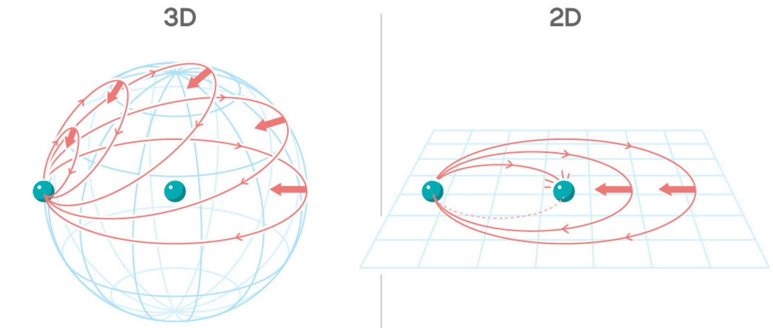
Topological Quantum Computing

Xiangdong Zeng
June 25, 2025

Anyons

- 3D case
  - Paths are topologically equivalent:
    $|\Psi(\lambda_2)\rangle = |\Psi(\lambda_1)\rangle = |\Psi(0)\rangle$
  - Encircle = exchange²: $|\Psi(\lambda_2)\rangle = R^2 \, |\Psi(0)\rangle$
  - $R = \pm 1$: bosons and fermions



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In 2D, the loop gets caught on the other particle.

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  - Encircle = exchange²: $|\Psi(\lambda_2)\rangle = R^2\,|\Psi(0)\rangle$
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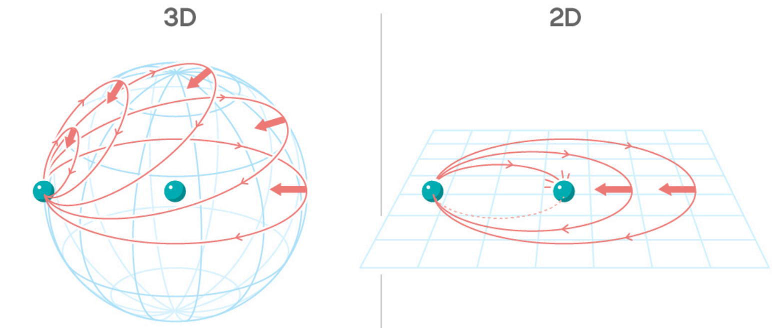
- 2D case
  - No longer topologically trivial:
    $|\Psi(\lambda_2)\rangle \neq |\Psi(\lambda_1)\rangle = |\Psi(0)\rangle$
  - $R = \mathrm{e}^{i\theta}$: abelian anyons
  - R is a unitary matrix: non-abelian anyons



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Topological states

- Symmetry-protected topological (SPT) states
  - Topological only given that some protecting symmetry
  - Do not support anyons as intrinsic quasiparticle excitations, unless ...
  - Having **defects**
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- (Intrinsic) topological order
  - Exhibit **long-range entanglement**
  - Give rise to **ground state degeneracy** and **topological entanglement entropy**
  - Examples: fractional quantum Hall states, spin liquids

Anyon models: fusion and braiding

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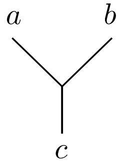
- Vector space 

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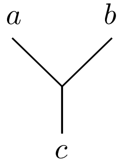
- Vector space 

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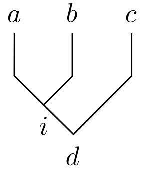
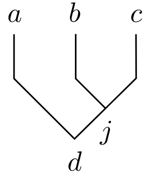
- F-move: basis transformation in $\mathbb{V}_{d}^{abc}$



$= \sum_{j} [F_{d}^{abc}]_{ij}$


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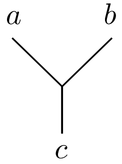
- Vector space 

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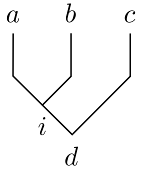

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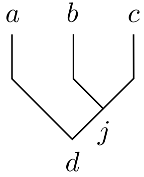
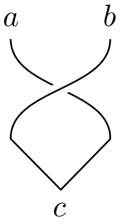
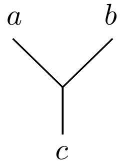


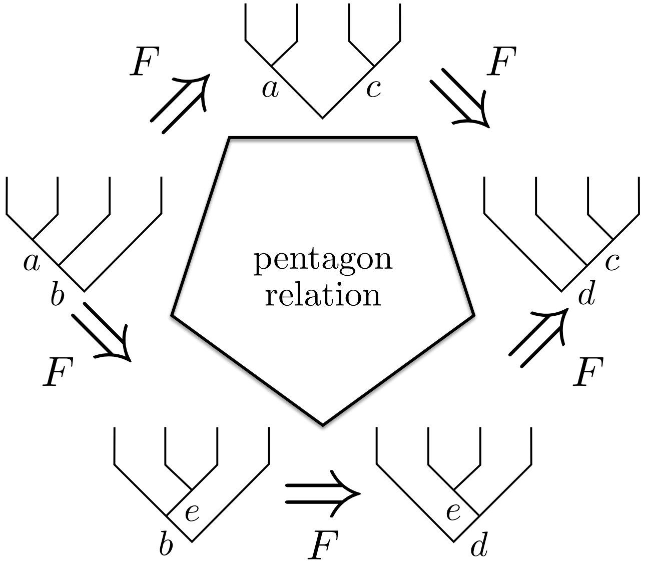
- R-move: exchanging two anyons

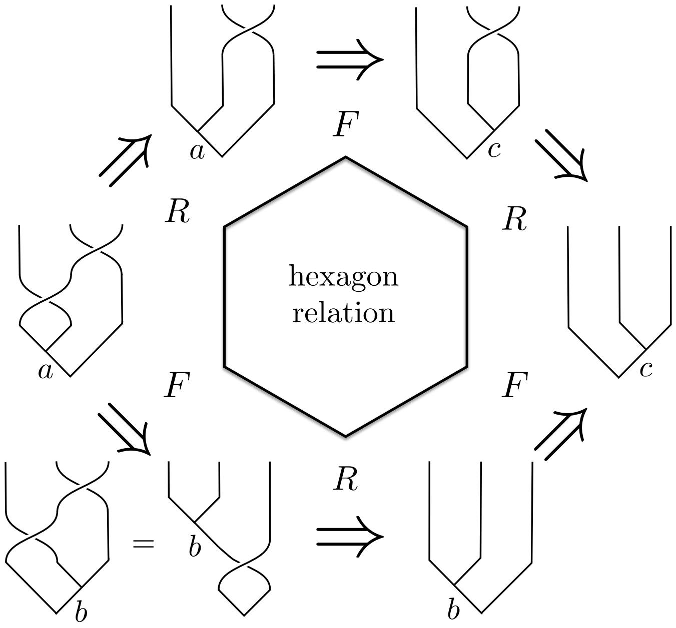


$= R_{c}^{ab}$



Anyon models: consistency relation





Mathematical structure (1)

- A **monoidal (tensor) category** is a category $\mathcal{C}$ with:
  - Tensor product: a bi-functor $\otimes \colon \mathcal{C} \times \mathcal{C} \to \mathcal{C}$
  - Unit object: $\mathbf{1} \in \mathcal{C}$
  - Natural isomorphisms:
    - Associator: $\alpha_{x,y,z} \colon (x \otimes y) \otimes z \xrightarrow{\sim} x \otimes (y \otimes z)$
    - Left unitor: $\lambda_x \colon \mathbf{1} \otimes x \xrightarrow{\sim} x$
    - Right unitor: $\rho_x \colon x \otimes \mathbf{1} \xrightarrow{\sim} x$
  - The following two diagrams commute

$$
\begin{tikzcd}
(x \otimes \mathbf{1}) \otimes y \arrow[rr, "\alpha_{x,\mathbf{1},y}"] \arrow[dr, "\rho_x \otimes \mathrm{id}_y"'] & & x \otimes (\mathbf{1} \otimes y) \arrow[dl, "\mathrm{id}_x \otimes \lambda_y"] \\
& x \otimes y &
\end{tikzcd}
$$

Triangle equation

$$
\begin{tikzcd}
& \big((w \otimes x) \otimes y\big) \otimes z \arrow[ddl, "\alpha_{w \otimes x,y,z}"'] \arrow[dr, "\alpha_{w,z,y} \otimes \mathrm{id}_z"] & \\
& & \big(w \otimes (x \otimes y)\big) \otimes z \arrow[dd, "\alpha_{w,x \otimes y,z}"] \\
(w \otimes x) \otimes (y \otimes z) \arrow[ddr, "\alpha_{w,x,y \otimes z}"'] & & \\
& & w \otimes \big((x \otimes y) \otimes z\big) \arrow[dl, "\mathrm{id}_w \otimes \alpha_{x,y,z}"] \\
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- MacLane's coherence theorem:
  - Every monoidal category is equivalent to a strict one
  - "$\xrightarrow{\sim}$" becomes "$=$"

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Mathematical structure (2)

- A **braided monoidal category** is a monoidal category $\mathcal{C}$ with:
  - Braiding: $\sigma_{x,y} \colon x \otimes y \xrightarrow{\sim} y \otimes x$
  - The following two diagrams commute

$$\begin{tikzcd}
x \otimes (y \otimes z) \arrow[r, "\alpha^{-1}_{x,y,z}"] \arrow[d, "\sigma_{x,y\otimes z}"'] & (x \otimes y) \otimes z \arrow[r, "\sigma_{x,y}\otimes \mathrm{id}_z"] & (y \otimes x) \otimes z \arrow[d, "\alpha_{y,x,z}"] \\
(y \otimes z) \otimes x & y \otimes (z \otimes x) \arrow[l, "\alpha^{-1}_{y,z,x}"] & y \otimes (x \otimes z) \arrow[l, "\mathrm{id}_y \otimes \sigma_{x,z}"]
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- A **fusion category** is
  - A (rigid semisimple $\mathbb{C}$-linear) monoidal category
  - With finitely many simple objects and finite dimensional spaces of morphisms
  - Unit object $\mathbf{1}$ is simple

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- A **fusion category** is
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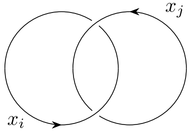
- A **modular tensor category** is
  - A braided monoidal fusion category
  - With S matrix $s_{ij} =$ 

non-degenerate

Examples

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- *R*-matrix (braiding):
  - $R = \begin{pmatrix} R^{1}_{\tau\tau} & 0 \\ 0 & R^{\tau}_{\tau\tau} \end{pmatrix} = \begin{pmatrix} e^{4\pi i/5} & 0 \\ 0 & e^{-3\pi i/5} \end{pmatrix}$

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- **Ising** anyon model
- Anyons: $\mathbf{1}$, ψ, σ
- Fusion rules:
  - $\psi \otimes \psi = \mathbf{1} + \sigma,$
    $\psi \otimes \sigma = \sigma \otimes \psi = \sigma,$
    $\sigma \otimes \sigma = \mathbf{1} + \psi$
  - Same as Ising CFT
- F-matrix:
  - $F = F^{\sigma}_{\sigma\sigma\sigma} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$
- R-matrix:
  - $R = \begin{pmatrix} R^{1}_{\sigma\sigma} & 0 \\ 0 & R^{\psi}_{\sigma\sigma} \end{pmatrix} = e^{-\pi i/8} \begin{pmatrix} 1 & 0 \\ 0 & e^{\pi i/2} \end{pmatrix}$

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  - Protect against local perturbations
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  - $|01\rangle = |\sigma\sigma; \mathbf{1}\rangle\ |\sigma\sigma; \psi\rangle\ |\sigma\sigma; \psi\rangle$
  - $|10\rangle = |\sigma\sigma; \psi\rangle\ |\sigma\sigma; \psi\rangle\ |\sigma\sigma; \mathbf{1}\rangle$
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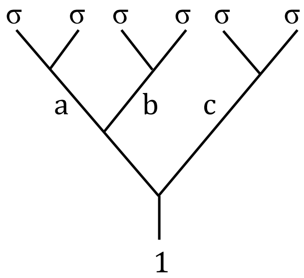

	a	b	c
$\vert 00\rangle$	1	1	1
$\vert 10\rangle$	ψ	ψ	1
$\vert 01\rangle$	1	ψ	ψ
$\vert 11\rangle$	ψ	1	ψ

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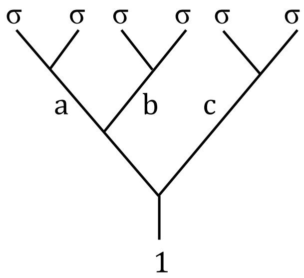
- Measurements: fusion outcomes
  - Detecting energy difference between $\mathbf{1}$ (vacuum) and ψ (a massive particle)



	a	b	c
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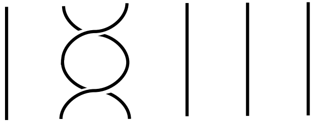
$\|01\rangle$	1	ψ	ψ
$\|11\rangle$	ψ	1	ψ

Quantum computation with anyons (2)

- Gates: braiding
- Single-qubit gates:
  - $X_1 = (R_{23})^2 = F^{-1}R^2F \otimes \mathbb{I}$
  - $Z_1 = (R_{12})^2 = R^2 \otimes \mathbb{I}$
  - $U_{H,1} = R_{12}R_{23}R_{12} = RF^{-1}RFR \otimes \mathbb{I}$
  - R_{ij}: exchange of anyons i and j
- Two-qubit gate:
  - $U_{CZ} = (R_{12})^{-1}R_{34}(R_{56})^{-1}$

$X_1 = (R_{23})^2$



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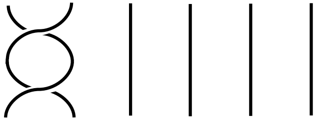
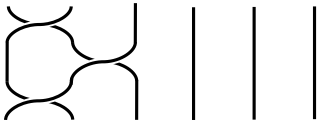
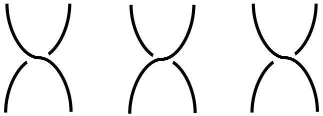


$U_{H,1} = R_{12}R_{23}R_{12}$



$U_{CZ} = R_{12}^{-1}R_{56}^{-1}R_{34}$



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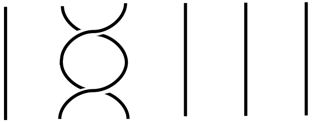
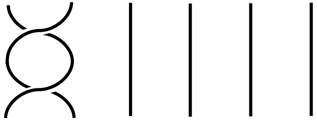
  - $U_{CZ} = (R_{12})^{-1}R_{34}(R_{56})^{-1}$
- Non-Clifford gate (necessary for universality):
  - $U = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\Delta Et} \end{pmatrix}$
  - Lift the degeneracy of the fusion channels by ΔE so it's not topologically protected

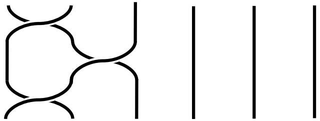
$X_1 = (R_{23})^2$



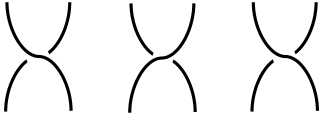
$Z_1 = (R_{12})^2$



$U_{H,1} = R_{12}R_{23}R_{12}$



$U_{CZ} = R_{12}^{-1}R_{56}^{-1}R_{34}$



Majorana zero modes: Majorana fermions

- Creation and annihilation operators:
  - Bosons: $[b, b^\dagger] = bb^\dagger - b^\dagger b = 1$
  - Fermions: $\{f, f^\dagger\} = ff^\dagger + f^\dagger f = 1$
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- Majorana Fermion is its own anti-particle
  - Neutrino?

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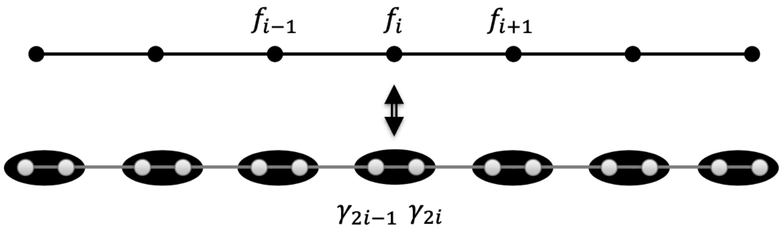
Majorana zero modes: Kitaev chain (1)



- Hamiltonian: $H = \sum_{j=1}^{L} \left[-t(f_j^\dagger f_{j+1} + f_{j+1}^\dagger f_j) - \mu \left(f_j^\dagger f_j - \frac{1}{2} \right) + \left(\Delta_p f_j f_{j+1} + \Delta_p^* f_{j+1}^\dagger f_j^\dagger \right) \right]$
  - t: tunnelling amplitude, μ: chemical potential, $\Delta_p = |\Delta_p| e^{i\theta}$: superconducting pairing potential
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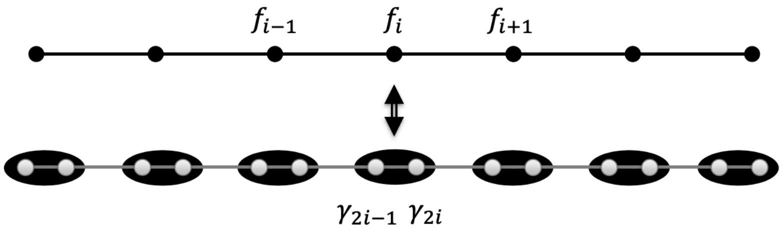

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- Use Majorana operators: $H = \sum_{j=1}^{L} \frac{i}{2} \left[-\mu \gamma_{2j-1}\gamma_{2j} + \left(t + |\Delta_p| \right) \gamma_{2j}\gamma_{2j+1} + \left(-t + |\Delta_p| \right) \gamma_{2j-1}\gamma_{2j+2} \right]$
  - Nearest and third-nearest neighbor interaction

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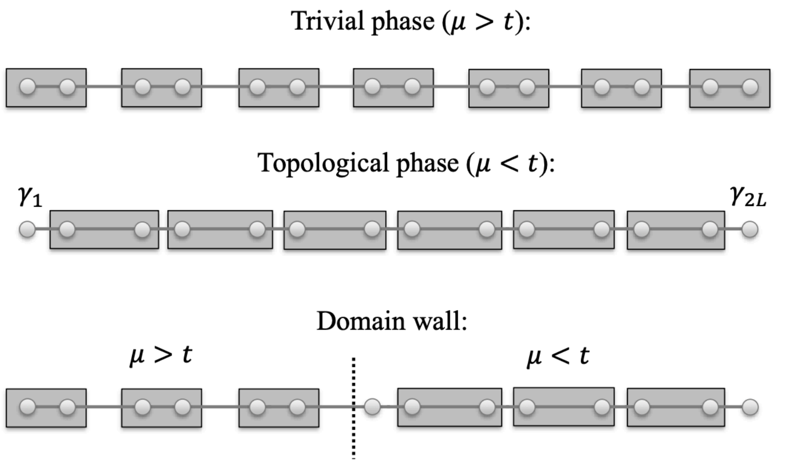

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    - γ_1 and γ_{2L}: Majorana zero modes
    - $[d^\dagger d,\ H] = 0$: two-fold degeneracy
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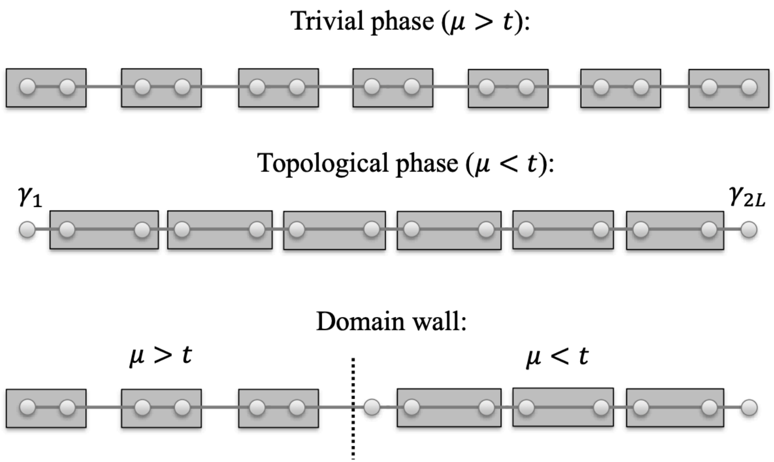

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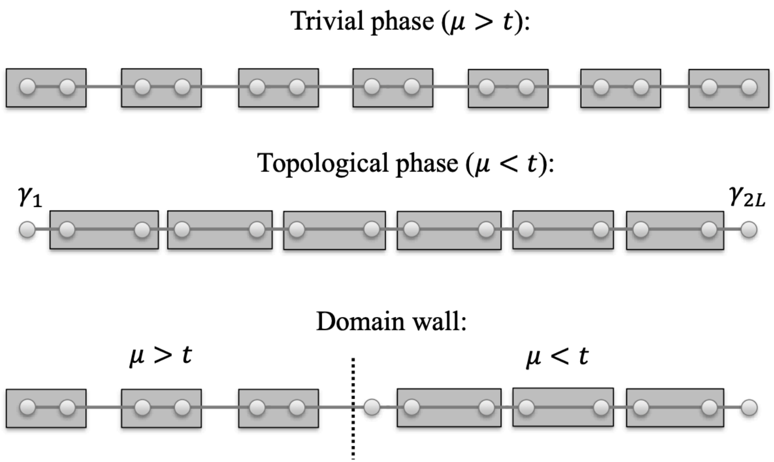
- Domain wall separating the phases also binds a localized Majorana mode



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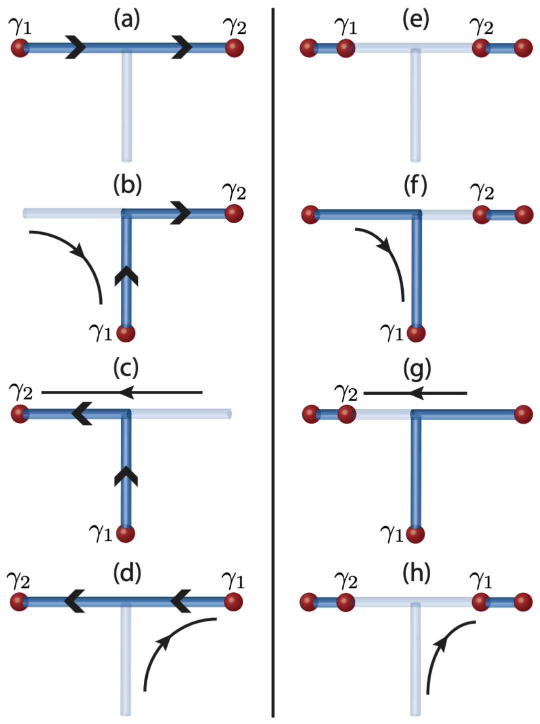
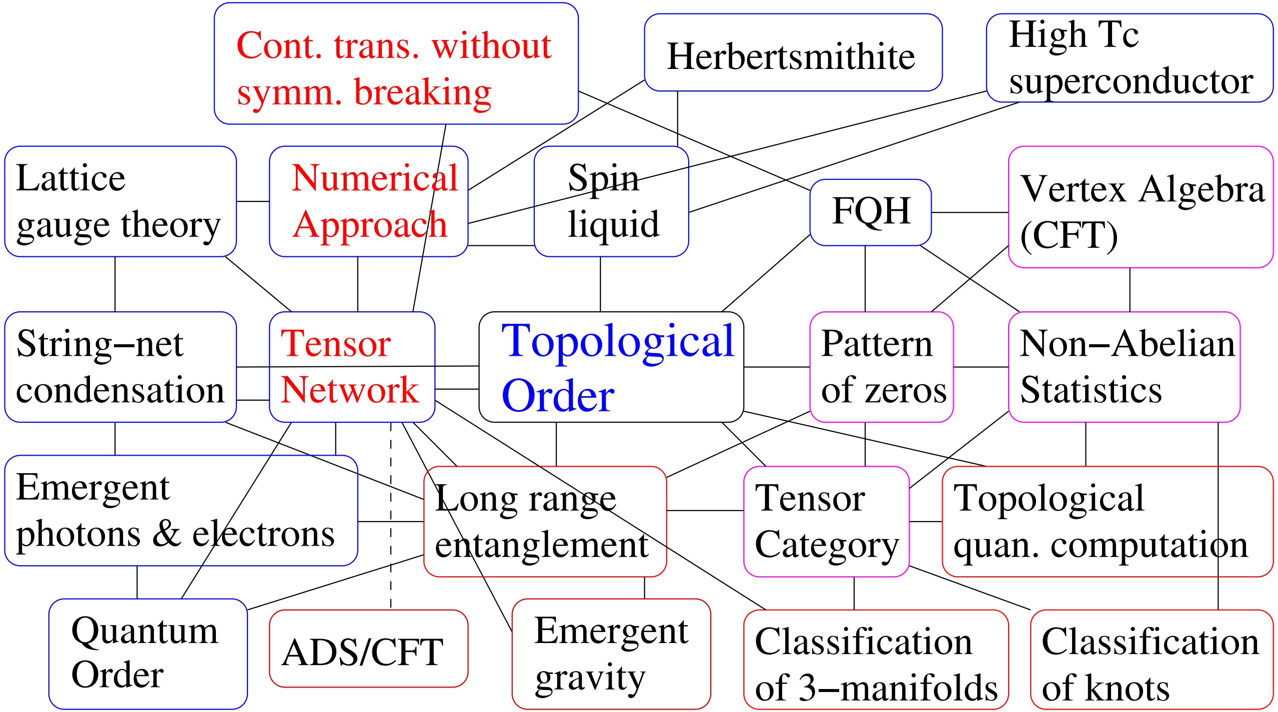
- Braiding: T-junction



The paradigm of topological order



From Xiao-Gang Wen (arXiv:0903.1069)

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