutions:
  - $\mathrm{size}(\tilde{M}) = 40 \times 29,\ \mathrm{rank}(\tilde{M}) = 28$ → only has one solution
  - No non-trivial generalized free field

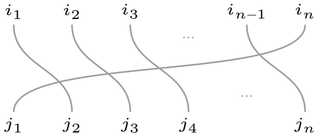
Tensor network representations of Virasoro & Kac–Moody algebra

Review of 2d CFT

- OPE of primary field ϕ
  - $T(z)\phi(w,\bar{z}) \sim \frac{h}{(z-w)^2}\phi(w,\bar{z}) + \frac{1}{z-w}\partial_w\phi(w,\bar{z})$
  - $T(z)T(w) \sim \frac{c/2}{(z-w)^4} + \frac{2}{(z-w)^2}T(w) + \frac{1}{z-w}\partial_w T(w)$
  - h: conformal dimension, c: central charge
- Virasoro algebra
  - Mode expansion of energy-momentum tensor: $L_n = \frac{1}{2\pi \mathrm{i}}\oint z^{n+1}T(z)\,\mathrm{d}z$
  - $[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}m(m^2-1)\delta_{m+n,0}$
- Kac–Moody algebra
  - Mode expansion of current operator: $J_n^\alpha = \frac{1}{2\pi \mathrm{i}}\oint z^{n+1}J^\alpha(z)\,\mathrm{d}z$
  - $[J_m^\alpha, J_n^\beta] = \mathrm{i}\sum_\gamma f^{\alpha\beta\gamma}J_{m+n}^\gamma + km\delta^{\alpha\beta}\delta_{m+n,0}$, $[L_m, J_n^\alpha] = -nJ_{m+n}^\alpha$

Torus partition function

- $Z = \operatorname{tr}\left[\exp(-2\pi\tau_2 H)\exp(2\pi\mathrm{i}\tau_1 P)\right] = \sum_\alpha \exp\left[-2\pi\tau_2\left(\Delta_\alpha - \frac{c}{12}\right) + 2\pi\mathrm{i}\tau_1 s_\alpha\right]$
  - $\tau = \tau_1 + \mathrm{i}\tau_2$: torus parameter
  - Δ_α: scaling dimension, s_α: conformal spin
- Lattice approximation on $m \times n$ grid
  - $Z = \sum_\alpha \exp\left[-2\pi\frac{m}{n}\left(\Delta_\alpha - \frac{c}{12}\right) + mnf + \mathcal{O}\left(\frac{m}{n^\gamma}\right)\right] = \operatorname{tr} M^m$
  - Eigenvalue of M: $\lambda_\alpha = \exp\left[-\frac{2\pi}{n}\left(\Delta_\alpha - \frac{c}{12}\right) + nf + \mathcal{O}\left(\frac{1}{n^\gamma}\right)\right]$
  - Fix $\Delta_{\mathbb{1}} = 0$ and $\Delta_T = 2$: $\Delta_\alpha = \frac{2}{\log\lambda_0 - \log\lambda_T}(\log\lambda_0 - \log\lambda_\alpha)$
- Translation operator: $P_{i_1 i_2\cdots i_n, j_1 j_2\cdots j_n} =$



Construction of Virasoro & Kac–Moody algebra

- Lattice Virasoro operator:
  $L_n \sim \sum_{j=1}^{N} \mathrm{e}^{\mathrm{i}jn\frac{2\pi}{N}} T(j)$
- Lattice Kac–Moody operator:
  $J_n \sim \sum_{j=1}^{N} \mathrm{e}^{\mathrm{i}jn\frac{2\pi}{N}} J(j)$
- Algorithm:
  (a) Build tensor network with A_{ijkl}
  (b) Calculate $|\phi_T\rangle$ or $|\phi_J\rangle$ from cylinder eigenstates
  (c) Reshape $|\phi_T\rangle$ / $|\phi_J\rangle$ to T / J
  (d) Insert T / J into a new cylinder with factor $\mathrm{e}^{\pm 2\pi \mathrm{i}jn/N}$

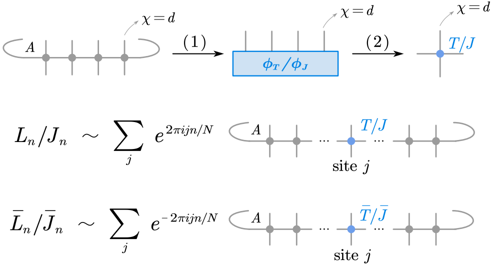
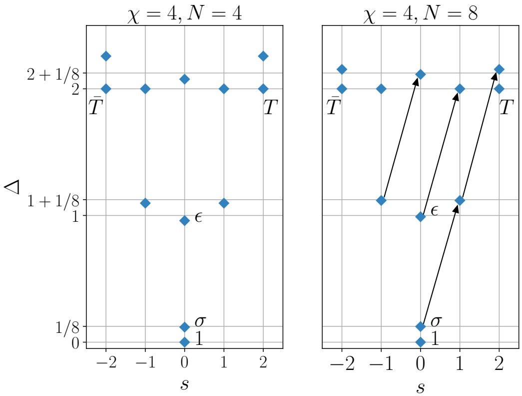


Image credit: Wang et al., *Phys. Rev. B* **106**: 115116 (2022)

Example: Virasoro algebra in Ising model

- Tensor unit:
  - $A_{ijkl} = \mathrm{e}^{-\beta(\sigma_i\sigma_j+\sigma_j\sigma_k+\sigma_k\sigma_l+\sigma_l\sigma_i)}$
  - Use blocking or TRG/TNR
- Can be verified by applying L_n on cylinder eigenstates $|\phi_\alpha\rangle$ and checking $\langle\phi_\beta|L_n|\phi_\alpha\rangle$
- Numerical results:
  - $N = 8, \chi = 4$ cylinder
  - $\frac{\|\langle\phi_\beta|L_n|\phi_\alpha\rangle\|}{\||\phi_\beta\rangle\|\cdot\|L_n|\phi_\alpha\rangle\|} \gtrsim 0.9, \quad n = \pm 1, \pm 2.$



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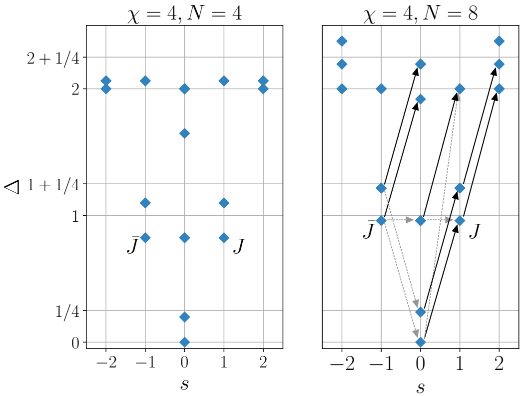
Example: Kac–Moody algebra in dimer model

- Tensor unit:
  - $B_{1111,2211,2121,1212,2222} = 1$
  - $B_{1122} = 2$
- Analysis of errors:
  - Mixing of different Kac–Moody towers: lowest eigenstate is polluted by other primary states
  - Mixing of lowering/raising actions within same towers: holomorphic and anti-holomorphic part are not fully separated



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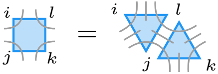
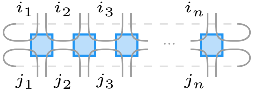
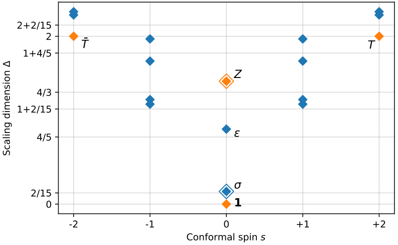
Fibonacci model: CFT spectrum

- Tensor unit and transfer matrix:
  - $A_{ijkl} =$ 

  - $\tilde{M}_{i_1 \cdots i_n, j_1 \cdots j_n} =$ 

  - Eliminate phases: $M = \tilde{M}\tilde{M}^{\dagger}$
- Use matrix-free linear operator methods to solve the eigensystem
- Reduce level-crossing: assume $\Delta = A + B/n$ and optimize fitting results



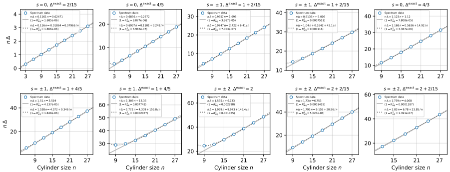
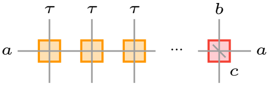
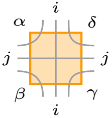
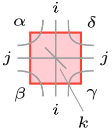
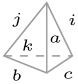


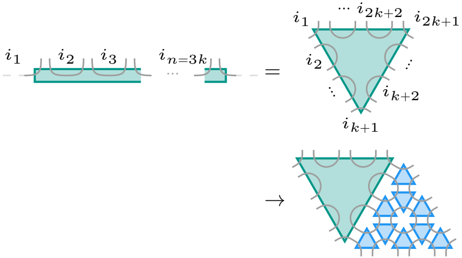
Image credit: Zeng et al., *Phys. Rev. B* **107**: 245146 (2023)

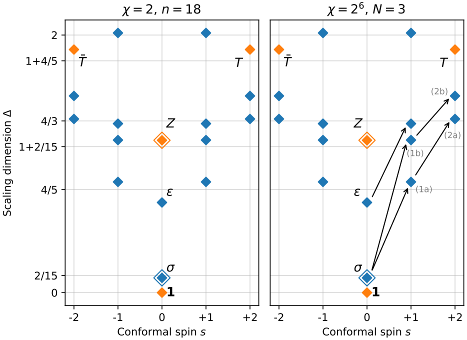
Fibonacci model: topological projectors

- Project transfer matrix to certain topological sector of the CFT
- How to identify T:
  - Descendant of the vacuum state → using the idempotent for trivial sector
- Tensor network representation:
  - $\mathcal{P}_{\mathbb{1}} = \frac{1}{\sqrt{5}}\left(\frac{1}{\phi}\mathcal{T}^{\mathbb{1}}_{\mathbb{1}\mathbb{1}} + \sqrt{\phi}\,\mathcal{T}^{\tau}_{\tau\mathbb{1}}\right), \quad \mathcal{T}^{c}_{ab} =$ 
  -  $= (d_\alpha d_\beta d_\gamma d_\delta)^{\frac{1}{4}} G^{\beta i \gamma}_{j\alpha\delta},$  $= (d_\alpha d_\beta d_\gamma d_\delta d_i d_j d_k)^{\frac{1}{4}} G^{k\beta\delta}_{ij\alpha} G^{i\gamma\beta}_{kj\delta}$
  - $G^{abc}_{ijk} = \frac{1}{\sqrt{d_j d_c}}\left[F^{aik}_{b}\right]_{jc} = \frac{1}{\sqrt{d_a d_b d_c d_i d_j d_k}}$ 

Fibonacci model: Virasoro algebra

- Reshape T:





- Cylinder size:
  - Eigenstates: $n = 18, \chi = 2$
  - Virasoro operators: $N = 3, \chi = 2^{n/3}$

Image credit: Zeng et al., *Phys. Rev. B* **107**: 245146 (2023)

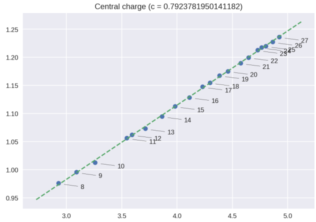
Summary & outlook

- RG operators determined by topological orders realize a holographic tensor network, and the eigenstates give rise to the boundary conditions
- Operator pushing in holographic tensor networks gives the condition for the existence of generalized free fields in the bulk
- Tensor network representation of Virasoro and Kac–Moody algebra can be built from cylinder eigenstates without explicitly knowing the Hamiltonian
- Possible future directions:
    - Generalization to higher dimensions and more complicated models
    - More (numerical and analytical) evidence for AdS/CFT correspondence

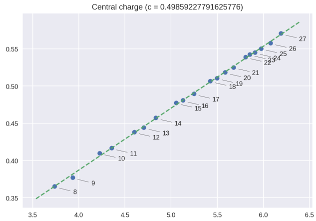
Thank you!

Application: calculation of central charge

- Use A_{ijkl} as iMPO unit and perform iTEBD → $\{\Gamma, \lambda\}$
- Physical quantities:
  - Correlation length: $\xi = -1/\log|\lambda_2/\lambda_1|$
  - Entanglement entropy: $S_A = \sum_i \lambda_i^2 \log \lambda_i^2$
- Fit with $S_A \sim \frac{c}{6}\log\xi$ for different bond dimension χ



Fibonacci: $c \simeq 0.792 \pm 0.004$



Ising: $c \simeq 0.499 \pm 0.004$

Example: $\mathcal{A}_{k+1}$ model

- Boundary state:

$$\langle\Omega| = \sum_{\{a\}} \prod_{a_i} T^{a_1 a_2 a_3} \langle\{a\}|\,, \quad T^{a_1 a_2 a_3} = \begin{cases} 1, & a_1 = a_3 = \frac{1}{2},\, a_2 = 0; \\ r, & a_1 = a_3 = \frac{1}{2},\, a_2 = 1; \\ 0, & \text{otherwise} \end{cases}$$

- Fixed points
  - Classified by Frobenius algebra
  - T will eventually flow to a *topological* fixed point for finite χ
  - *Conformal* fixed point occurs when T starts to converge to the topological one

Numerical results

- Fixed point tensor (at $\chi = 1$)
- Small r
  - Non-vanishing component: T^{000}
  - Frobenius algebra: $\mathcal{A}_0 = \{0\}$
- Large r
  - Non-vanishing component:
    $$T^{000} = T^{110} = T^{101} = T^{011}$$
  - Also exists T^{111} at $k > 2$
  - Frobenius algebra: $\mathcal{A}_1 = \{0, 1\}$
  - Ratio: $\dfrac{T^{000}}{T^{111}} \simeq \begin{cases} 1.43463, & k = 3; \\ 1.18921, & k = 4 \end{cases}$

- Critical coupling: $r_c = \dfrac{\sqrt[4]{2\cos\left(\frac{2\pi}{k+2}\right)+1}}{\sqrt{2\cos\left(\frac{\pi}{k+2}\right)+1}}$
- Recover to about 1 significant figure

k	2	3	4	5	6
r_c^n	0.60-0.61	0.67-0.68	0.69-0.70	0.71-0.72	0.72-0.73
r_c^a	0.643594	0.697043	0.719471	0.731426	0.738656

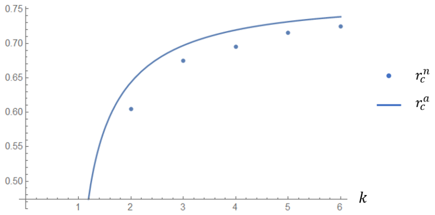
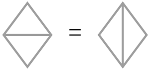


Image credit: Chen et al., arXiv:2210.12127 (2022)

Generalization: 1+1d

- Dijkgraaf–Witten model characterized by group G
- Triangulation: $\alpha_2(g_1, g_2) \in H^2(G, U(1))$, $g_1 \times g_2 = g_3$
- Associativity condition:
  - $\alpha(g_1, g_2)\,\alpha(g_1 g_2, g_3) = \alpha(g_1, g_2 g_3)\,\alpha(g_2, g_3)$



  - Convert the boundary circle with $2N$ edges to N edges

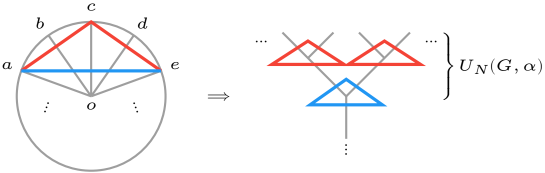


Image credit: Chen et al., arXiv:2210.12127 (2022)

Generalization: 3+1d

- 3+1d $\mathbb{Z}_2$ Dijkgraaf–Witten model (toric code) → 3d Ising model
- RG procedure:
  - Smallest self-repeating unit: 2×2×2 cubes
  - Eliminate edge centers → eliminate face centers → eliminate body centers