Supersymmetry (SUSY) in Quantum Mechanics

量子力学中的超对称

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1D harmonic oscillator

Hamiltonian (with $\hbar = 1$):

$$\hat{H} = \frac{1}{2}\left(\hat{p}^2 + \omega^2 \hat{x}^2\right)$$

Raising & lowering operators:

$$\begin{cases} \hat{a} = \dfrac{1}{\sqrt{2\omega}}(\hat{p} - \mathrm{i}\omega\hat{x}) \\ \hat{a}^\dagger = \dfrac{1}{\sqrt{2\omega}}(\hat{p} + \mathrm{i}\omega\hat{x}) \end{cases} \implies \left[\hat{a},\, \hat{a}^\dagger\right] = 1$$

Then

$$\hat{H} = \frac{\omega}{2}\left(\hat{a}^\dagger \hat{a} + \hat{a}^\dagger \hat{a}\right) = \omega\left(\hat{a}^\dagger \hat{a} + \frac{1}{2}\right) \equiv \omega\left(\hat{N} + \frac{1}{2}\right)$$

where

$$\hat{N}\,|n\rangle = n\,|n\rangle$$

2D harmonic oscillator

Hamiltonian:

$$\hat{H} = \frac{1}{2}\left(\hat{p}_x^2 + \hat{p}_y^2\right) + \frac{\omega^2}{2}\left(\hat{x}^2 + \hat{y}^2\right) \equiv \hat{H}_x + \hat{H}_y$$

Raising & lowering operators:

$$\begin{cases} \hat{a}_x = \dfrac{1}{\sqrt{2\omega}}\left(\hat{p}_x - \mathrm{i}\omega\hat{x}\right) \\ \hat{a}_y = \dfrac{1}{\sqrt{2\omega}}\left(\hat{p}_y - \mathrm{i}\omega\hat{y}\right) \end{cases} \implies \begin{aligned} \hat{H} &= \omega\left(\hat{a}_x^\dagger\,\hat{a}_x + \hat{a}_y^\dagger\,\hat{a}_y + 1\right) \\ &\equiv \omega\left(\hat{N} + 1\right) \end{aligned}$$

where

$$\hat{N}\left|n_x, n_y\right\rangle = \left(n_x + n_y\right)\left|n_x, n_y\right\rangle$$

Supersymmetrical oscillator

x & y oscillator $\rightarrow$ bosonic & fermionic oscillator

(Anti-)commutation relations:

$$\begin{cases} \text{Bosonic:} & \left[\hat{a},\, \hat{a}^\dagger\right] = 1 \\ \text{Fermionic:} & \left\{\hat{c},\, \hat{c}^\dagger\right\} = 1 \end{cases} \quad \text{and} \quad \left[\hat{a},\, \hat{c}\right] = 0$$

Hamiltonian:

$$\begin{cases} \hat{H}_\mathrm{B} = \dfrac{\omega}{2}\left\{\hat{a}^\dagger,\, \hat{a}\right\} \\ \hat{H}_\mathrm{F} = \dfrac{\omega}{2}\left[\hat{c}^\dagger,\, \hat{c}\right] \end{cases} \Longrightarrow \quad \begin{aligned} \hat{H}_\mathrm{S} = \hat{H}_\mathrm{B} + \hat{H}_\mathrm{F} &= \omega\left(\hat{a}^\dagger\hat{a} + \hat{c}^\dagger\hat{c}\right) \\ &\equiv \omega\hat{N} \end{aligned}$$

$\hat{Q}$ operators & superalgebra

Superalgebra for N-dimension:

$$\left[\hat{Q}_i,\, \hat{H}_{\mathrm{S}}\right] = 0, \quad \left\{\hat{Q}_i,\, \hat{Q}_j\right\} = 2\delta_{ij}\hat{H}_{\mathrm{S}}, \quad \hat{Q}_i^\dagger = \hat{Q}_i$$

In 2D:

$$\hat{Q}_1 = \sqrt{\omega}\left(\hat{a}^\dagger \hat{c} + \hat{a}\hat{c}^\dagger\right), \quad \hat{Q}_2 = \mathrm{i}\sqrt{\omega}\left(\hat{a}^\dagger \hat{c} - \hat{a}\hat{c}^\dagger\right)$$

Define non-hermitian $\hat{Q}$ and $\hat{Q}^\dagger$ — “supercharges”

$$\hat{Q} = \frac{\mathrm{i}}{2}\left(\hat{Q}_1 + \mathrm{i}\hat{Q}_2\right) = \mathrm{i}\sqrt{\omega}\,\hat{a}\hat{c}^\dagger$$

Note that

$$\begin{cases} \hat{a}\left|n_{\mathrm{B}},\, n_{\mathrm{F}}\right\rangle \propto \left|n_{\mathrm{B}} - 1,\, n_{\mathrm{F}}\right\rangle \\ \hat{c}^\dagger\left|n_{\mathrm{B}},\, n_{\mathrm{F}}\right\rangle \propto \left|n_{\mathrm{B}},\, n_{\mathrm{F}} + 1\right\rangle \end{cases} \Longrightarrow \hat{Q}\left|n_{\mathrm{B}},\, n_{\mathrm{F}}\right\rangle \propto \left|n_{\mathrm{B}} - 1,\, n_{\mathrm{F}} + 1\right\rangle$$

Superpotentials

Schrödinger's equation (bosonic):

$$\hat{H}_{\mathrm{B}}\psi_0 = \left[-\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}x^2} + V_{\mathrm{B}}(x)\right]\psi_0 = 0$$

Let

$$\hat{H}_{\mathrm{B}} = \hat{A}^\dagger \hat{A} \implies \begin{cases} \hat{A} = \dfrac{1}{\sqrt{2}}\dfrac{\mathrm{d}}{\mathrm{d}x} + W(x) \\ \hat{A}^\dagger = -\dfrac{1}{\sqrt{2}}\dfrac{\mathrm{d}}{\mathrm{d}x} + W(x) \end{cases}$$

$$\implies V_{\mathrm{B}}(x) = W^2(x) - \frac{1}{\sqrt{2}}W'(x)$$

$$\implies W(x) = -\frac{1}{\sqrt{2}}\frac{\psi_0'}{\psi_0}$$

$W(x)$ — "superpotential"

Superpotentials — continued

$$\hat{H}_{\mathrm{B}} = \hat{A}^{\dagger}\hat{A} \implies V_{\mathrm{B}}(x) = W^2(x) - \frac{1}{\sqrt{2}}W'(x)$$

Reverse the order of $\hat{A}$ and $\hat{A}^{\dagger}$:

$$\hat{H}_{\mathrm{F}} = \hat{A}\hat{A}^{\dagger} \implies V_{\mathrm{F}}(x) = W^2(x) + \frac{1}{\sqrt{2}}W'(x)$$

Relationship between eigenstates:

$$\begin{cases} \hat{H}_{\mathrm{F}}\left(\hat{A}\psi_n^{\mathrm{B}}\right) = E_n^{\mathrm{B}}\left(\hat{A}\psi_n^{\mathrm{B}}\right) \\ \hat{H}_{\mathrm{B}}\left(\hat{A}^{\dagger}\psi_n^{\mathrm{F}}\right) = E_n^{\mathrm{F}}\left(\hat{A}^{\dagger}\psi_n^{\mathrm{F}}\right) \end{cases} \implies \begin{cases} \psi_n^{\mathrm{F}} = \left(E_{n+1}^{\mathrm{B}}\right)^{-1/2} \cdot \left(\hat{A}\psi_{n+1}^{\mathrm{B}}\right) \\ \psi_{n+1}^{\mathrm{B}} = \left(E_n^{\mathrm{F}}\right)^{-1/2} \cdot \left(\hat{A}^{\dagger}\psi_n^{\mathrm{F}}\right) \end{cases}$$

and $\quad E_n^{\mathrm{F}} = E_{n+1}^{\mathrm{B}}$

Application: Coulomb potential (1)

Write $\hat{A}$ and $\hat{A}^\dagger$ in:

$$\begin{cases} \hat{A} = \dfrac{1}{\sqrt{2}}\left(\mathrm{i}\hat{p} + v\right) \\ \hat{A}^\dagger = \dfrac{1}{\sqrt{2}}\left(-\mathrm{i}\hat{p} + v\right) \end{cases}$$

where

$$\left[v,\, \hat{p}\right] = \mathrm{i}v'$$

and

$$\left[\hat{A},\, \hat{A}^\dagger\right] = v'$$

Then the partner Hamiltonians will be

$$\begin{cases} \hat{H}_\mathrm{F} = \hat{A}\hat{A}^\dagger = \dfrac{1}{2}\left(\hat{p}^2 + v^2 + v'\right) \\ \hat{H}_\mathrm{B} = \hat{A}^\dagger\hat{A} = \dfrac{1}{2}\left(\hat{p}^2 + v^2 - v'\right) \end{cases}$$

Introduce two new Hamiltonians:

$$\begin{cases} \hat{H}_1 = \dfrac{1}{2}\hat{p}^2 + V_1 \\ \hat{H}_2 = \dfrac{1}{2}\hat{p}^2 + V_2 \end{cases}$$

Express them in

$$\begin{cases} \hat{H}_1 = \hat{H}_\mathrm{F} + E_{1,1} \\ \hat{H}_2 = \hat{H}_\mathrm{B} + E_{1,1} \end{cases}$$

Solve for v and $\hat{A}$:

$$v = \frac{\psi'_{1,1}}{\psi_{1,1}}, \quad \hat{A} = \frac{1}{\sqrt{2}}\left(\frac{\mathrm{d}}{\mathrm{d}x} + \frac{\psi'_{1,1}}{\psi_{1,1}}\right)$$

$$\implies \hat{H}_1 - \hat{H}_2 = V_1 - V_2 = v' = \left(\ln \psi_{1,1}\right)''$$

Application: Coulomb potential (2)

Radial equation:

$$\left[-\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}r^2}-\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}+\frac{l(l+1)}{2r^2}-\frac{1}{r}\right]R_{nl}(r)$$
$$=E_{nl}R_{nl}$$

Let

$$R_{nl}=u_{nl}(r)/r$$
$$\hat{h}_l=-\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}r^2}+\underbrace{\frac{l(l+1)}{2r^2}-\frac{1}{r}}_{V_l}$$

The first solution is then

$$\begin{cases} u_{1,l}(r)=N_{1,l}\cdot r^{l+1}\exp\left(\sqrt{-2E_{1,l}}\cdot r\right) \\ E_{1,0}=-\frac{1}{2}\ \text{(a.u.)} \end{cases}$$

Use the formula in previous page:

$$V_l-V_s=\frac{\mathrm{d}^2}{\mathrm{d}r^2}\left[\ln u_{1,l}(r)\right]$$

Then we can find

$$V_s=V_l-\frac{\mathrm{d}^2}{\mathrm{d}r^2}\left[\ln u_{1,l}(r)\right]=V_{l+1}$$

i.e. V_l and V_{l+1} (or $\hat{h}_l$ and $\hat{h}_{l+1}$) are supersymmmetric partners.

Repeat it, and we will obtain recursion relation for energy levels:

$$E_{nl}=E_{n-1,l+1}=E_{n-2,l+2}=\cdots=E_{1,l+n-1}$$

Application: Coulomb potential (3)

Analogy to $\hat{H}_1$ and $\hat{H}_2$,

$$\begin{cases} \hat{l}_{l+1} = \hat{A}_{l+1}\hat{A}^\dagger_{l+1} + E_{1,\,l+1} \\ \hat{l}_{l+1} = \hat{A}_l\hat{A}^\dagger_l + E_{1,\,l} \end{cases}$$

Note that $\hat{A}$ and $\hat{A}^\dagger$ can be expressed with $\hat{p}$ and v, then

$$v_{l+1}^2 + \frac{\mathrm{d}v_{l+1}}{\mathrm{d}r} + 2E_{1,\,l+1} = v_l^2 - \frac{\mathrm{d}v_l}{\mathrm{d}r} + 2E_{1,\,l}$$

Calculate v_l with

$$v_l = \frac{\mathrm{d}}{\mathrm{d}r}\ln u_{1,\,l} \implies v_l = \frac{l+1}{r} - \sqrt{-2E_{1,l}}$$

Then

$$\frac{E_{1,\,l}}{E_{1,\,l+1}} = \left(\frac{l+2}{l+1}\right)^2 \implies E_n = E_{1,l} = -\frac{1}{2}n^2\,(\text{a.u.})$$

1. T. Wellman. "An Introduction to Supersymmetry in Quantum Mechanical Systems", 2003.
2. Valance, A., T. J. Morgan, and H. Bergeron. "Eigensolution of the Coulomb Hamiltonian via supersymmetry." *American Journal of Physics*, 1990, **58**(5): 487–491.

Questions?