QEC 101

Introduction to quantum error correction

Xiangdong Zeng

November 15, 2024

Invitation: QR code

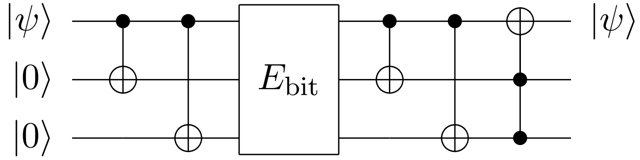
- Basic idea: using **redundant** information
- Error correction level: 7–30%
- Reed–Solomon codes
    - Polynomials over finite fields (Galois fields)
- 回形针 PaperClip：Vol.120 二维码的秘密

Quantum errors

- Bloch sphere state: $|\psi\rangle = \cos\frac{\theta}{2}|0\rangle + e^{i\phi}\sin\frac{\theta}{2}|1\rangle$
- General coherent error: $U(\delta\theta,\delta\phi)|\psi\rangle = \cos\frac{\theta+\delta\theta}{2}|0\rangle + e^{i(\phi+\delta\phi)}\sin\frac{\theta+\delta\theta}{2}|1\rangle$
  - Rotation on Bloch sphere
- Write in Pauli matrices: $U(\delta\theta,\delta\phi) = \alpha_I I + \alpha_X X + \alpha_Z Z + \alpha_Y Y \; (= \alpha_{XZ} XZ)$
- Two types: X-error ($|0\rangle \leftrightarrow |1\rangle$) and Z-error ($|0\rangle \to |0\rangle, |1\rangle \to -|1\rangle$)
- Challenges:
  - No-cloning theorem: $U_{\text{clone}}(|\psi\rangle \otimes |0\rangle) \not\to |\psi\rangle \otimes |\psi\rangle$
  - Detect both X and Z errors simultaneously
  - Wavefunction collapse
  - Continuous error in θ and ϕ
- General procedure:
  - Syndrome extraction: measure $U|\psi\rangle$ and project to subspace (e.g. $X|\psi\rangle$)
  - Apply correspoding operator (e.g. X^{-1}) to recover $|\psi\rangle$

Bit flip code (1)

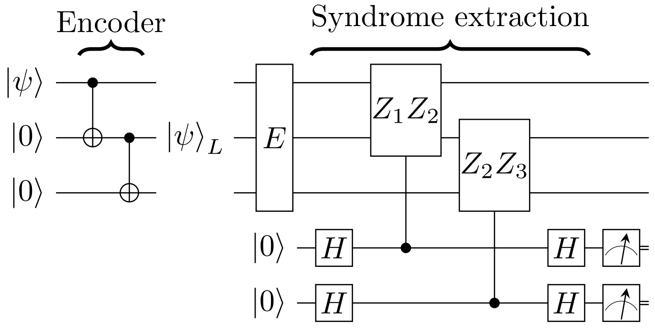
- Initial state: $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$
- Bit flip error: $P(|0\rangle \leftrightarrow |1\rangle) = p$
- No-cloning theorem:
  - $|\psi\rangle \not\rightarrow |\psi\psi\psi\rangle$
  - $|0\rangle \rightarrow |0_L\rangle = |000\rangle$, $|1\rangle \rightarrow |1_L\rangle = |111\rangle$
    - Logical qubits: $|x_L\rangle$, ancilla qubits: $|x00\rangle$
- Syndrome diagnosis:
  - $P_0 = |000\rangle\langle 000| + |111\rangle\langle 111|$: no error
  - $P_1 = |100\rangle\langle 100| + |011\rangle\langle 011|$: bit flip on qubit 1
  - $P_2 = |010\rangle\langle 010| + |101\rangle\langle 101|$: bit flip on qubit 2
  - $P_3 = |001\rangle\langle 001| + |110\rangle\langle 110|$: bit flip on qubit 3
- Correct error: apply $M = \sum_i X_i P_i$

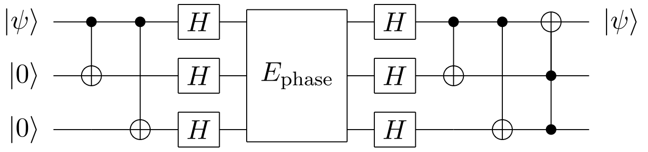


- Only single bit flip error can be corrected
- Fidelity:
  - $F_\psi = 1 - p$
  - $F_{\psi'} = (1-p)^3 + 3p(1-p)^2 > F_\psi$ for $p < \frac{1}{2}$

Bit flip code (2)

- Another way to do syndrome measurement:
  - $Z_1 Z_2 = Z \otimes Z \otimes I$: compare first two qubits
  - $Z_2 Z_3 = I \otimes Z \otimes Z$: compare last two qubits
- Syndrome table:
  - 00: no error
  - 01: bit flip on qubit 1
  - 10: bit flip on qubit 2
  - 11: bit flip on qubit 3
- Measurement **does not** change the state
- Sign flip code:
  - Dual between X and Z





Quantum code distance

- Classical:
  - Minimum Hamming distance between any two codewords
  - $d_{\text{classical}}(0000,\,1111) = 4$
- Quantum:
  - Shortest path to get from one state to another by using Pauli operators
  - Minimum size error that will go undetected
- $[[n, k, d]]$ notation:
  - n: number of **physical** qubits
  - k: number of **logical** qubits
  - d: code distance

Stabilizer formalism (1): we need some group theory!

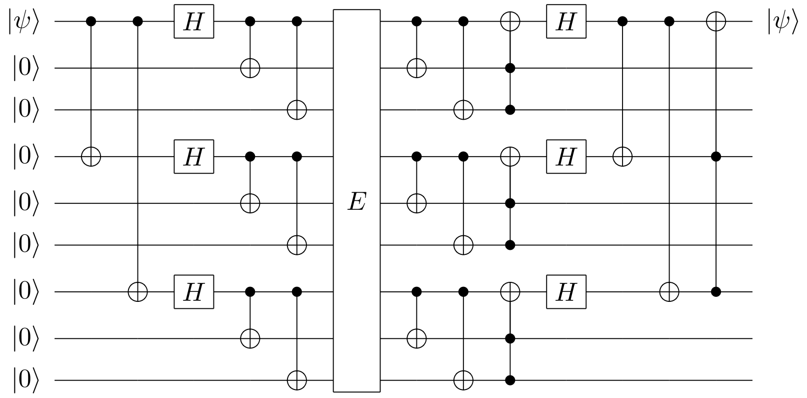
- Pauli group:
  - 1 qubit: $G_1 = \{\pm I, \pm iI, \pm X, \pm iX, \pm Y, \pm iY, \pm Z, \pm iZ\} = \langle X, Y, Z \rangle$
  - n qubits: $G_n = \{g_i \otimes \cdots \otimes g_n | g_i \in G_1\}$
- Stabilizer group: all elements that leave a state invariant
  - $\mathcal{S} = \{g \in G_n | g \left|\psi\right\rangle = \left|\psi\right\rangle\}$
  - $\mathcal{S}$ is Abelian group
- Example:
  - 3 qubits, $\mathcal{S} = \{I, Z_1 Z_2, Z_2 Z_3, Z_3 Z_1\} = \langle Z_1 Z_2, Z_2 Z_3 \rangle$
  - $Z_1 Z_2$ can stabilize $\{\left|000\right\rangle, \left|001\right\rangle, \left|110\right\rangle, \left|111\right\rangle\}$
  - $Z_2 Z_3$ can stabilize $\{\left|000\right\rangle, \left|100\right\rangle, \left|011\right\rangle, \left|111\right\rangle\}$
  - Hence $\mathcal{S}$ can stabilize $\{\left|000\right\rangle, \left|111\right\rangle\}$

Stabilizer formalism (2)

- Error $E \in \mathcal{E} \subset G_n$, measurement $M \in \mathcal{S}$
  - M and E anti-commute: $M(E|\psi\rangle) = -EM|\psi\rangle = -E|\psi\rangle$
    - Error will corrupt the encoded state
    - Can detect error E
  - M and E commute: $M(E|\psi\rangle) = EM|\psi\rangle = E|\psi\rangle$
    - Can't correct error E
- Stabilizer error-correcting condition:
  - Any errors $E_1, E_2 \in \mathcal{E}$ can be corrected, if
    - $E_1^\dagger E_2 \notin \mathcal{Z}(\mathcal{S})$ or
    - $E_1^\dagger E_2 \in \mathcal{S}$
  - Centralizer: $\mathcal{Z}(\mathcal{S}) = \{g \in G_n | gM = Mg, \forall M \in \mathcal{S}\}$
    - Commute with all elements in $\mathcal{S}$

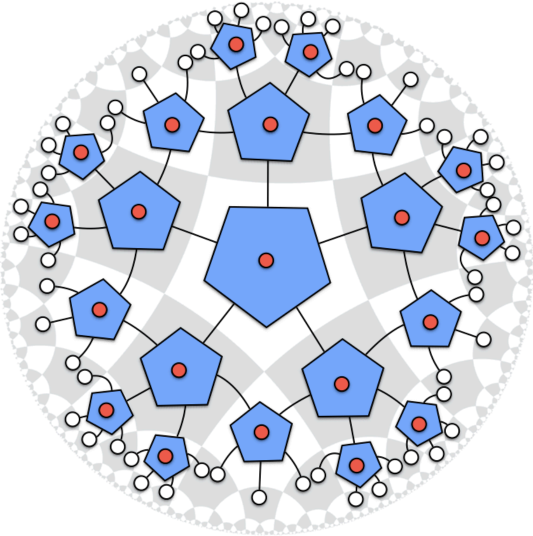
Shor code

- 9 physical qubits for 1 logical qubit
- Combine bit flip and sign flip codes
  - $Z_1Z_2,\ Z_2Z_3,\ \ldots,\ Z_8Z_9$ will detect X errors
  - $X_1X_2X_3X_4X_5X_6,\ X_4X_5X_6X_7X_8X_9$ will detect Z errors
- Stabilizer language ($[[9,1,3]]$ code):
  - $\mathcal{S} = \{Z_1Z_2,\ \ldots,\ Z_8Z_9,\ X_1\cdots X_6,\ X_4\cdots X_9\}$
  - Example:
    - $E = X_1Y_4 \notin \mathcal{Z}(\mathcal{S})$: anti-commute with Z_1Z_2, can be corrected
    - $E = Z_1Z_3 \in \mathcal{S}$: can be corrected



And more...

- Surface code: topological QEC
  - 2D lattice of qubits
  - Stabilizer: Pauli chains on the surface
  - Anyons, topological order, string-net condensation, categorical symmetries, etc
  - See arXiv:quant-ph/9707021 etc
- HaPPY code: holographic QEC
  - Encode k “bulk” logical qubits into n “boundary” physical qubits
  - Bulk-boundary correspondence, AdS/CFT duality, Ryu–Takayanagi formula, tensor networks, *p*-adic physics, etc
  - See arXiv:1503.06237 and 1802.01040



References

- 杂然赋流形丶. 可否用通俗的物理语言（少用一点数学...）解释下量子纠错码？
- 杂然赋流形丶. https://www.zhihu.com/question/29447951/answer/3547326663
- 魏朝晖. CCF量子计算 |07⟩ 量子纠错与量子电路（上）
- 魏朝晖. CCF量子计算 |08⟩ 量子纠错与量子电路（下）
- Wikipedia. *Quantum error correction*
- Wikipedia. *Stabilizer code*
- J. Roffe. *Quantum Error Correction: An Introductory Guide* (arXiv:1907.11157)