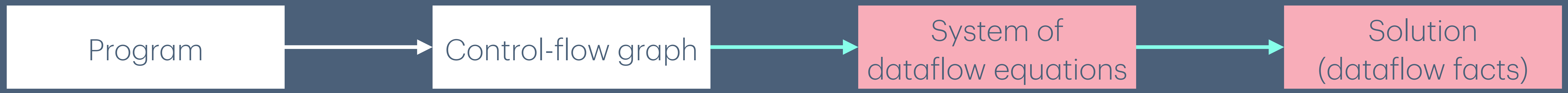
Newtonian Program Analysis of Probabilistic Programs

Di Wang and Thomas Reps

OOPSLA'24

2024.07.02

A Pipeline of Program Analysis



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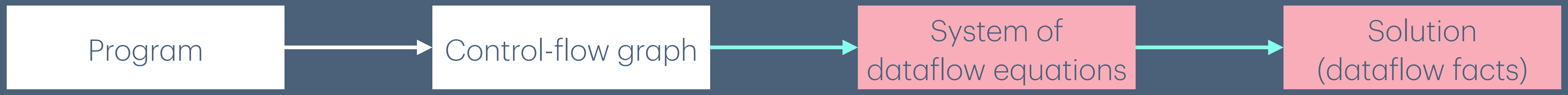

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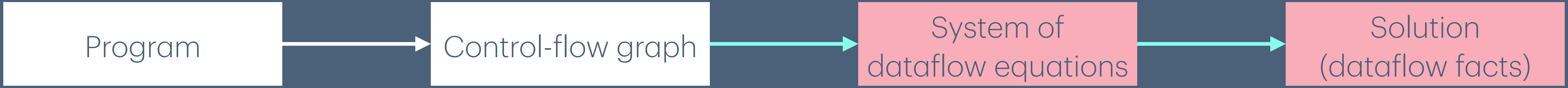
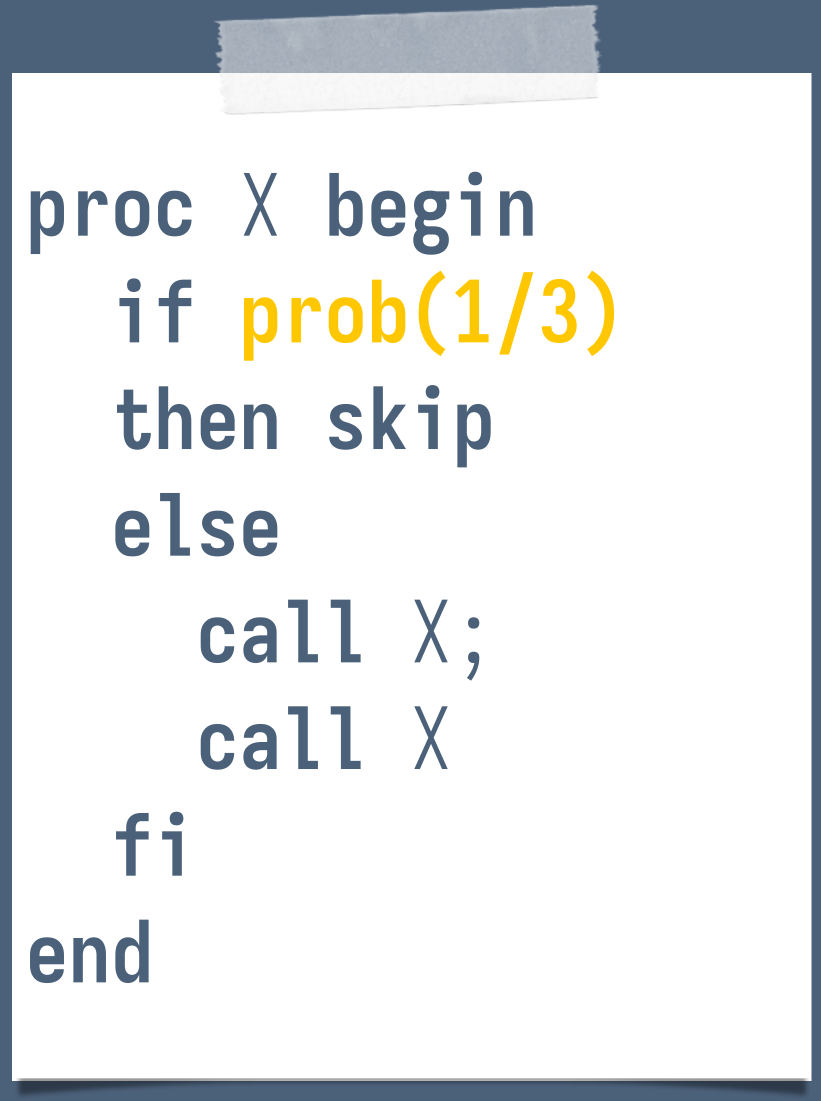


Program → Control-flow graph → System of dataflow equations → Solution (dataflow facts)



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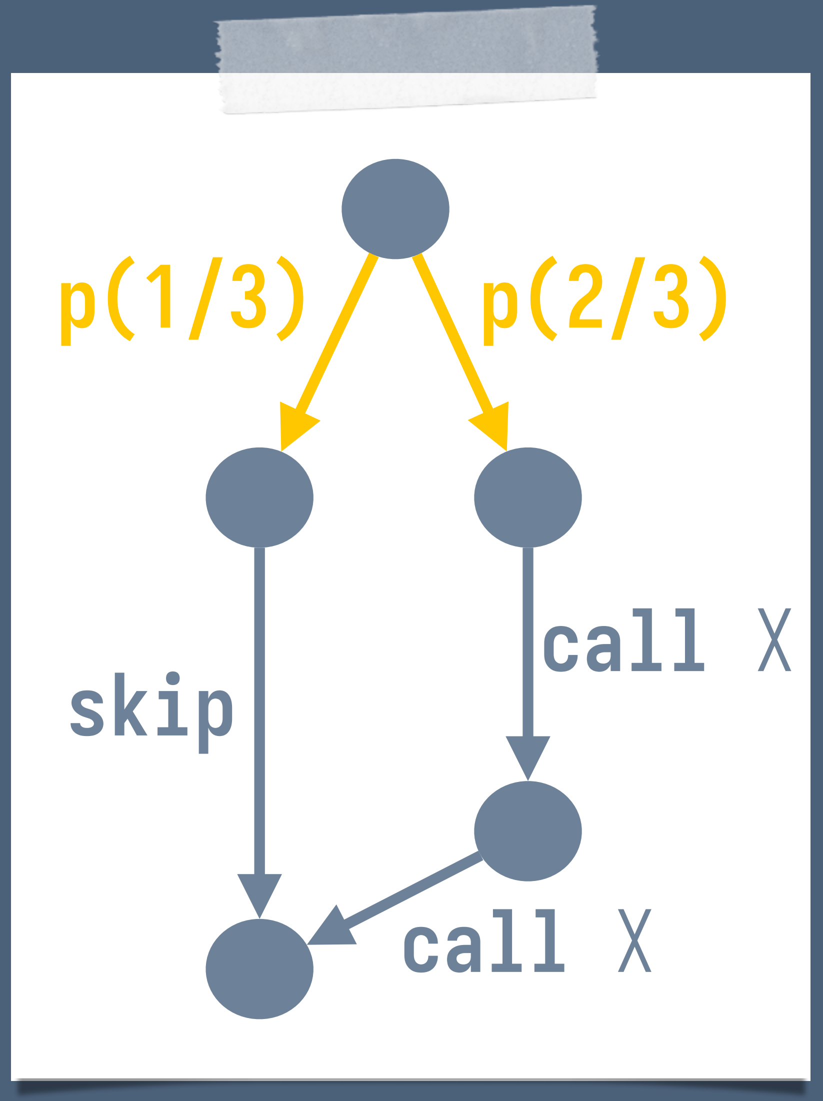

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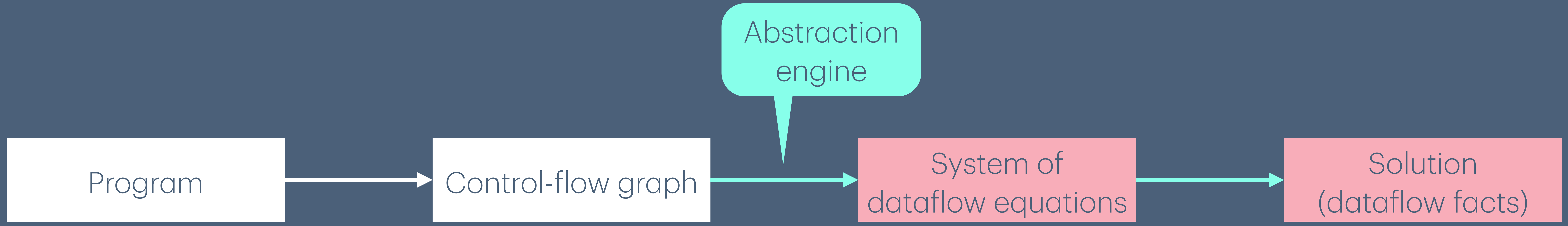

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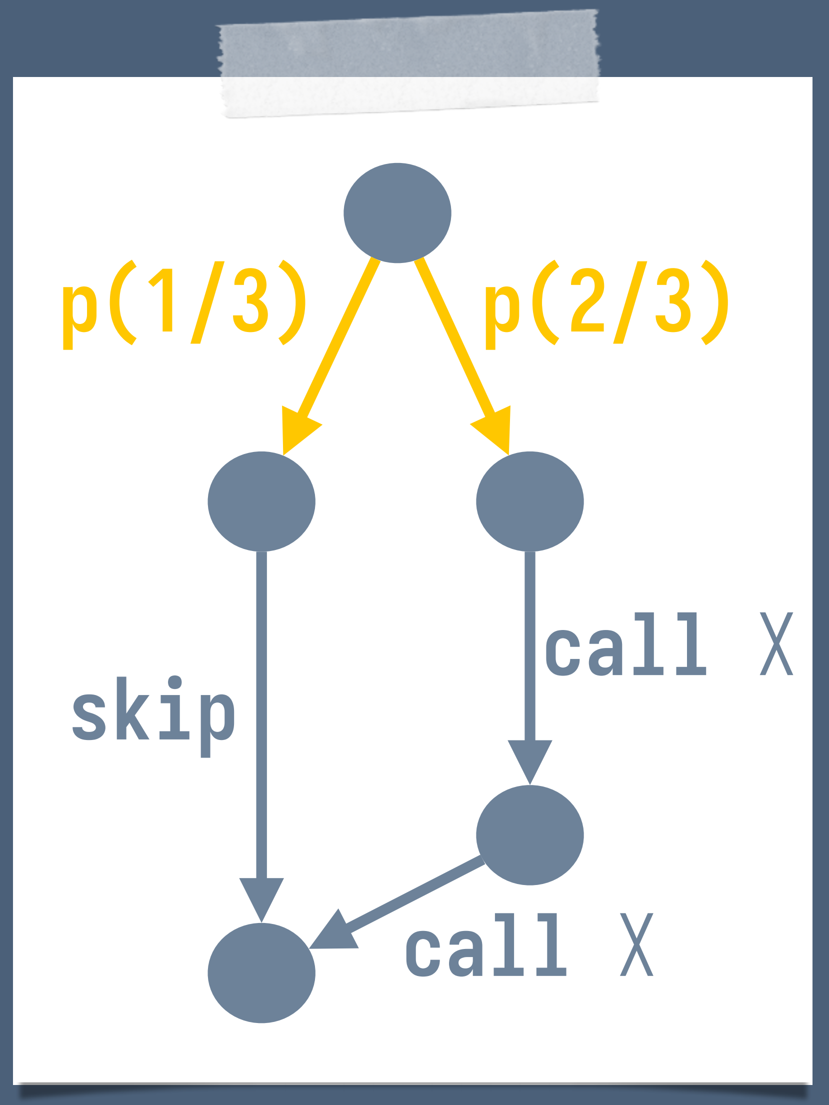

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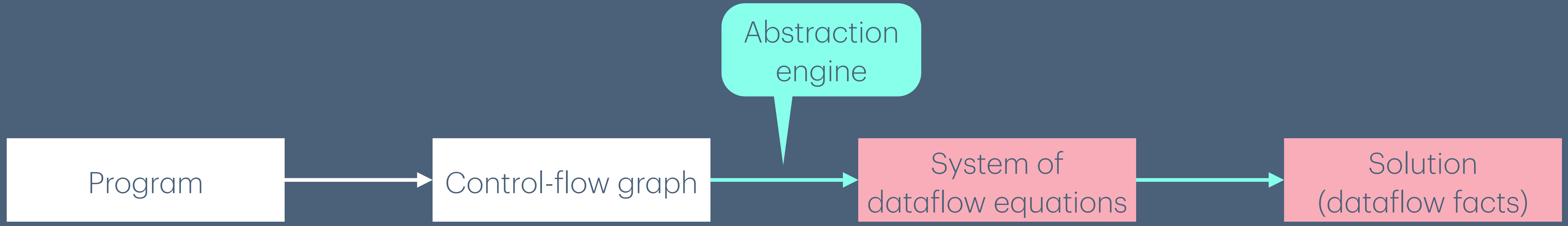

Abstraction engine

Program

Control-flow graph

System of dataflow equations

Solution (dataflow facts)

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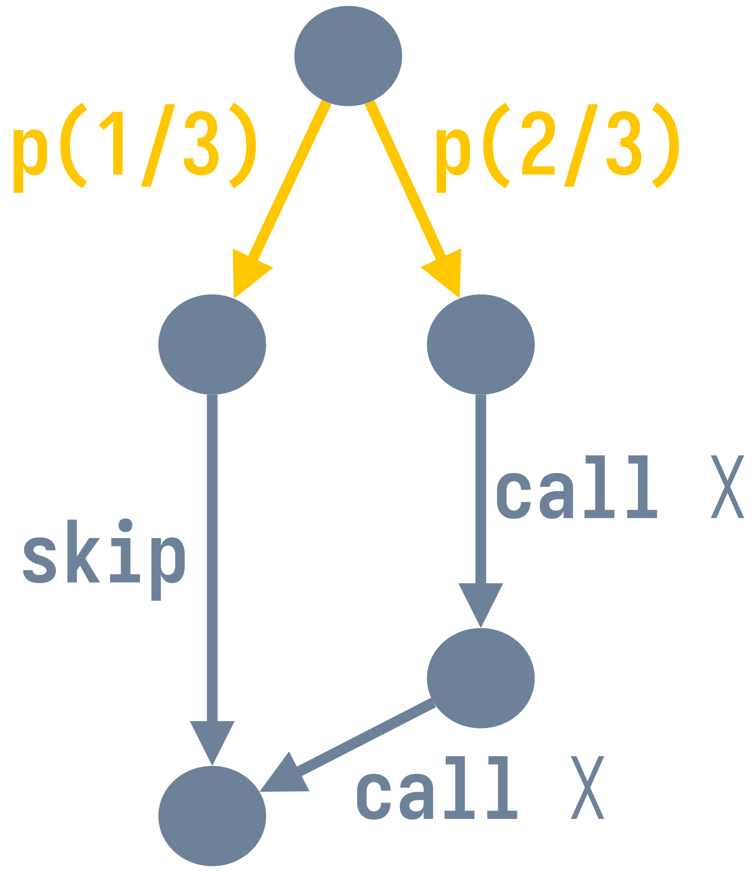

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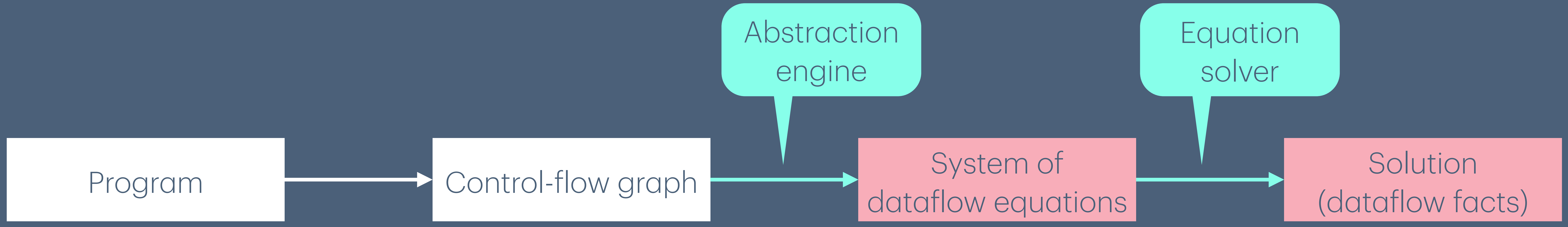

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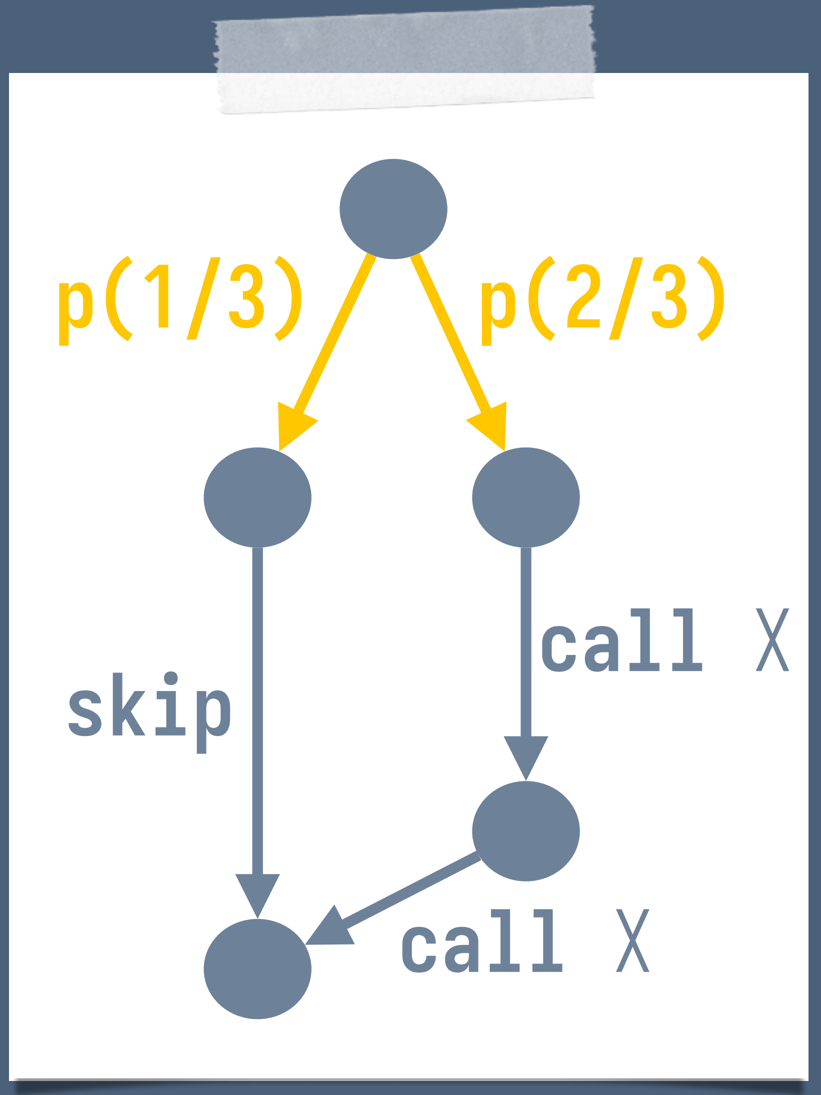

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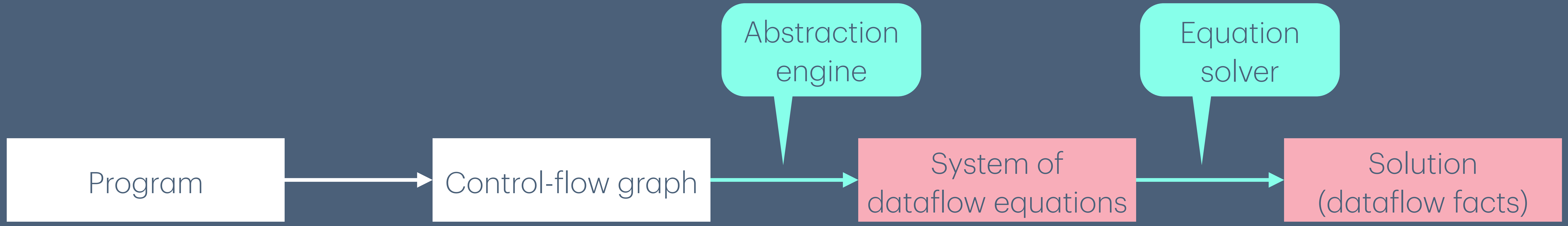

Abstraction engine

Equation solver

Program

Control-flow graph

System of dataflow equations

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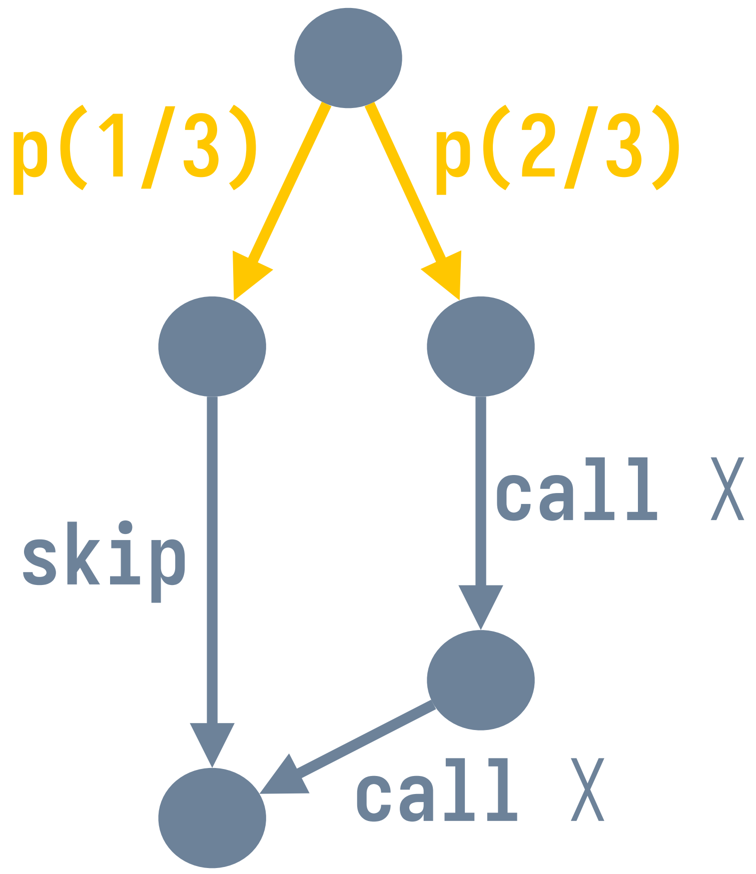

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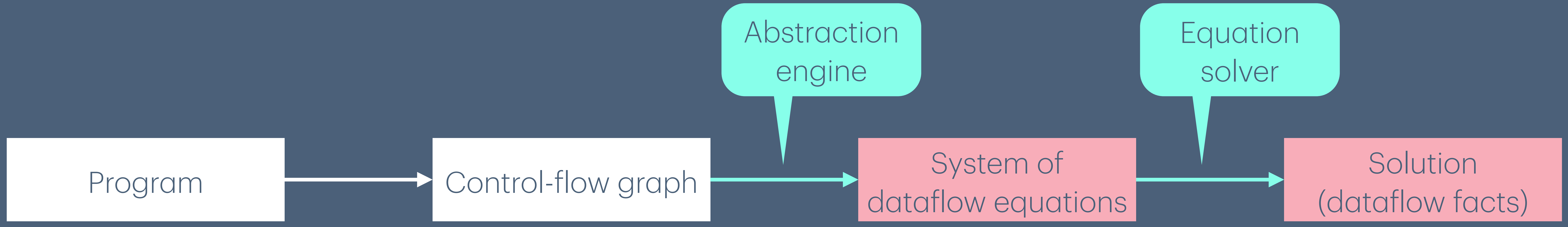

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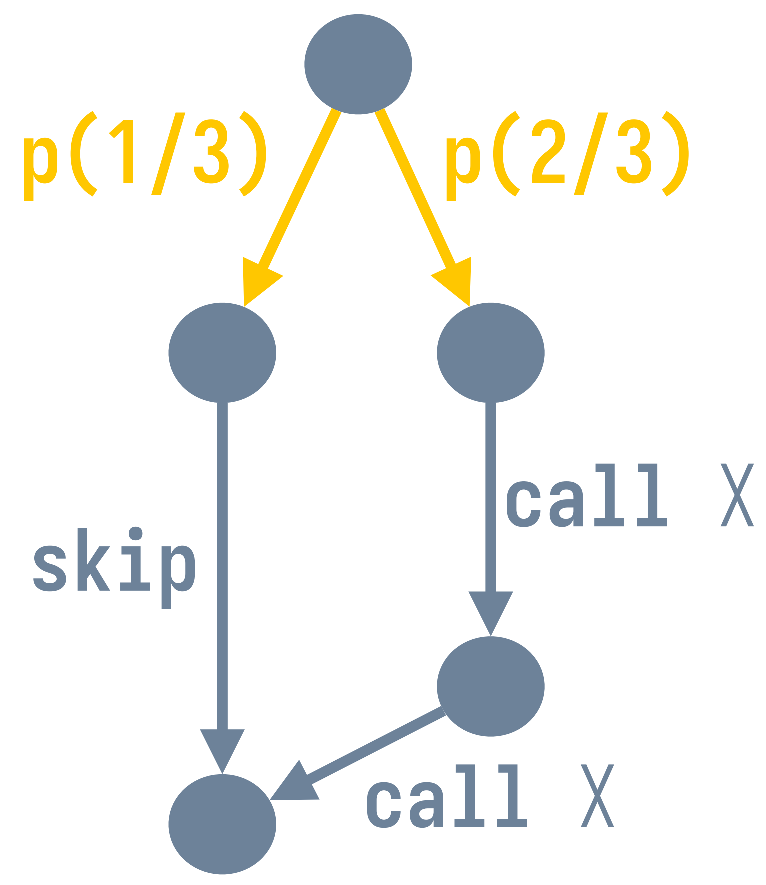

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Also a **procedure summary** for X

The Functional Approach

[Sharir and Pnueli 1981]

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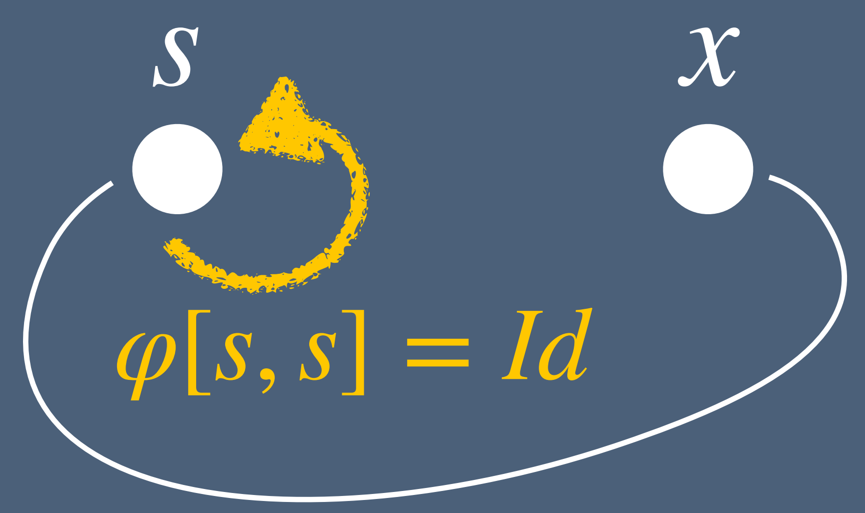

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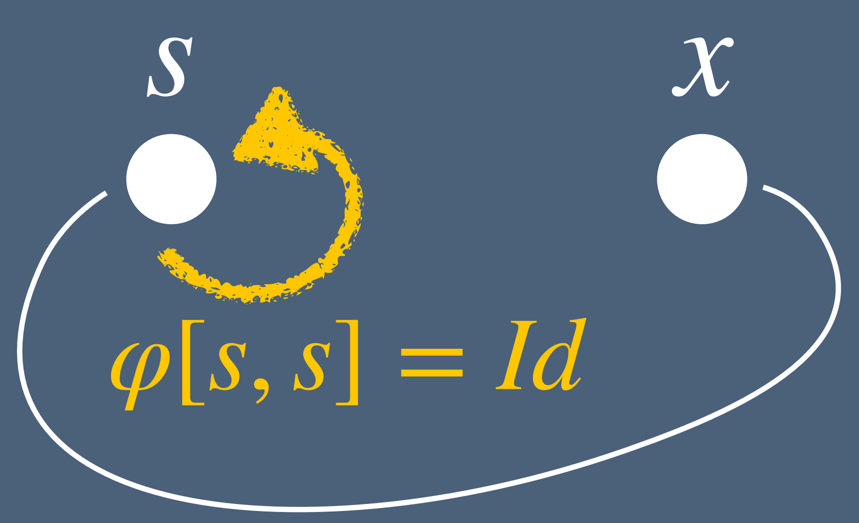

$$\varphi[s, s] = Id$$

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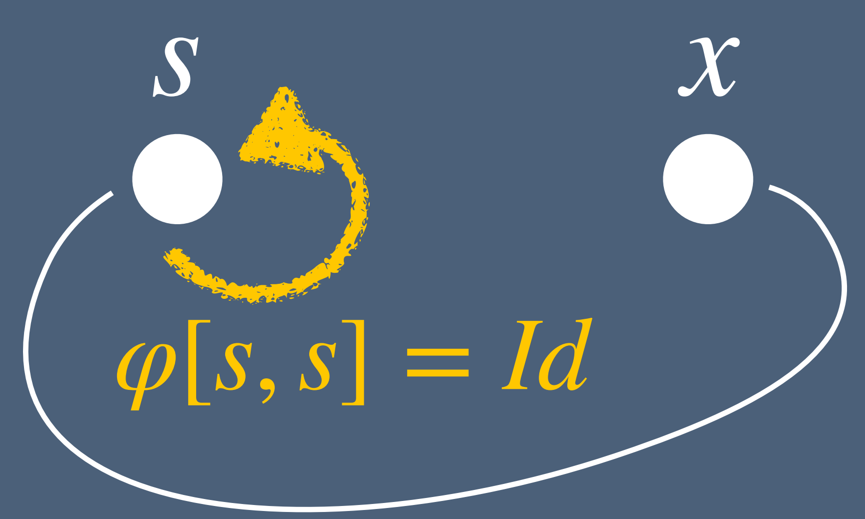

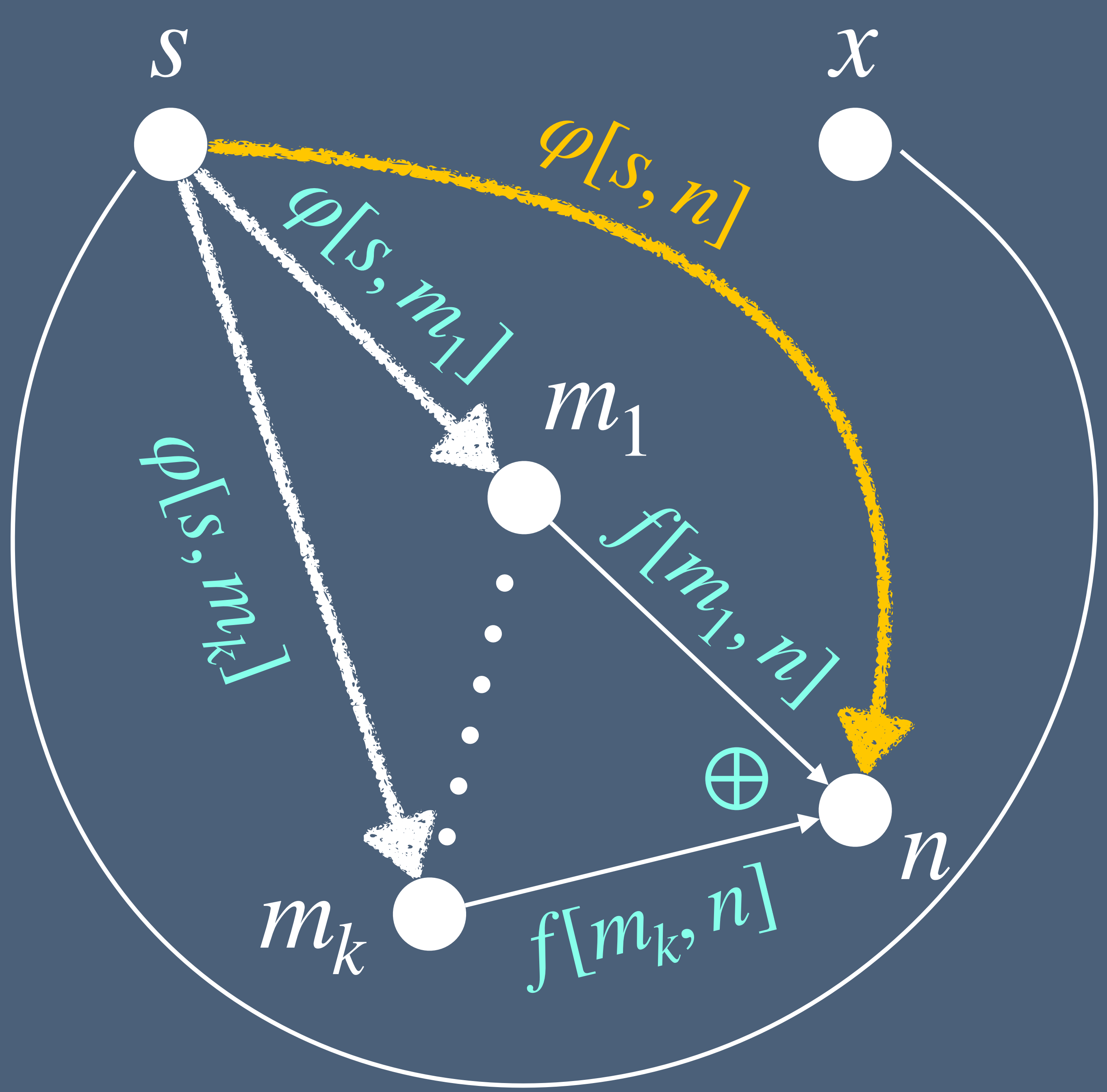

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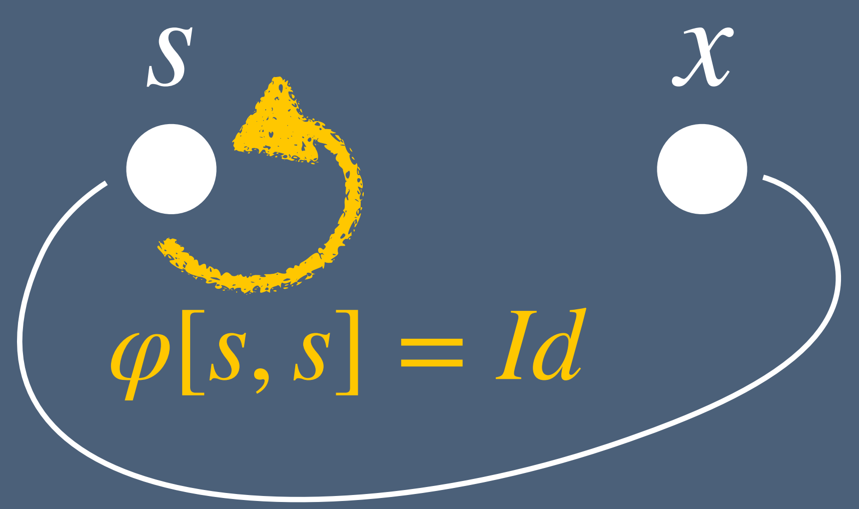

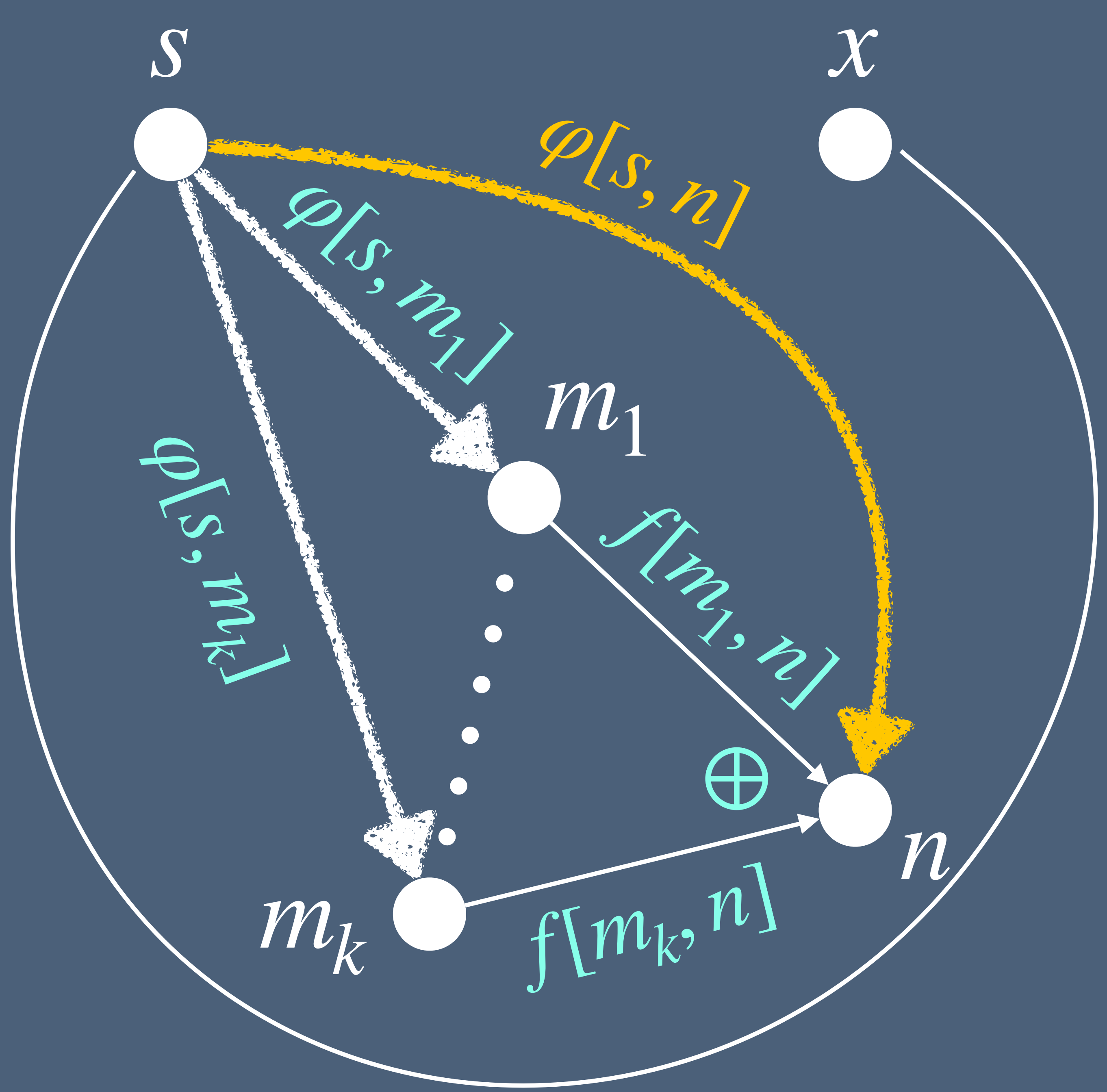

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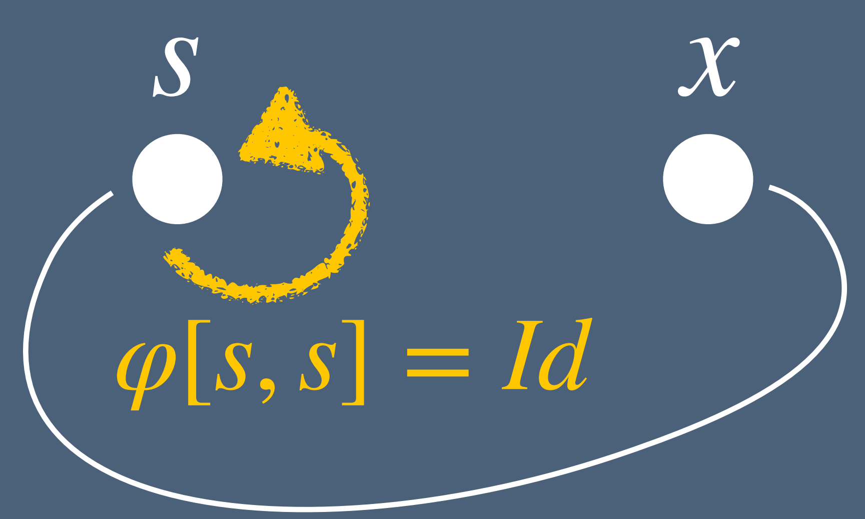

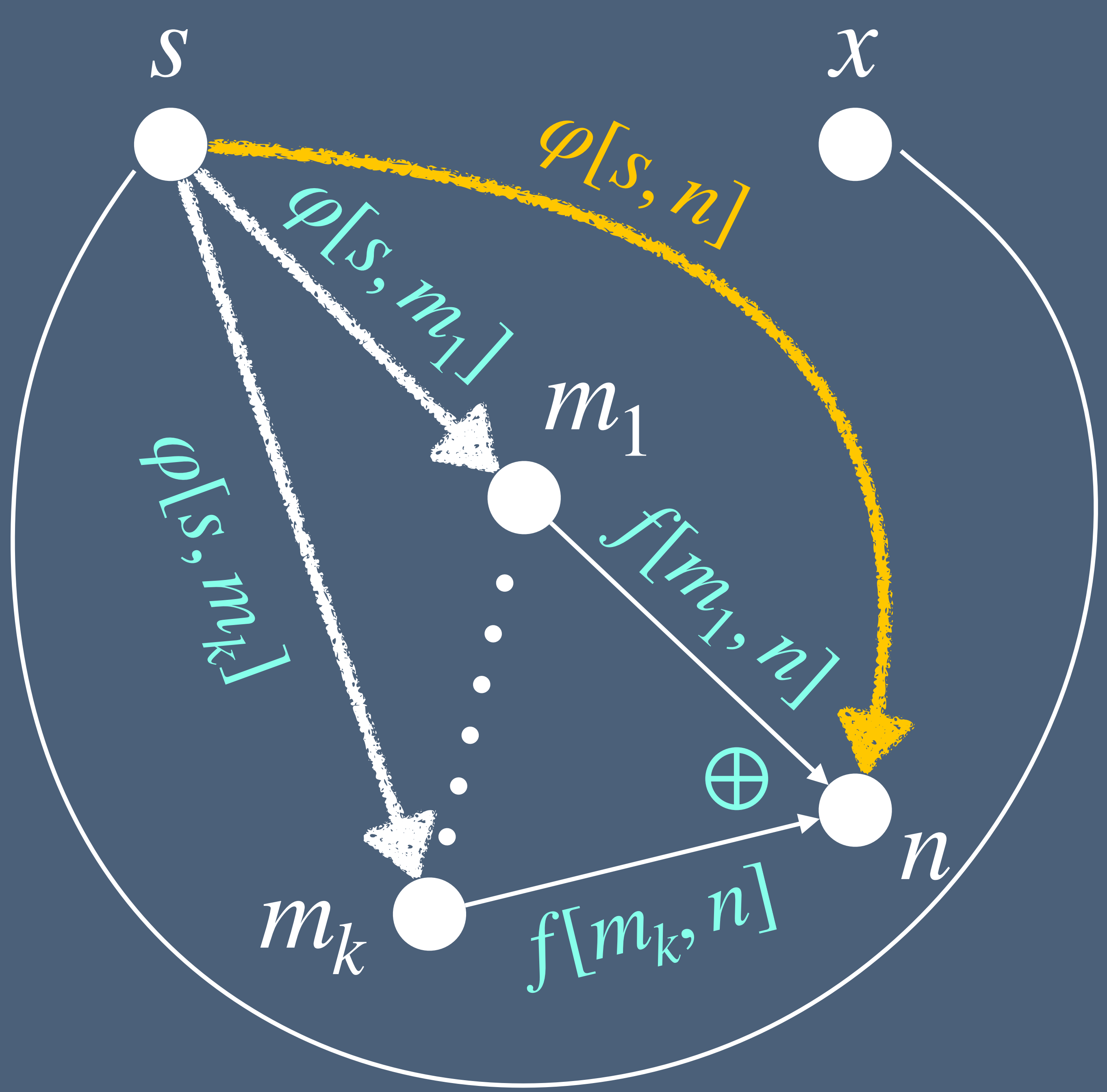

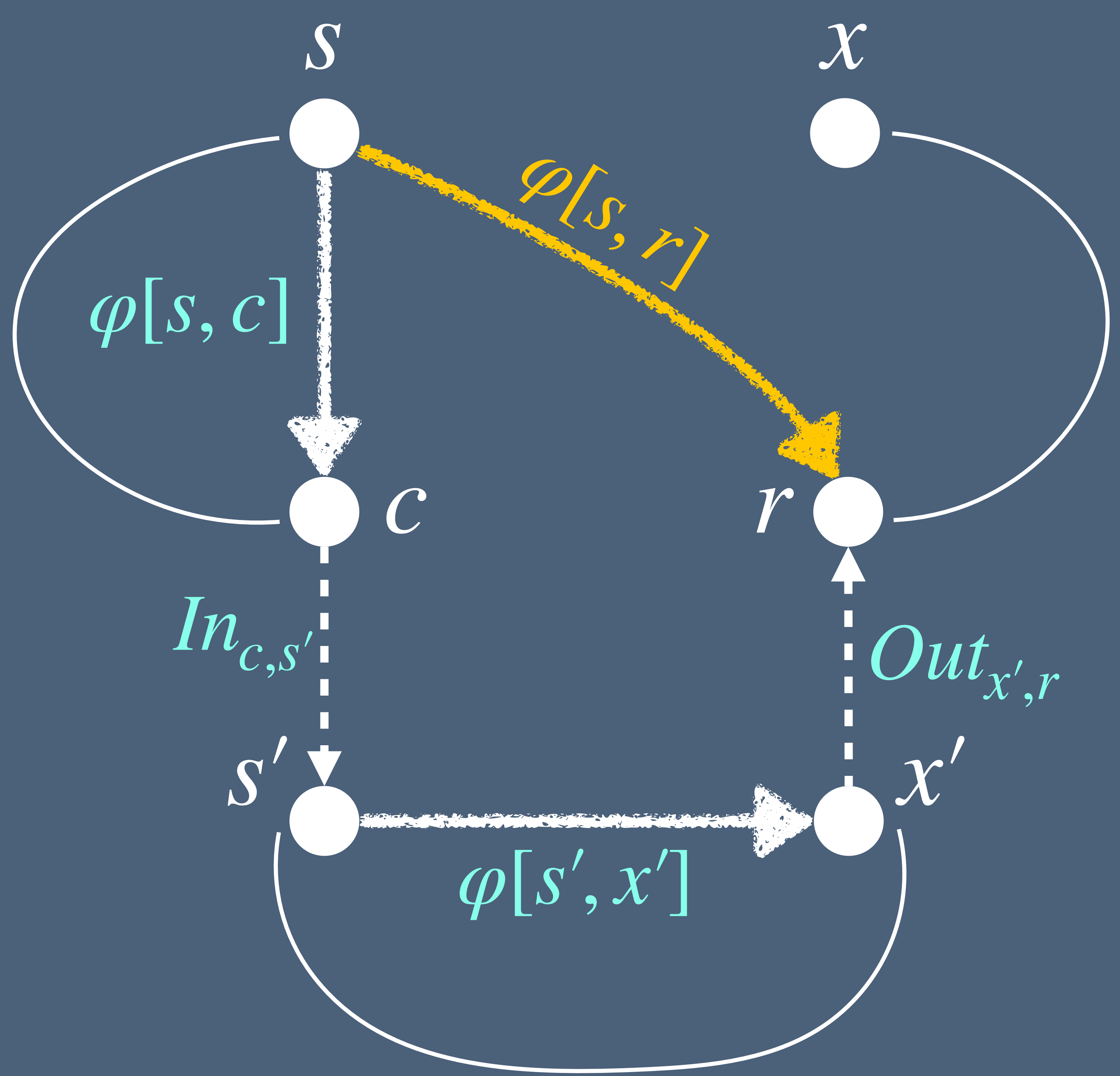

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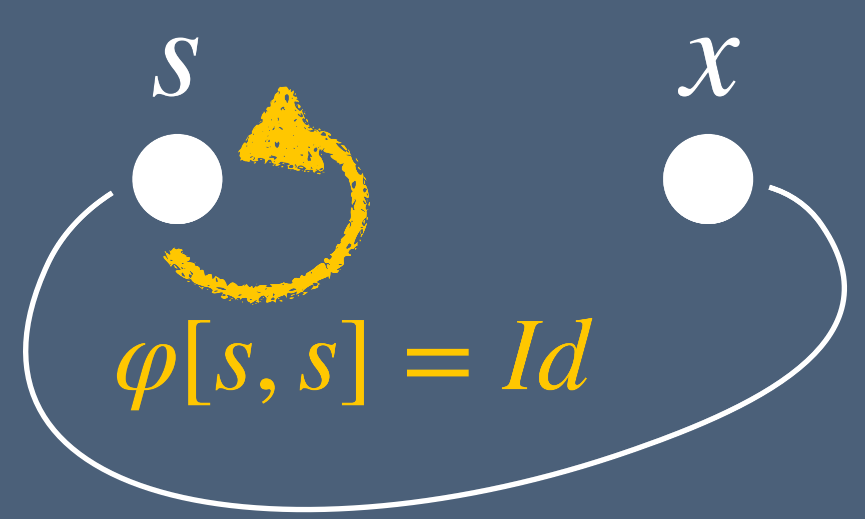

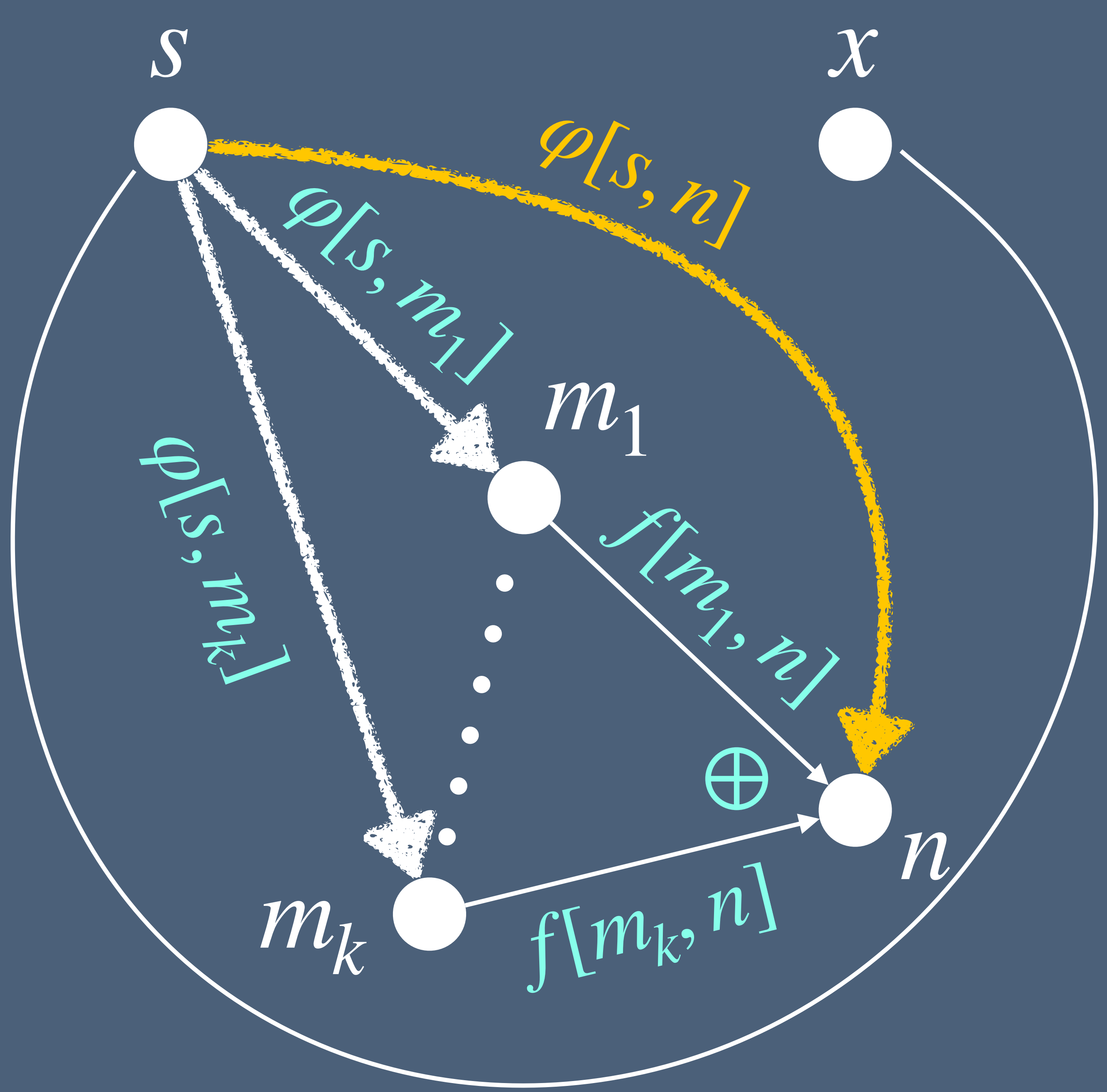

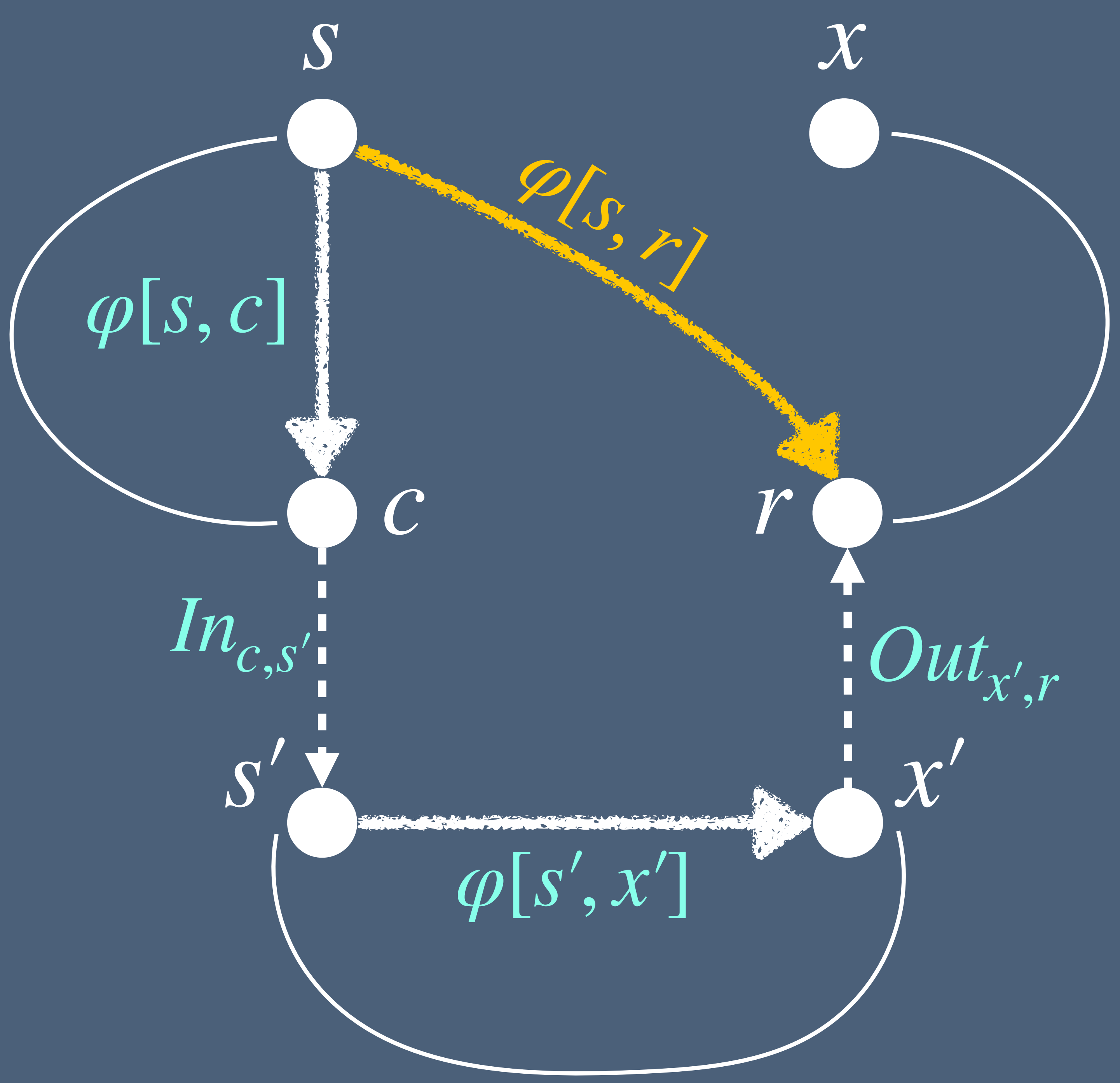

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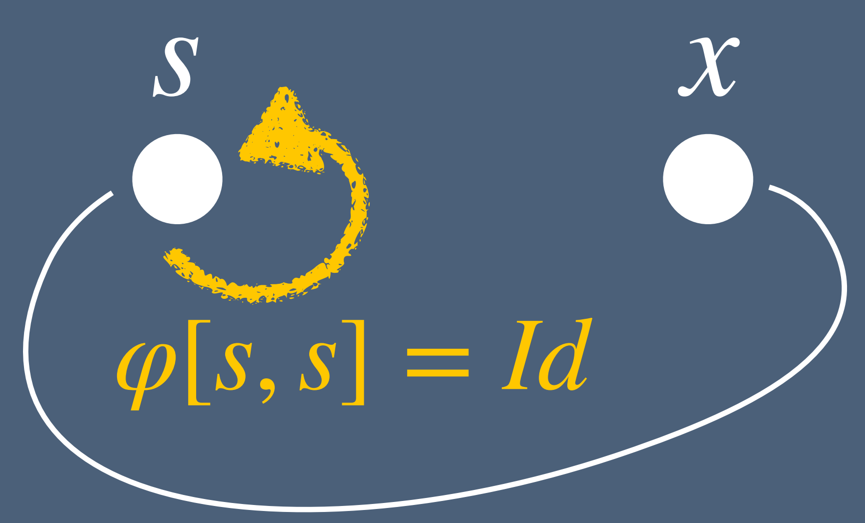

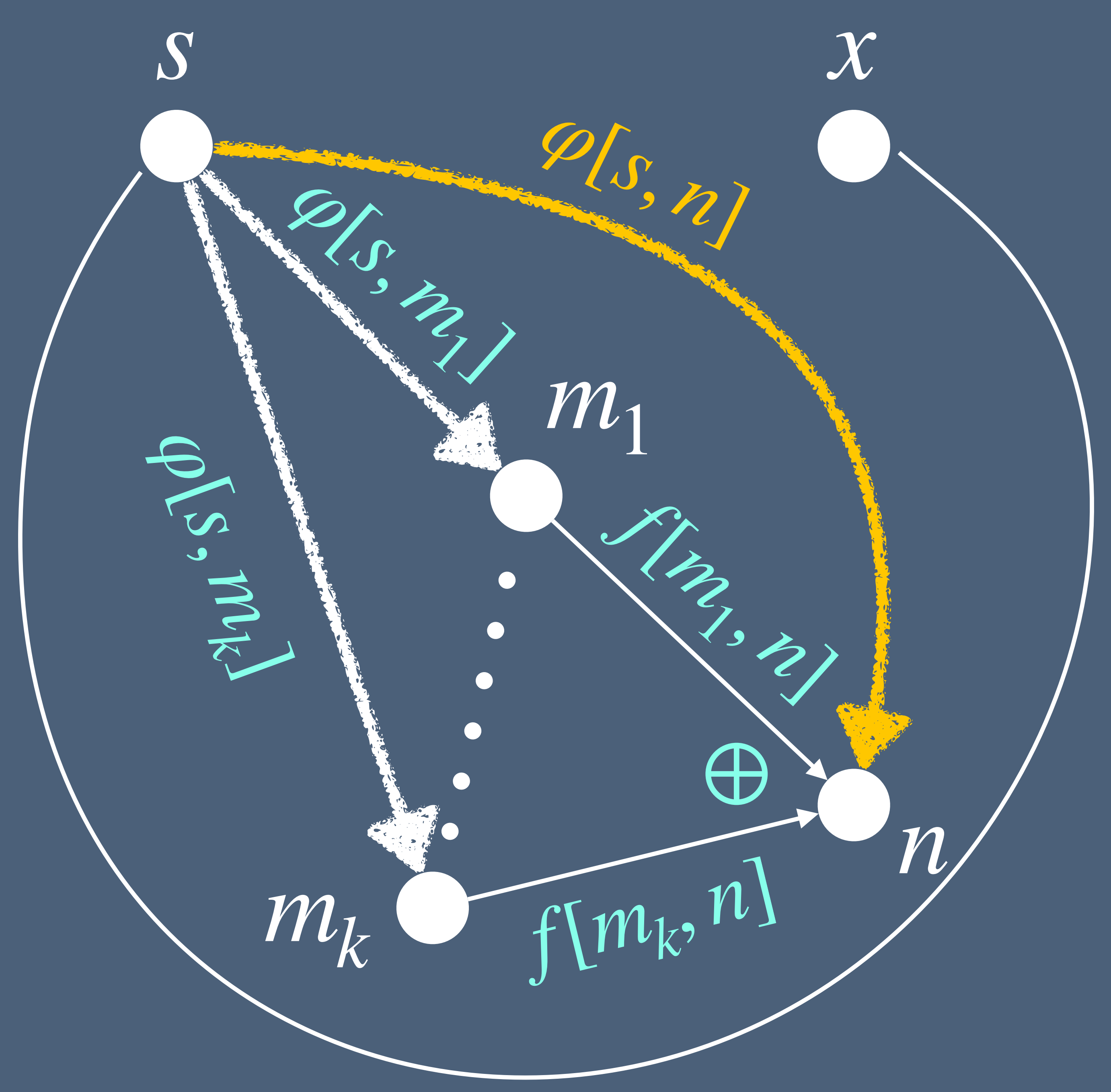

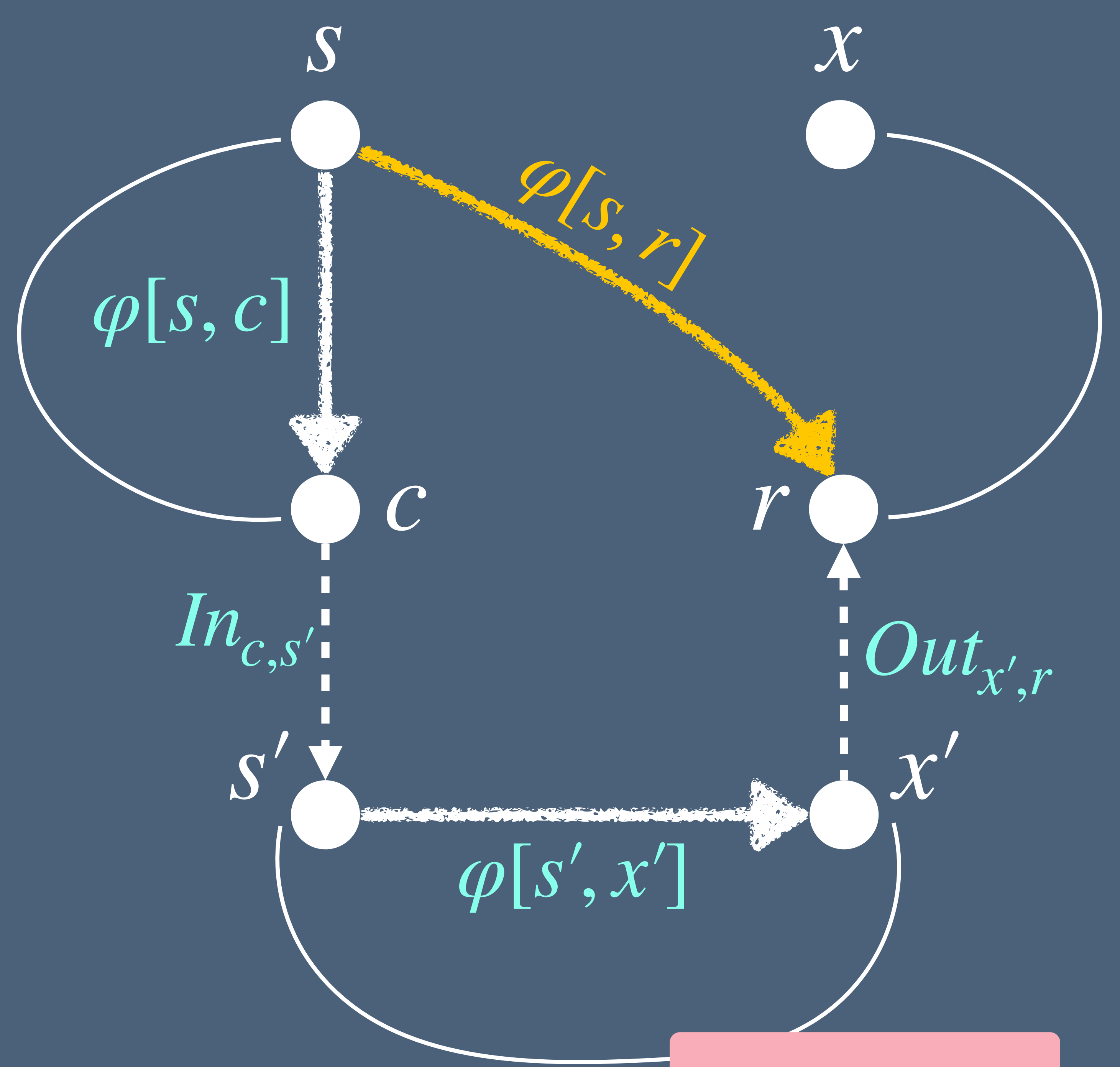

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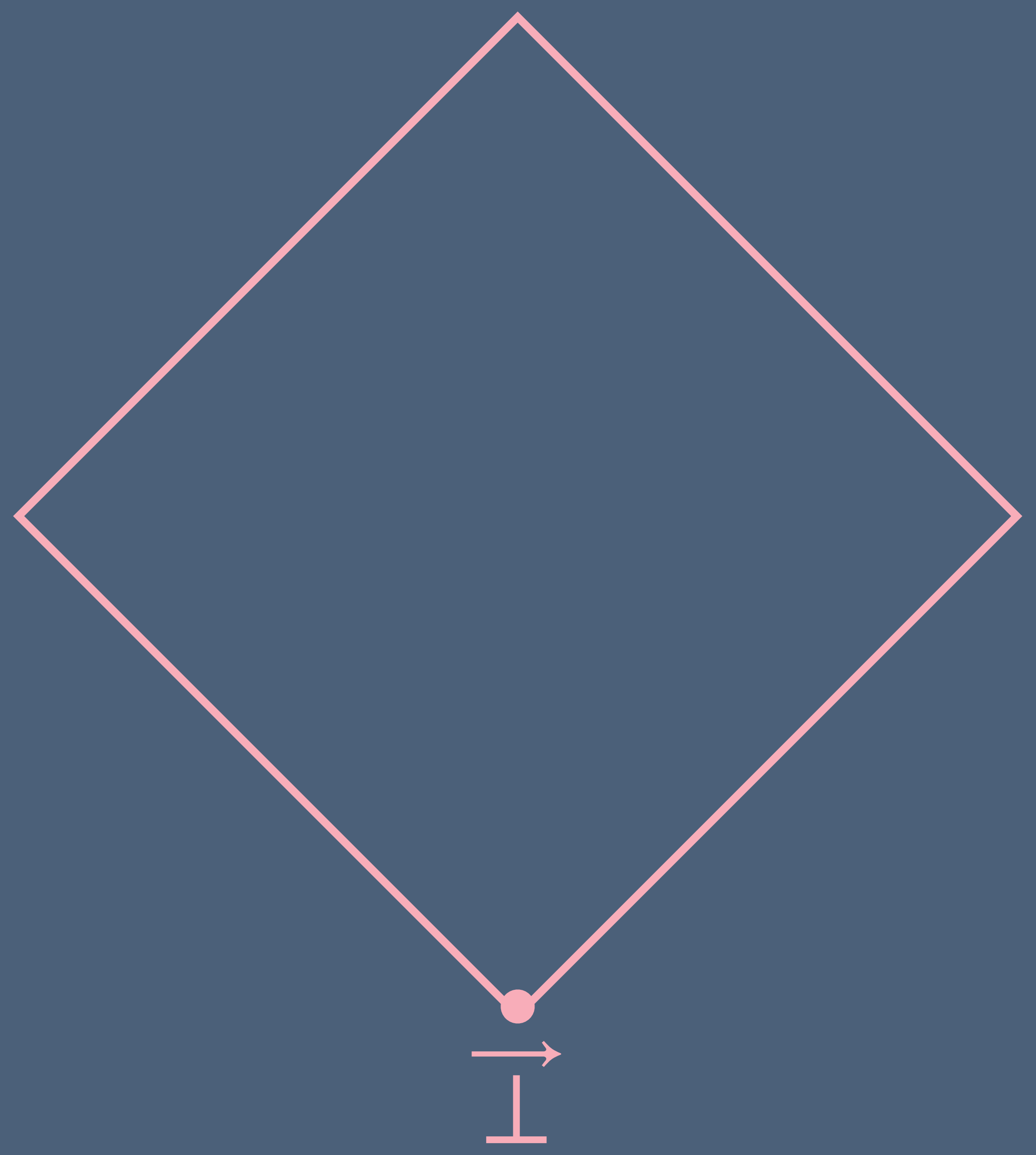

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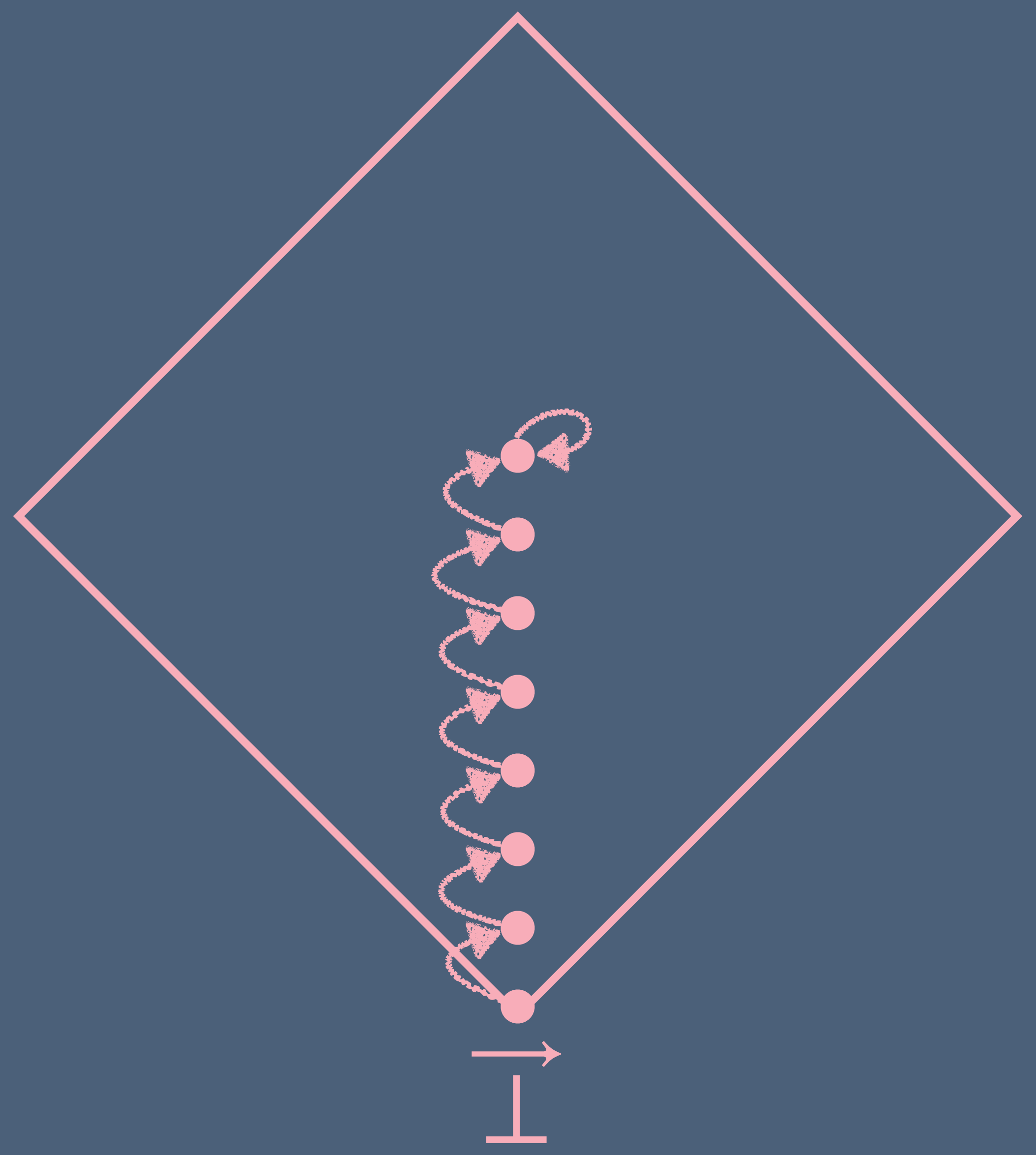

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$$\begin{aligned}
\kappa^{(0)} &= 0 \\
\kappa^{(1)} &= \frac{1}{3} \cdot 1 + \frac{2}{3} \cdot \kappa^{(0)} \cdot \kappa^{(0)} = \frac{1}{3} \\
\kappa^{(2)} &= \frac{1}{3} \cdot 1 + \frac{2}{3} \cdot \kappa^{(1)} \cdot \kappa^{(1)} = \frac{11}{27} \\
&\vdots \\
\kappa^{(\infty)} &= \frac{1}{2}
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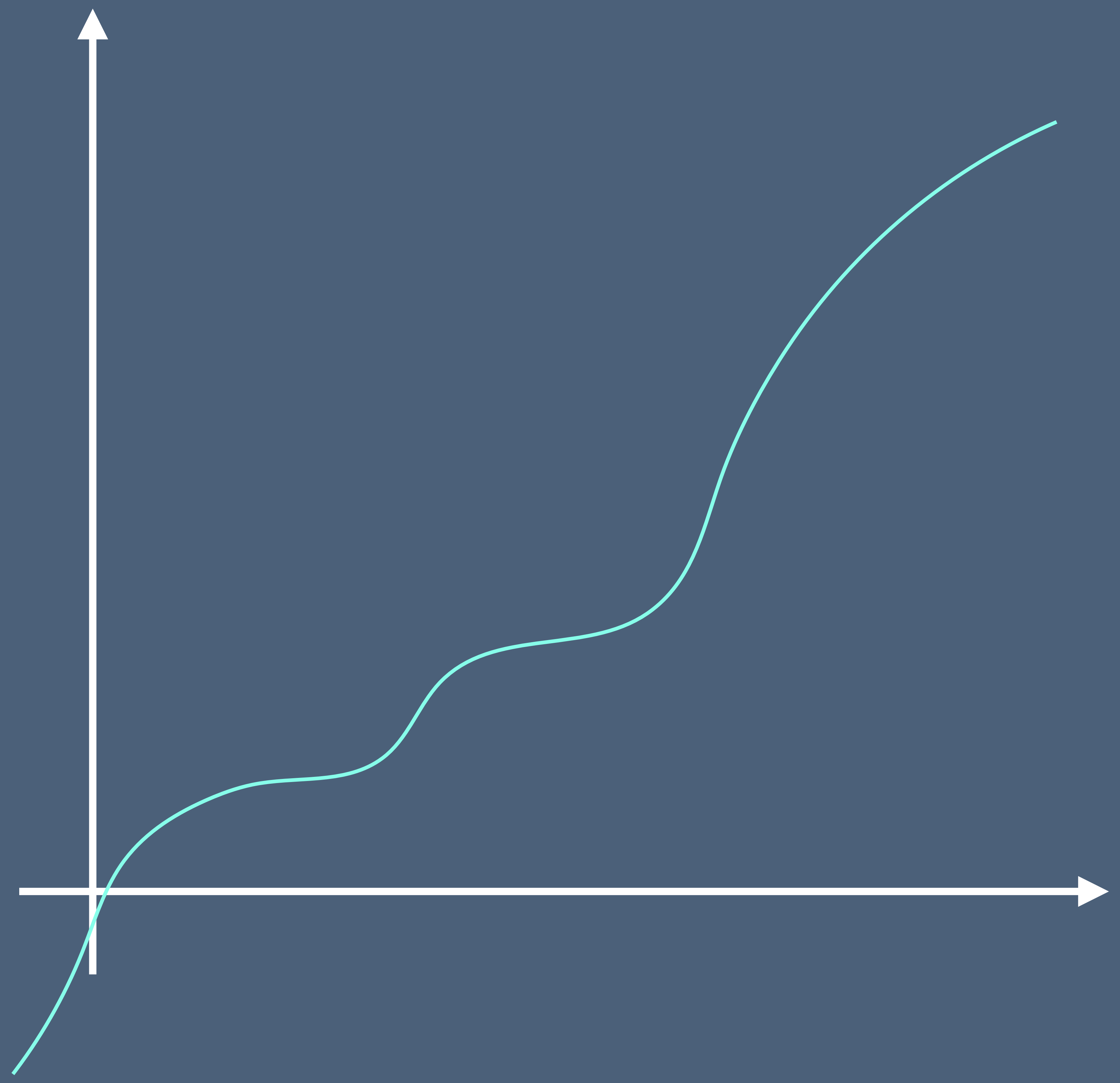

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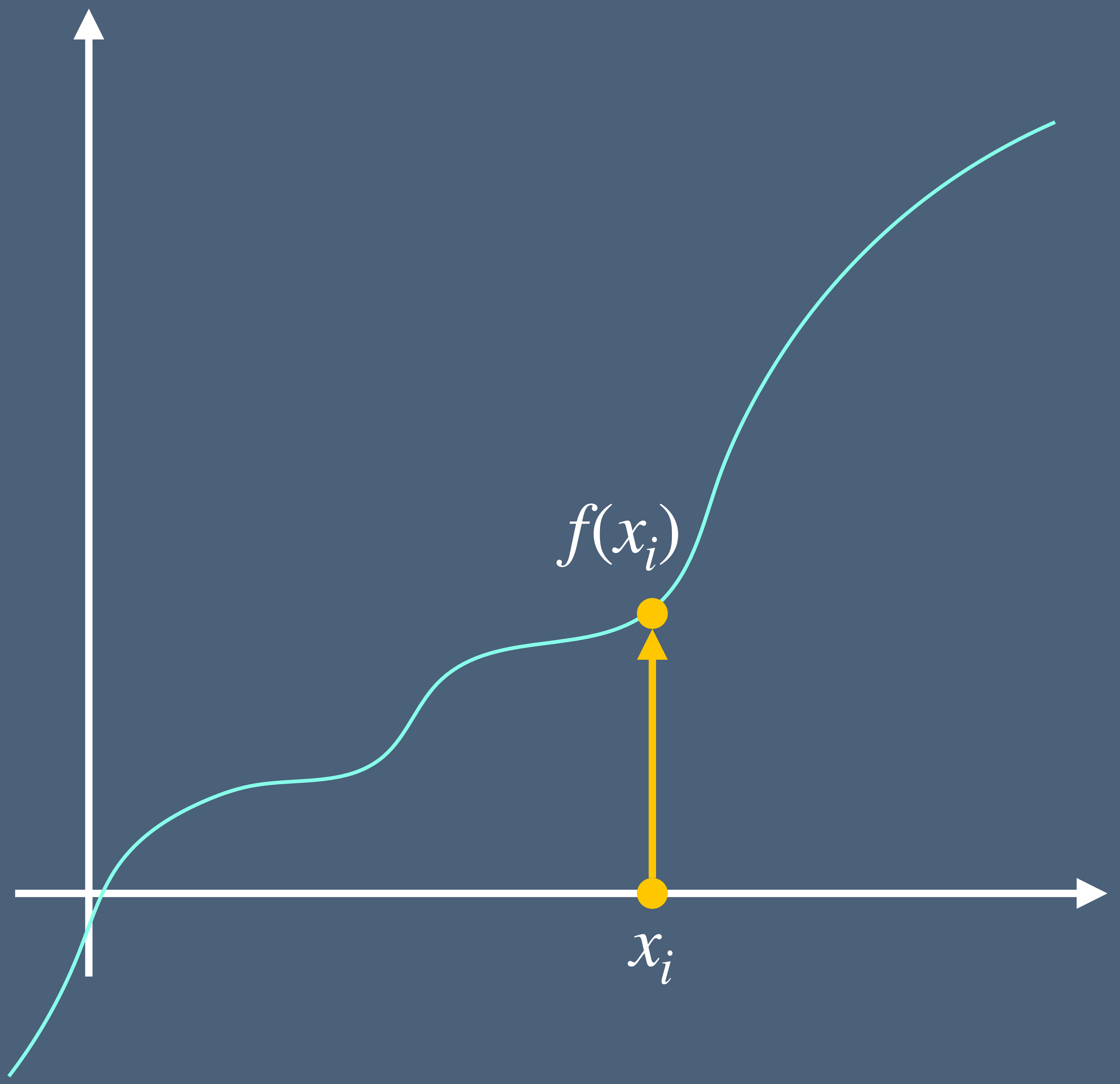

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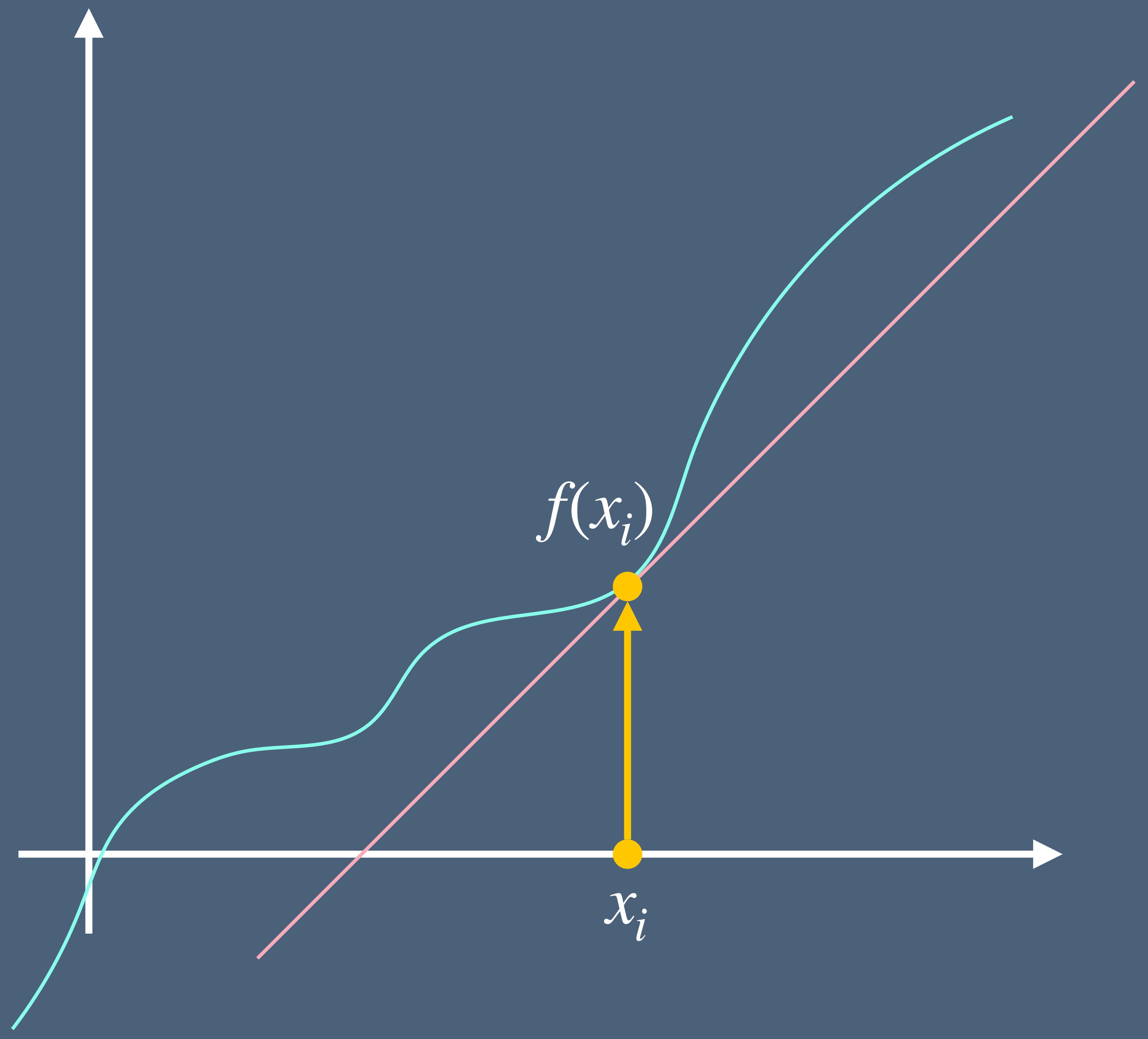

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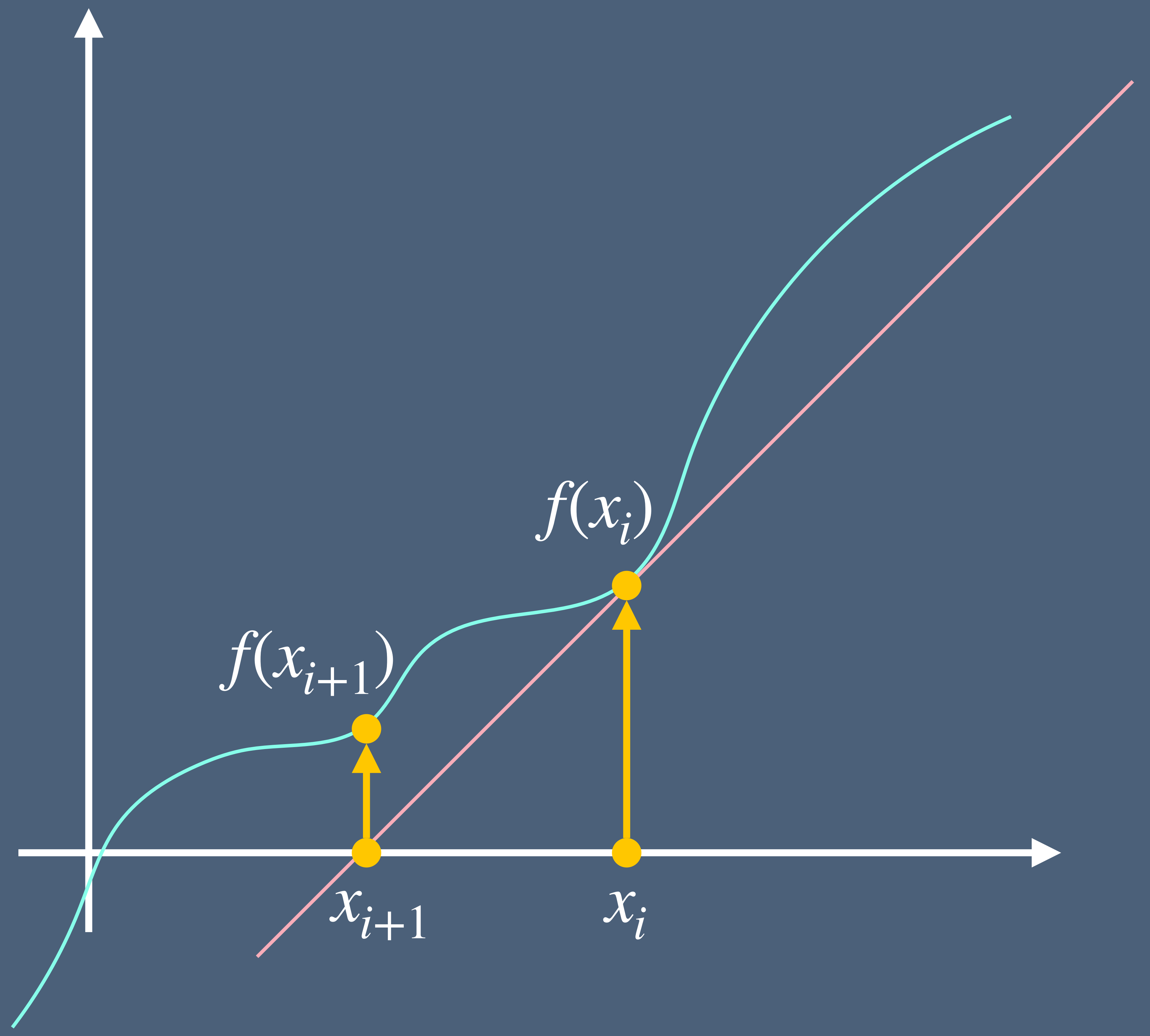

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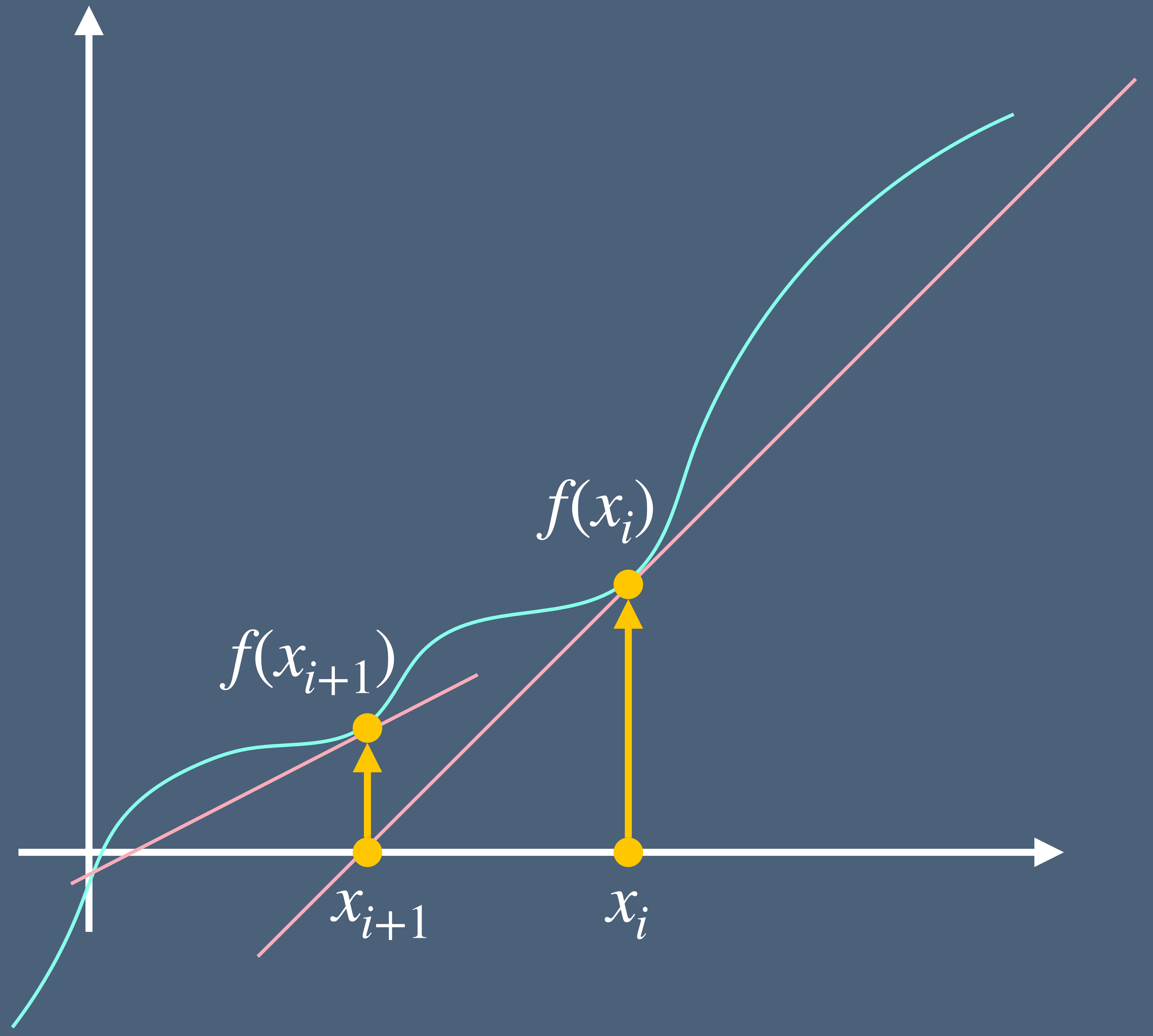

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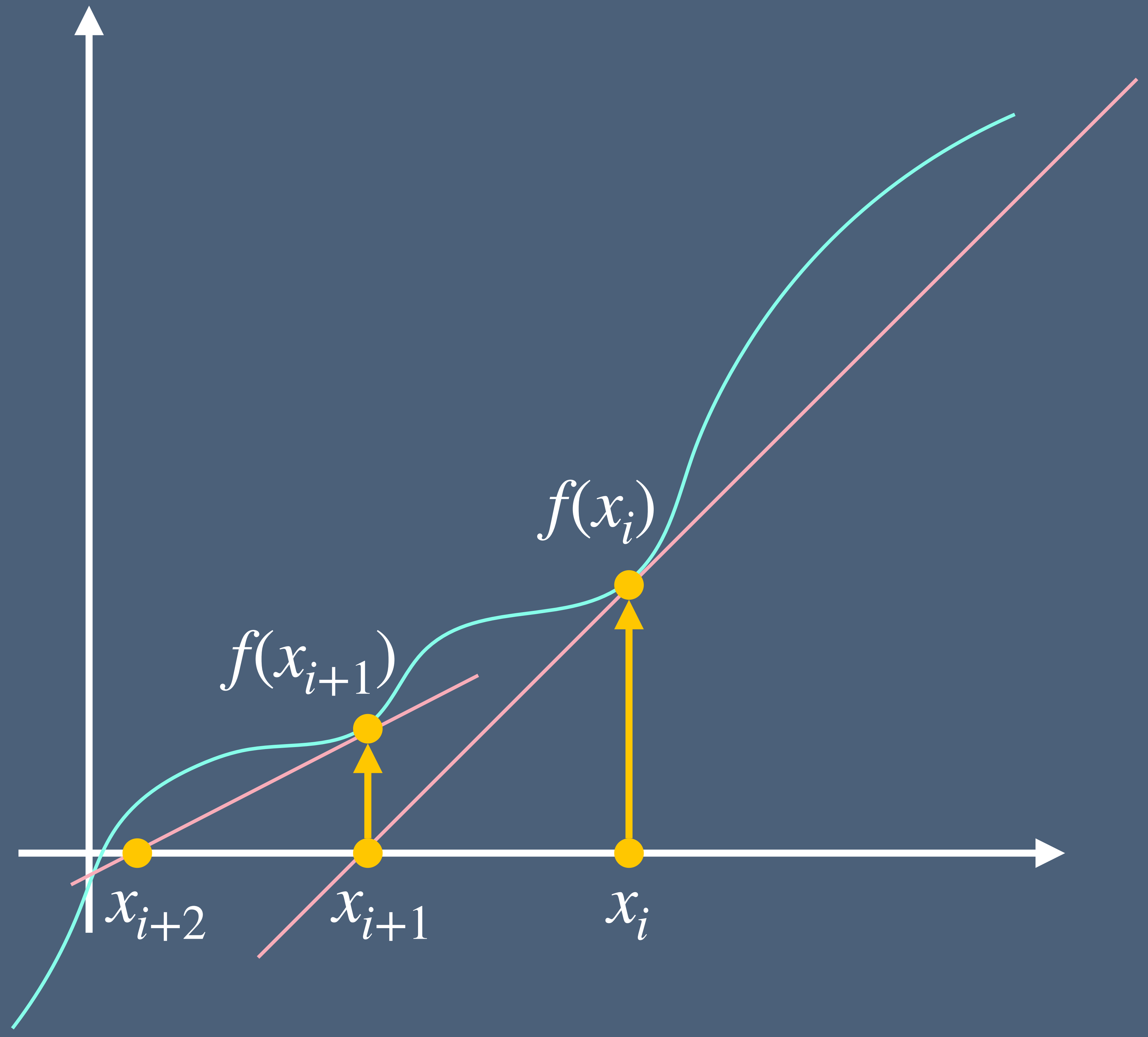

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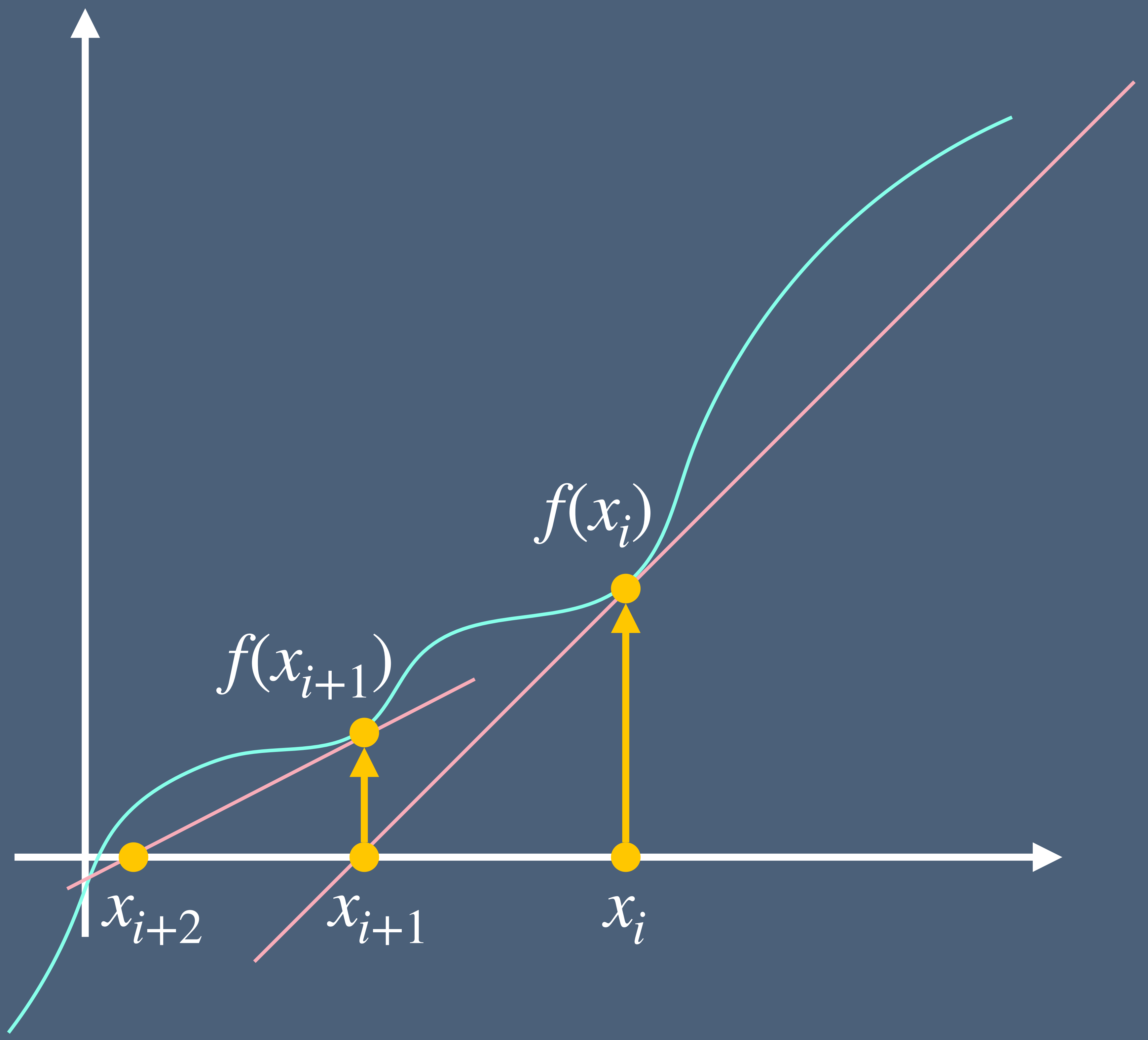
- A way to find **successively** better approximations of a root a function

- Given a function f, its derivative f' and an initial x_0, repeatedly perform $x_{i+1} = x_i - \dfrac{f(x_i)}{f'(x_i)}$



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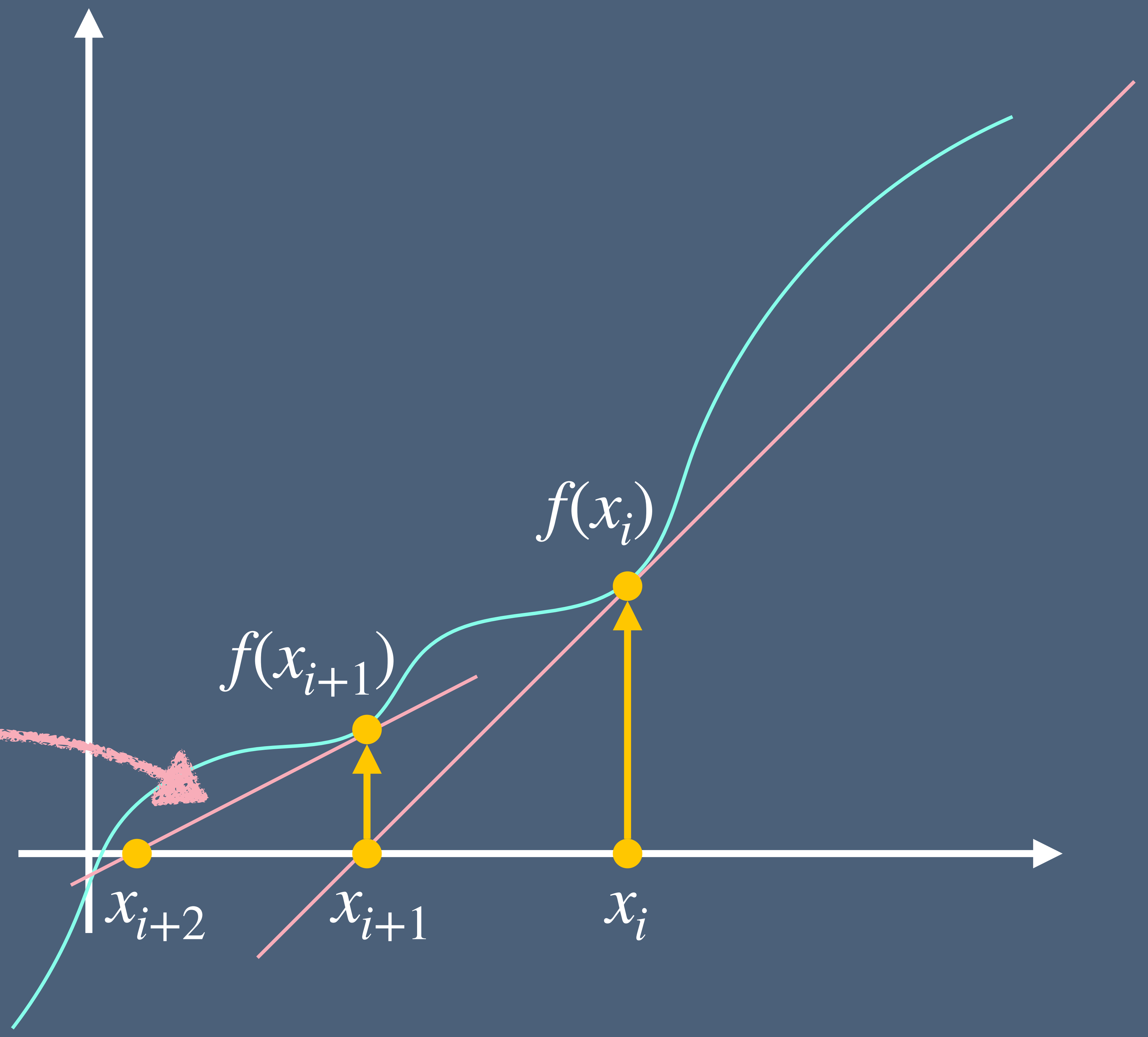
Create a **linear model** to find a better approximation



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$$f(X) = \frac{1}{3} \cdot 1 + \frac{2}{3} \cdot X \cdot X - X$$

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\nu^{(1)} &= \frac{2\nu^{(0)^2} - 1}{4\nu^{(0)} - 3} = \frac{1}{3} \\
\nu^{(2)} &= \frac{2\nu^{(1)^2} - 1}{4\nu^{(1)} - 3} = \frac{7}{15} \\
&\vdots \\
\nu^{(\infty)} &= \frac{1}{2}
\end{aligned}$$

Kleene vs Newton

which converges faster?

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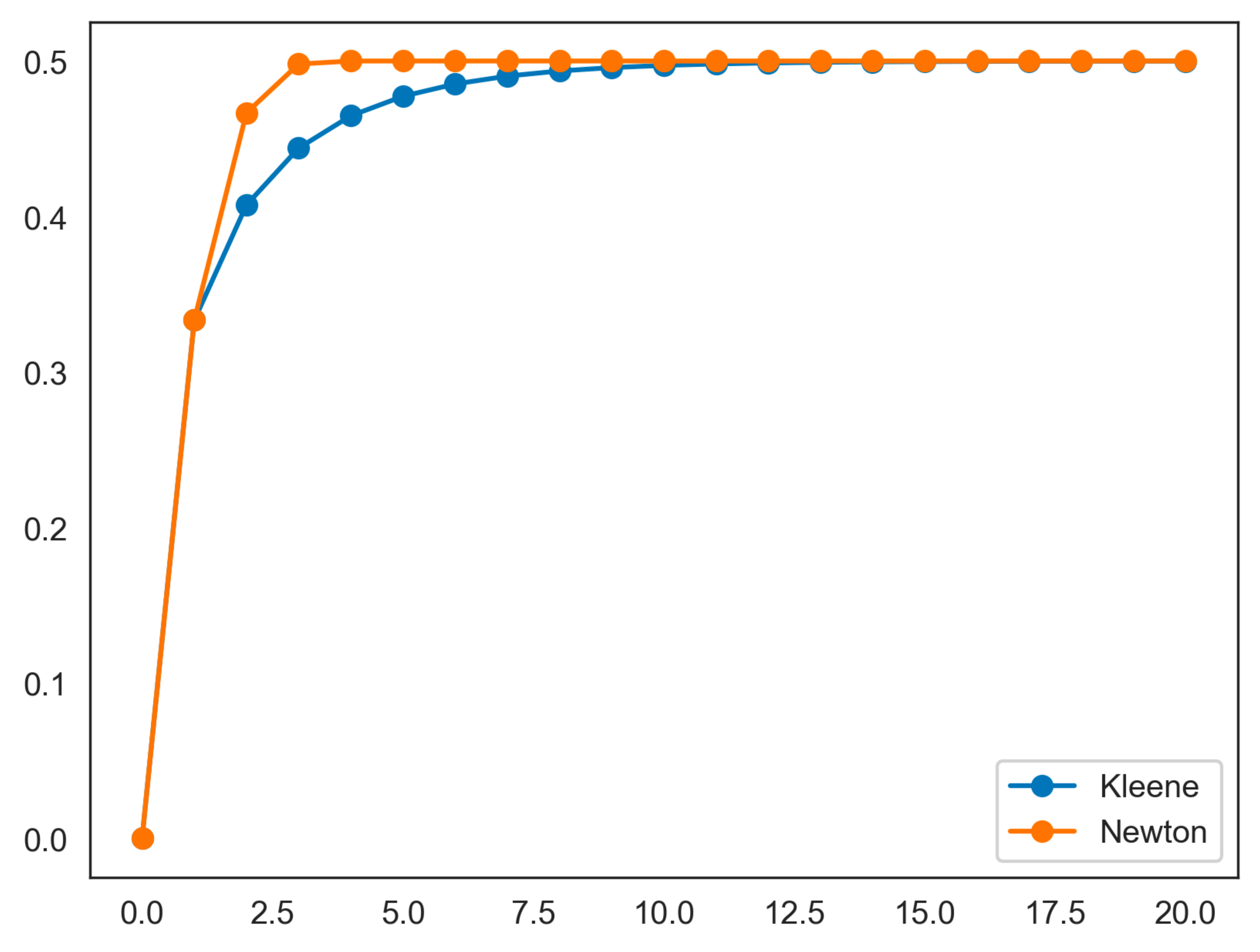

Newton's Method for Program Analysis

[Esparza, Kiefer, and Luttenberger 2008]

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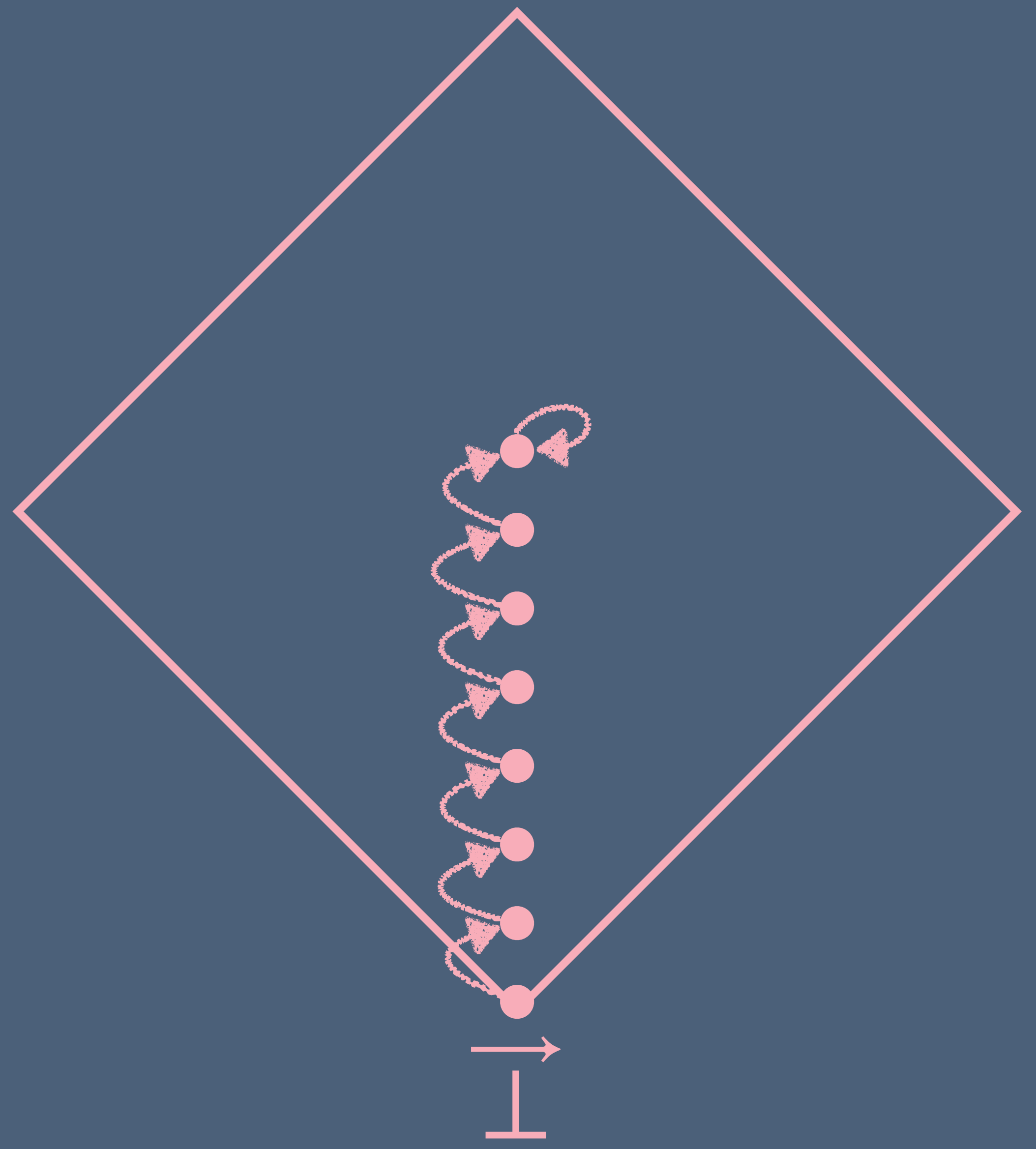

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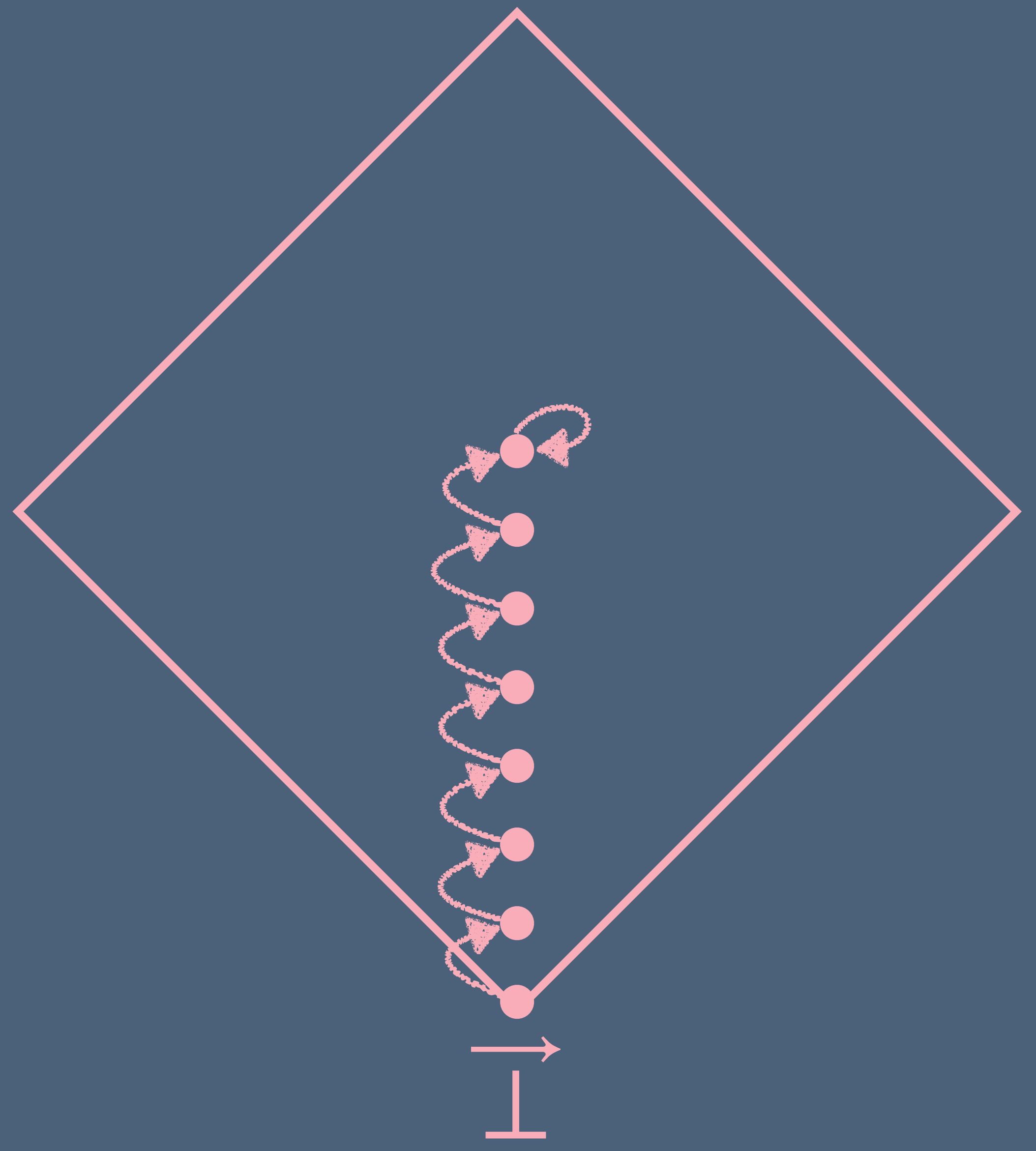

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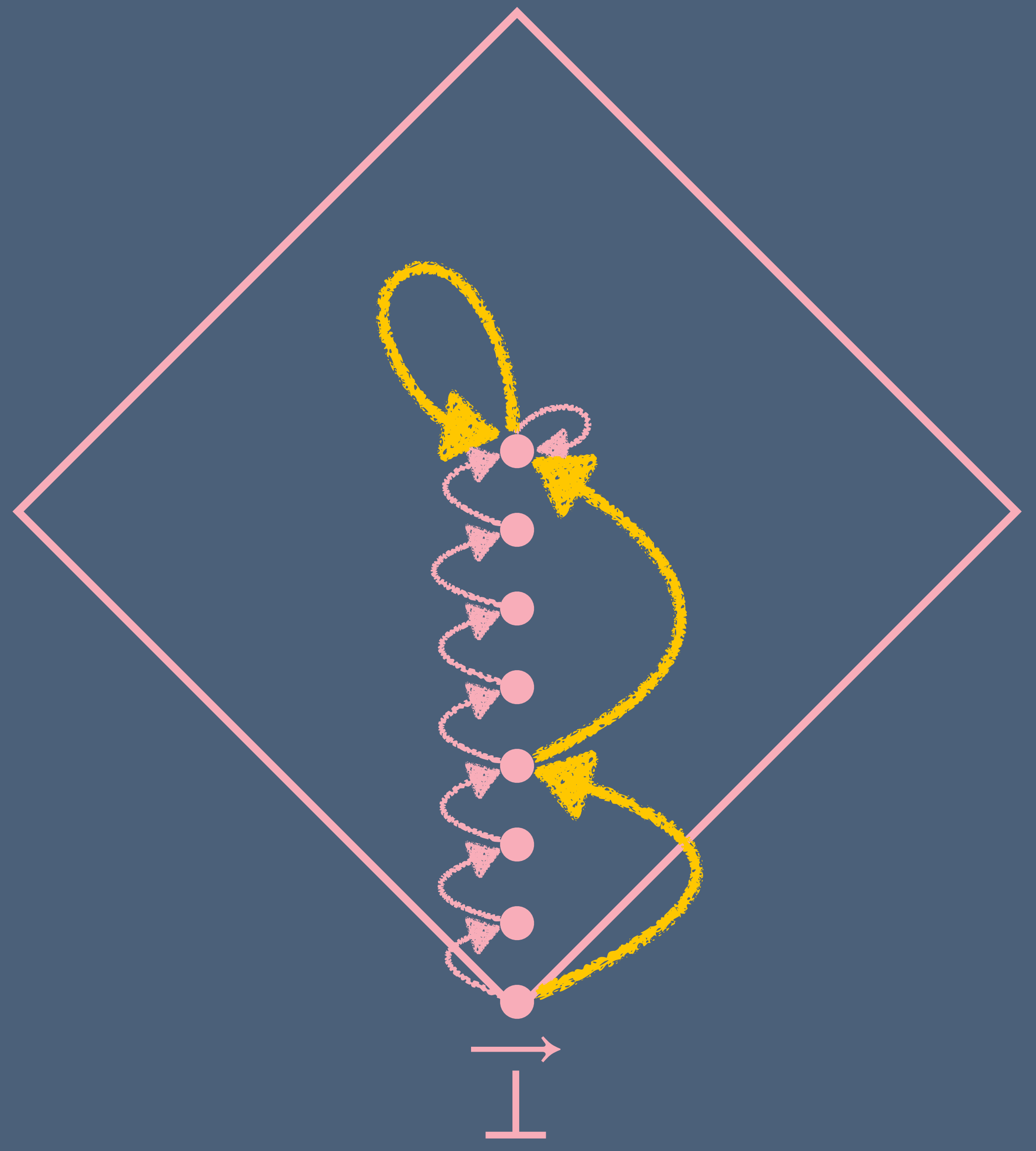

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Semiring constant $\underline{0}$

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$$b \otimes X \otimes X \otimes c \xrightarrow{\nu} (b \otimes Y \otimes \nu \otimes c) \oplus (b \otimes \nu \otimes Y \otimes c)$$

Termination-Probability Analysis

via Newton's Method for Program Analysis

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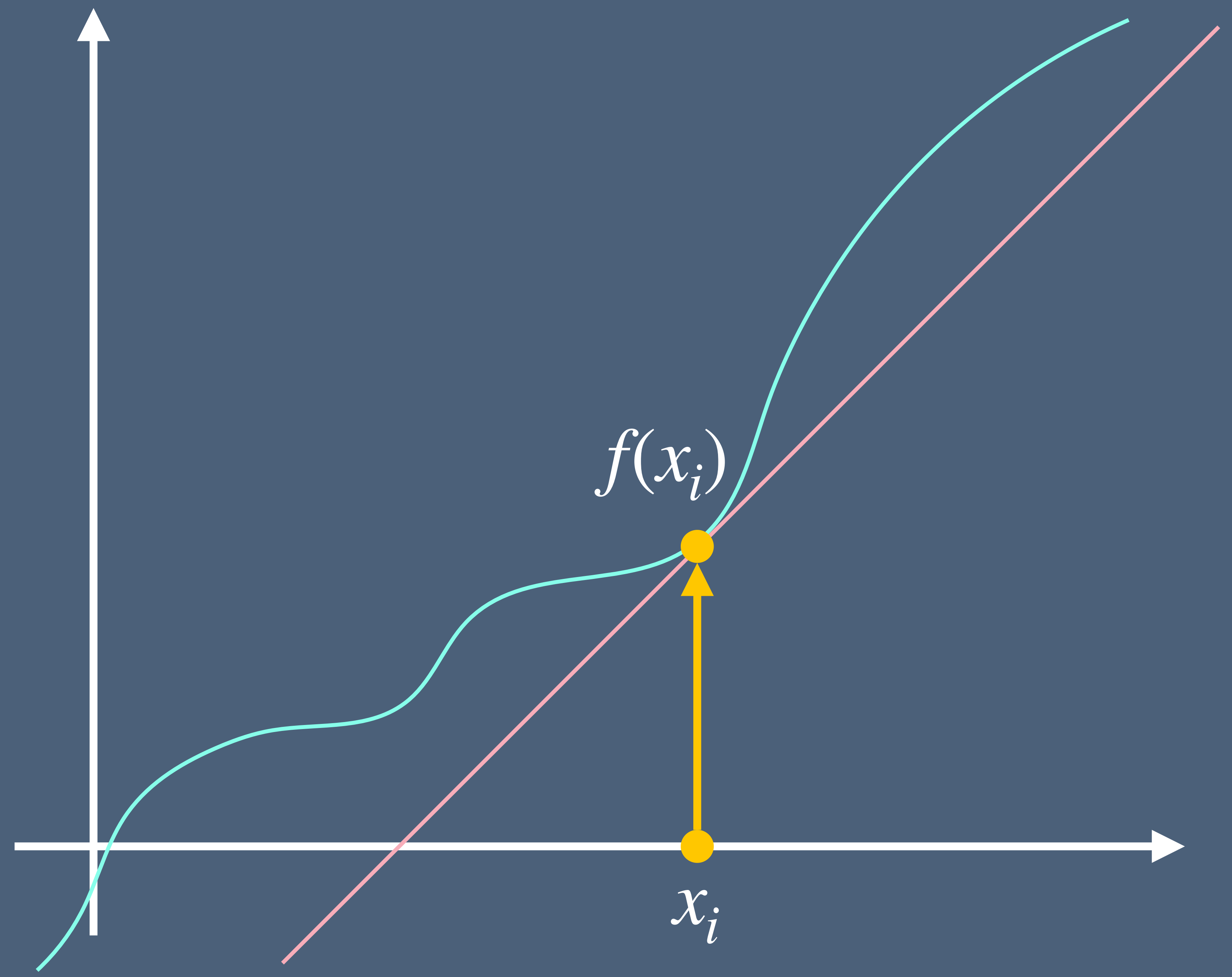

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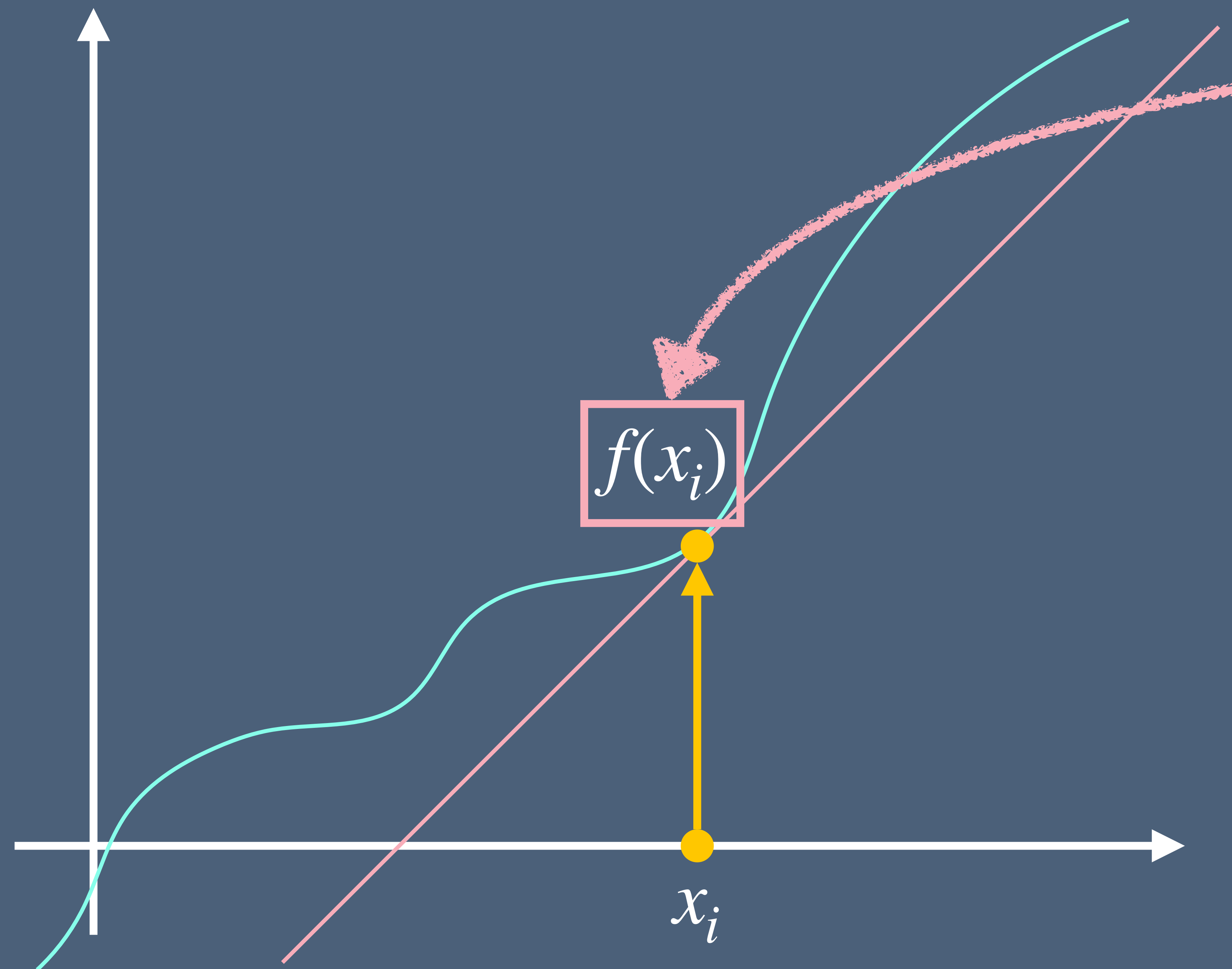

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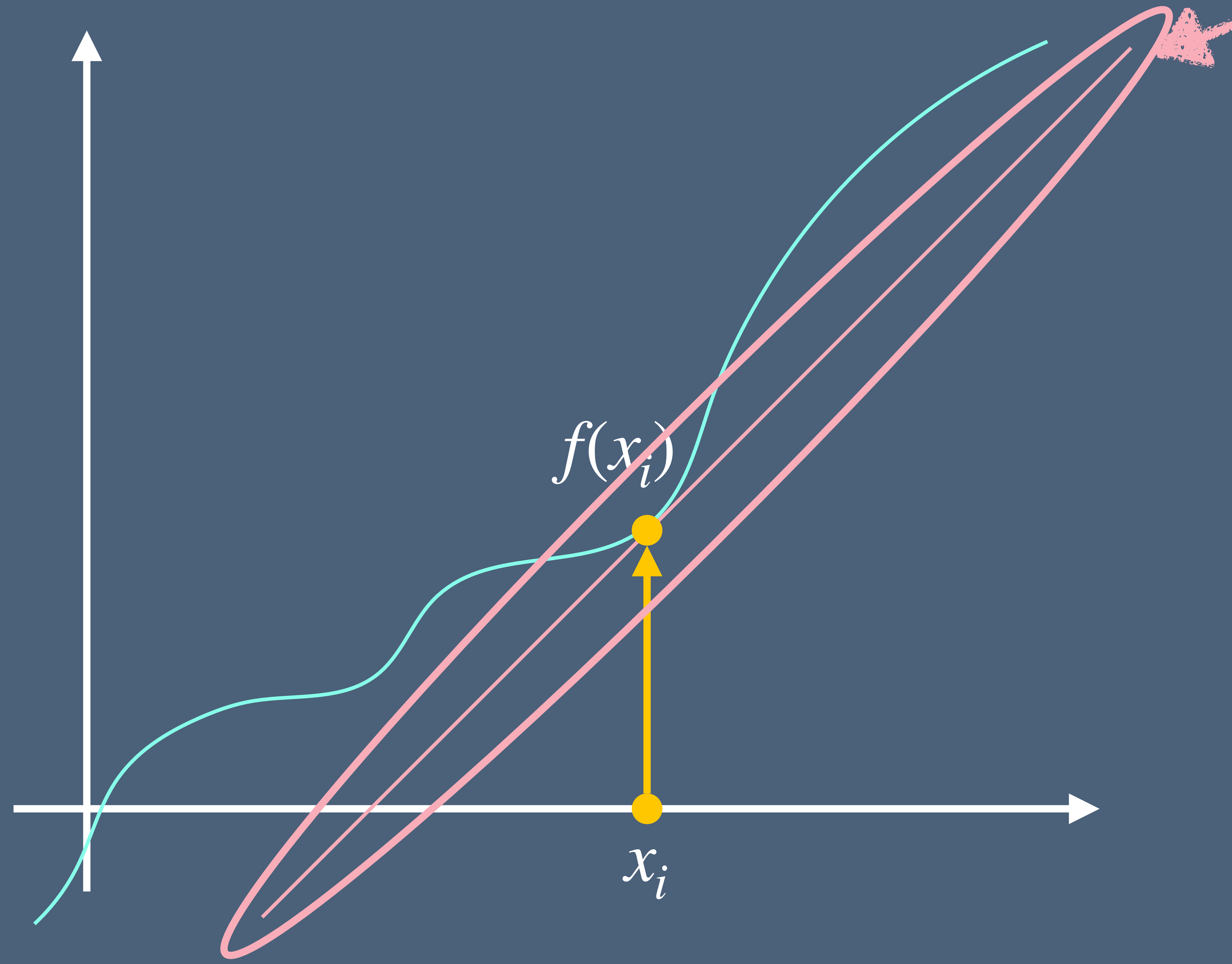

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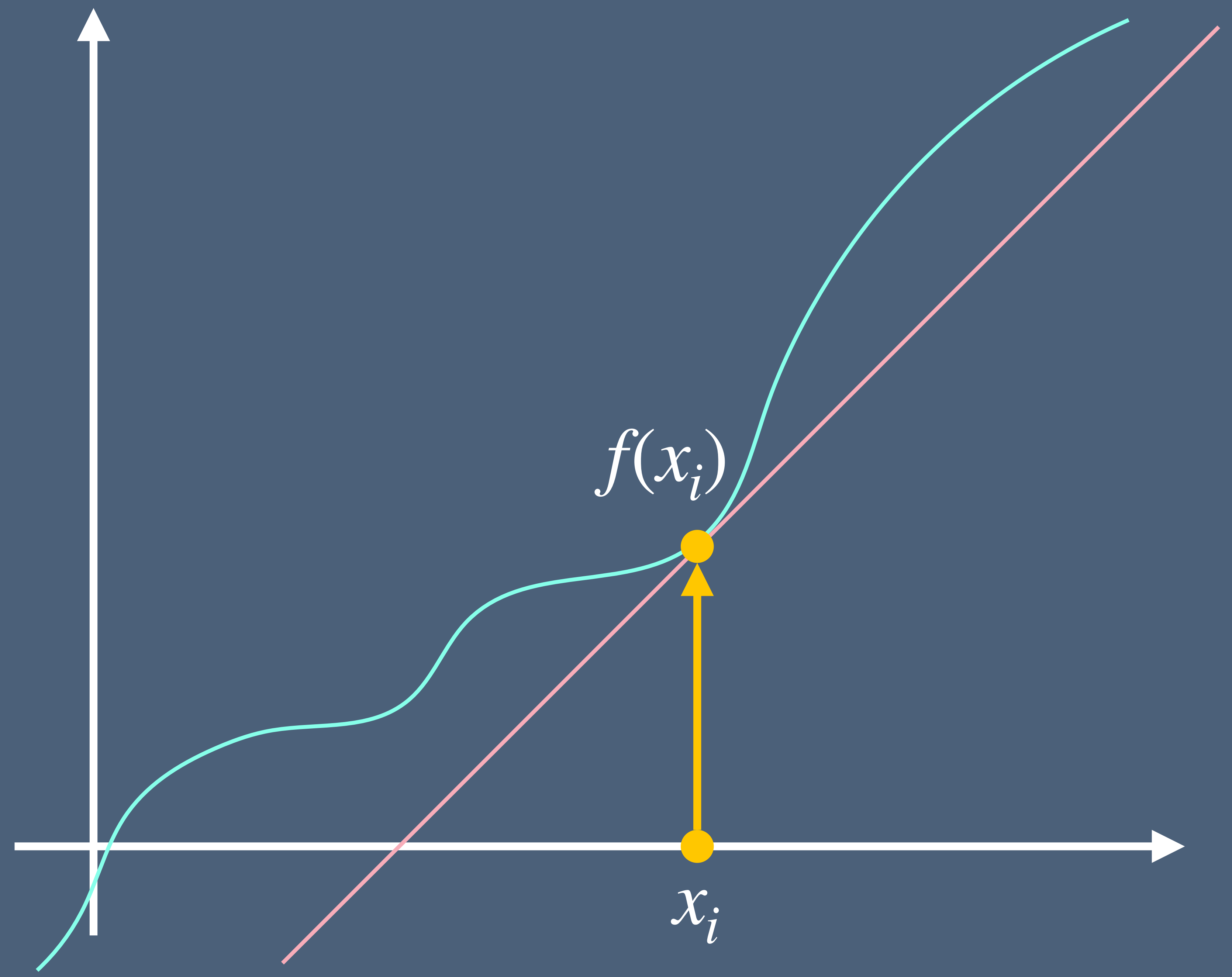
Each summand has only one variable

→

The equation becomes **linear**!



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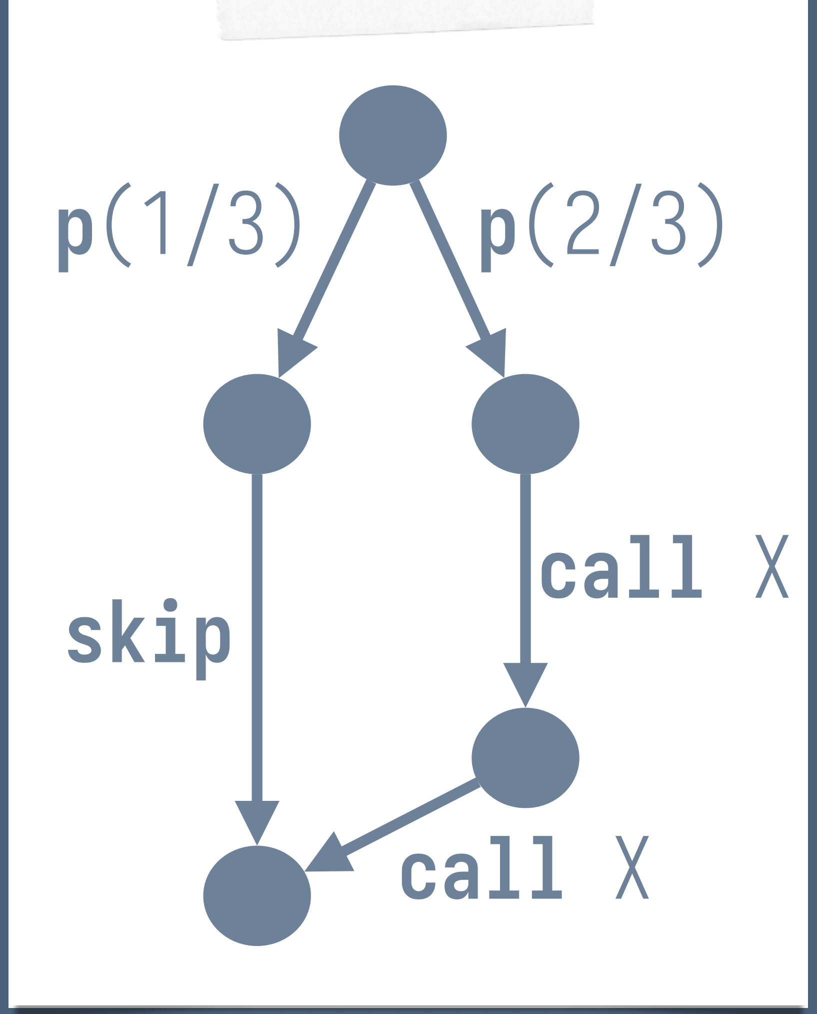

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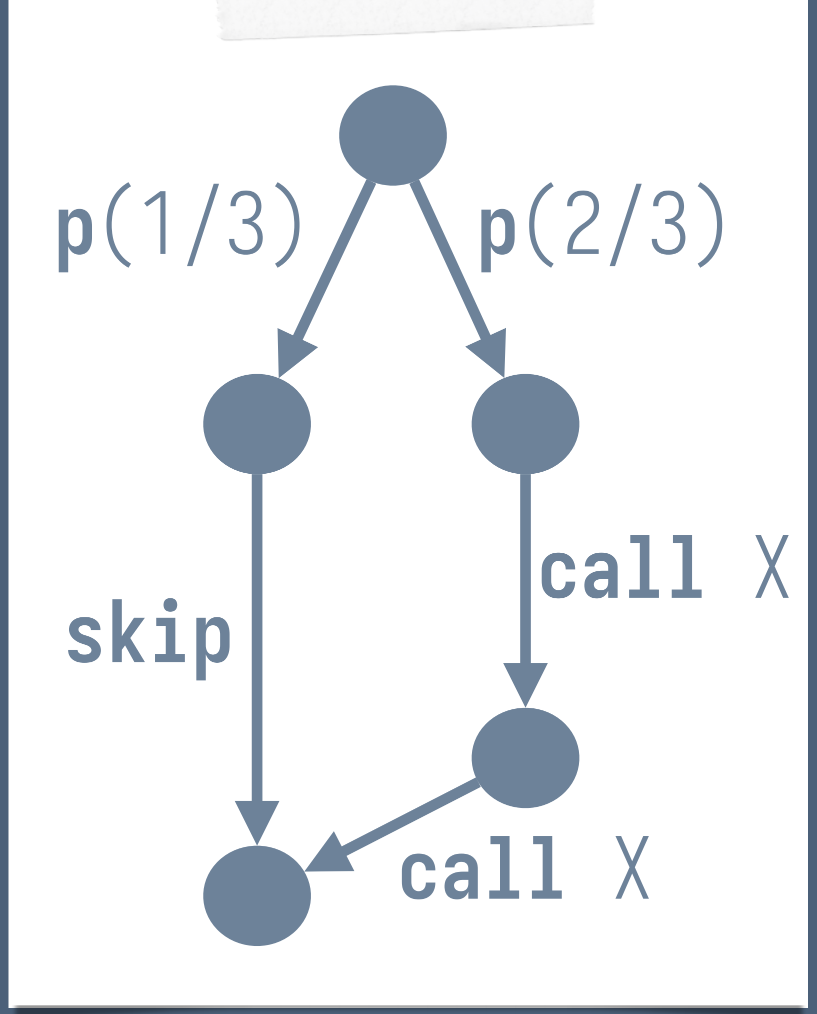

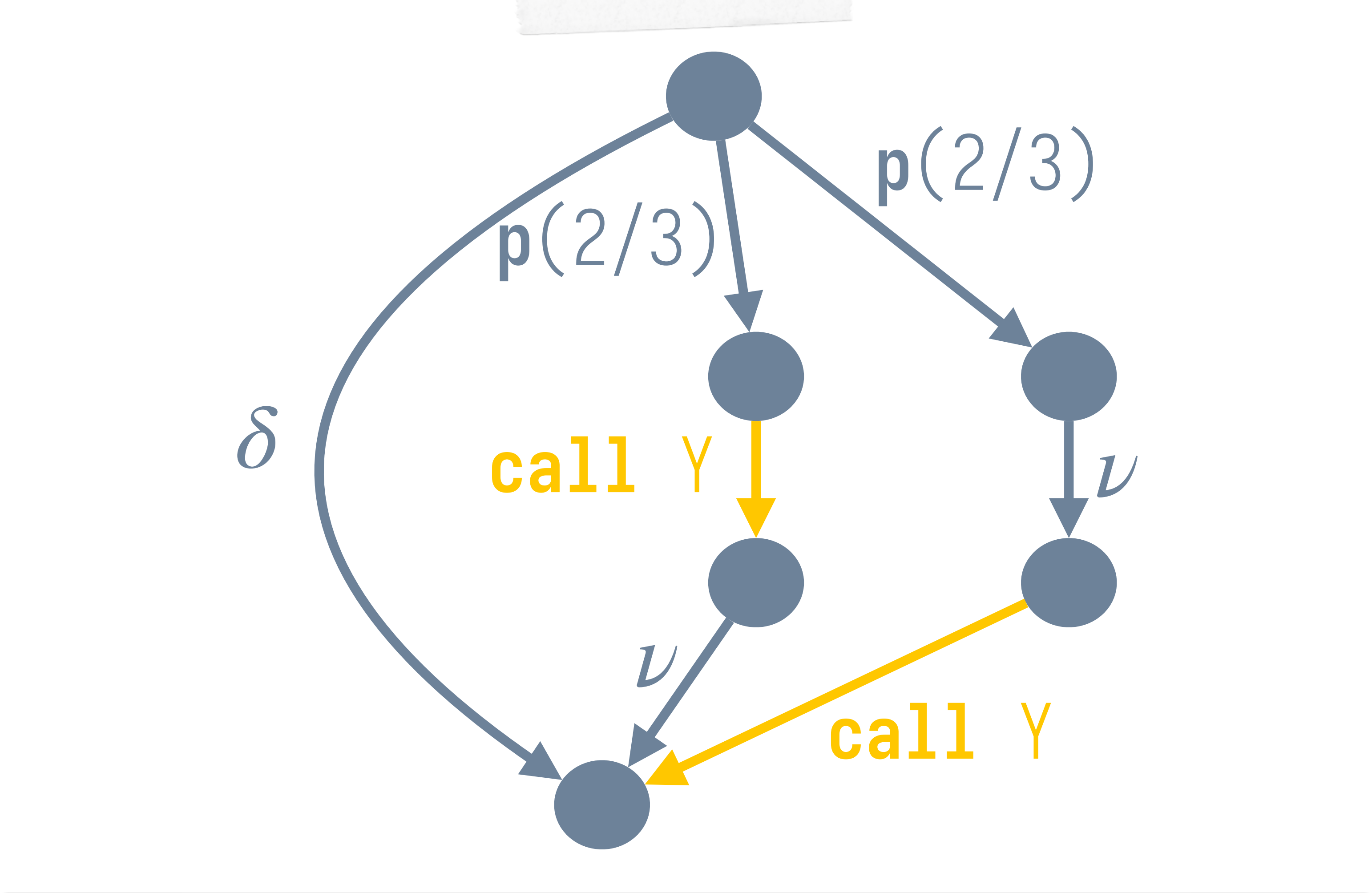

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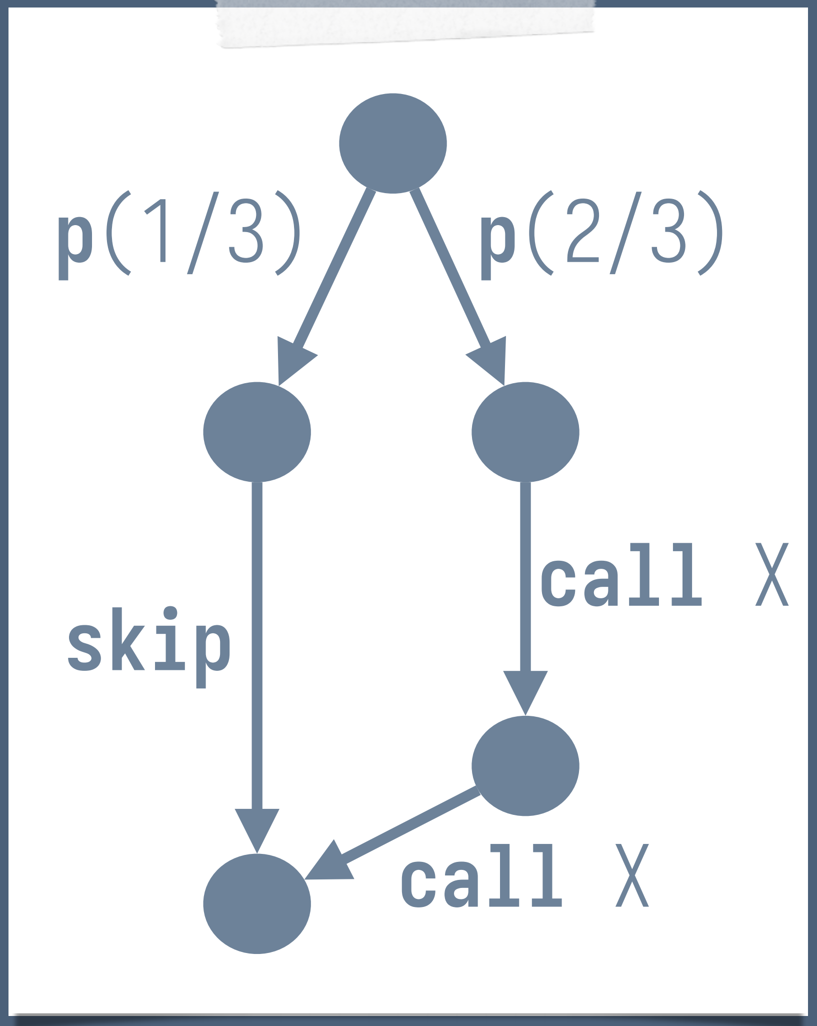
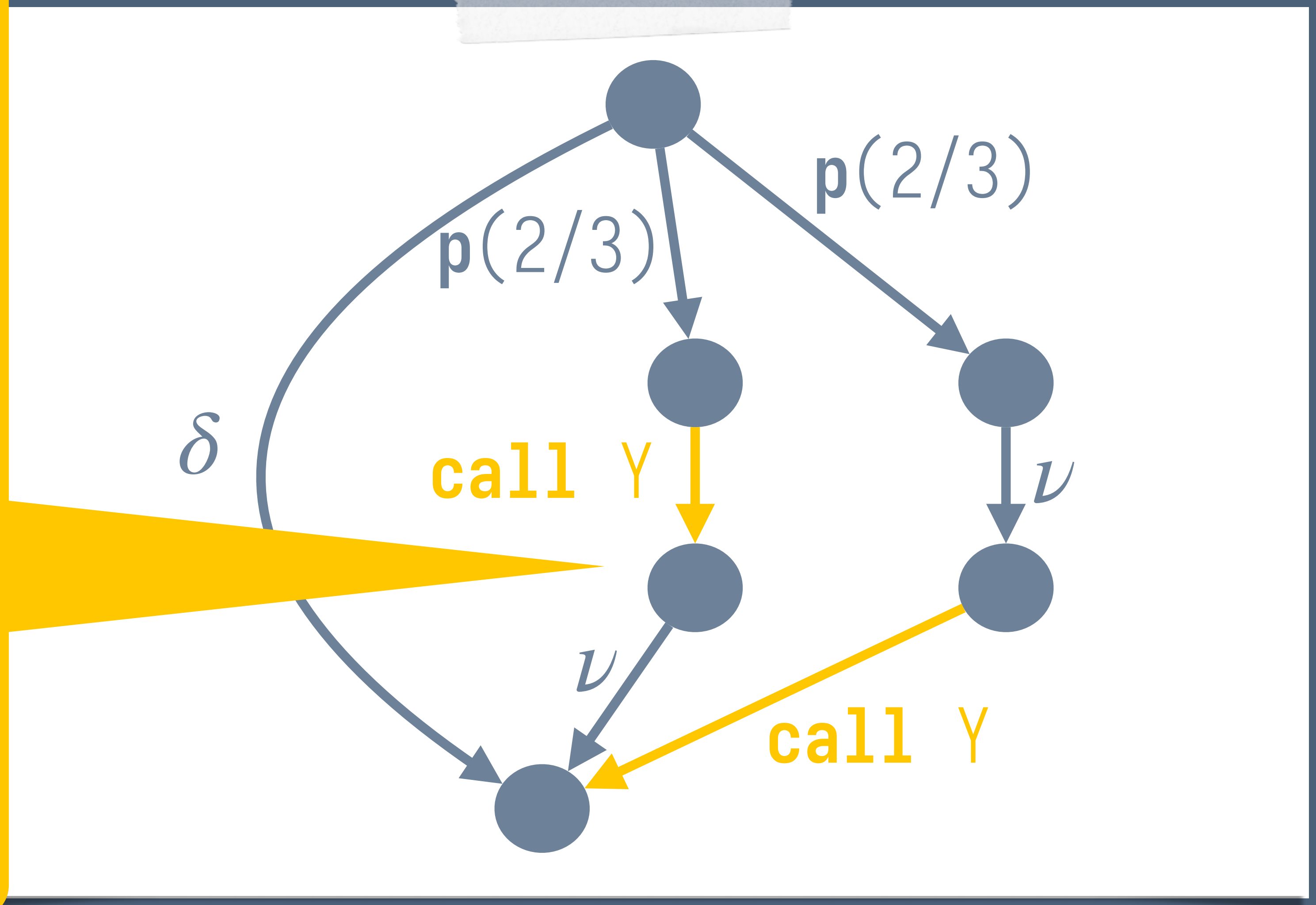


- At 1st call, perform **exploration**; at 2nd call, use the summary (ν)



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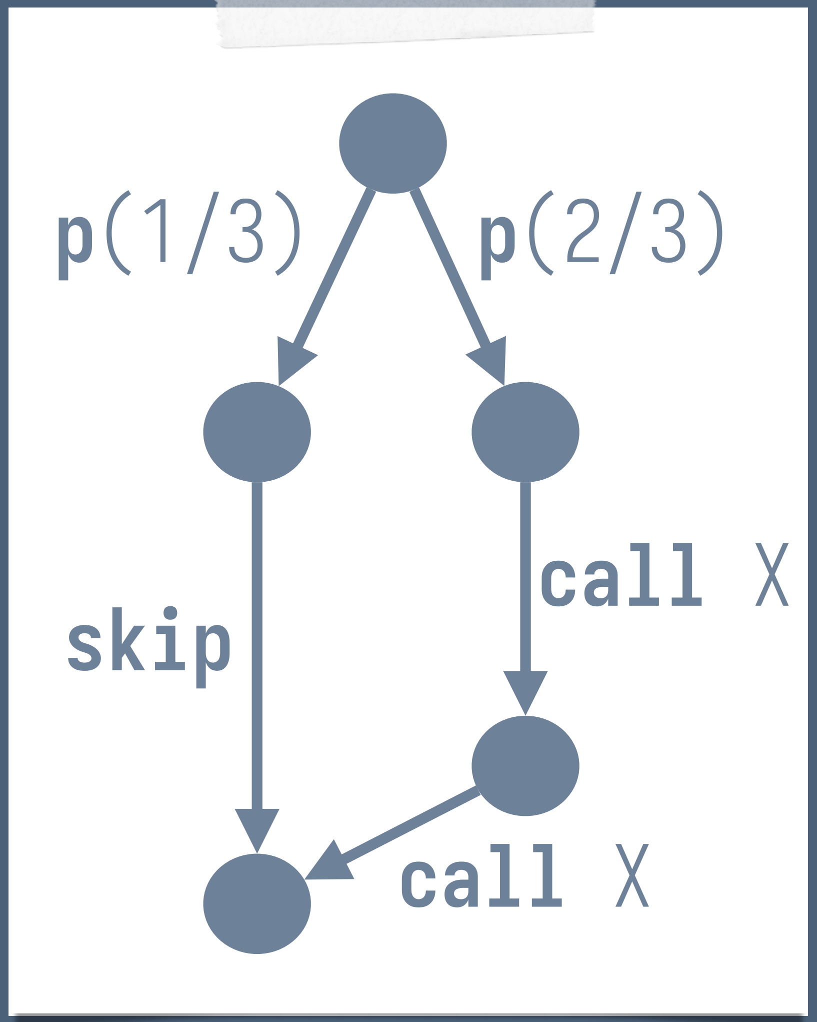

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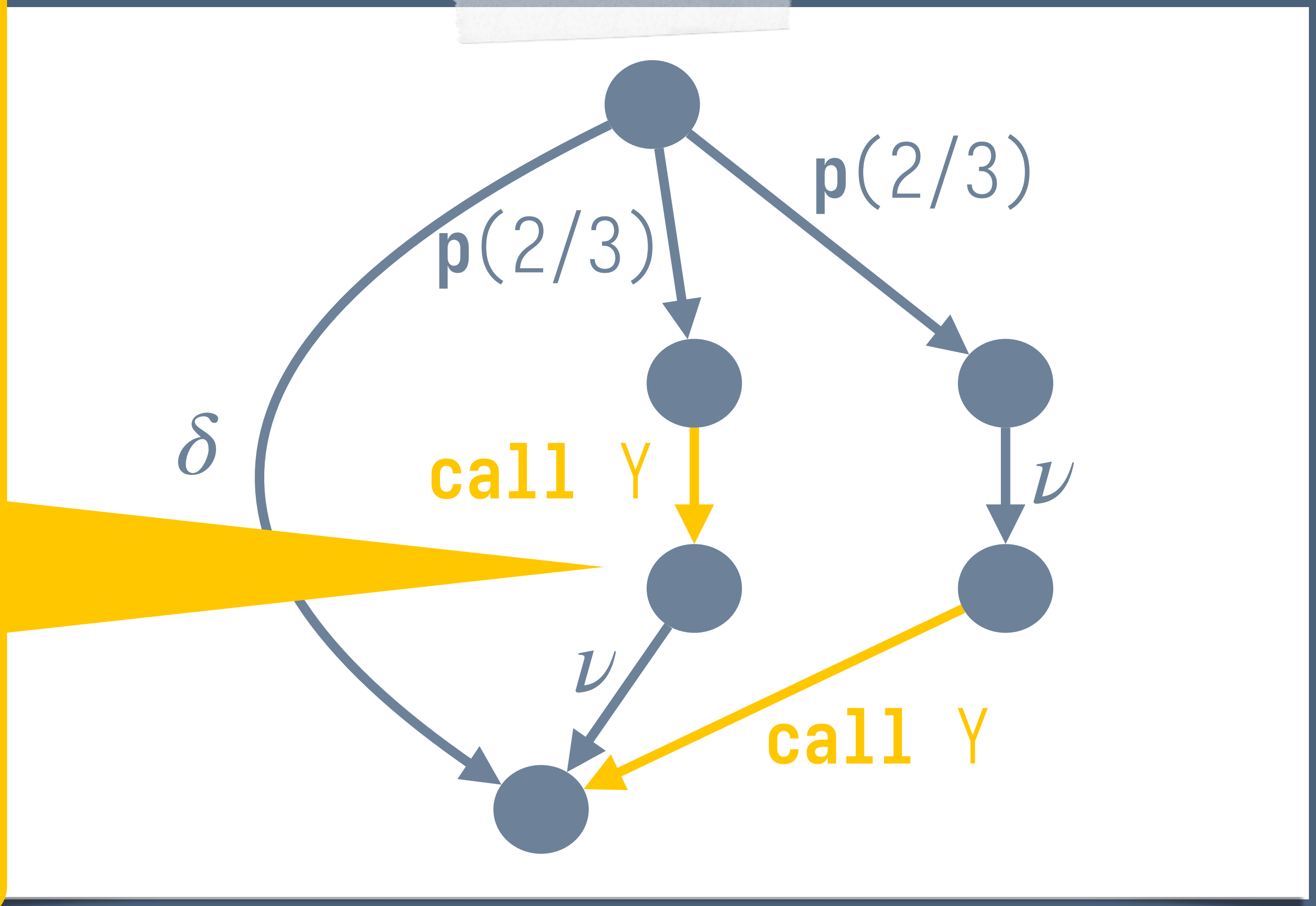

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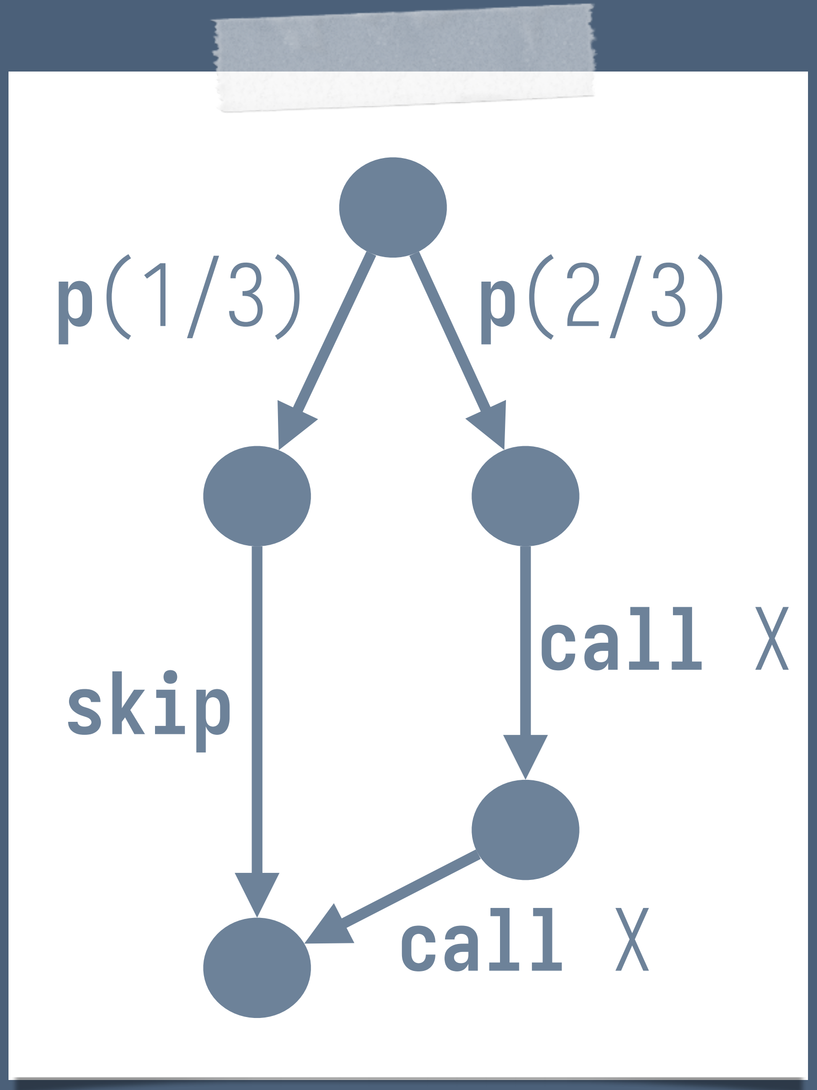

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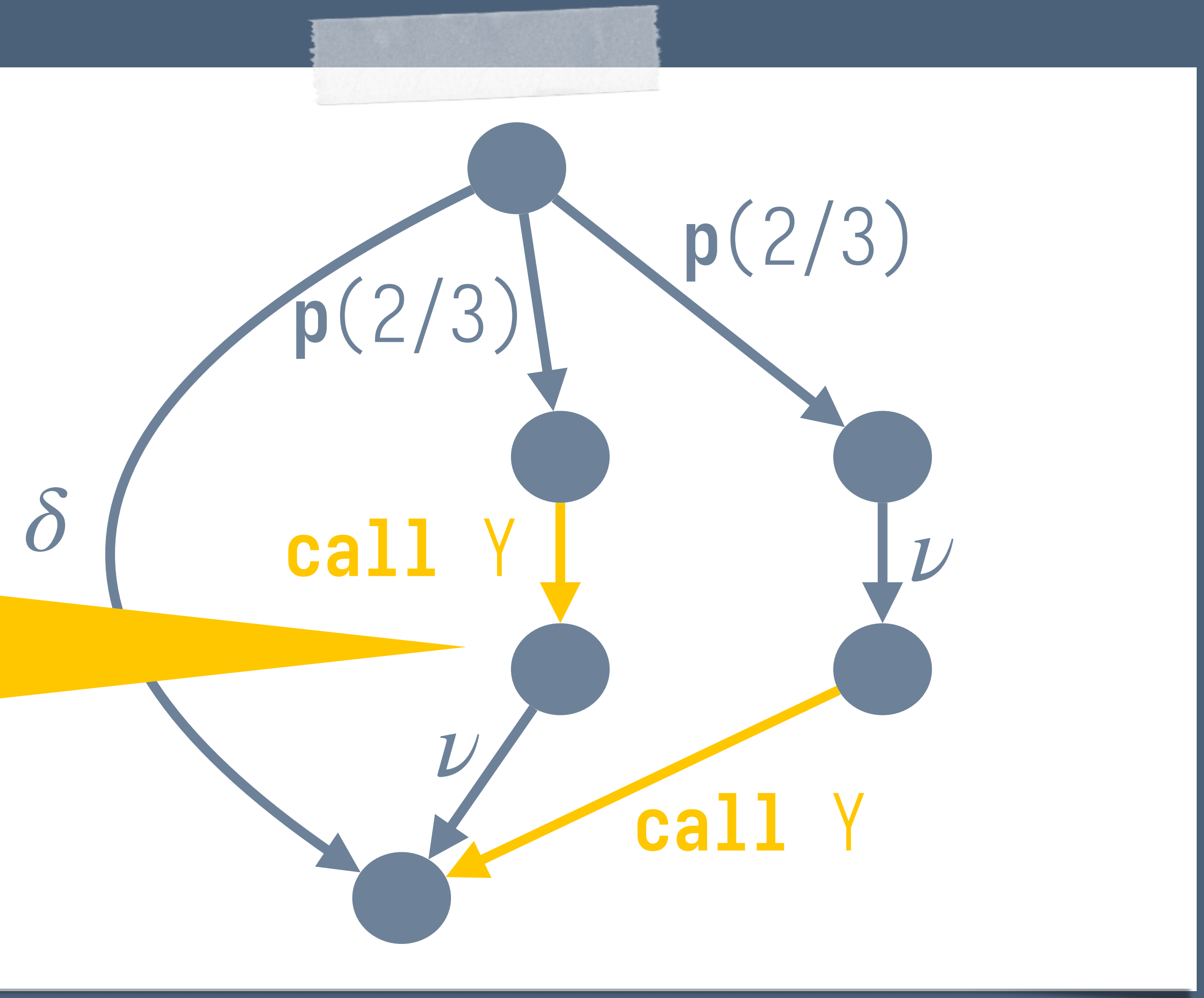
- Combine via $\oplus$



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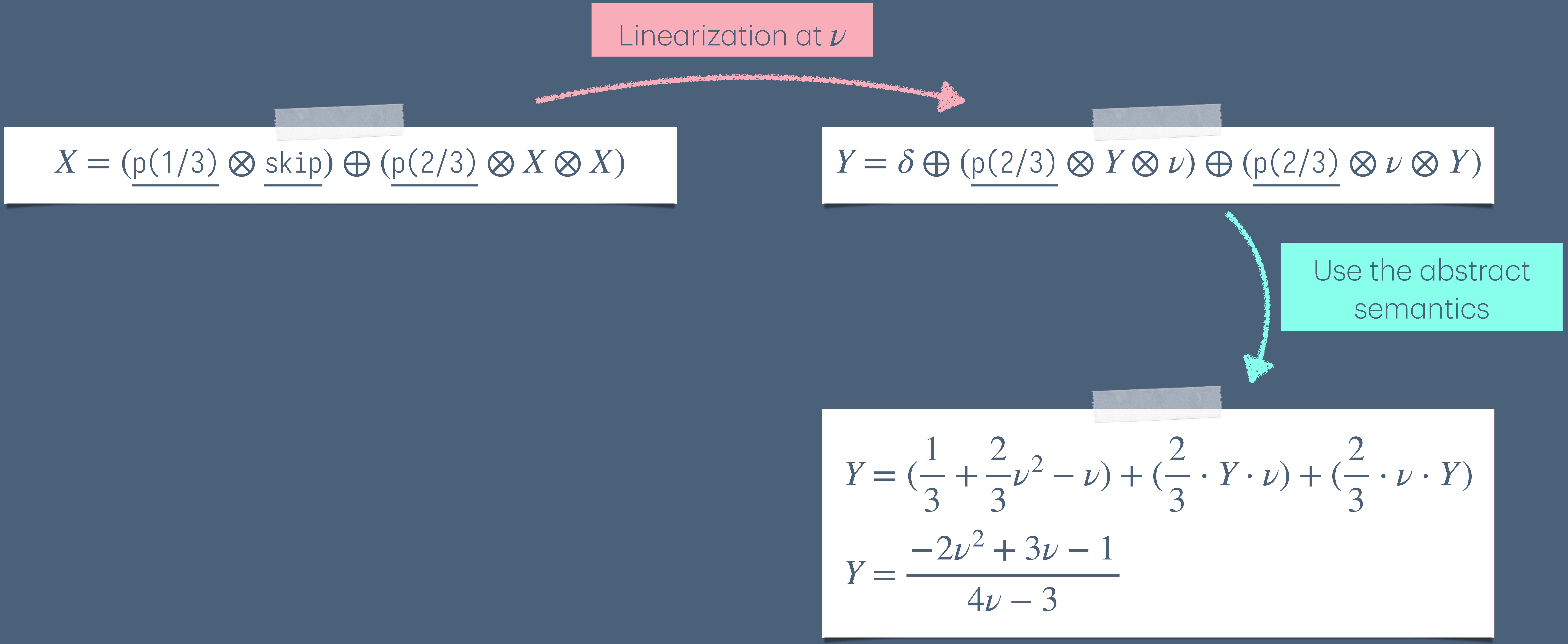

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Use the abstract semantics

$$Y = (\frac{1}{3} + \frac{2}{3}\nu^2 - \nu) + (\frac{2}{3} \cdot Y \cdot \nu) + (\frac{2}{3} \cdot \nu \cdot Y)$$

$$Y = \frac{-2\nu^2 + 3\nu - 1}{4\nu - 3}$$

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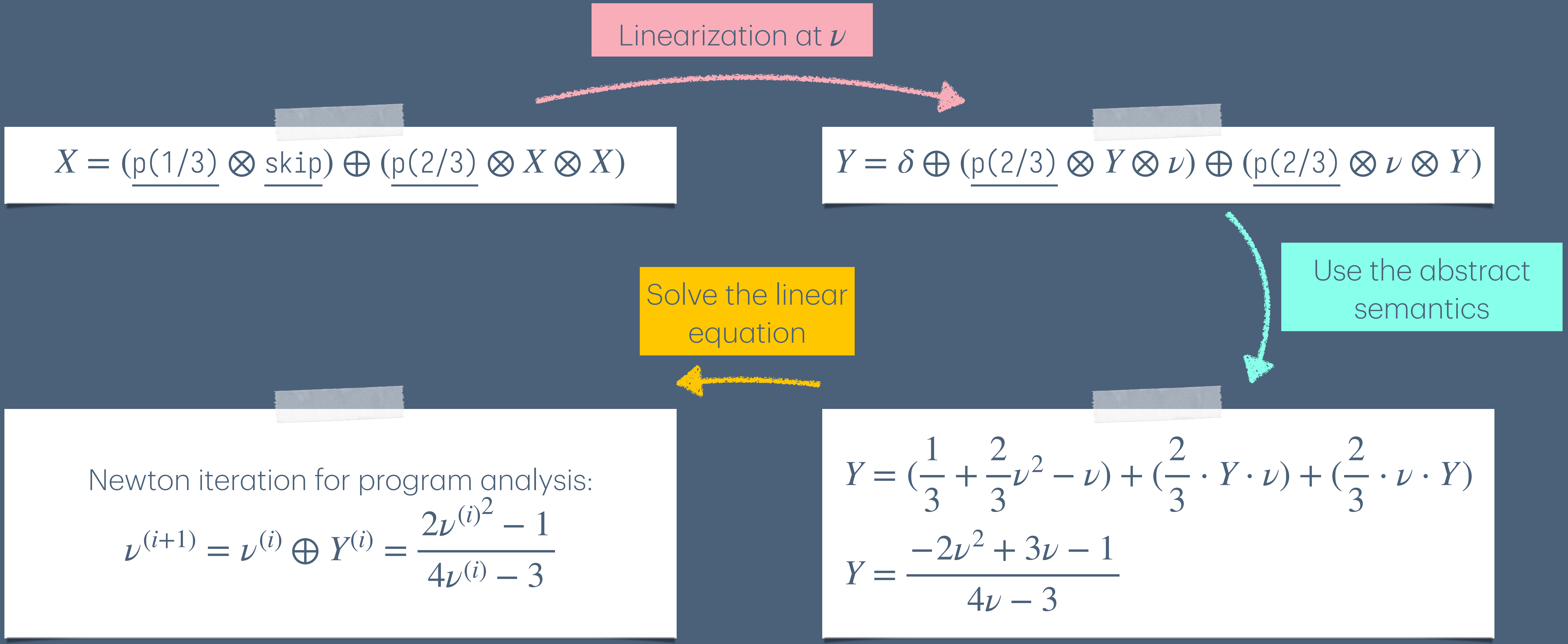

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Solve the linear equation

Newton iteration for program analysis:

$$\nu^{(i+1)} = \nu^{(i)} \oplus Y^{(i)} = \frac{2\nu^{(i)^2} - 1}{4\nu^{(i)} - 3}$$

So far so good?

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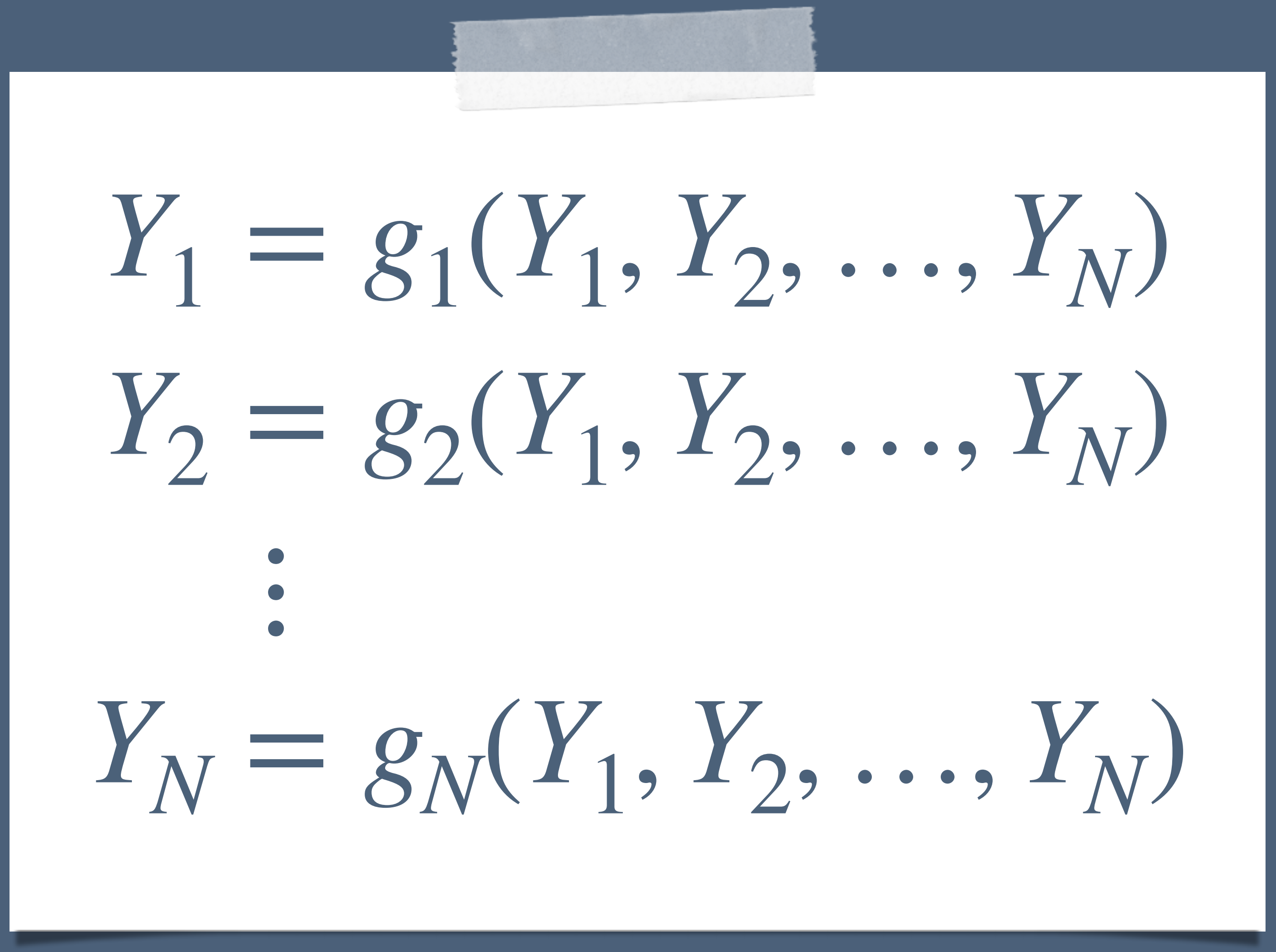
- Each Newton iteration generates a system of **linear** equations:



$$\begin{aligned} Y_1 &= g_1(Y_1, Y_2, \ldots, Y_N) \\ Y_2 &= g_2(Y_1, Y_2, \ldots, Y_N) \\ &\vdots \\ Y_N &= g_N(Y_1, Y_2, \ldots, Y_N) \end{aligned}$$

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Each g has the form:

$$a \oplus (b_1 \otimes Y_{i_1} \otimes c_1) \oplus (b_2 \otimes Y_{i_2} \otimes c_2) \oplus \ldots \oplus (b_k \otimes Y_{i_k} \otimes c_k)$$

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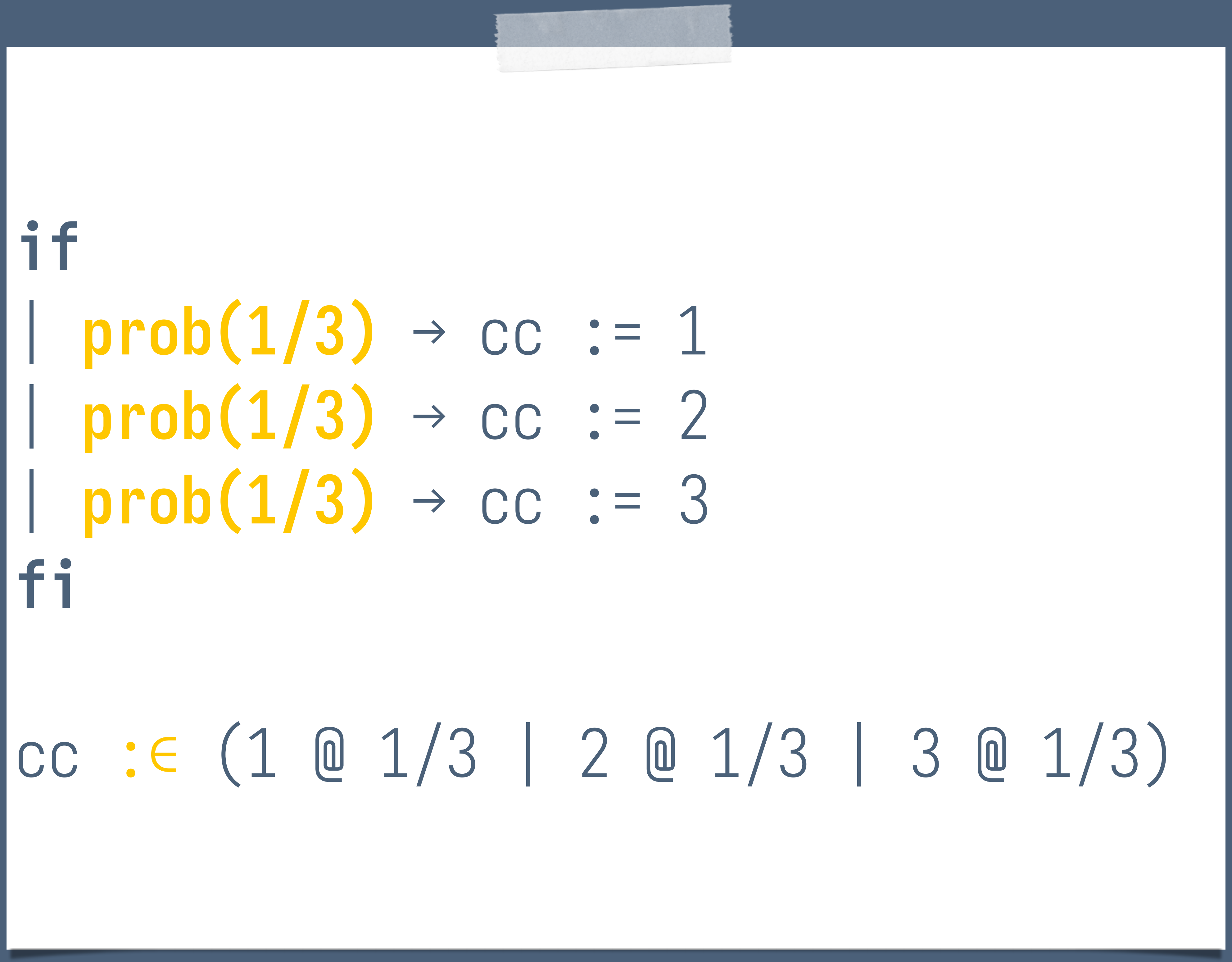
- However, Newton's method is efficient **only if one can solve linear equations efficiently**
  - [Reps, Turetsky, and Prabhu 2016] proposed a general solution that uses tensor products

Probabilistic Programs

Probabilistic Programs

- We have already seen probabilistic branching



```
if
| prob(1/3) → cc := 1
| prob(1/3) → cc := 2
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cc :∈ (1 @ 1/3 | 2 @ 1/3 | 3 @ 1/3)
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Probabilistic Programs

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- True randomness

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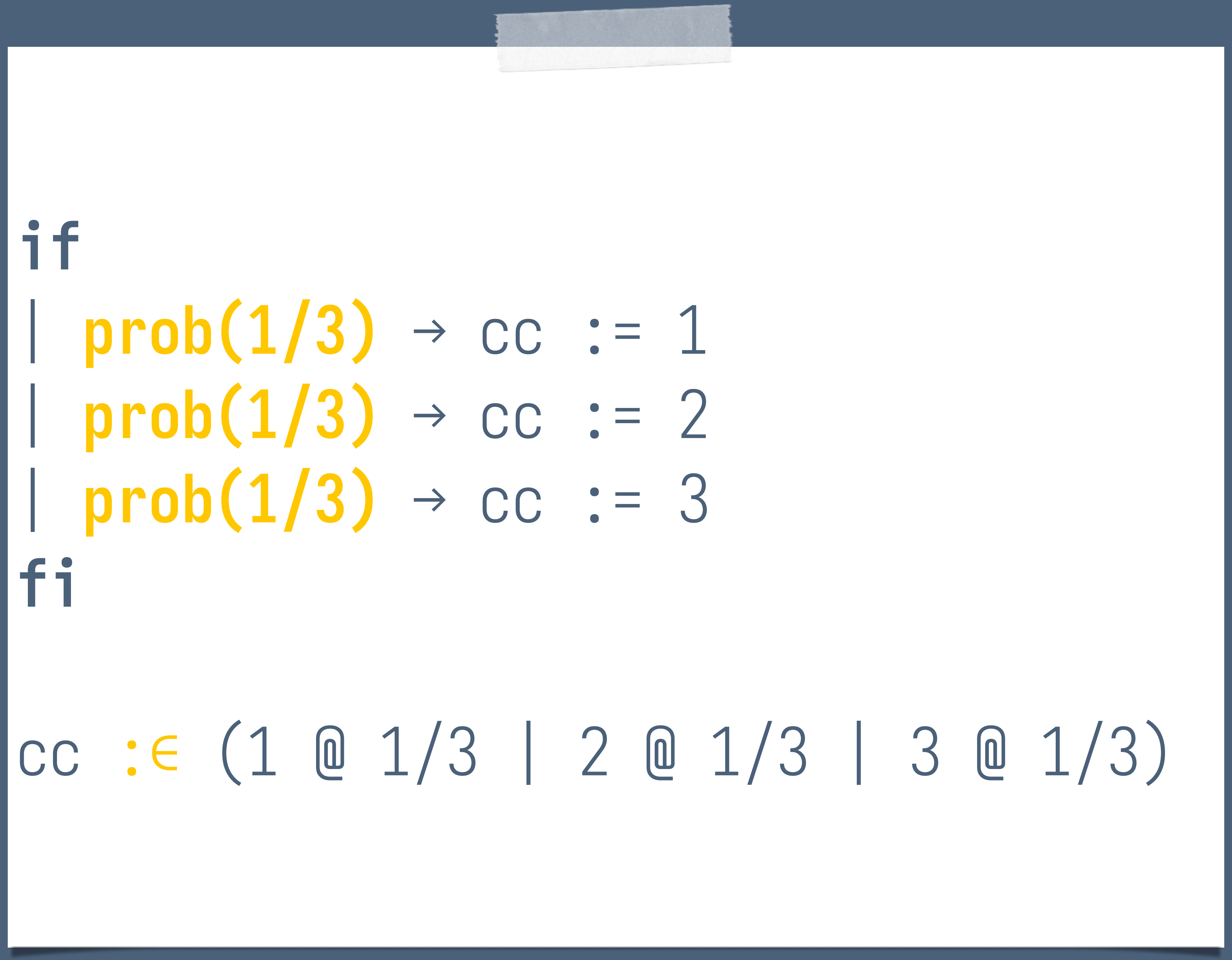

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- **Probabilistic nondeterminism**

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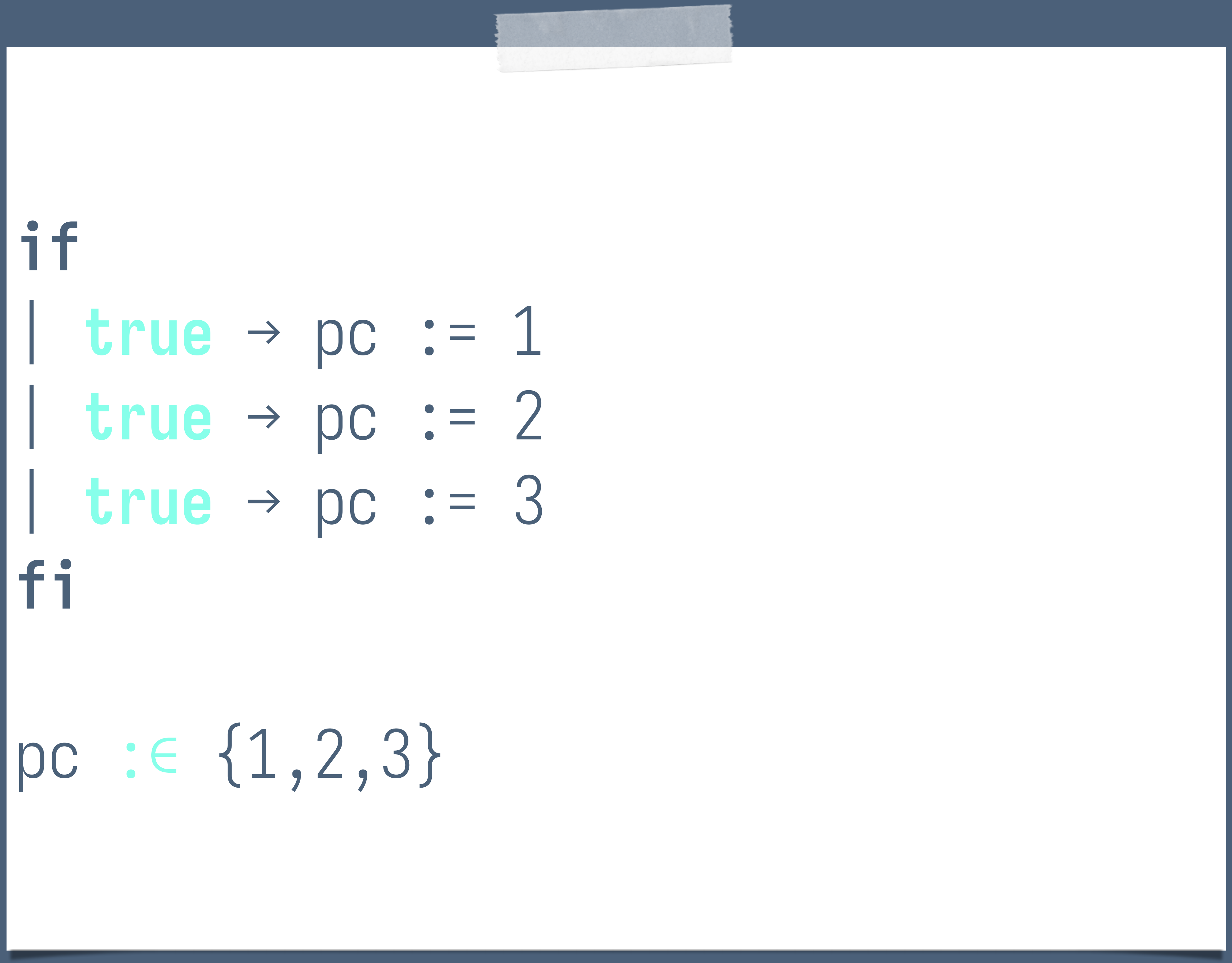
Probabilistic Programs

Probabilistic Programs

- There are also other kinds of branching



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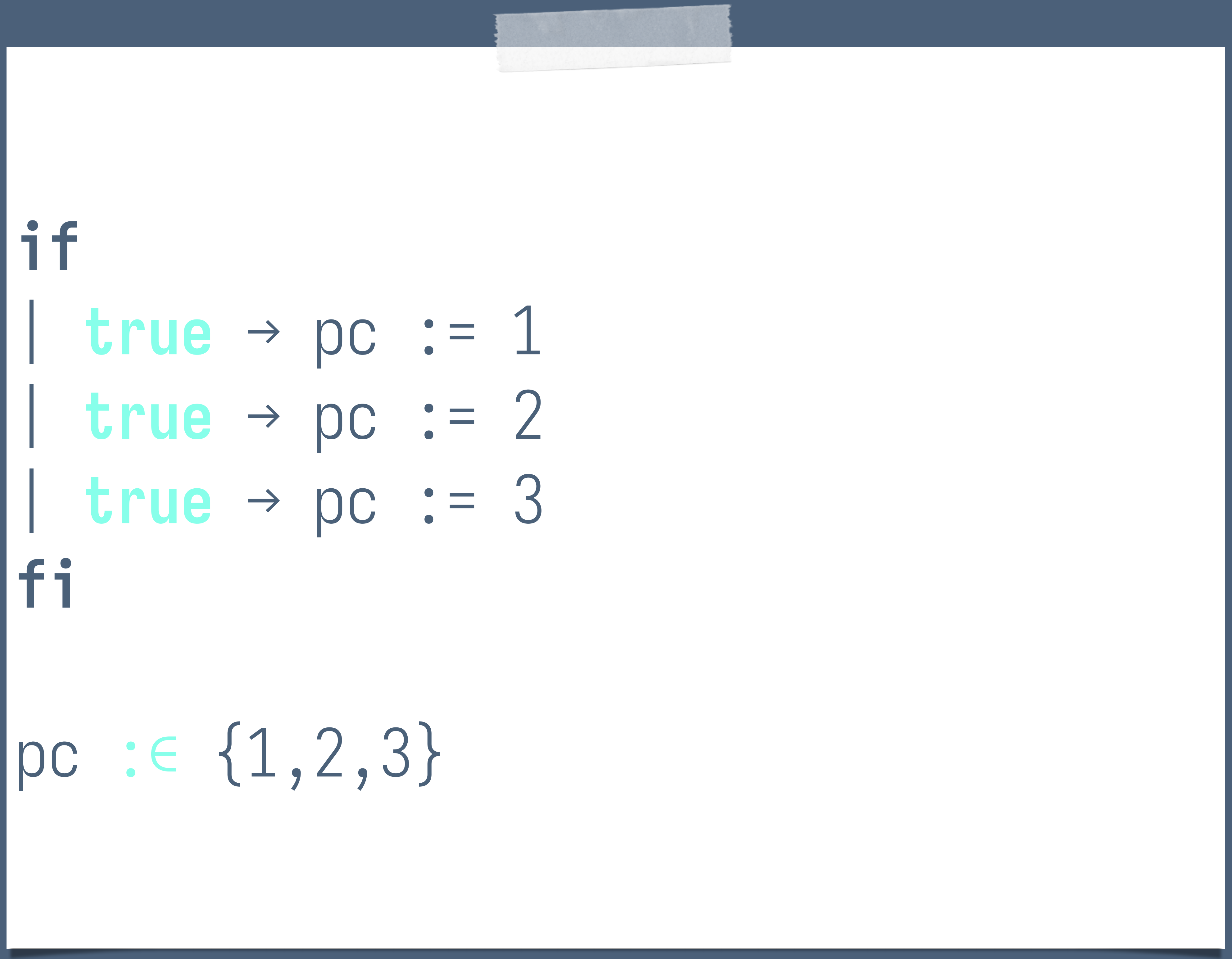
Probabilistic Programs

- There are also other kinds of branching
- Dijkstra's **Guarded Command Language** (GCL)



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Probabilistic Programs

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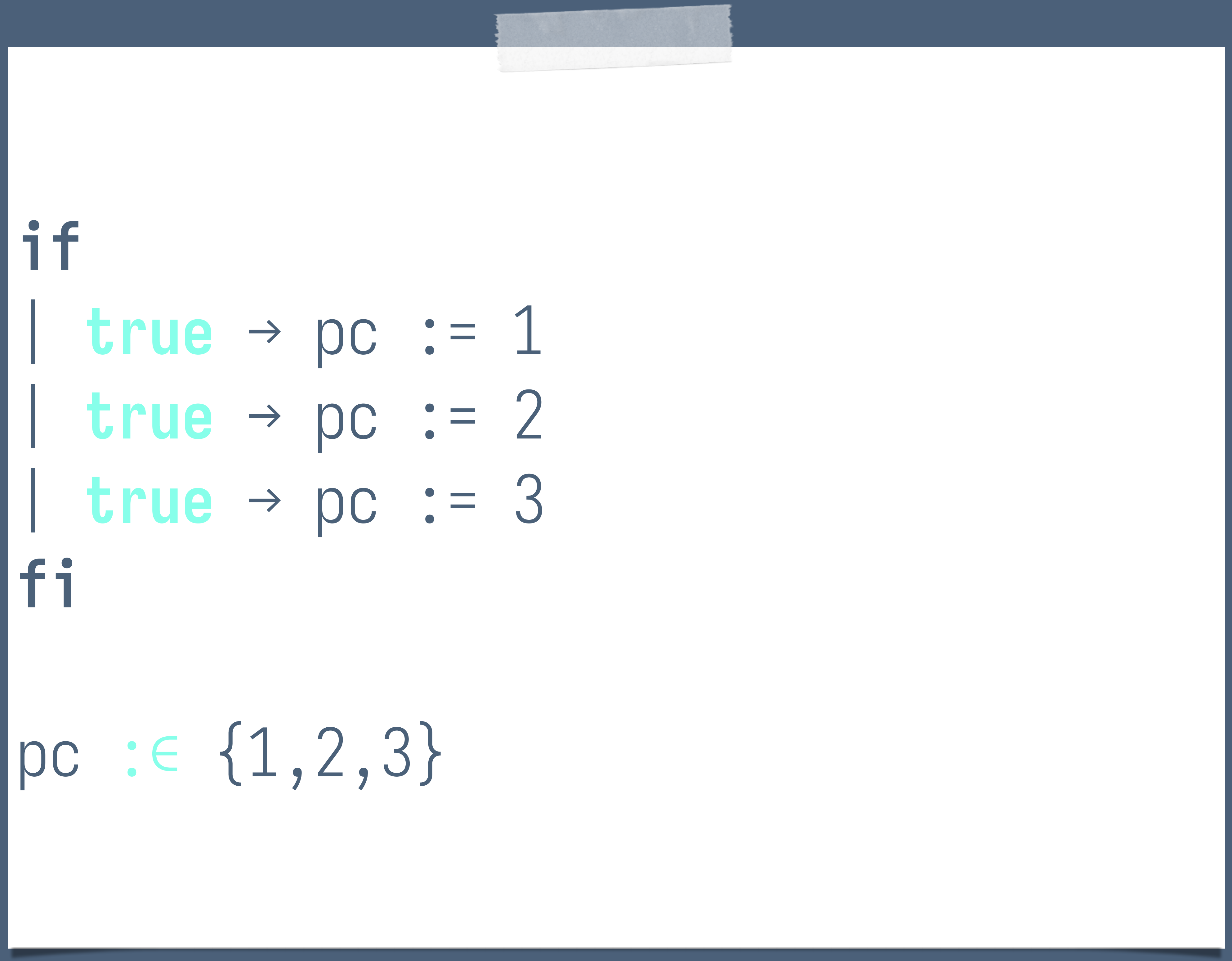

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Probabilistic Programs

- There are also other kinds of branching
- Dijkstra's **Guarded Command Language** (GCL)
- A set of execution paths
- **Demonic nondeterminism**

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```

The Monty-Hall Puzzle

as a probabilistic program

The Monty-Hall Puzzle

as a probabilistic program

- Programs can use multiple kinds of branching

The Monty-Hall Puzzle

as a probabilistic program

- Programs can use multiple kinds of branching
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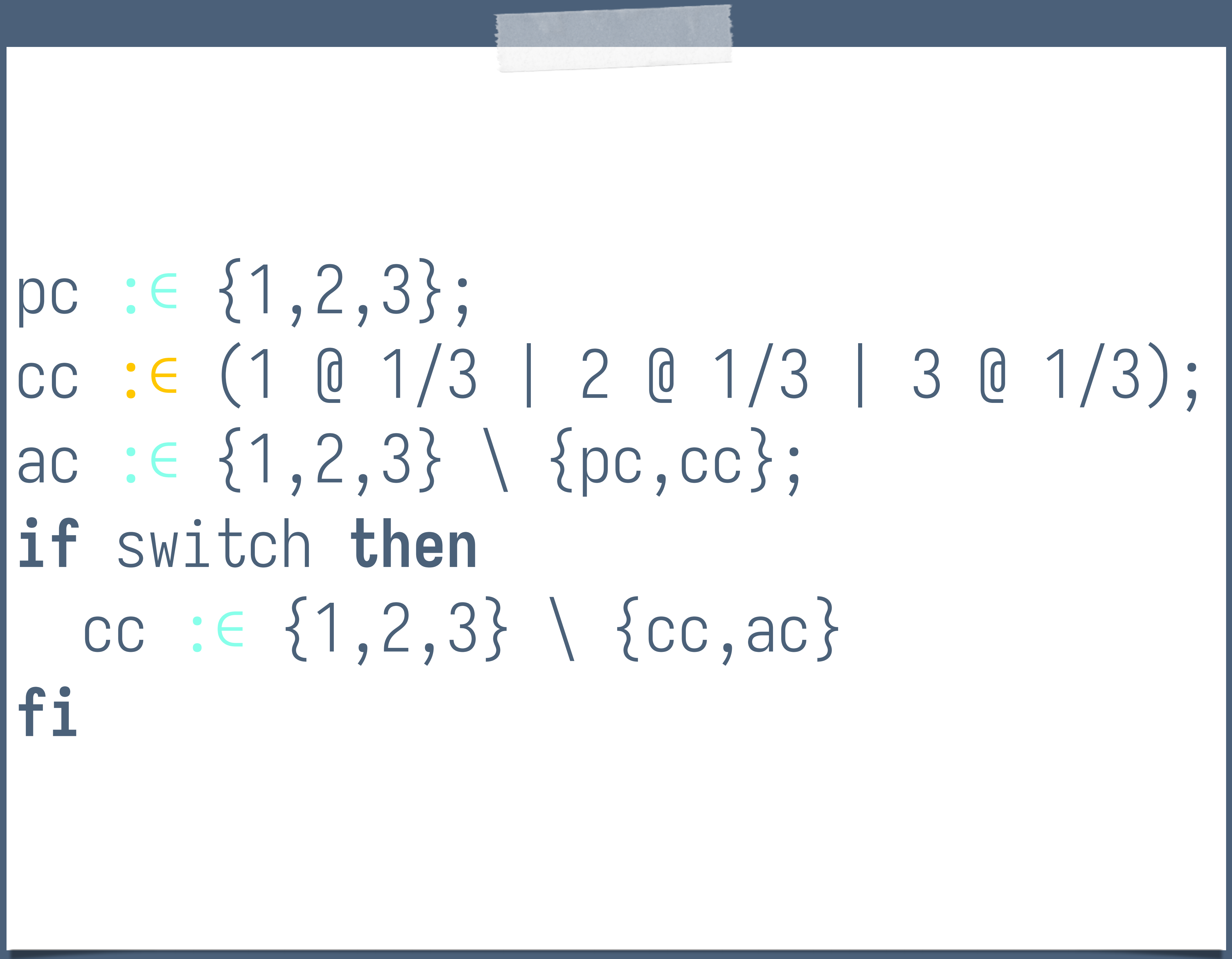

```
pc :∈ {1,2,3};
cc :∈ (1 @ 1/3 | 2 @ 1/3 | 3 @ 1/3);
ac :∈ {1,2,3} \ {pc,cc};
if switch then
  cc :∈ {1,2,3} \ {cc,ac}
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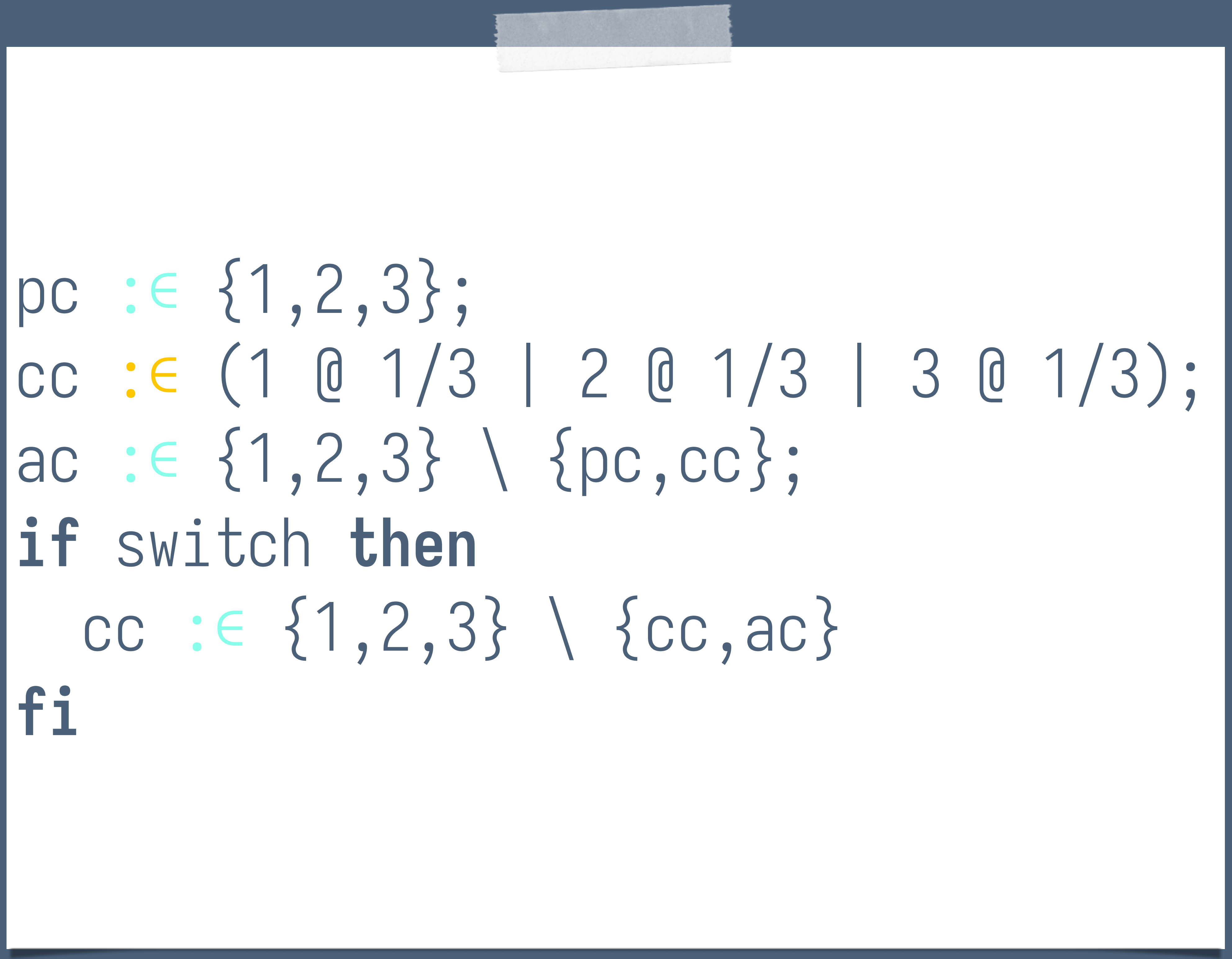
The Monty-Hall Puzzle

as a probabilistic program

- Programs can use multiple kinds of branching
- McIver and Morgan's **probabilistic Guarded Command Language** (pGCL)
- Combine three kinds of branching:
  - Probabilistic
  - Demonic
  - Conditional



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Termination-Probability Analysis

of Boolean programs

Termination-Probability Analysis

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- Problem: A semiring has **only one** combine (⊕) operation

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proc X begin
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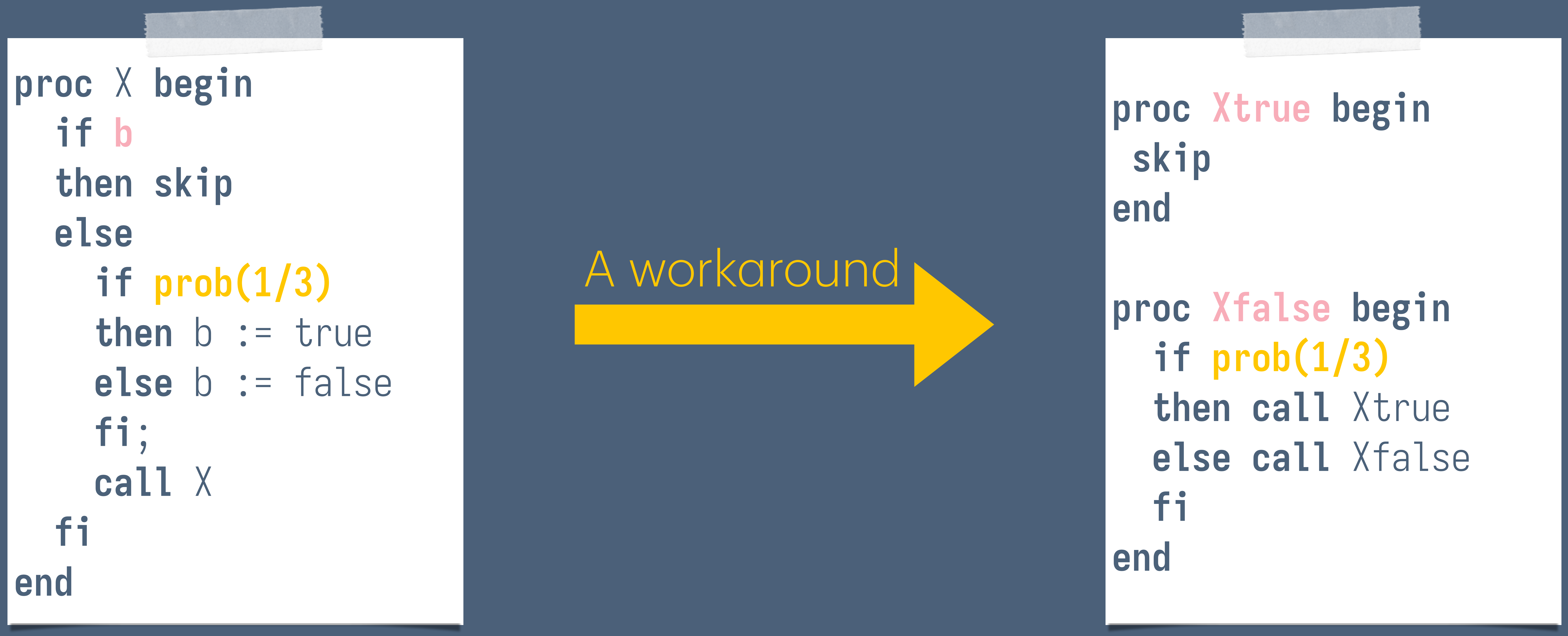

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A workaround

```
proc Xtrue begin
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end

proc Xfalse begin
  if prob(1/3)
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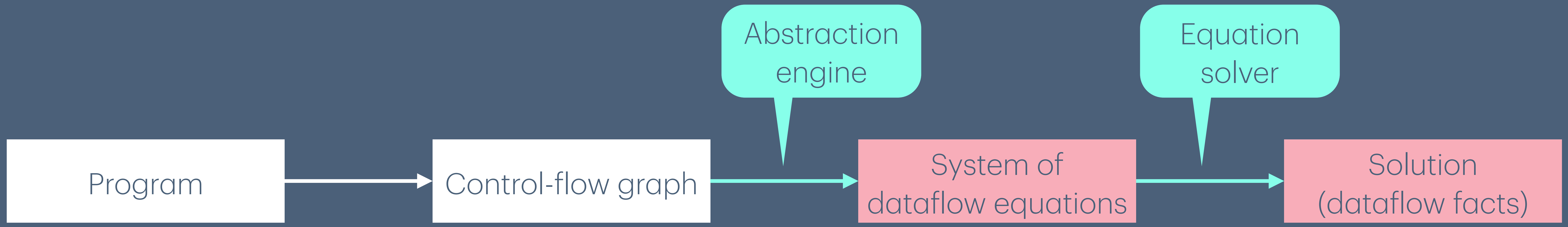
- Introduce extra procedures to encode different states
- Cannot handle **infinite state spaces**

Towards Multiple Combine Operations

Towards Multiple Combine Operations


Program

Control-flow graph

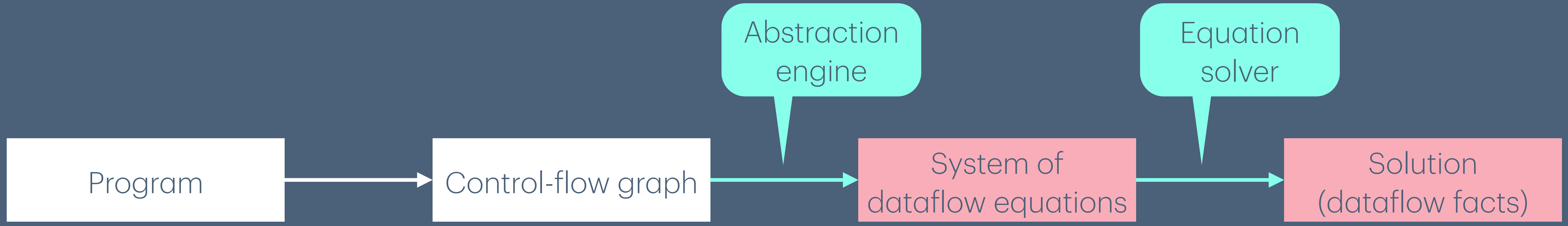
Abstraction engine

System of dataflow equations

Equation solver

Solution (dataflow facts)

Towards Multiple Combine Operations



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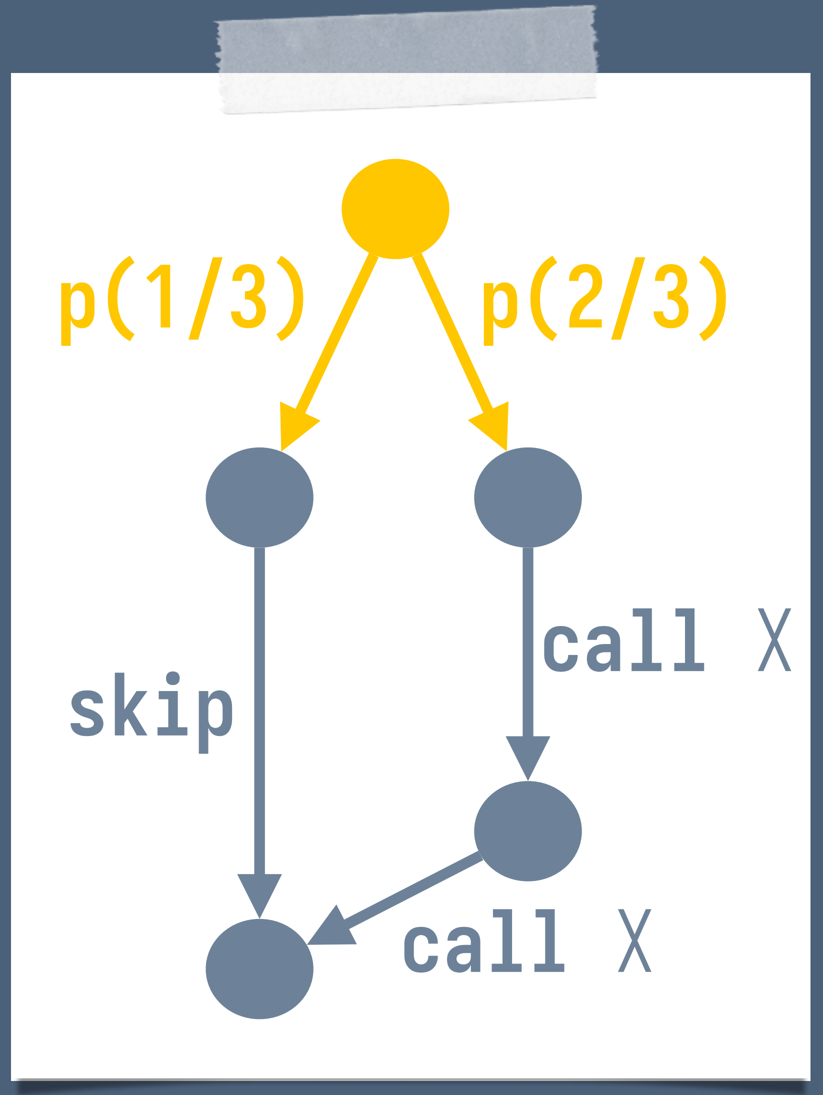
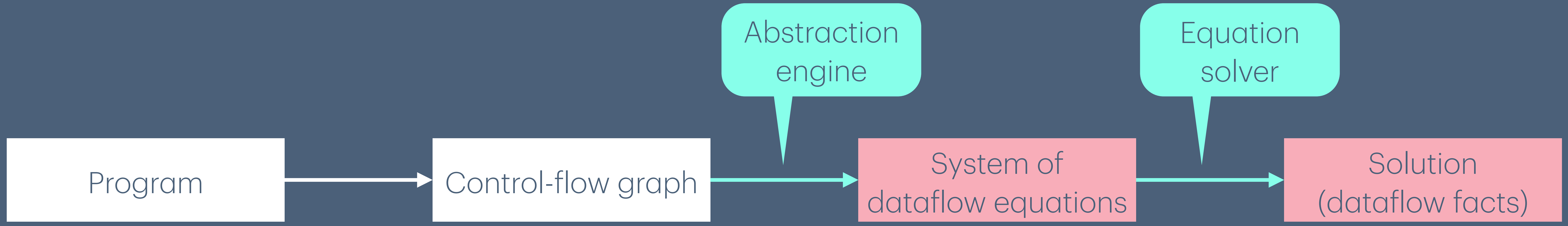


Towards Multiple Combine Operations



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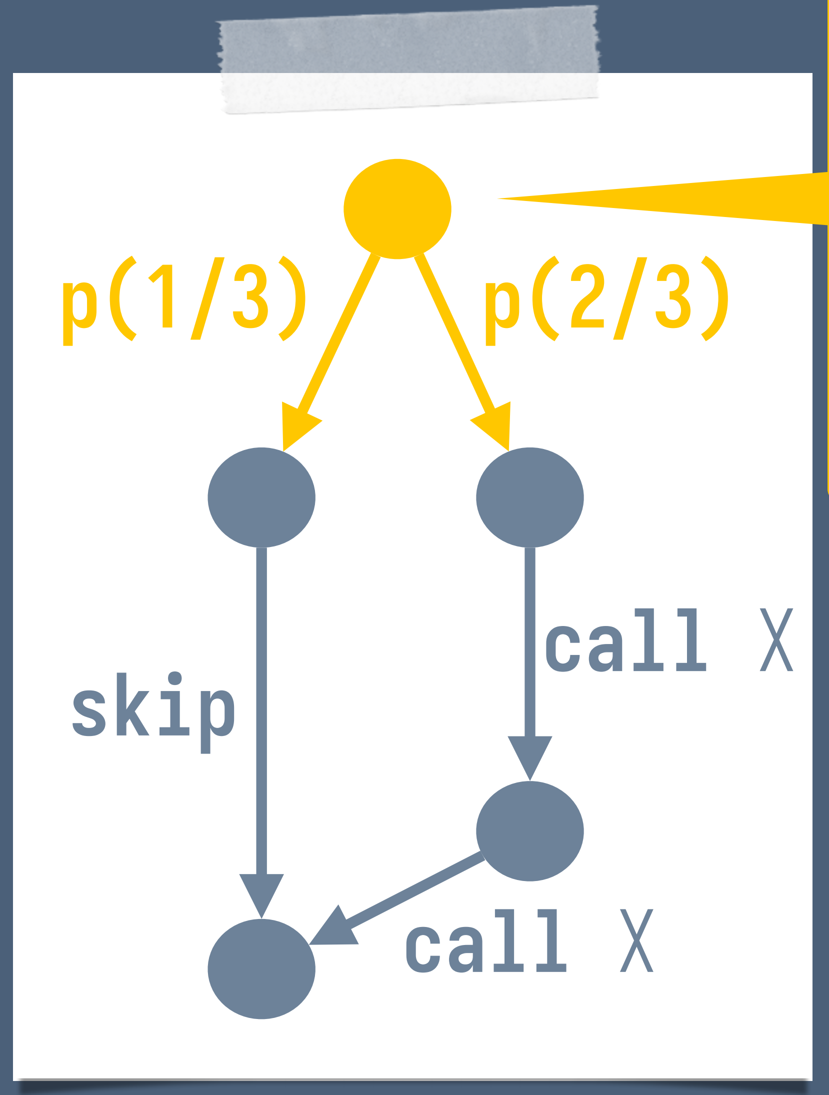
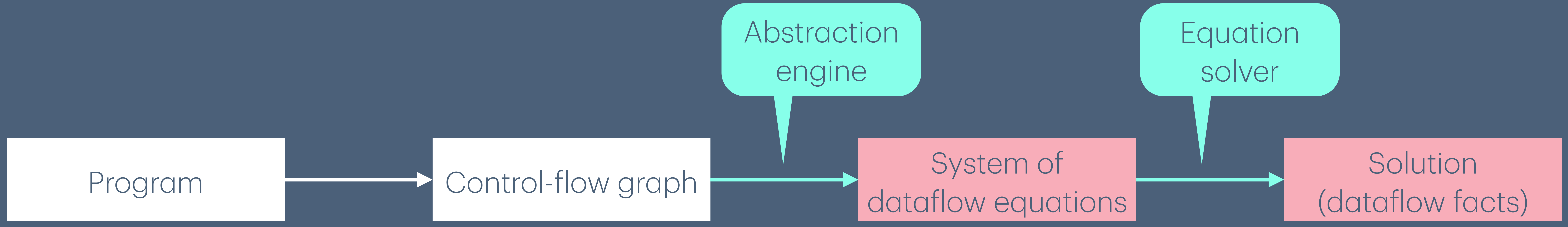
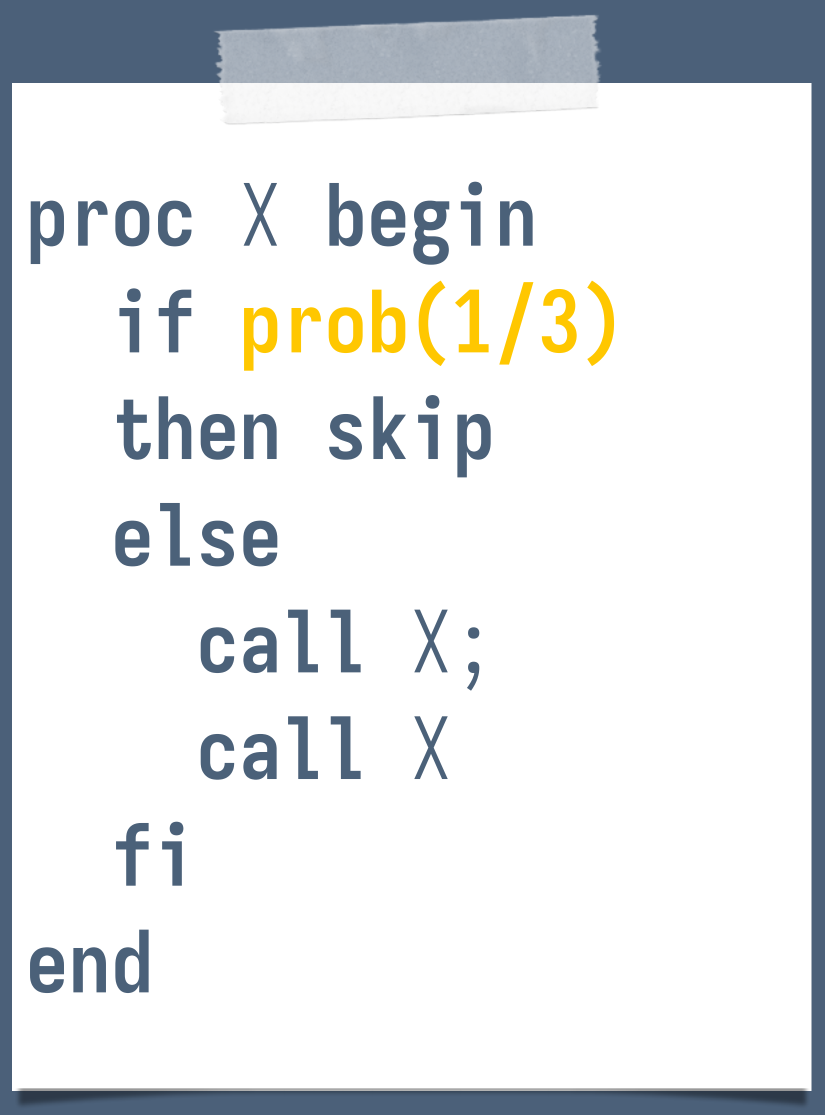


- Confluence is interpreted by $\oplus$, implicitly

Towards Multiple Combine Operations





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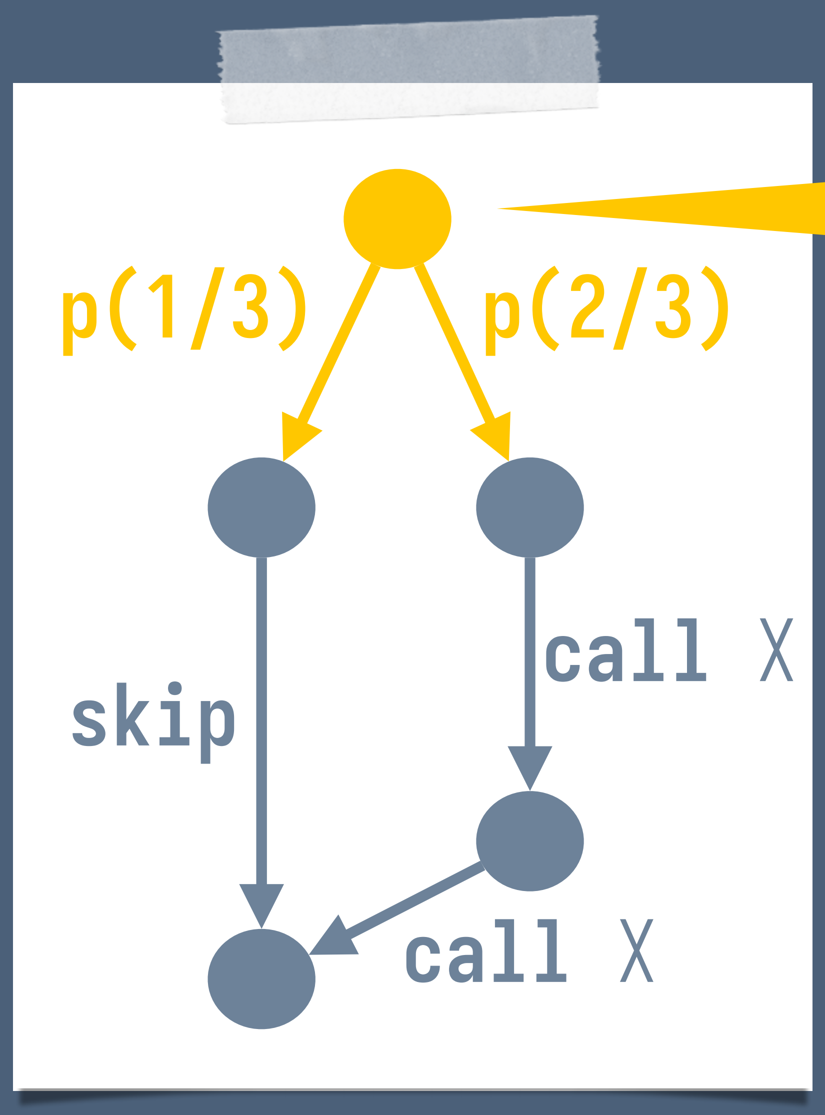
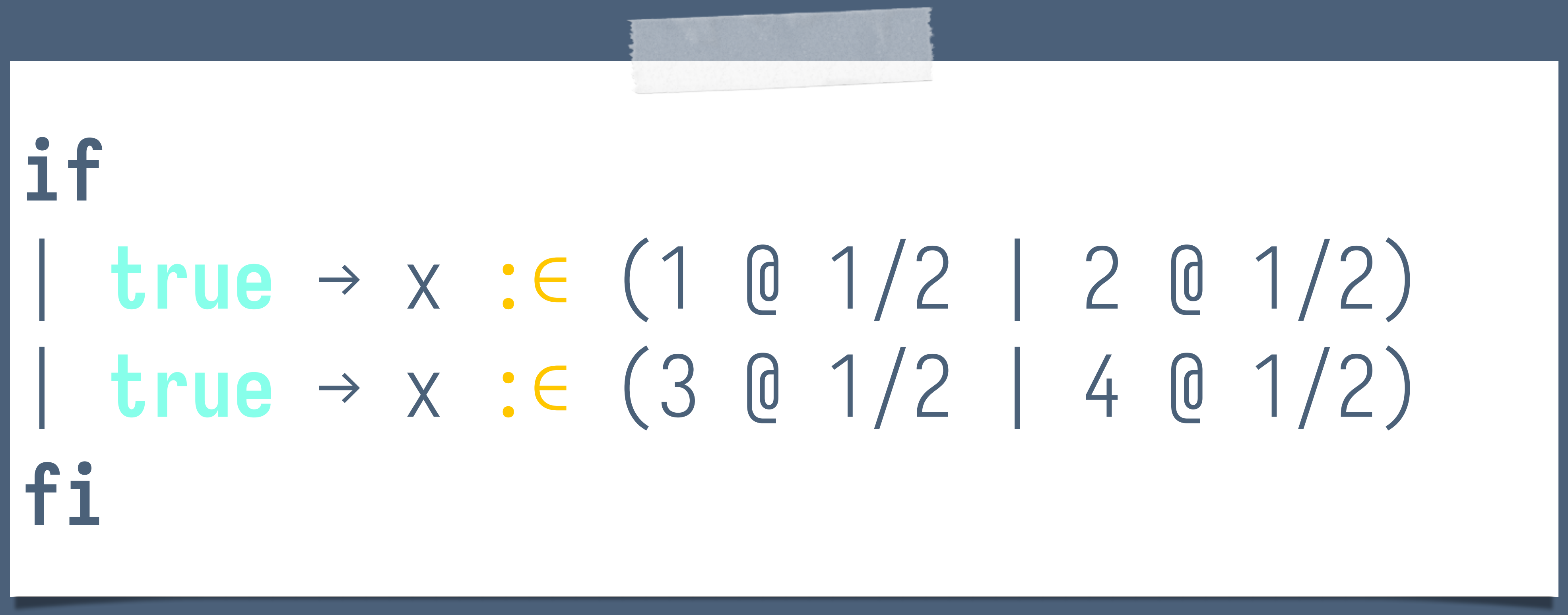


- Confluence is interpreted by ⊕, implicitly
- To support **multiple combine operations**, we need to first distinguish different confluences in the graph

Control-flow Hyper-graph

Control-flow Hyper-graph



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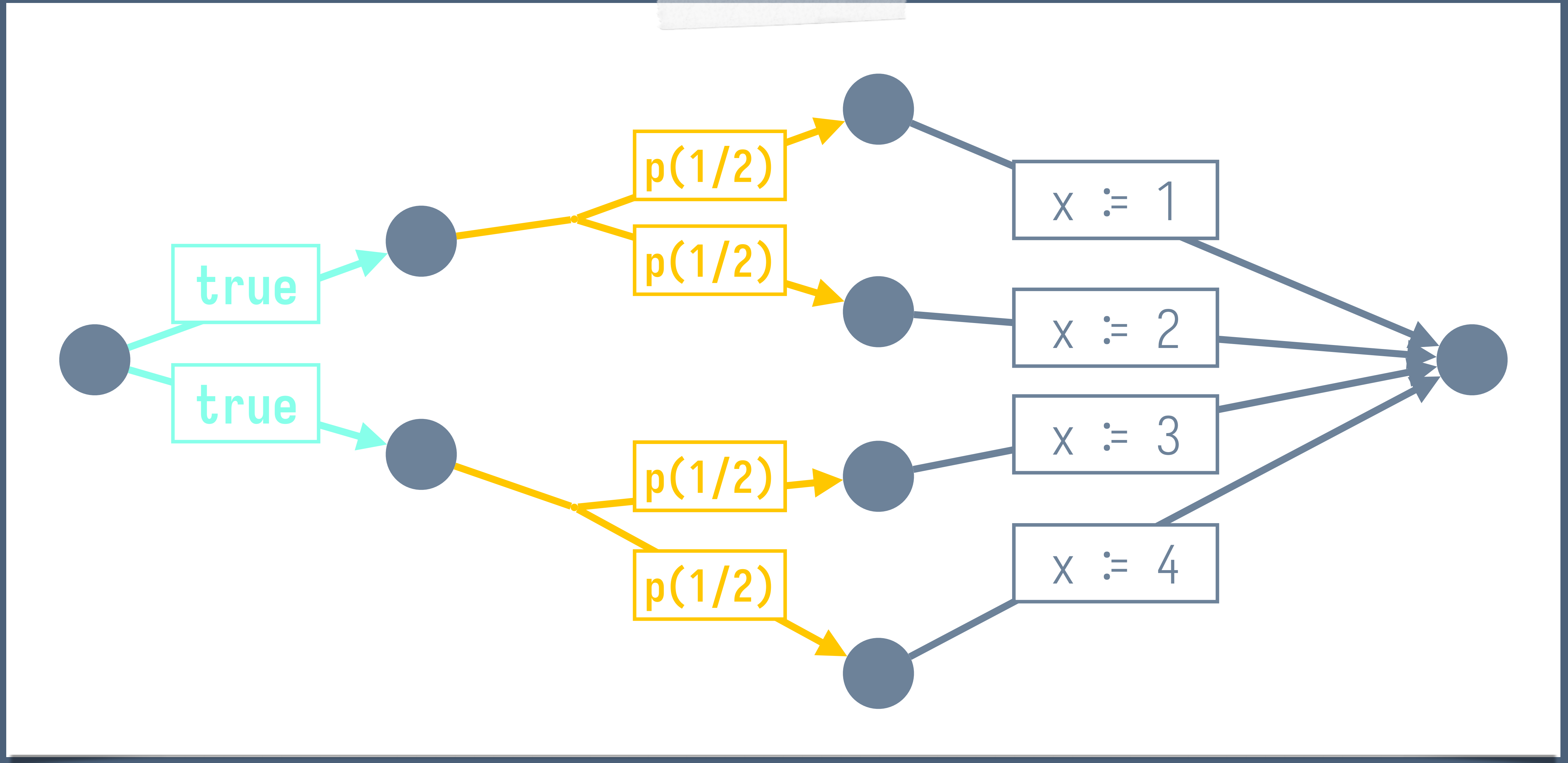

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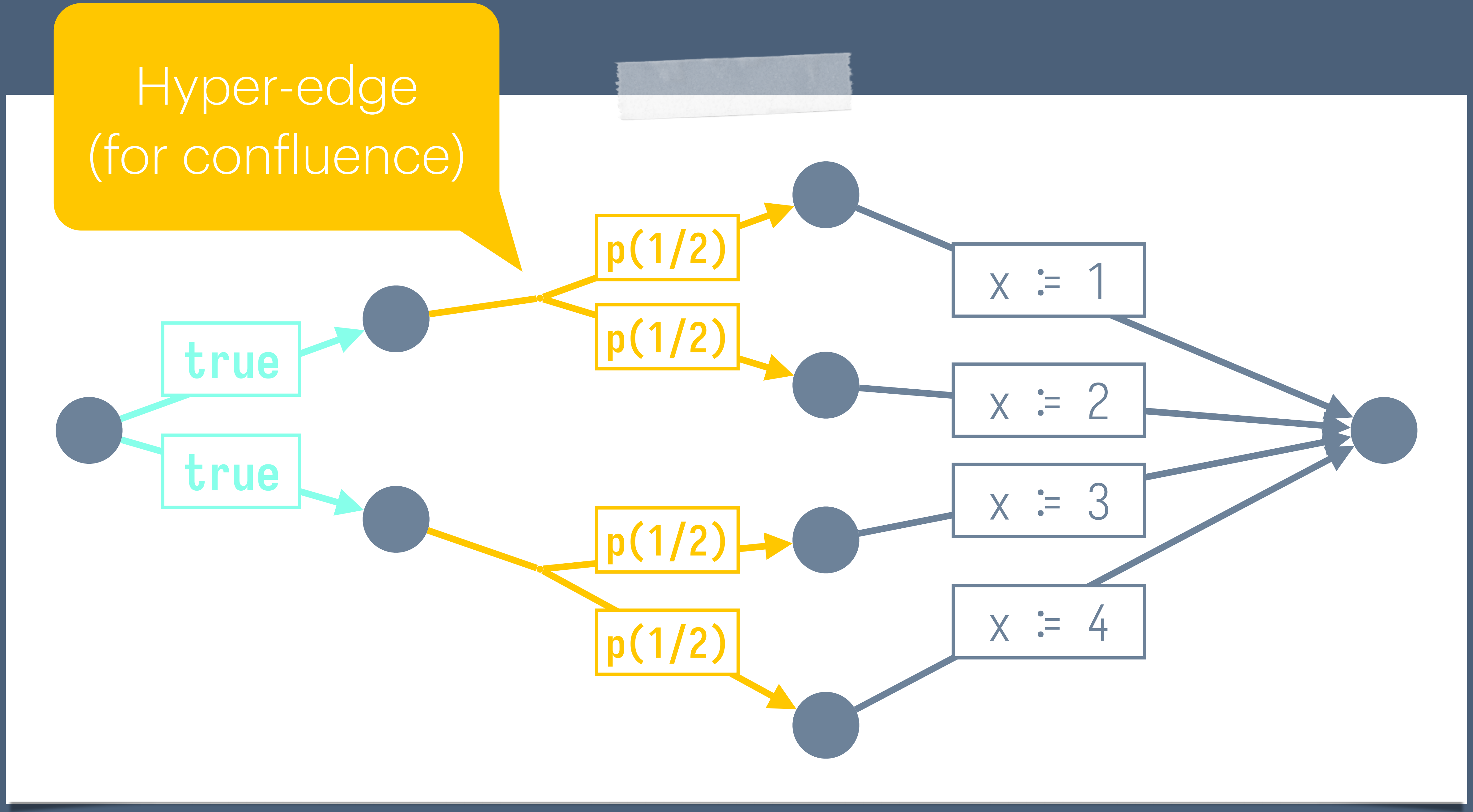

Hyper-edge
(for confluence)

Control-flow Hyper-graph

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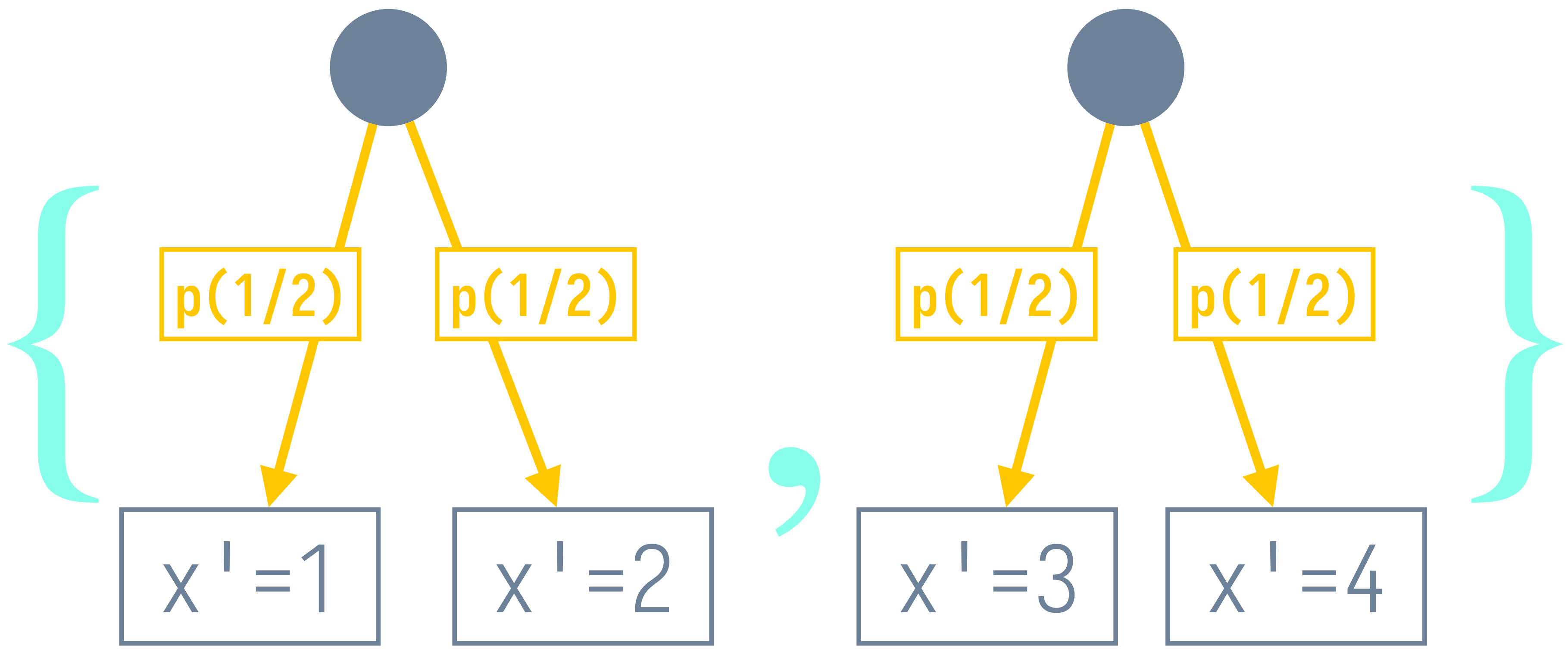

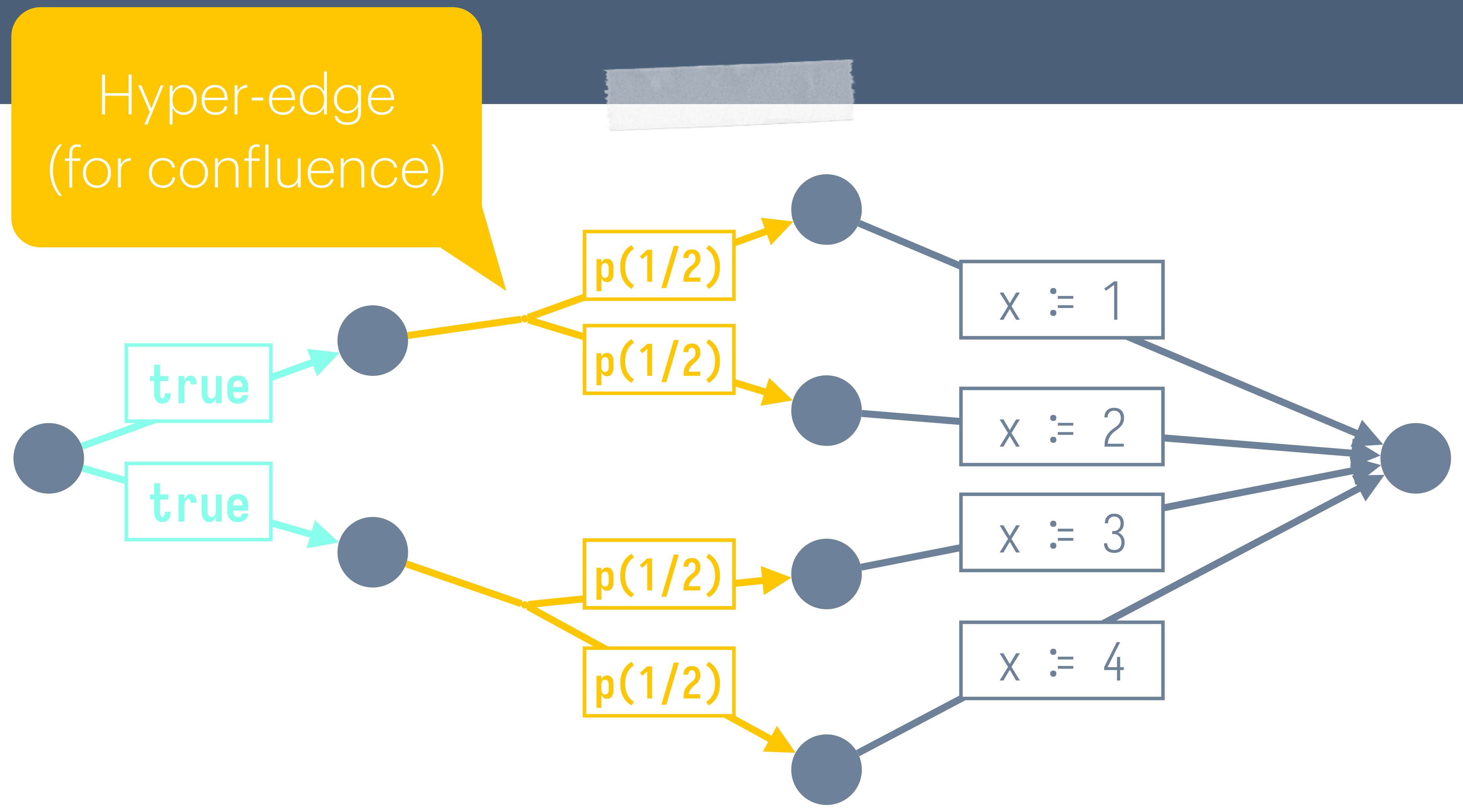
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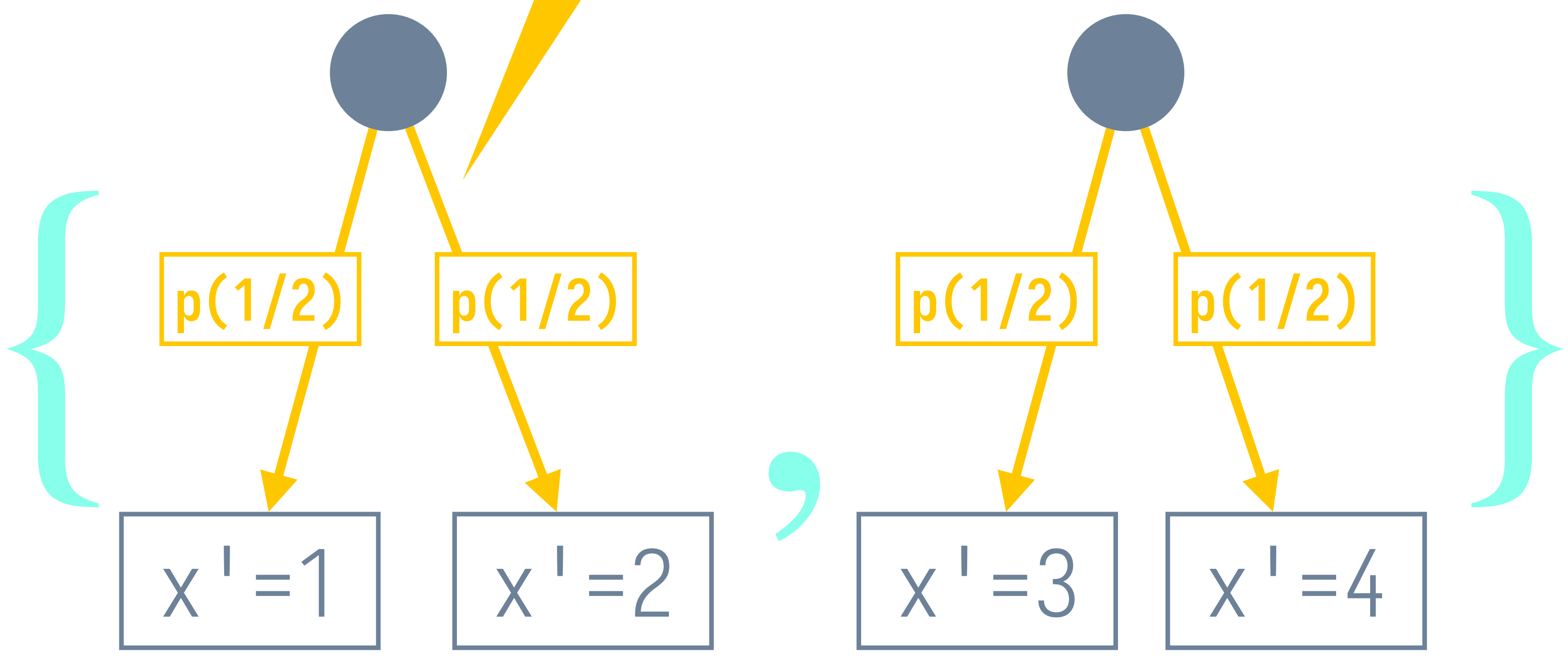
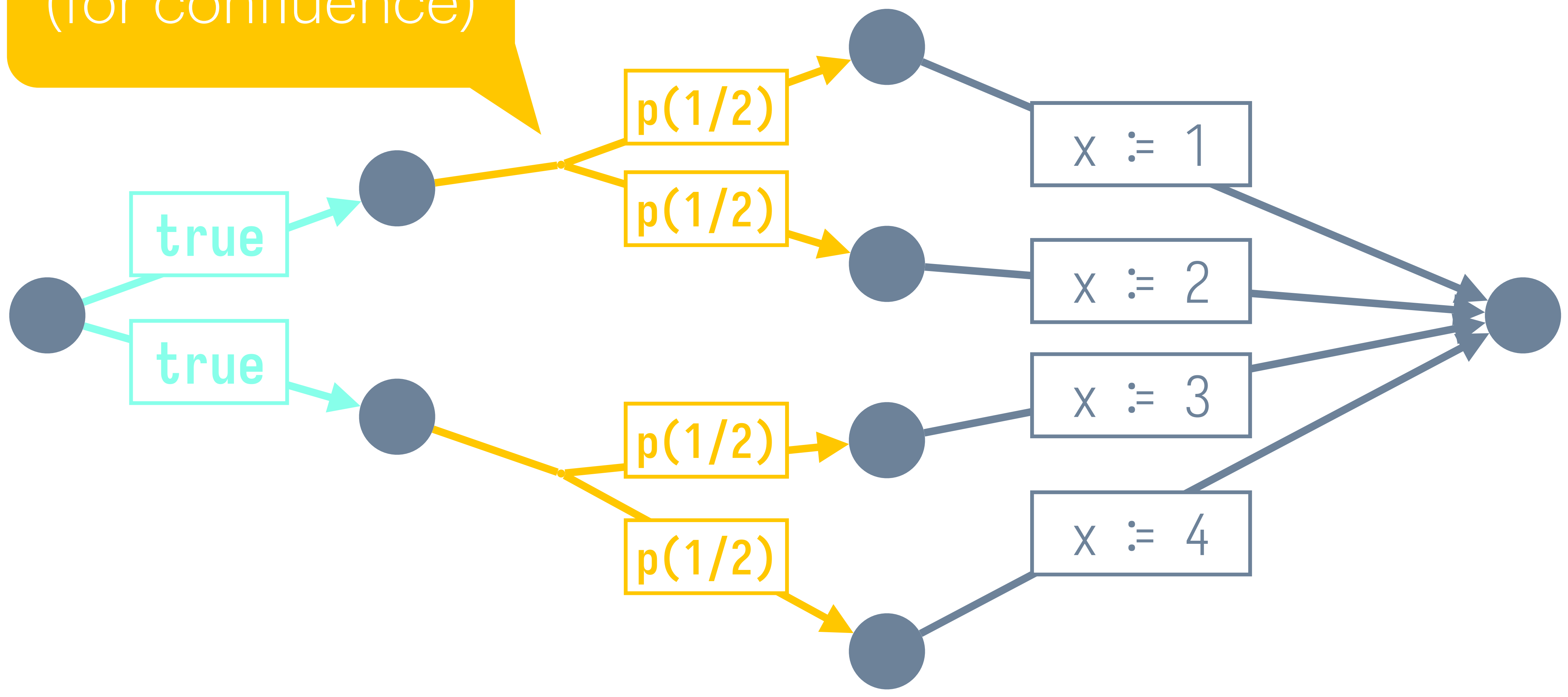


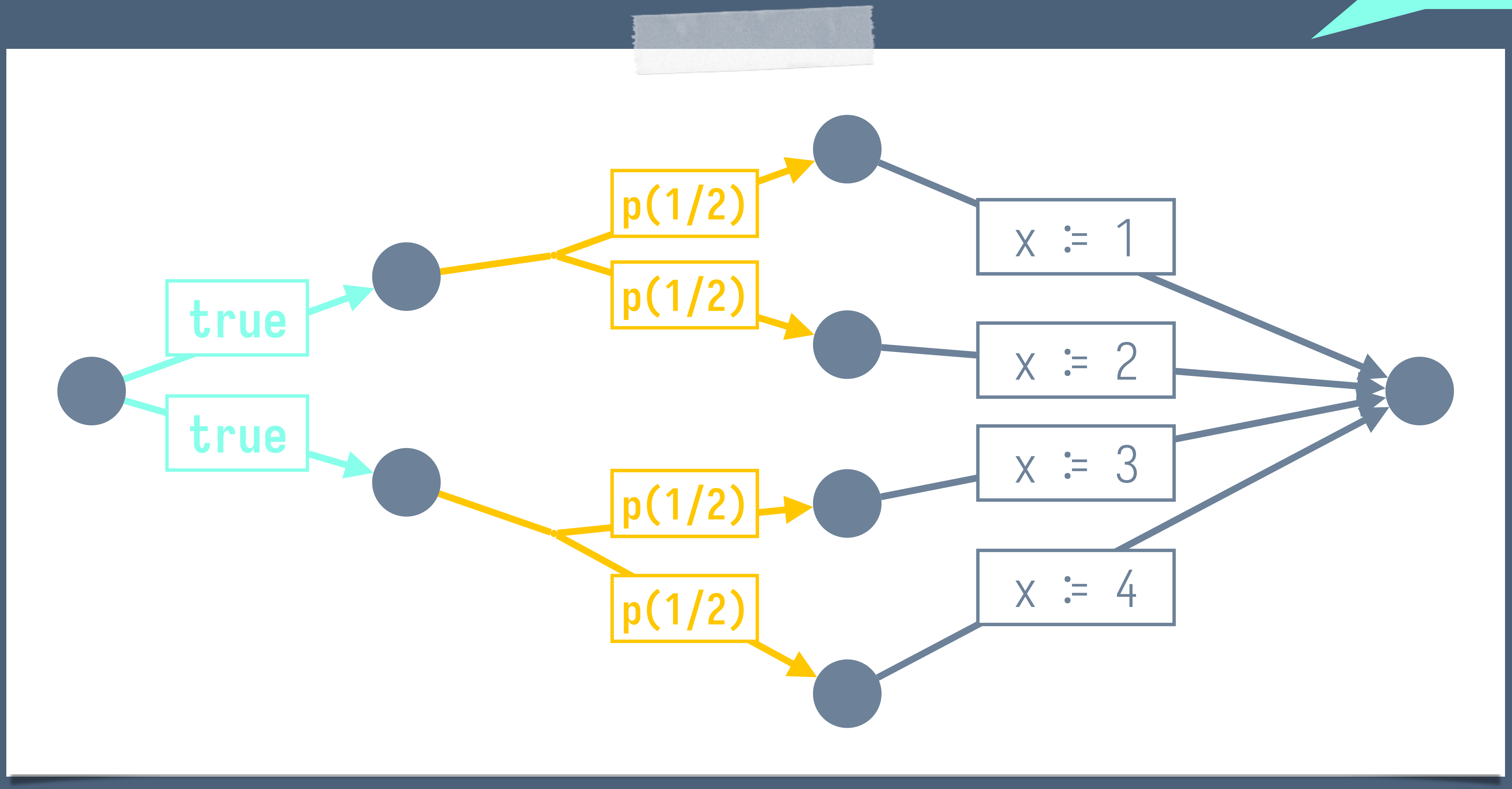
Hyper-path
(like a tree)

Hyper-edge
(for confluence)



Tree Expression

Control-flow hyper-graph



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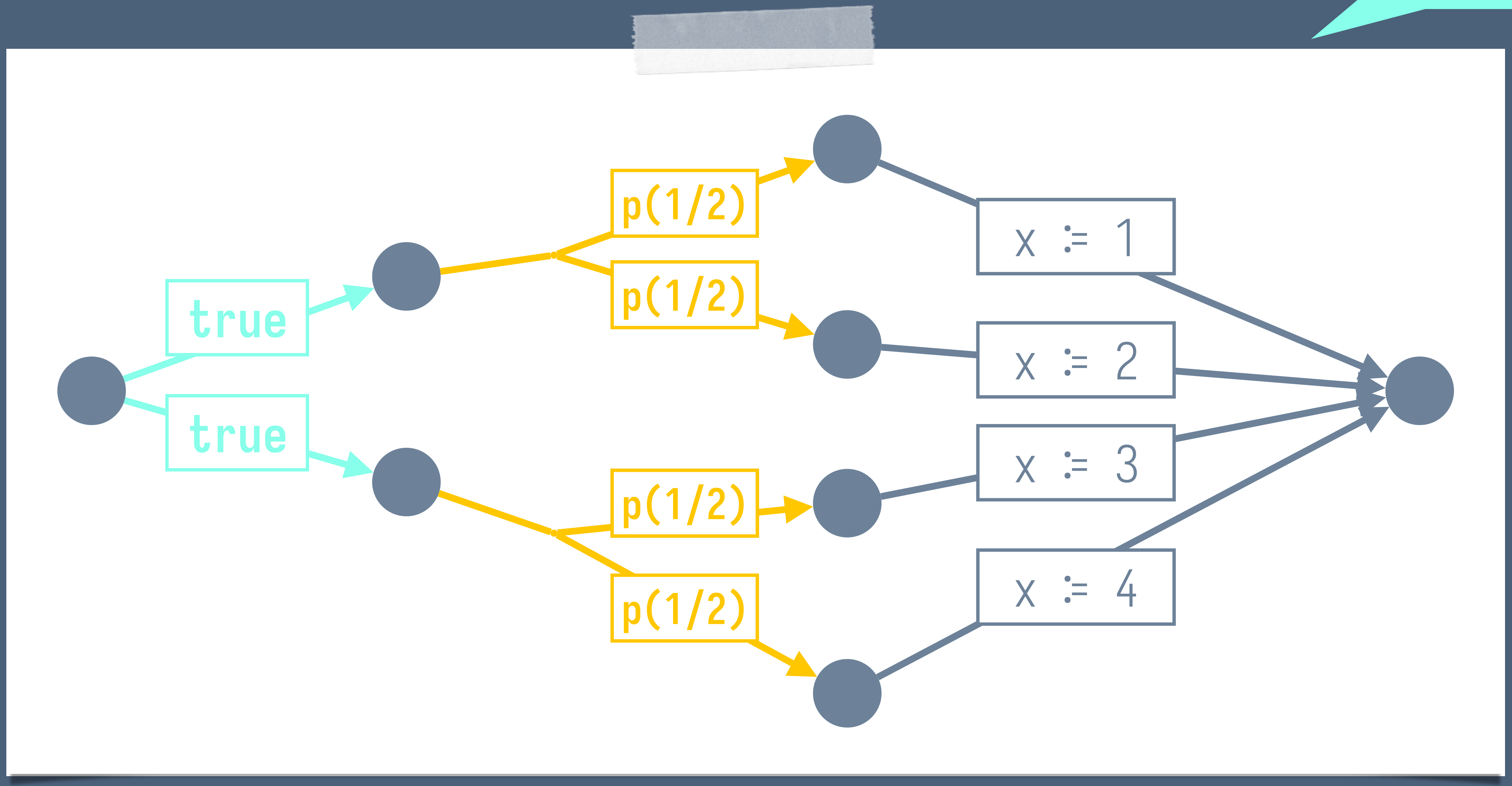
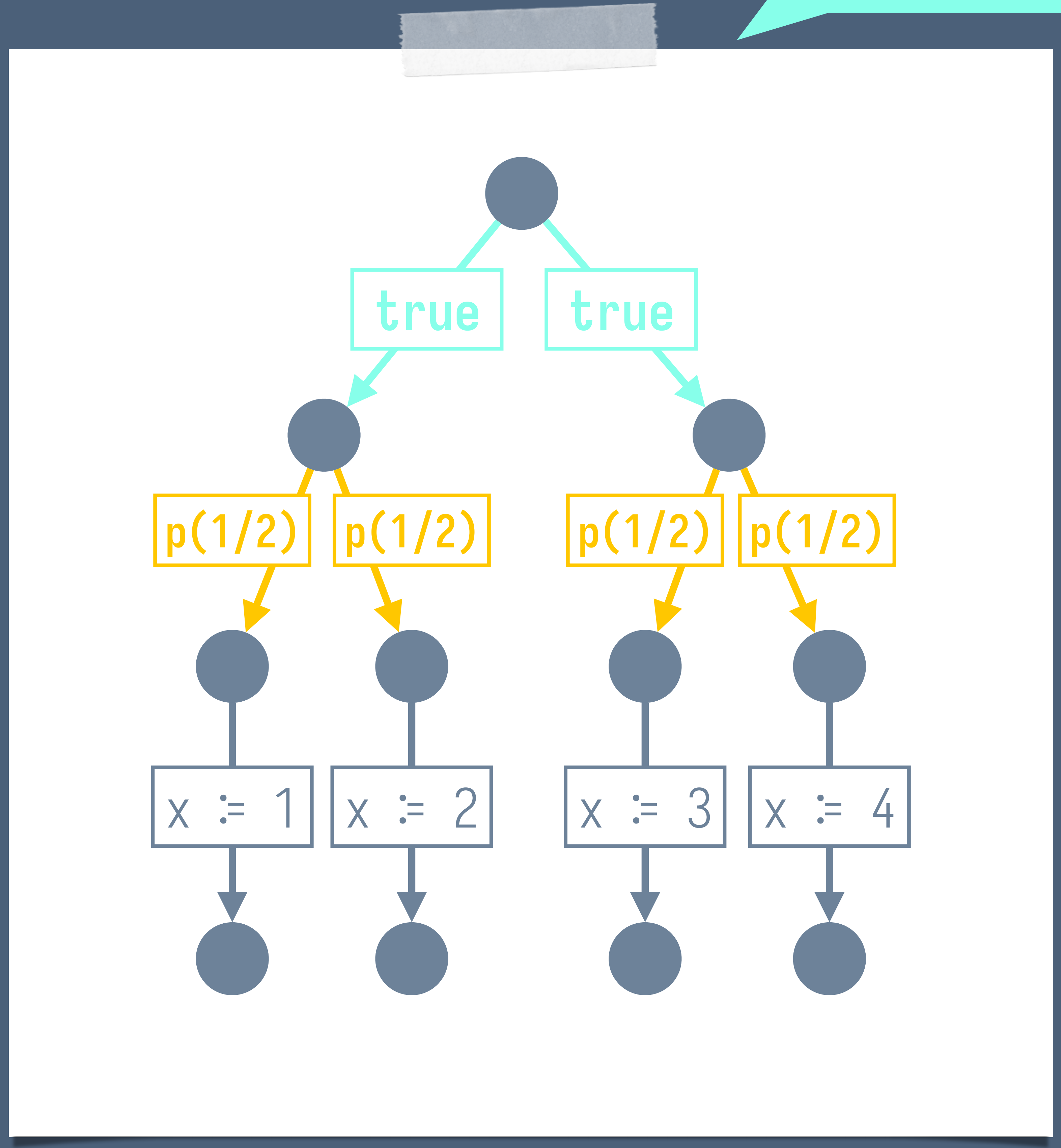


Tree expression (graphic)



Tree Expression

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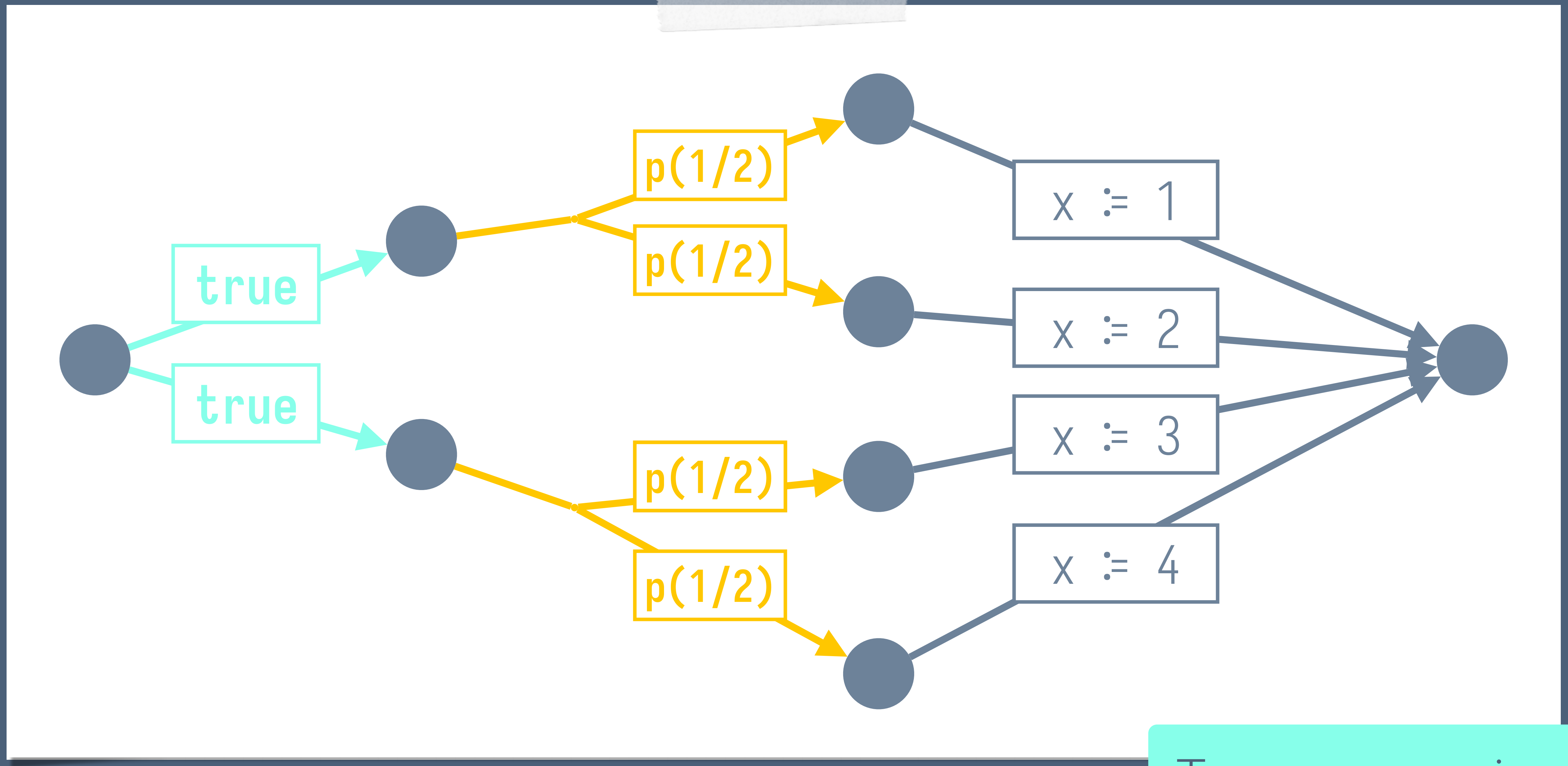

Tree expression (literal)

$$
\begin{aligned}
&ndet(prob[1/2](seq[\mathtt{x:=1}](\varepsilon), seq[\mathtt{x:=2}](\varepsilon)),\\
&\qquad prob[1/2](seq[\mathtt{x:=3}](\varepsilon), seq[\mathtt{x:=4}](\varepsilon)))
\end{aligned}
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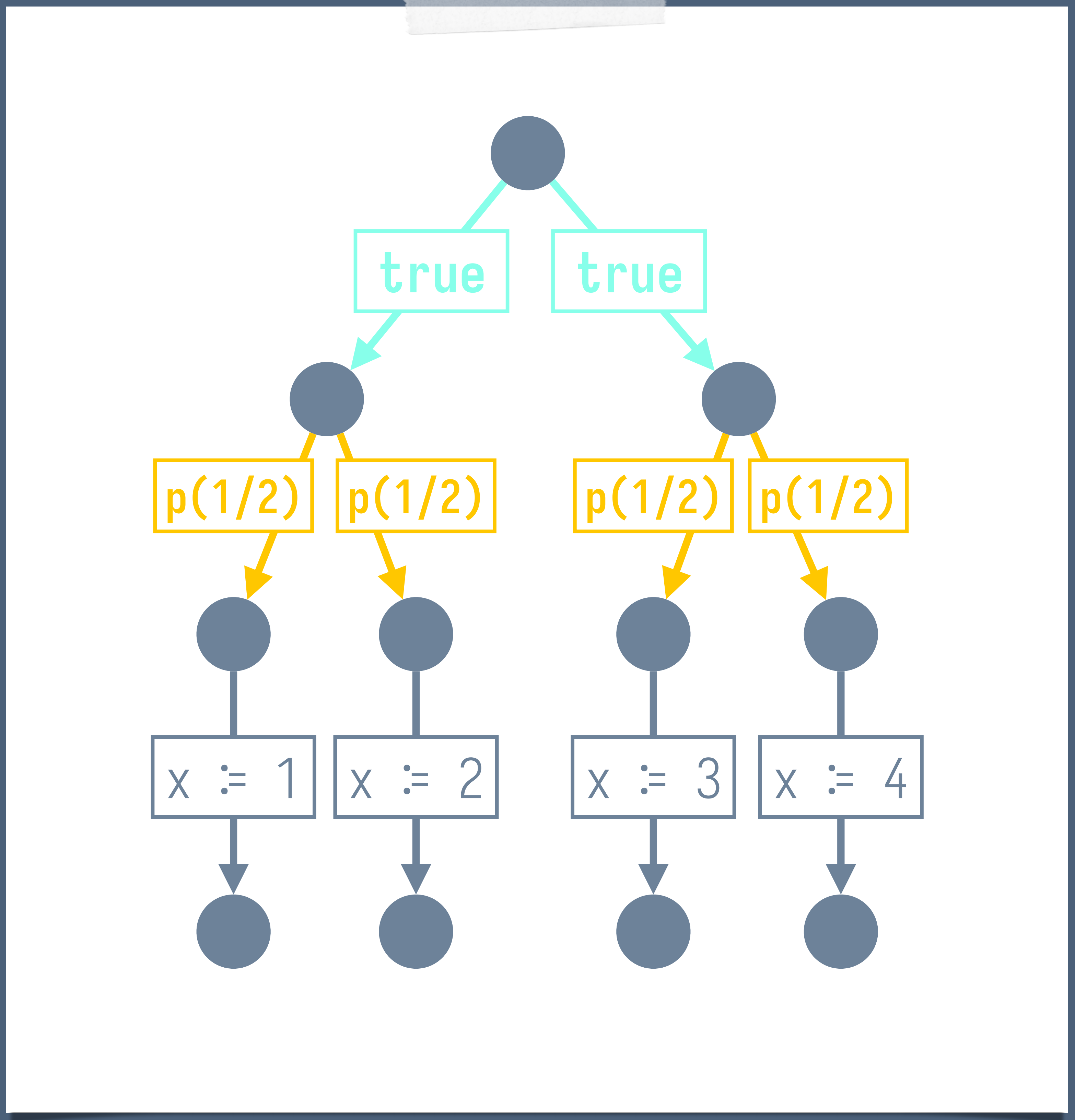

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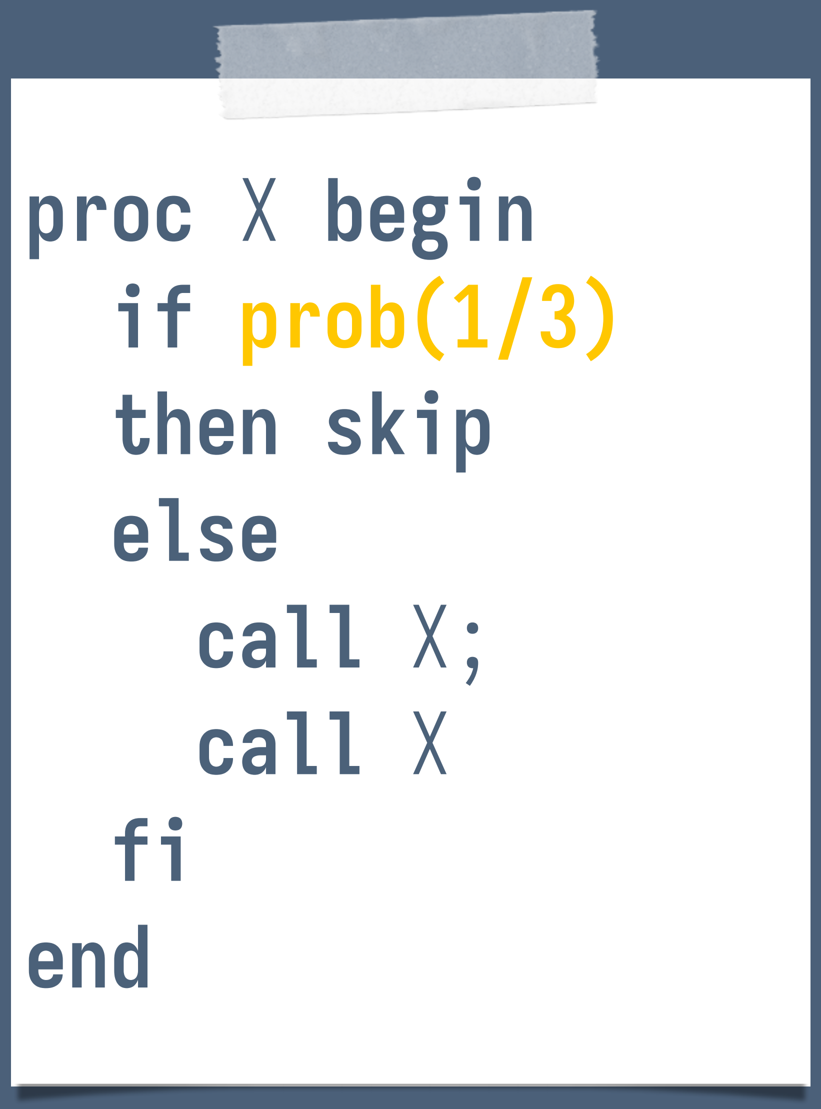

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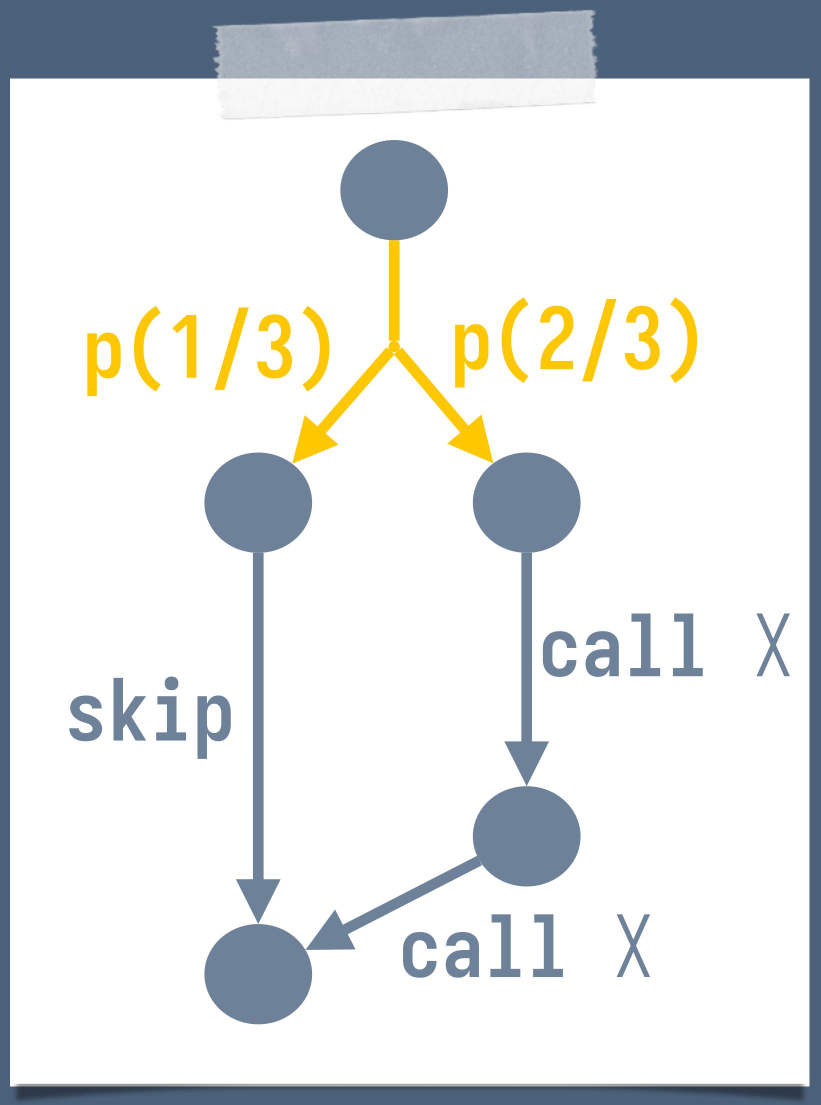

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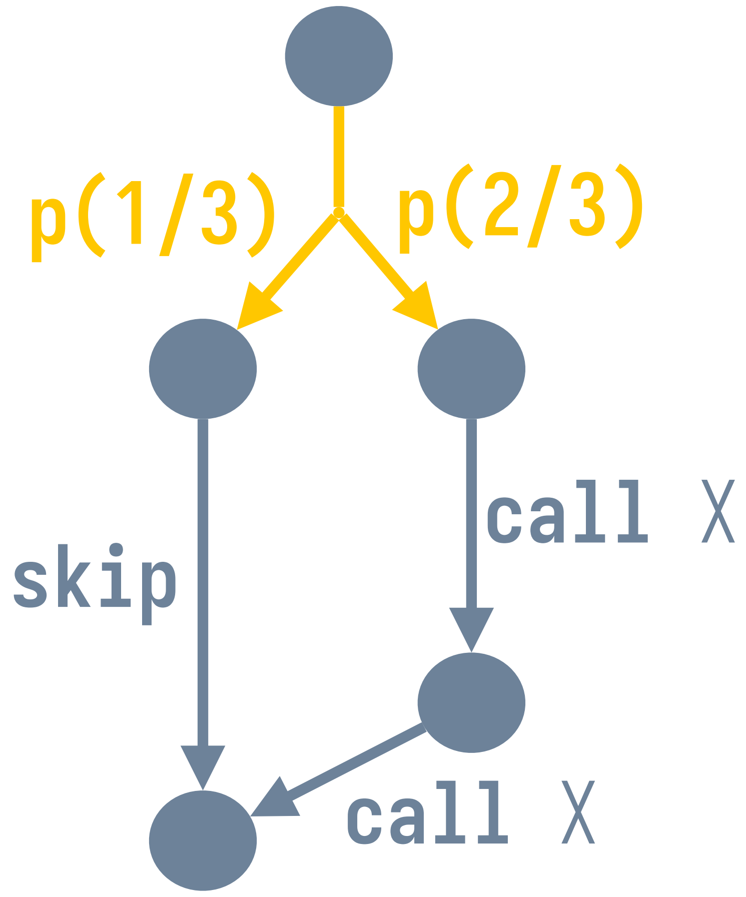

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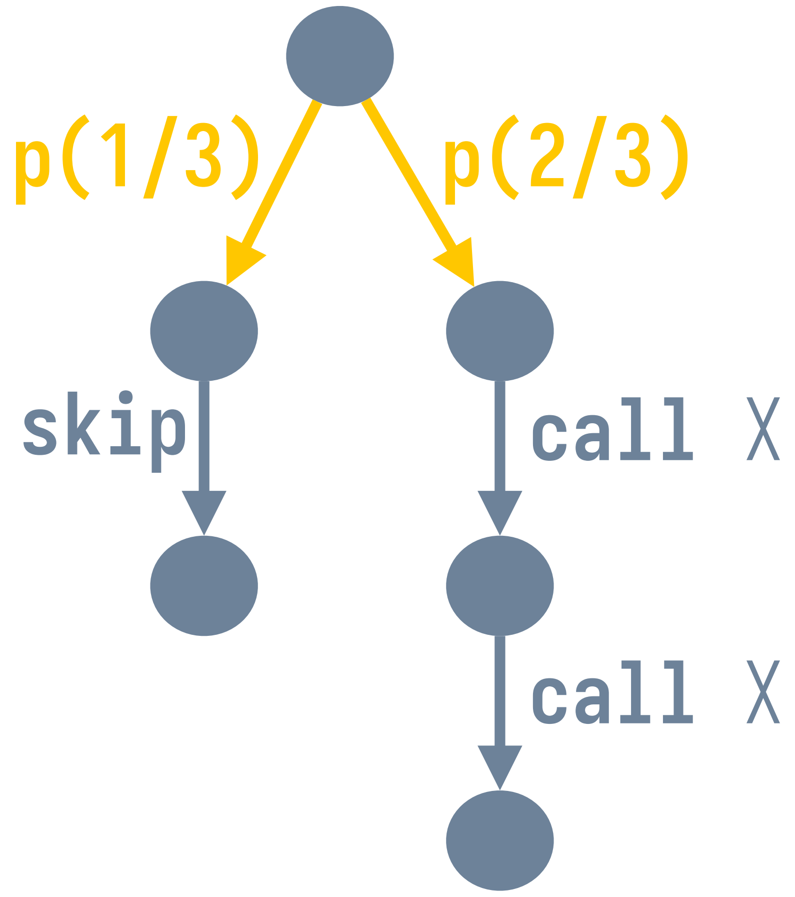

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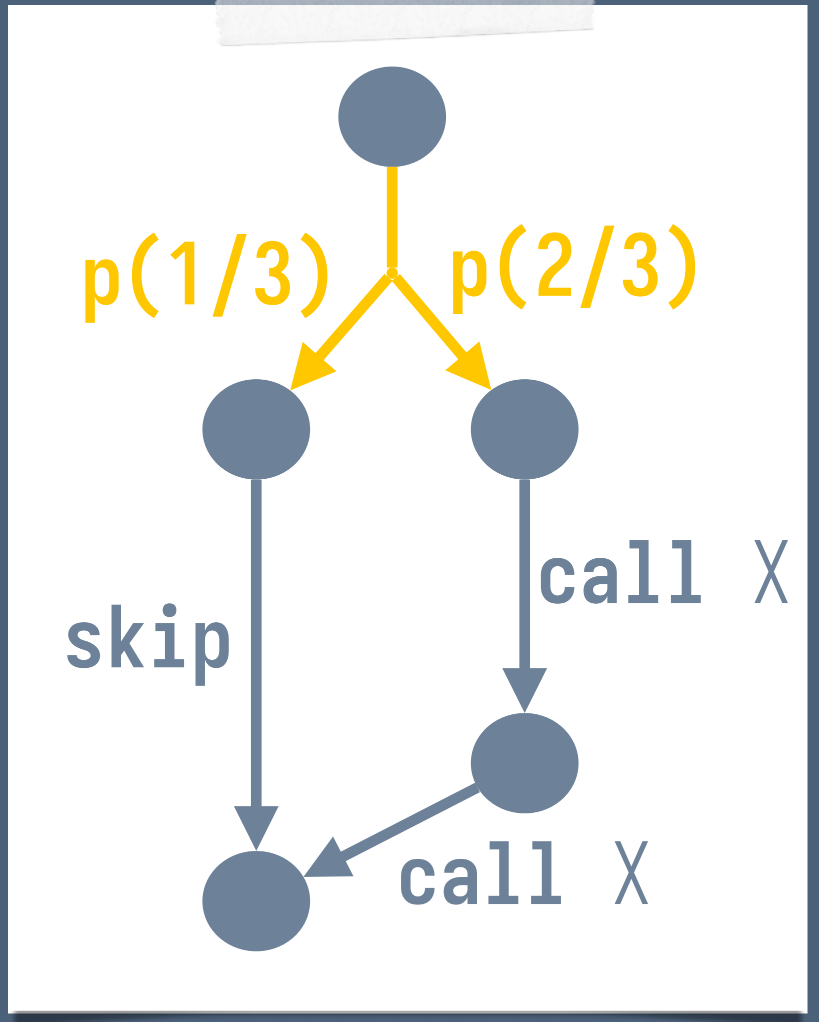

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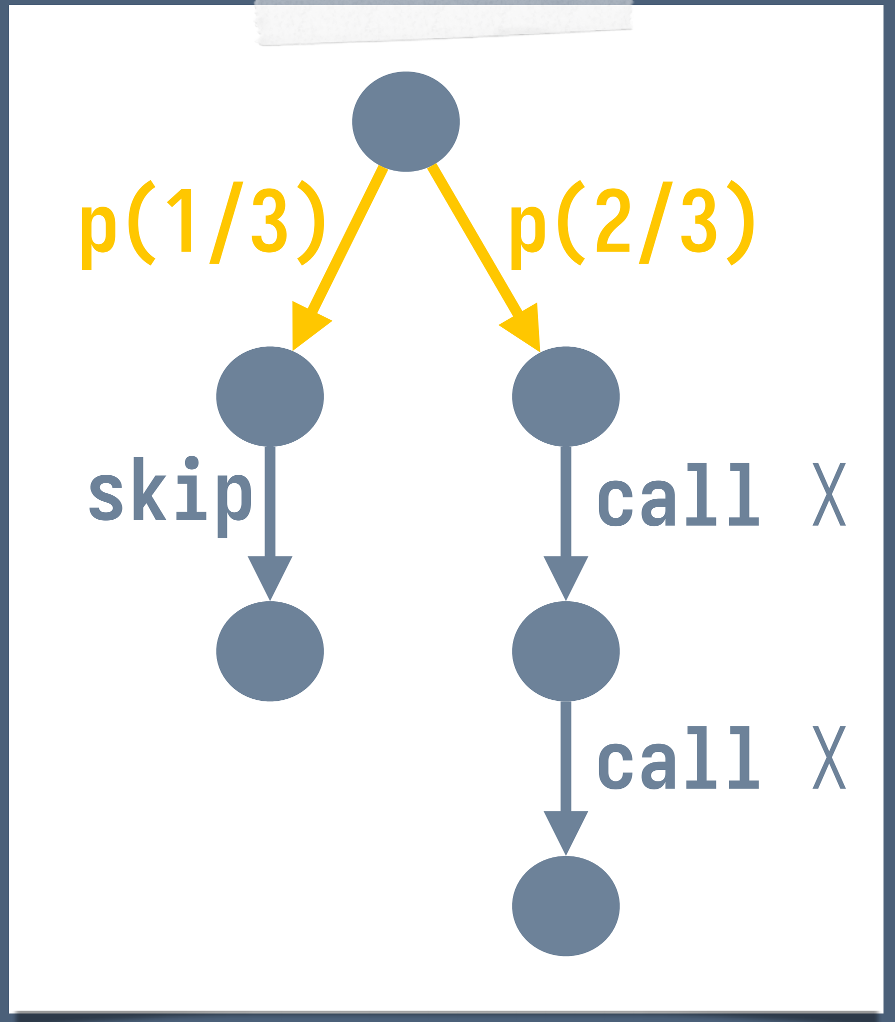
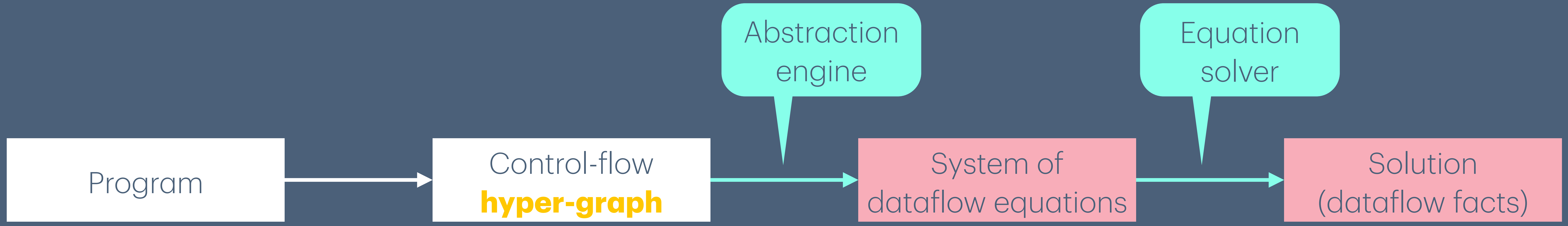


Tree expression (literal)

$$X = prob[1/3](seq[\mathtt{skip}](\varepsilon), call[X](call[X](\varepsilon)))$$

Towards Multiple Combine Operations



Abstraction engine

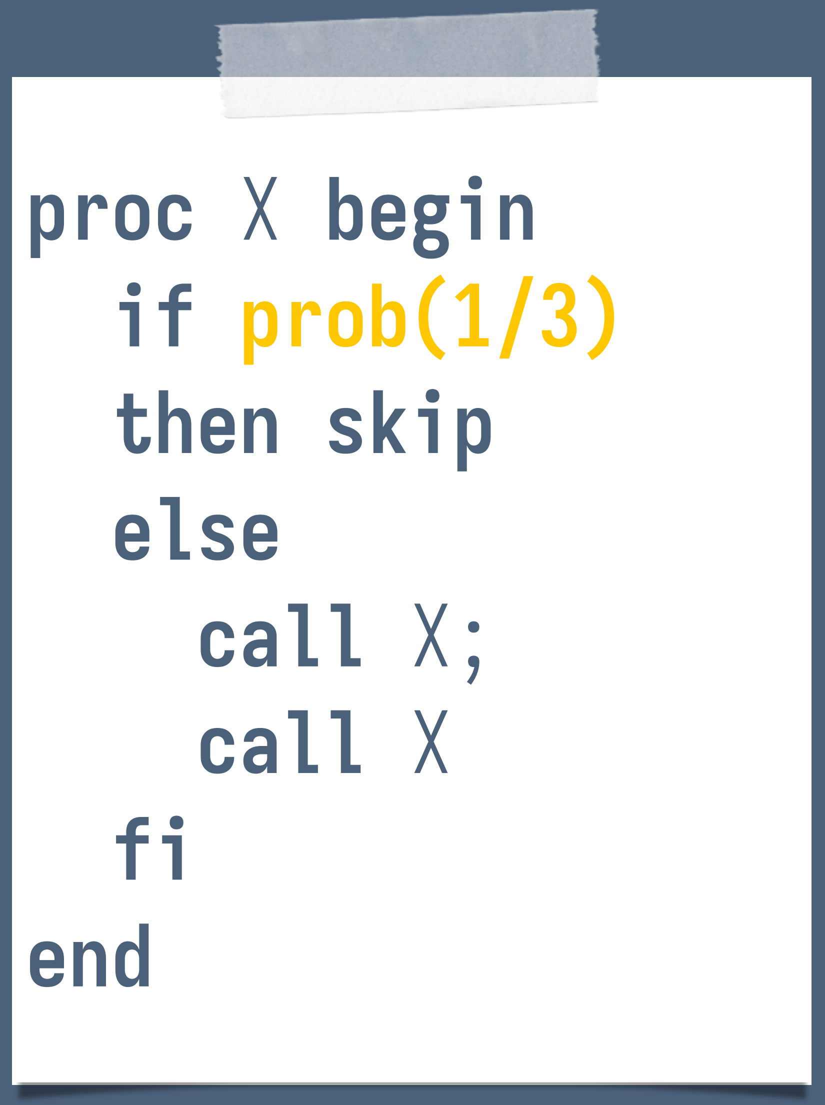
Equation solver

Program

Control-flow **hyper-graph**

System of dataflow equations

Solution (dataflow facts)



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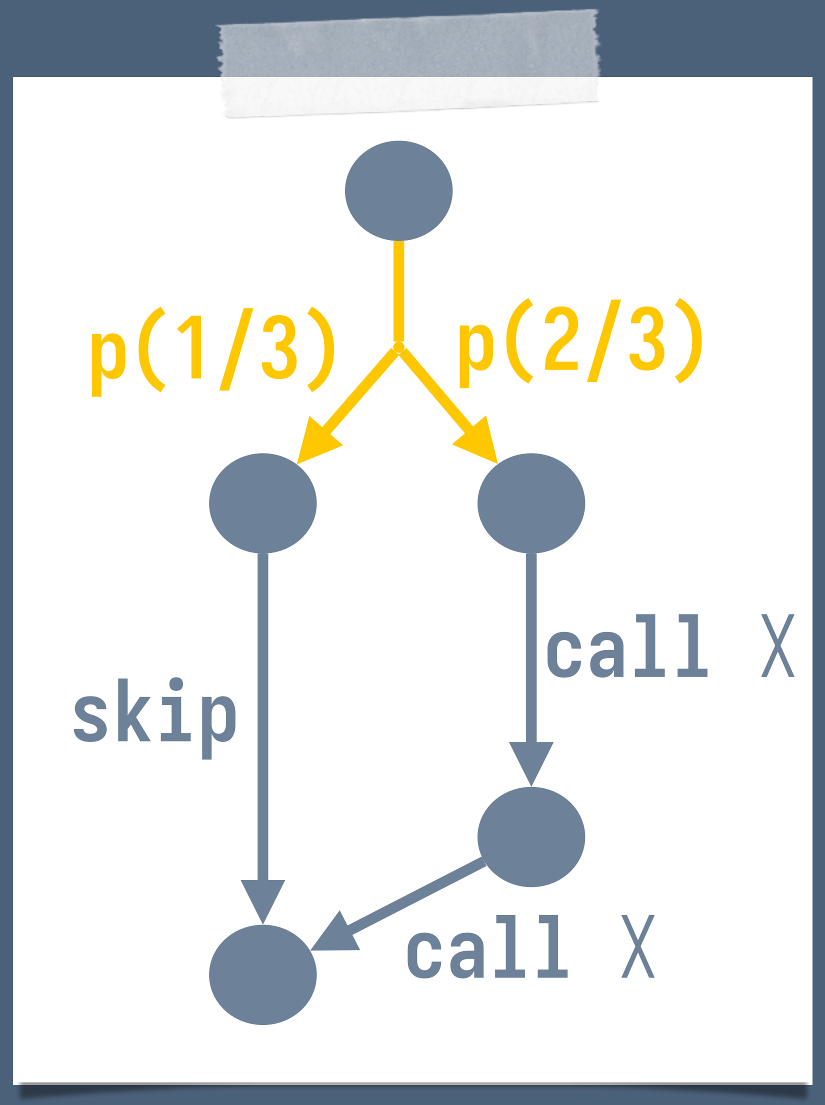

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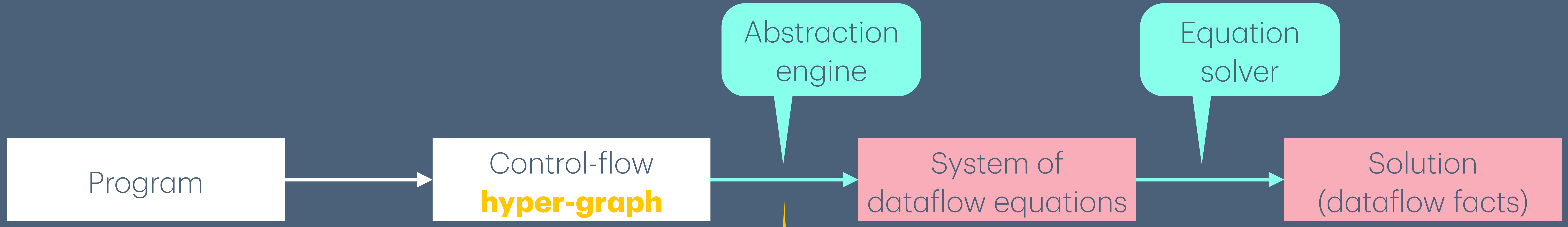
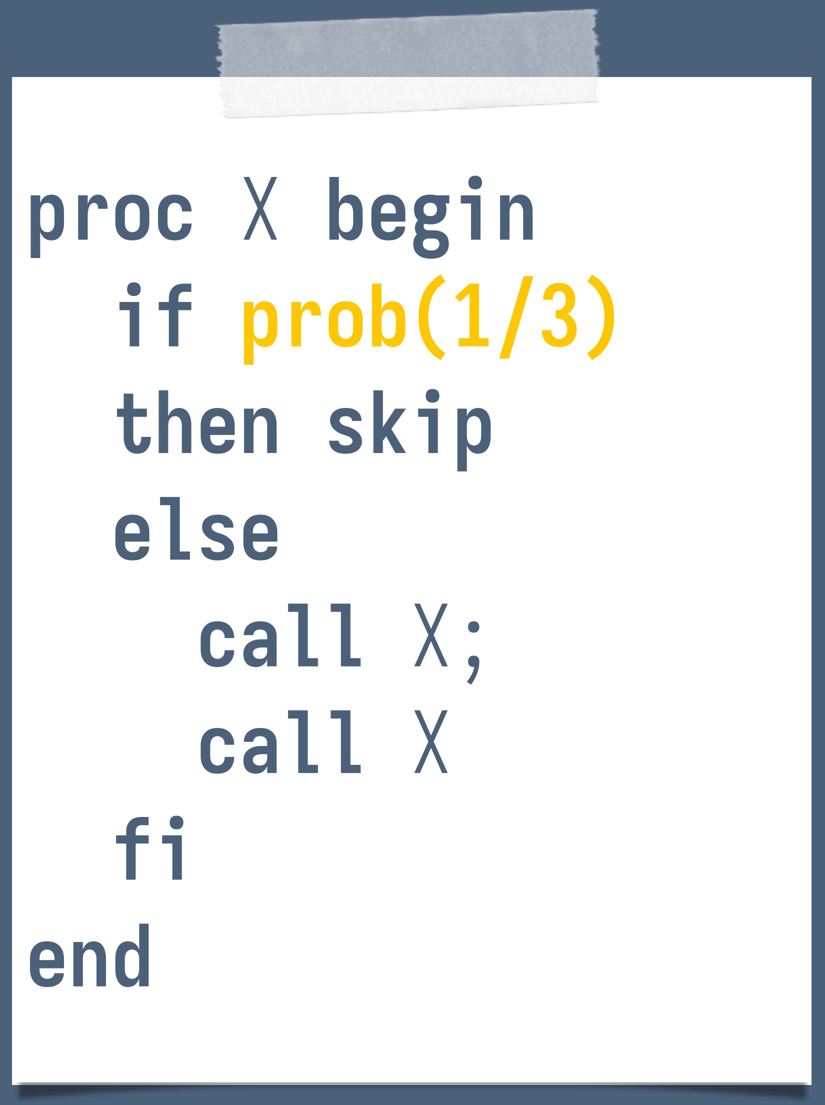


Abstraction engine

Equation solver

Program → Control-flow **hyper-graph** → System of dataflow equations → Solution (dataflow facts)

How to interpret tree expressions, algebraically?



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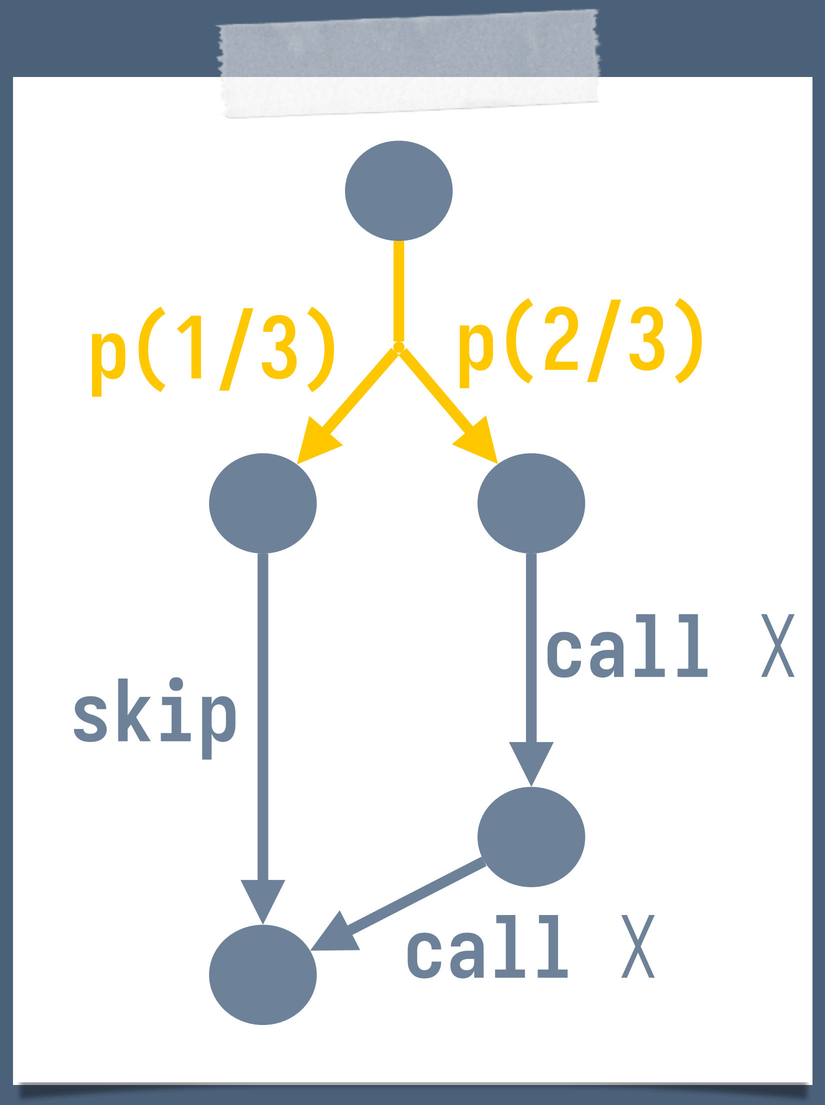

Markov Algebras

semirings + more combine operations

Markov Algebras

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$$\left\langle M, \oplus , \otimes , {}_{\phi}\oplus , \sqcap , \underline{0}, \underline{1} \right\rangle$$

Markov Algebras

semirings + more combine operations

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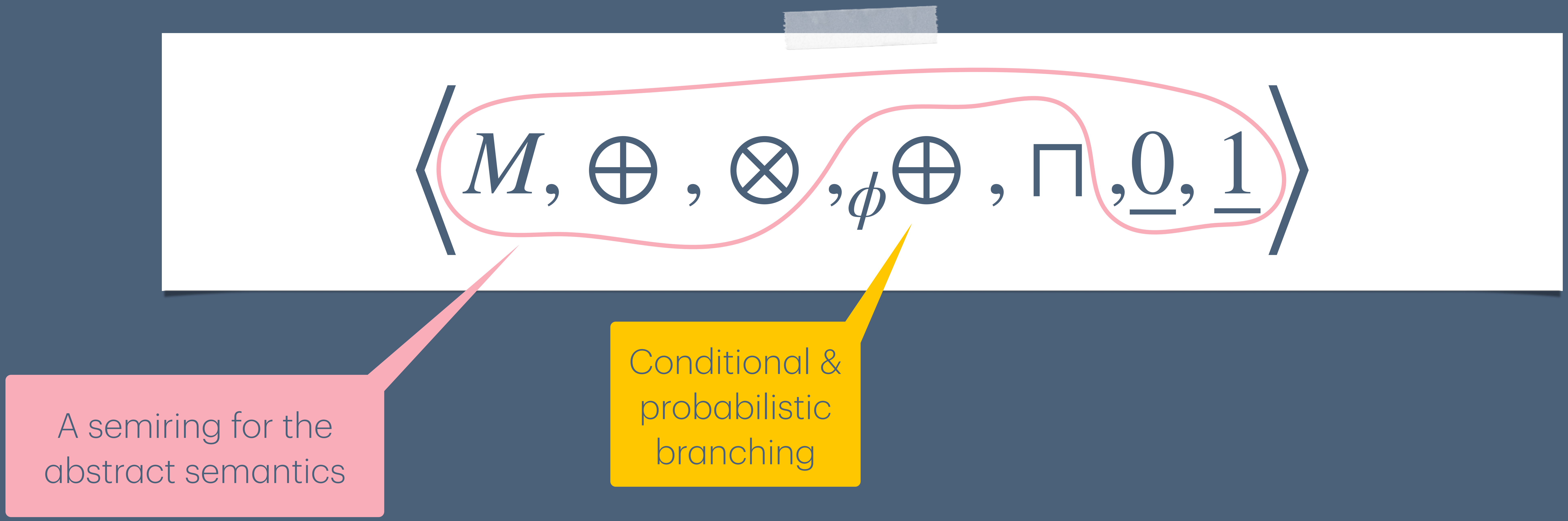
A semiring for the abstract semantics

Markov Algebras

semirings + more combine operations



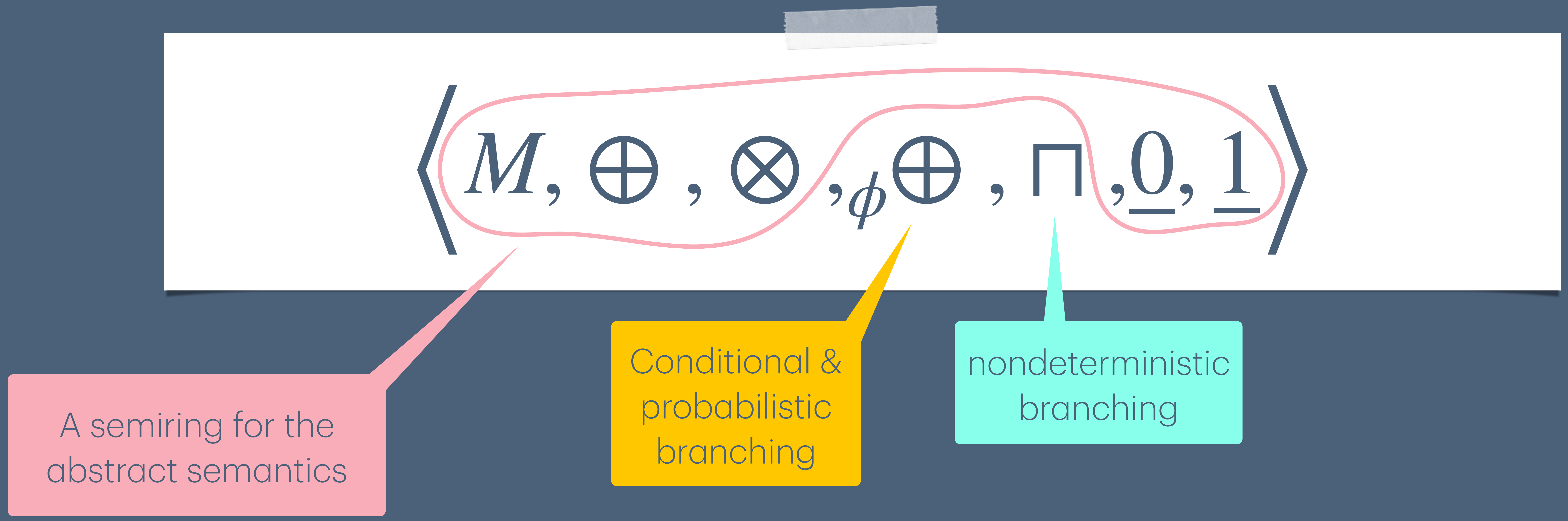
$$\left\langle M, \oplus, \otimes, {}_{\phi}\oplus, \sqcap, \underline{0}, \underline{1} \right\rangle$$

A semiring for the abstract semantics

Conditional & probabilistic branching

Markov Algebras

semirings + more combine operations



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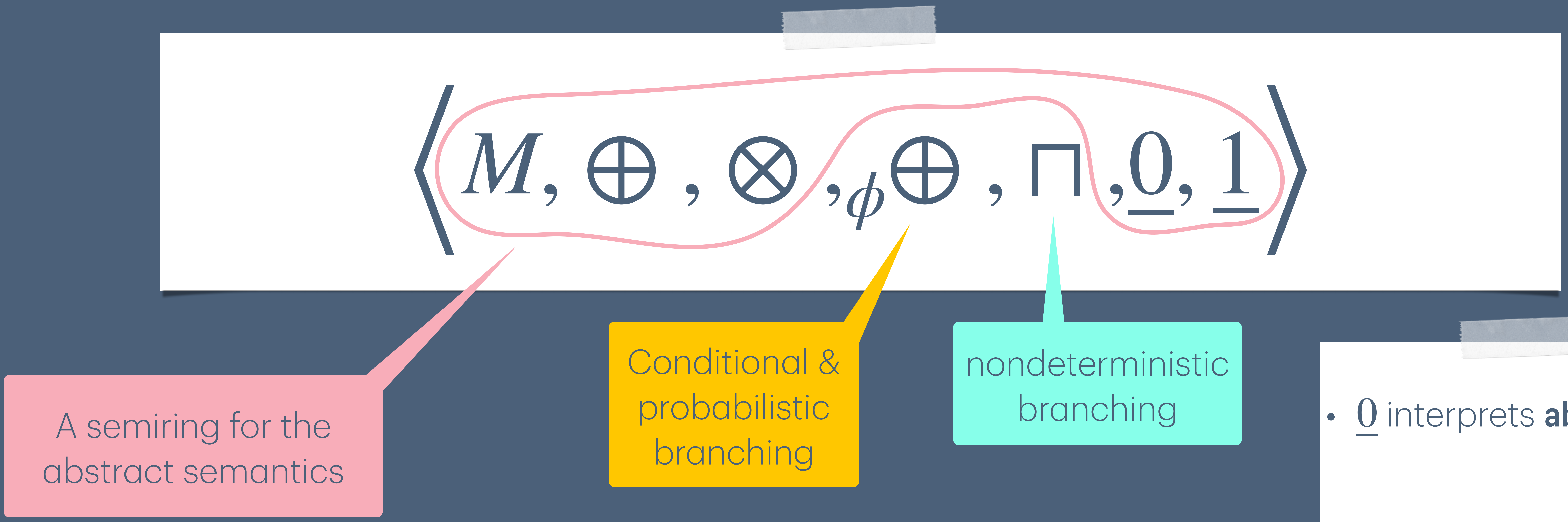

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- $\underline{0}$ interprets **`abort`**

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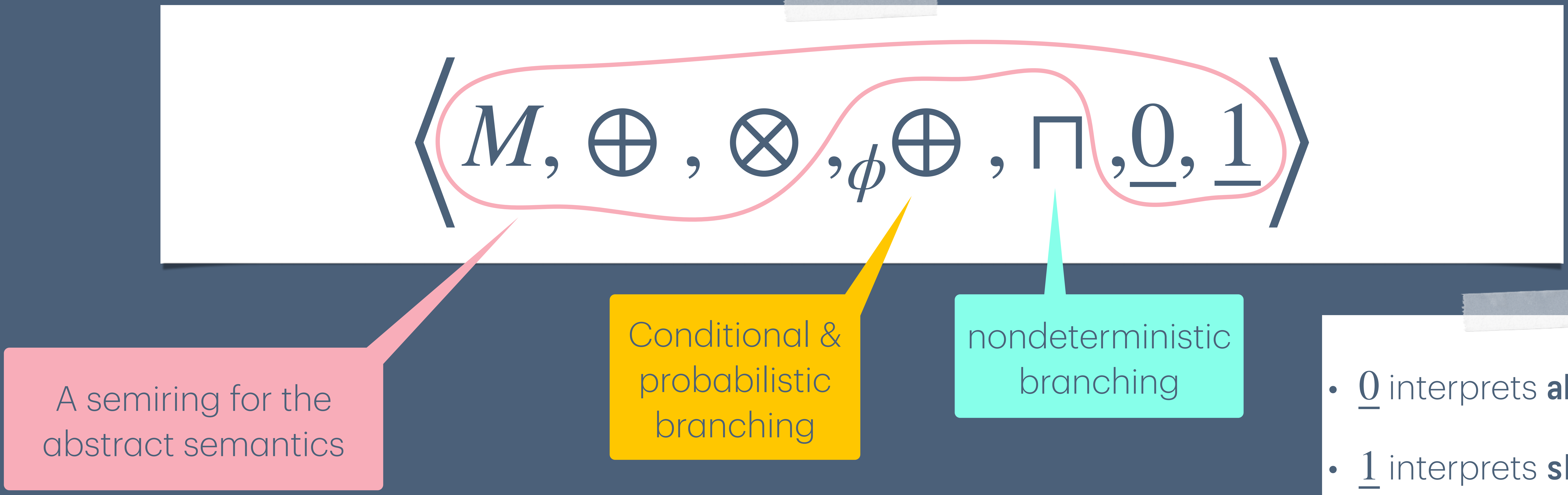

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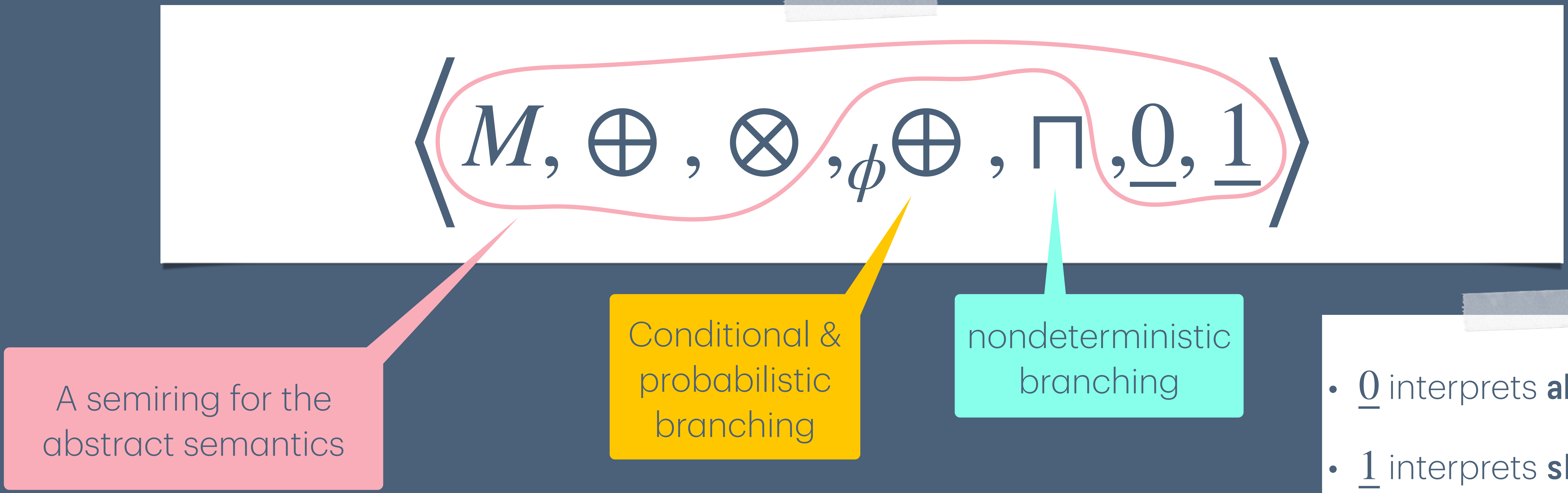

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A semiring for the abstract semantics

Conditional & probabilistic branching

nondeterministic branching

- $\underline{0}$ interprets **`abort`**
- $\underline{1}$ interprets **`skip`**
- Algebraic laws:
  - $a\,{}_{p}\!\oplus b = b\,{}_{1-p}\!\oplus a$
  - $a\,{}_{\varphi}\!\oplus b = b\,{}_{\neg\varphi}\!\oplus a$
  - ……

Interpretation of Tree Expressions

using a Markov Algebra

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For the termination-probability analysis:

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\begin{aligned}
a \oplus b &= a + b\\
a \otimes b &= a \cdot b\\
a\,{}_p\!\oplus b &= p \cdot a + (1-p) \cdot b\\
&\ldots
\end{aligned}
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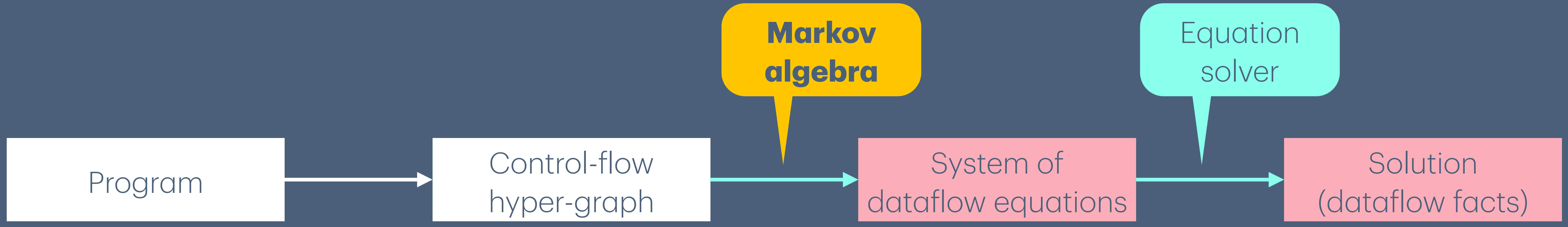
Towards Multiple Combine Operations



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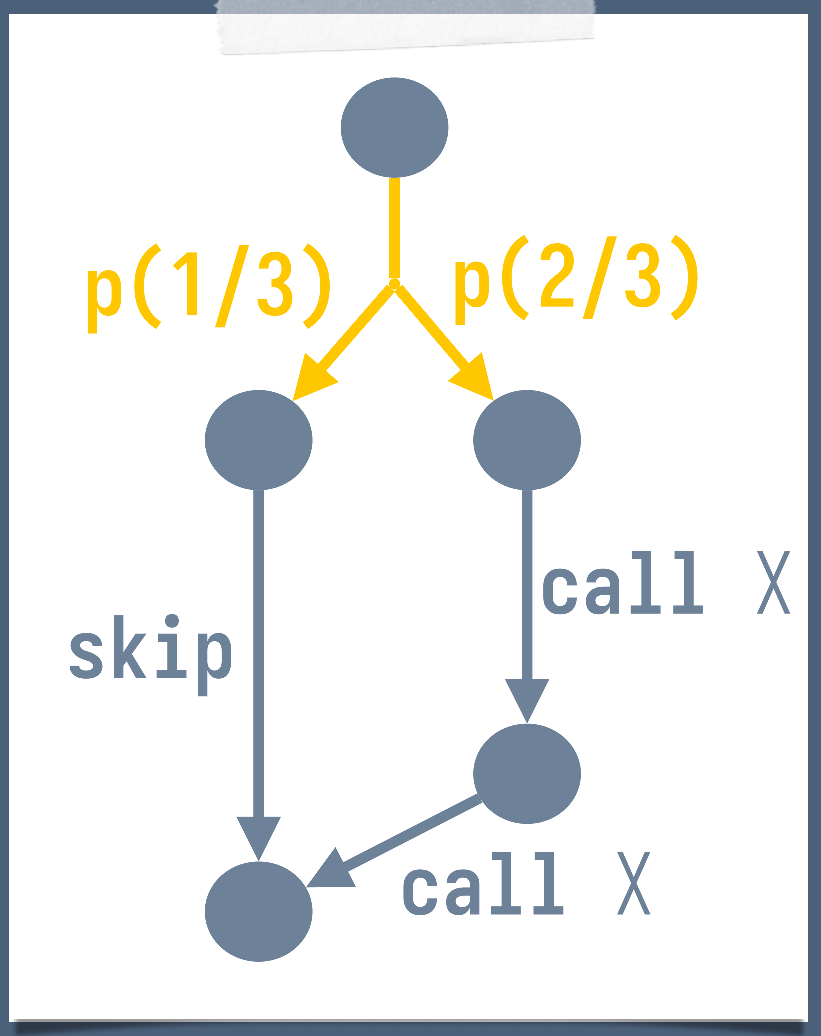

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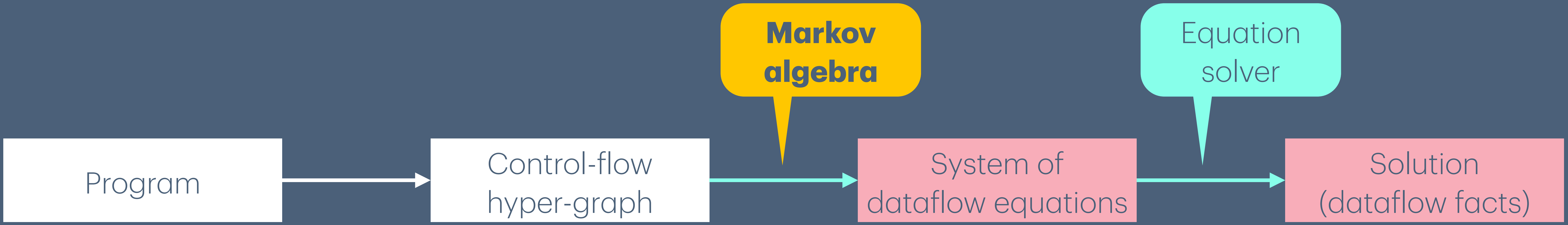

Markov algebra

Equation solver

Program

Control-flow hyper-graph

System of dataflow equations

Solution (dataflow facts)

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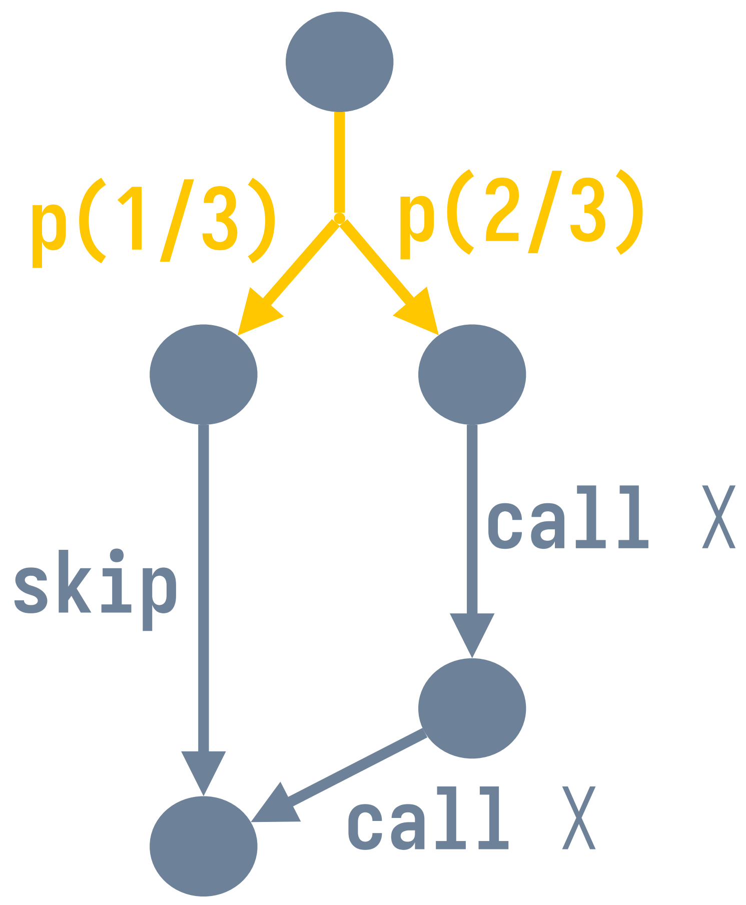

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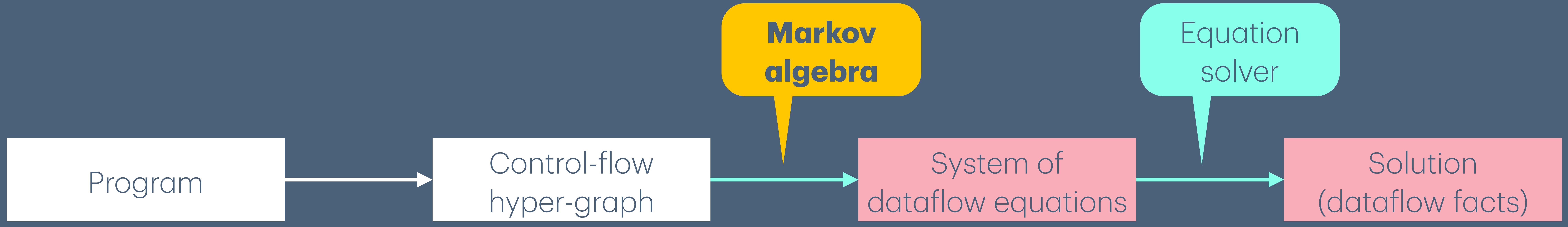

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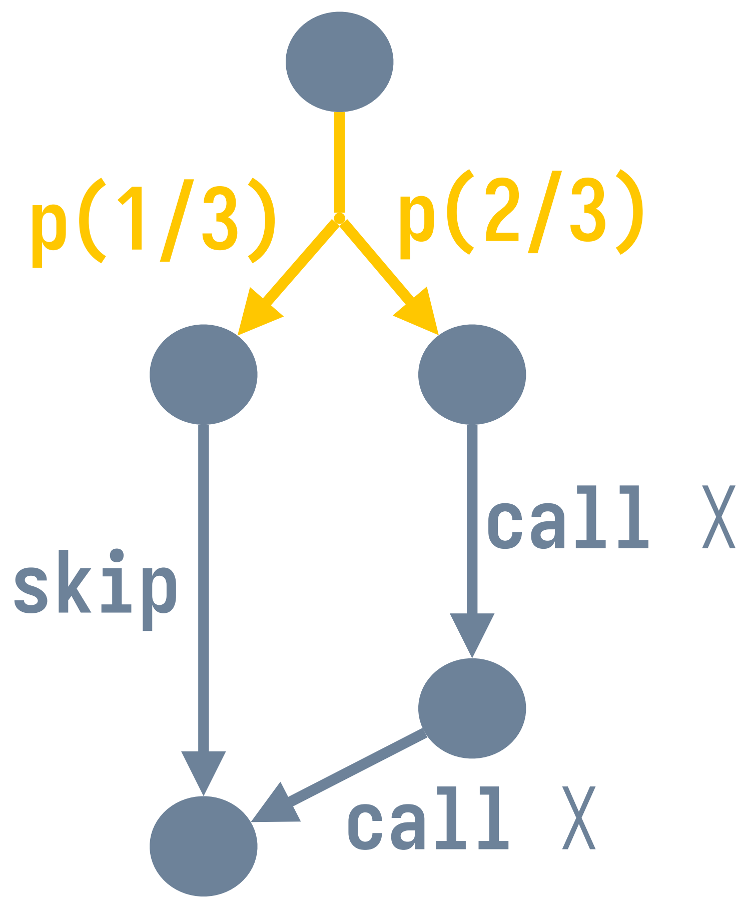

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How to solve such equations, algebraically?

Linearization of Tree Expressions

for Newton's Method

Linearization of Tree Expressions

for Newton's Method

- Syntactic linearization:

$$D(g \oplus h) = Dg \oplus Dh$$

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$$D(g {}_{\phi}\!\oplus h) = Dg {}_{\phi}\!\oplus Dh$$

$$D(g \sqcap h) = ((g \oplus Dg) \sqcap (h \oplus Dh)) \ominus (g \sqcap h)$$

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Carefully developed to render Newton's method sound

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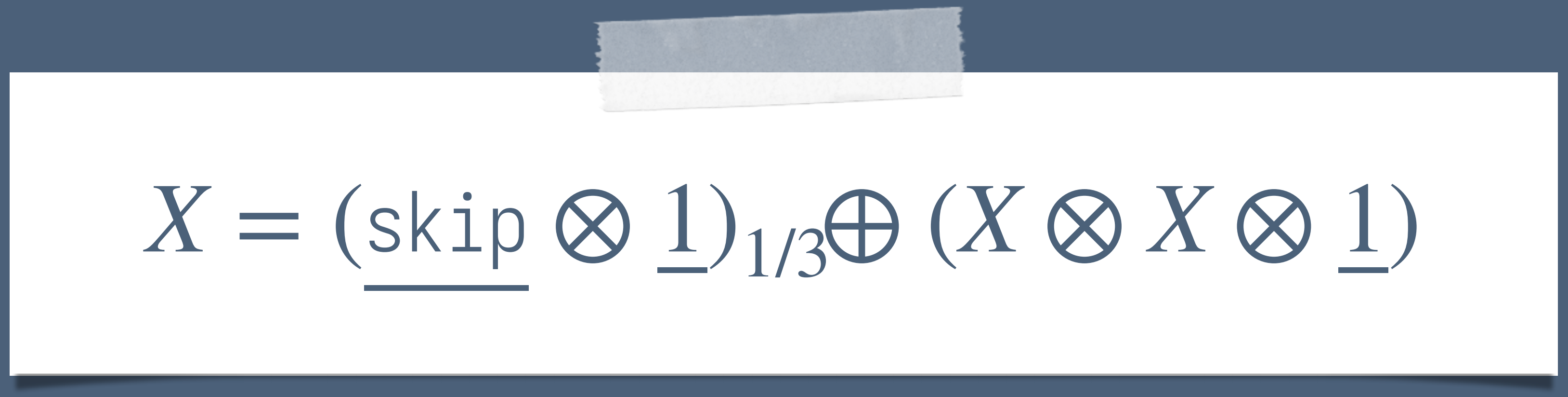

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Linearization at ν

$$Y = \delta \oplus (\underline{0}_{1/3}\!\oplus ((Y \otimes \nu) \oplus (\nu \otimes Y)))$$

where

$$\delta = ((\underline{\mathtt{skip}} \otimes \underline{1})_{1/3}\!\oplus (\nu \otimes \nu \otimes \underline{1})) \ominus \nu$$

Linearization of Tree Expressions

for Newton's Method

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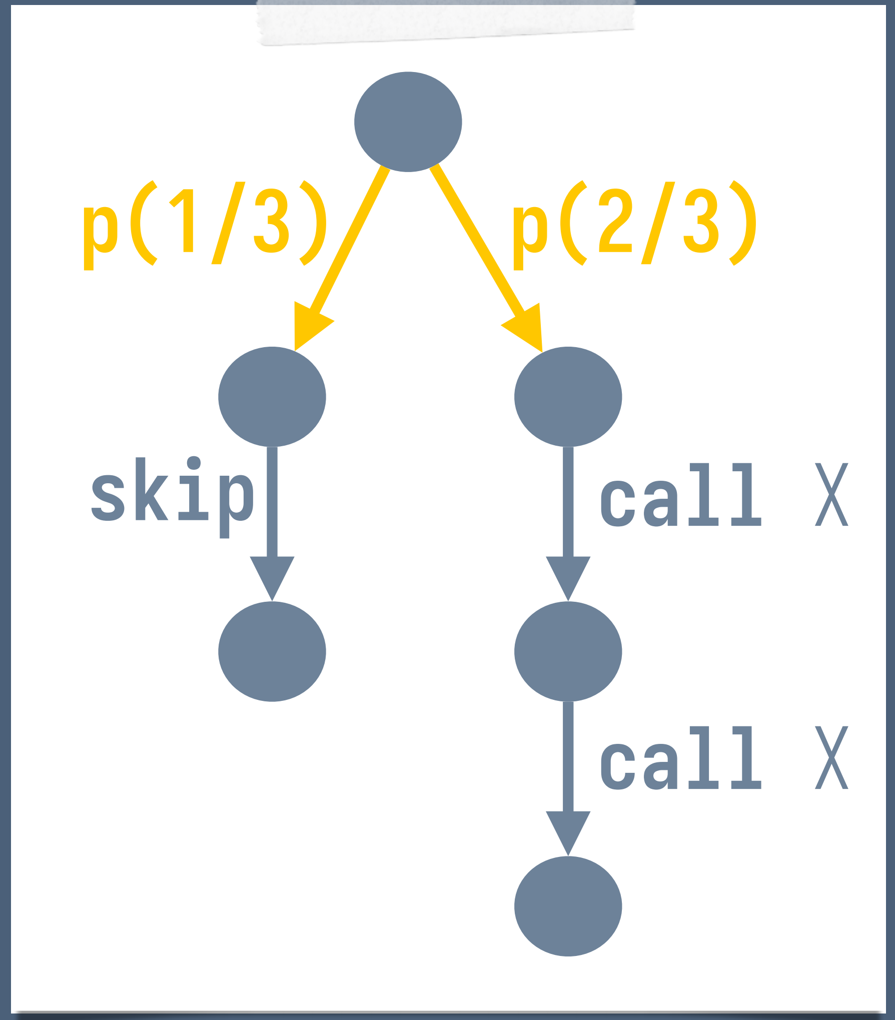

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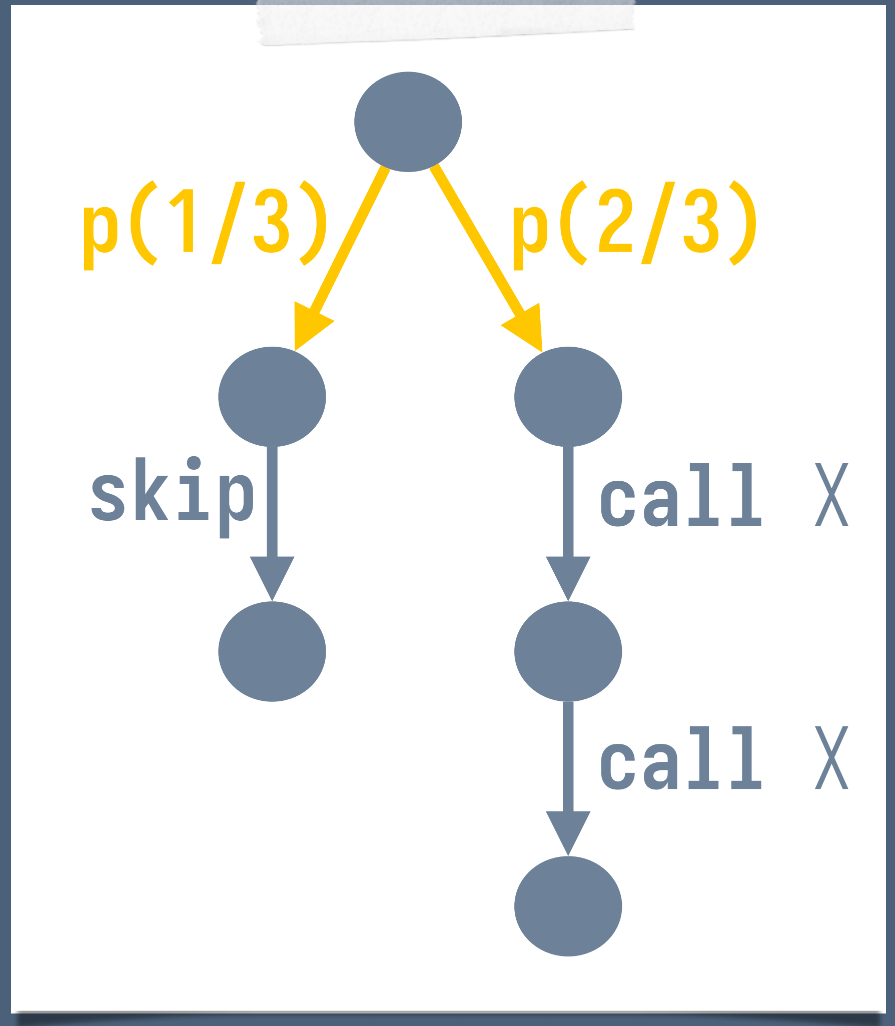

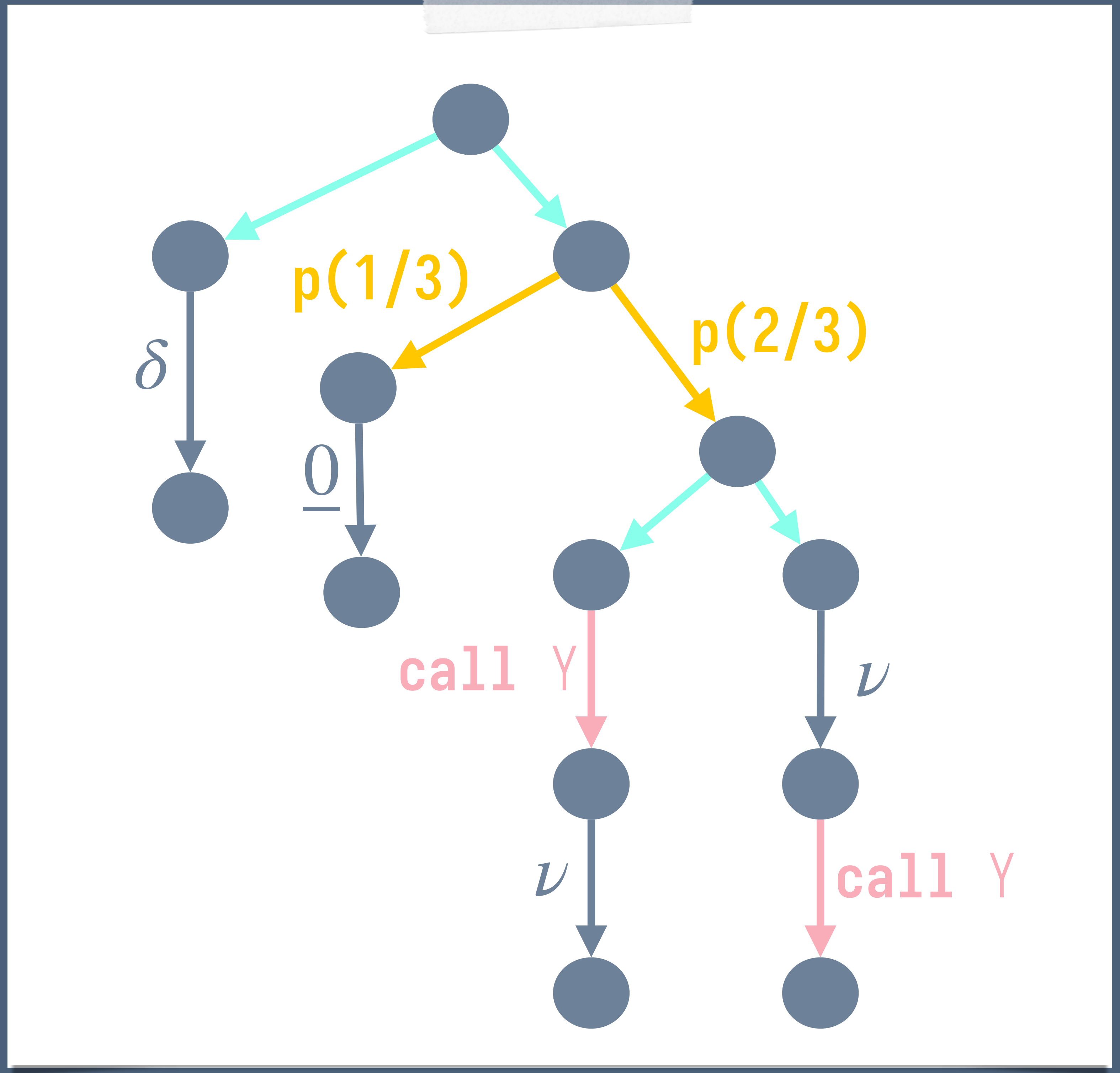

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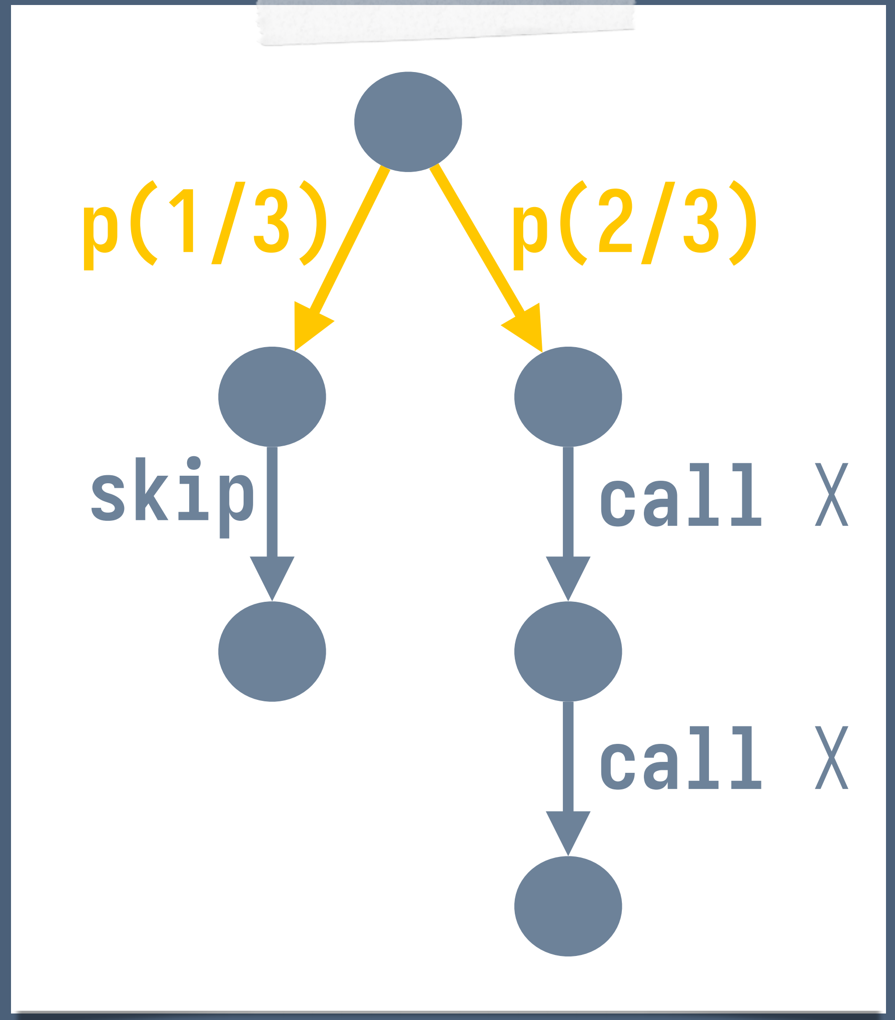
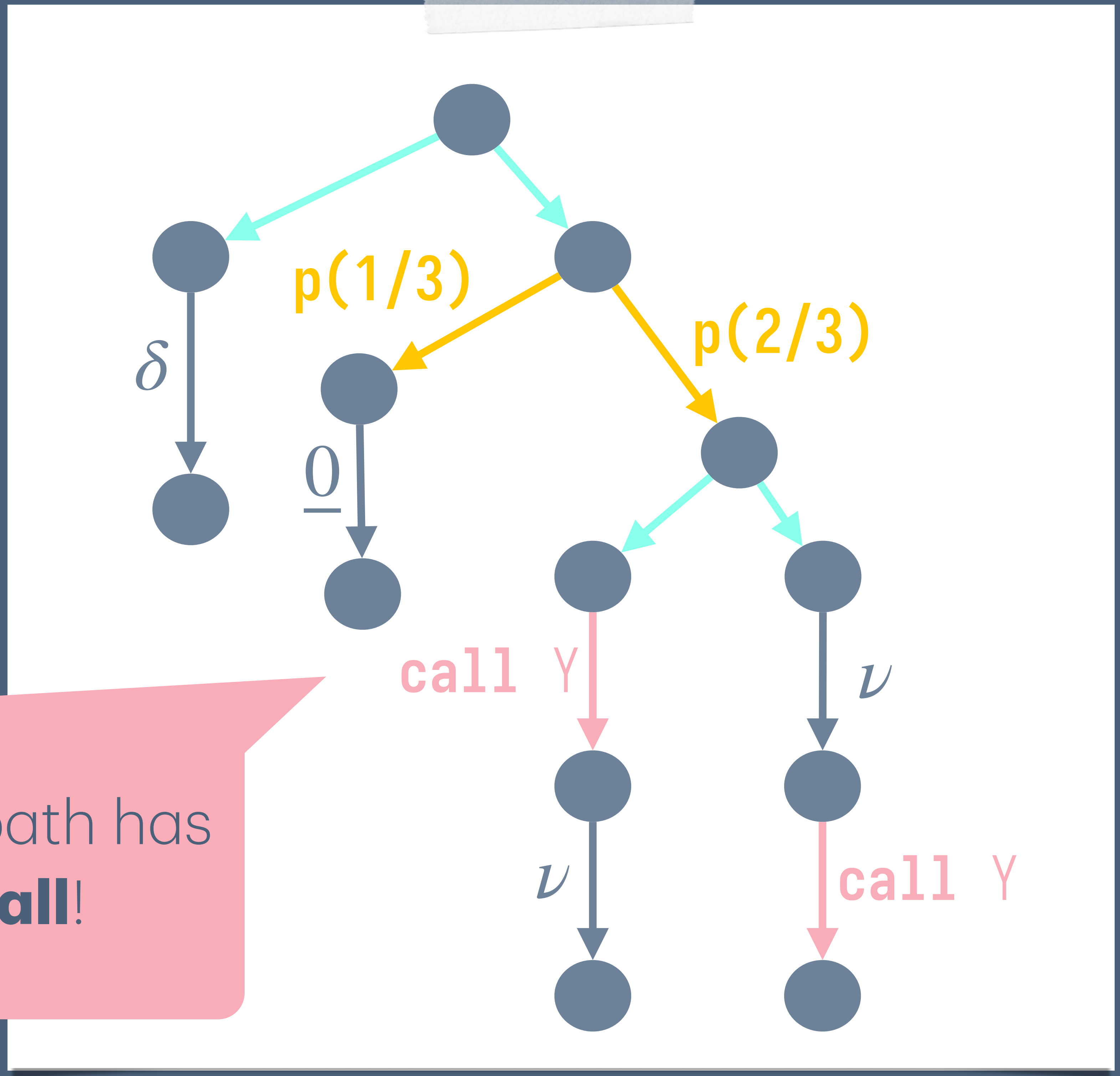
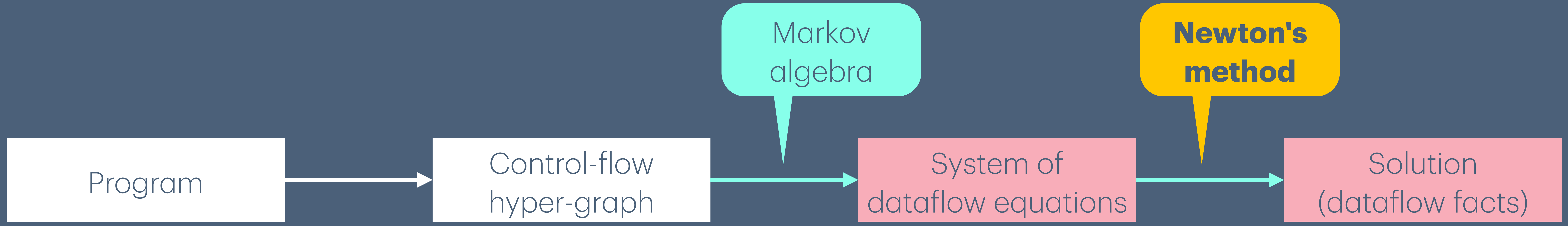




Every root-to-leaf path has **at most one call**!

Towards Multiple Combine Operations



Markov algebra

Newton's method

Program

Control-flow hyper-graph

System of dataflow equations

Solution (dataflow facts)

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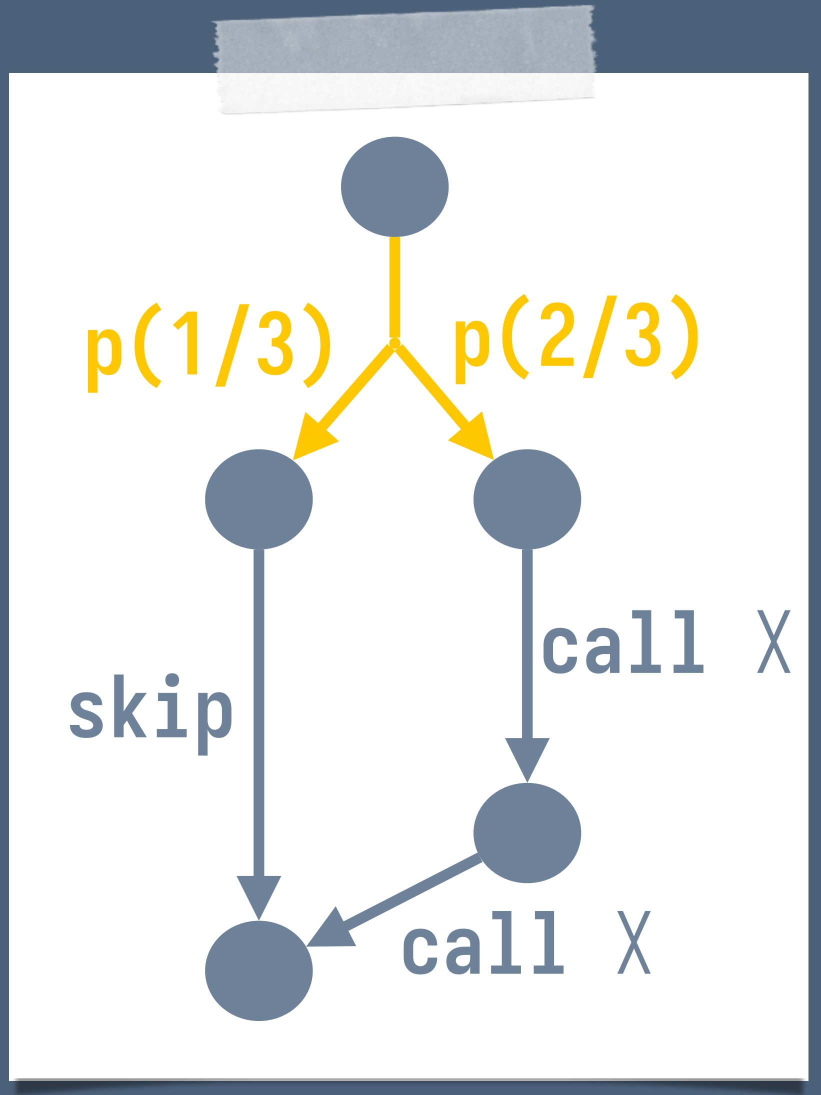

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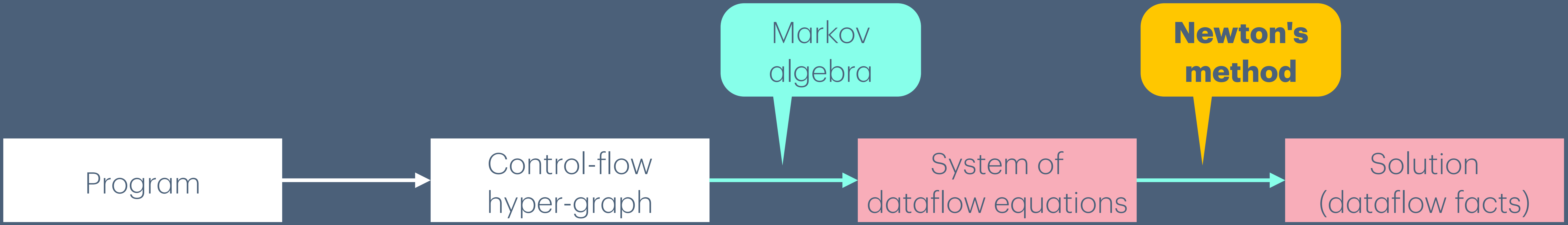

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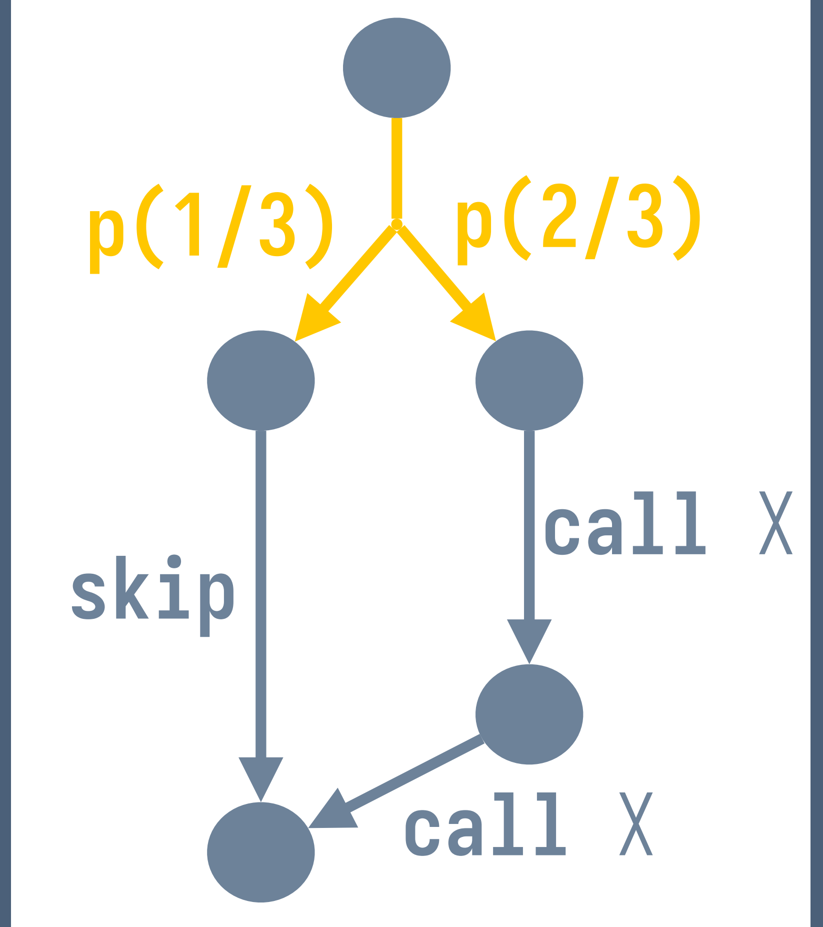

For Newton's method to be efficient, we require an **analysis-supplied strategy** for solving linearized equations

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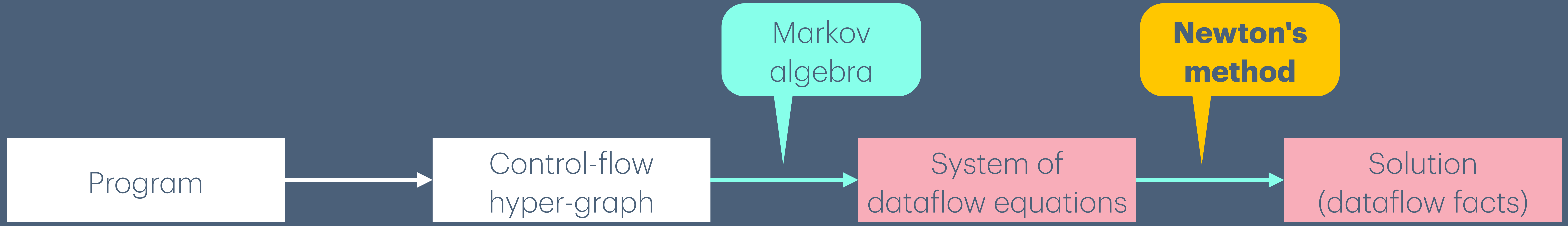

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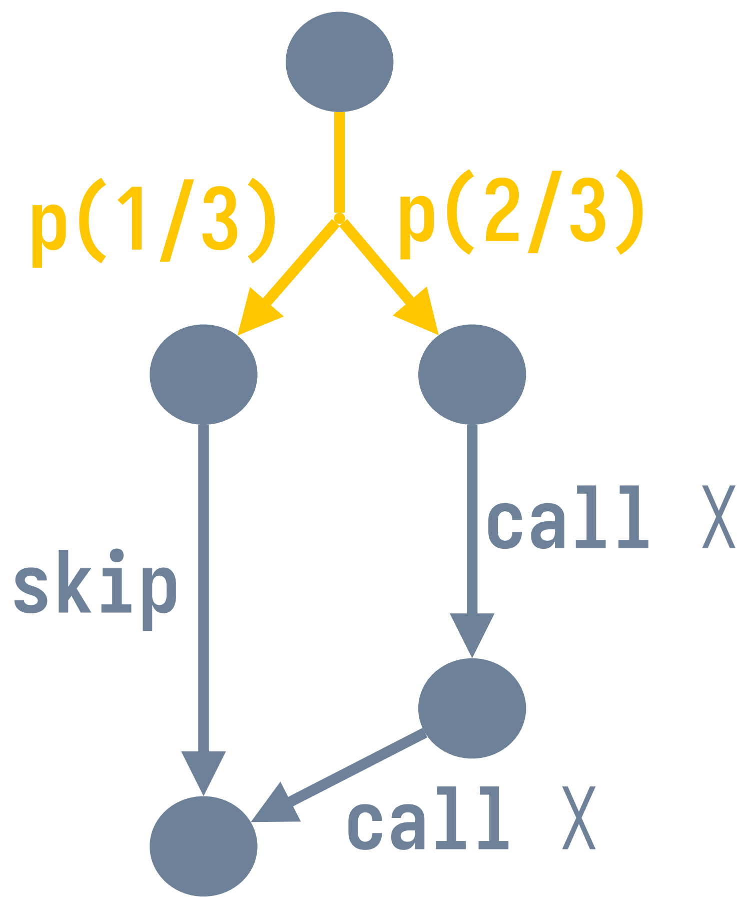

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For example, termination-probability analysis:

- LP solvers
- BDD/ADD-based solvers

Case Studies (Selected)

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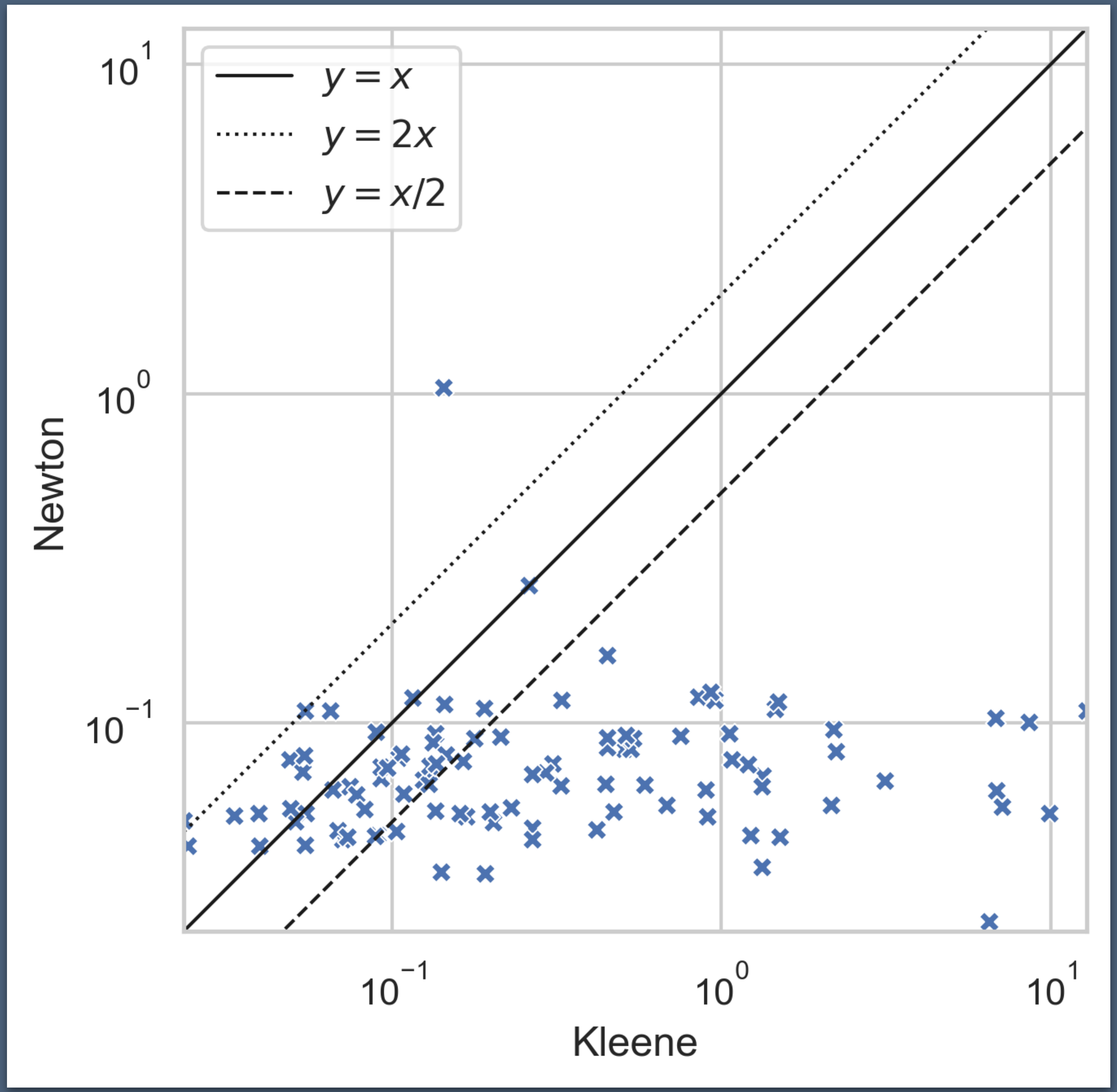

Termination-probability analysis

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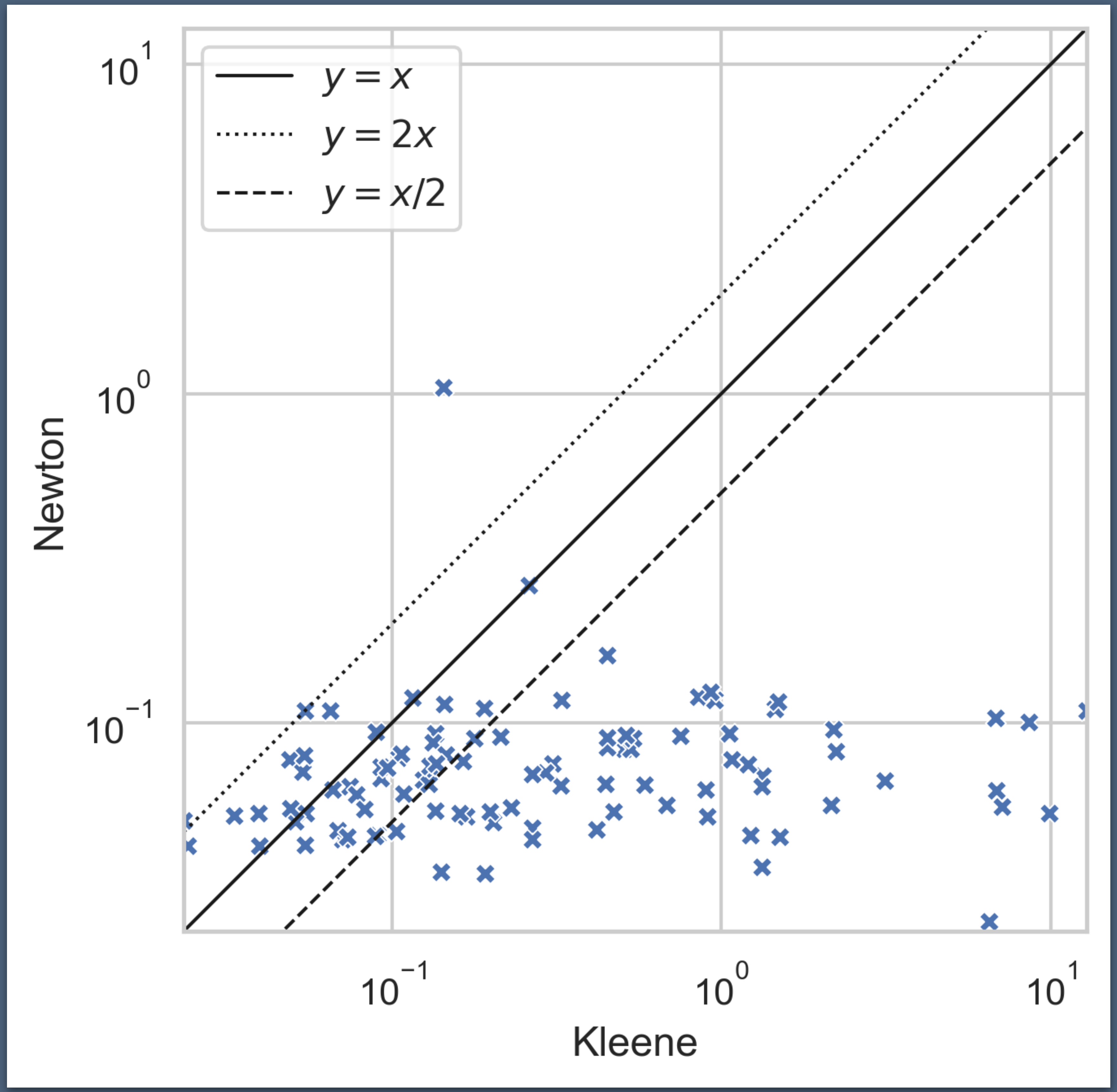

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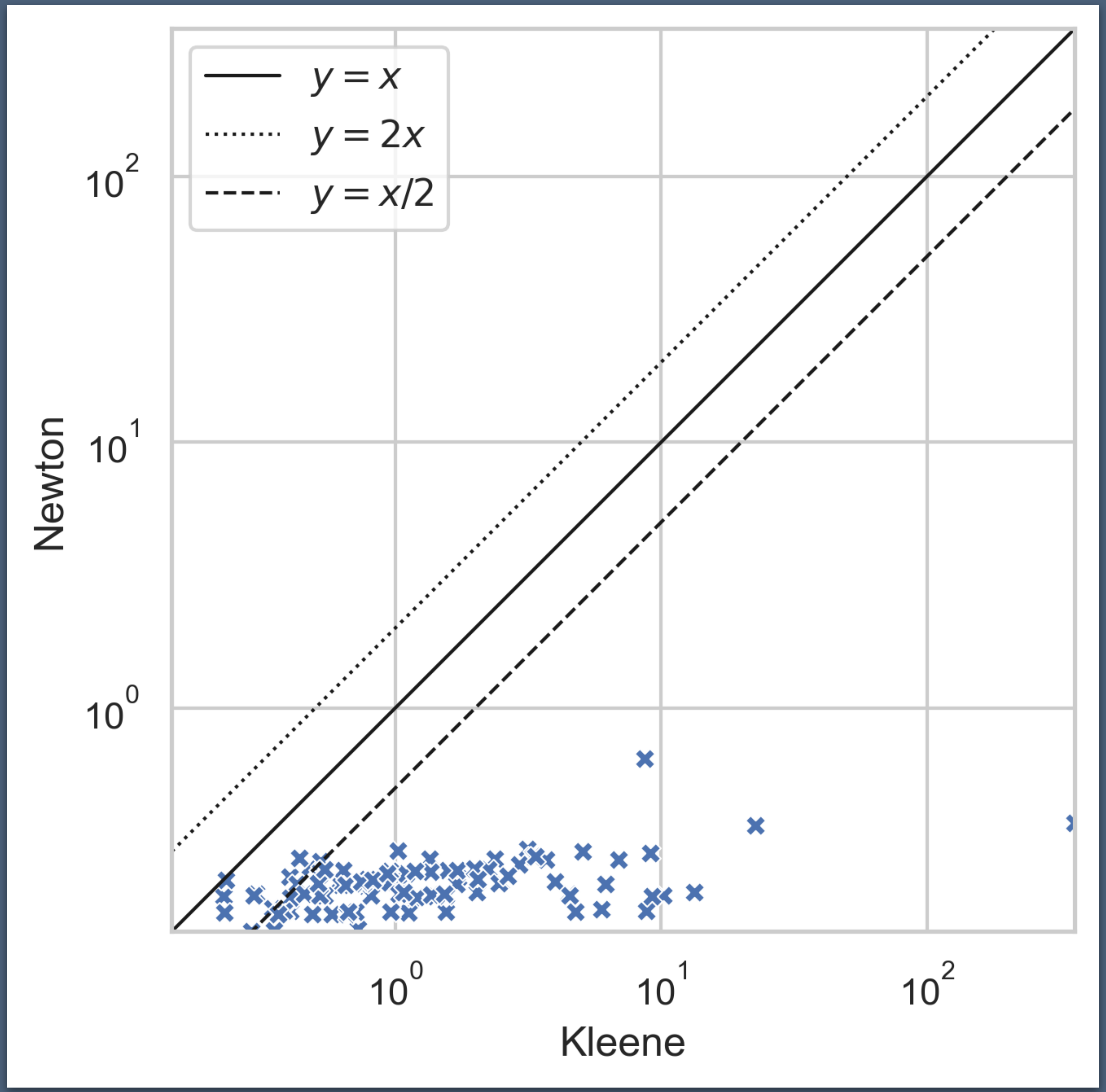
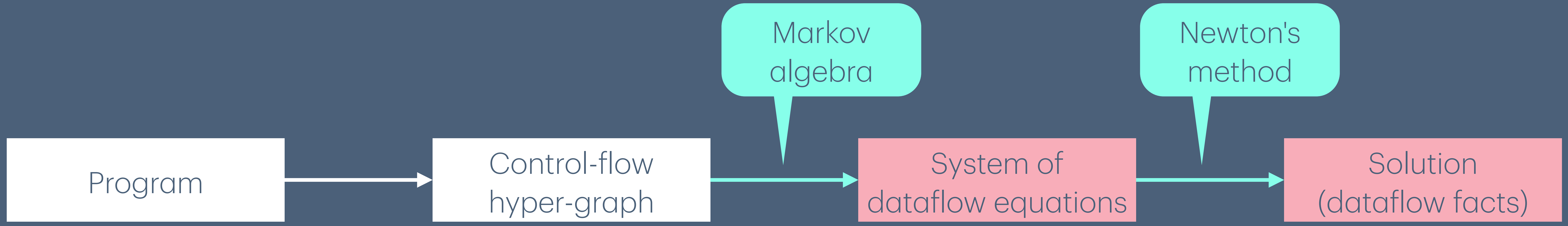


Moment-of-reward analysis

Summary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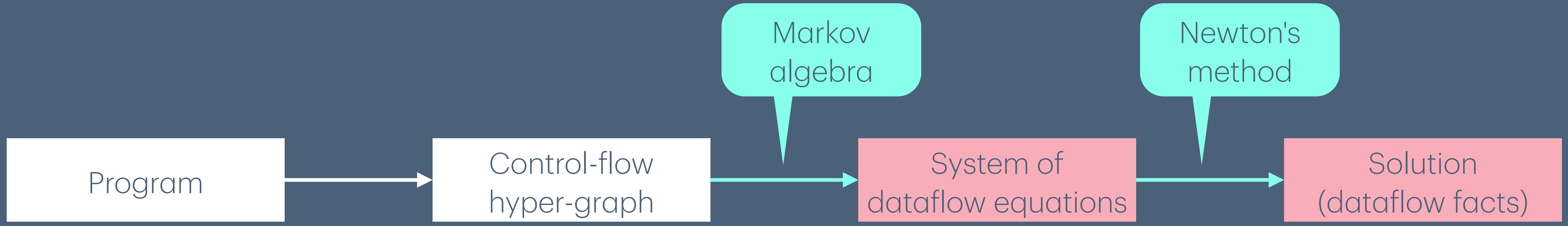

- Key takeaway: Extend Newtonian Program Analysis to support **more combine operations**

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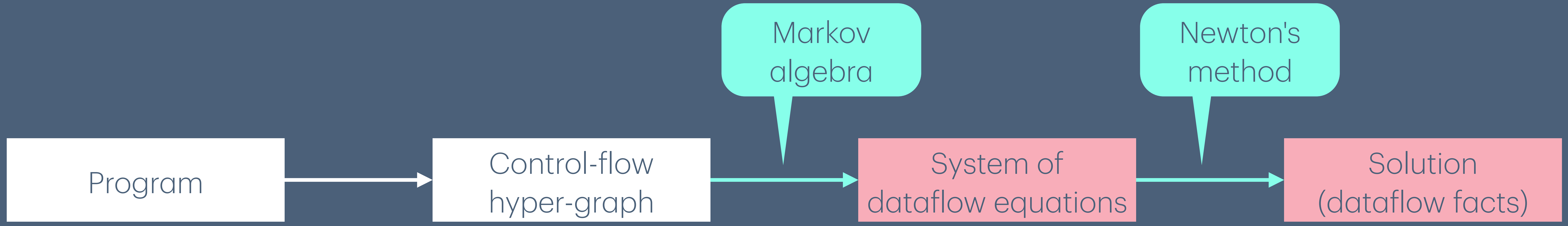

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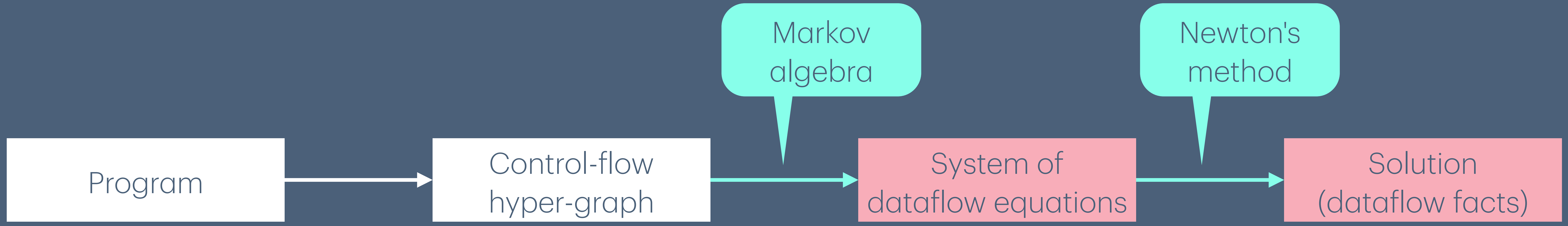

- Key takeaway: Extend Newtonian Program Analysis to support **more combine operations**
  - enabling analysis of programs with probabilistic, demonic, and conditional branching
- More in the paper:
  - Support of loops and unstructured control-flow
  - More case studies (e.g., expectation-invariant analysis)