DZ 3

(3.16) $f(x) = \ln(\ln(4-x))$

$$f'(x) = -\frac{1}{\ln(4-x)(4-x)}$$

$$f'(0) = -\frac{1}{(2\ln 2)} = -\frac{1}{8\ln 2}$$

$$f''(x) = (-1)\left(\frac{\left(\frac{4-x}{4-x}\right) - \ln(4-x)}{\ln^2(4-x)(4-x)^2}\right) = -\frac{1+\ln(4-x)}{\ln^2(4-x)(4-x)}$$

$$f''(0) = -\left(\frac{1+\ln 4}{(8\ln 2)^2}\right) = -\left(\frac{1+2\ln 2}{64\ln^2 2}\right)$$

$$f^{(3)}(x) = -\frac{(\ln(4-x)+1)'(\ln^2(4-x)x^2 - 8\ln^2(4-x)\cdot x + 16\ln^2(4-x)) - (\ln^2(4-x)(4-x)^2)'(\ln(4-x)+1)}{(\ln^2(4-x)(4-x)^2)^2}$$

$$= -\frac{\left(-\frac{1}{4-x}\right)(\ln^2(4-x)(4-x)^2) - (\ln^2(4-x)(4-x)^2)'(\ln(4-x)+1)}{\ln^4(4-x)(4-x)^4} =$$

$$= -\frac{2\ln^2(4-x)+3\ln(4-x)+2}{\ln^3(4-x)(4-x)^3}$$

$$f^{(3)}(0) = -\left(\frac{8\ln^2 2+6\ln 2+2}{512\ln^3 2}\right) = -\frac{4\ln^2 2+3\ln 2+1}{256\ln^3 2}$$

$$f(x) = \ln\ln 2 + \ln 2 - \frac{x}{8\ln 2} - \frac{(1+2\ln 2)x^2}{128\ln^2 2} - \frac{1+3\ln 2+4\ln^2 2}{1536\ln^3 2}.$$

(3.22)
$$\lim_{x\to 0}\frac{\frac{1}{\operatorname{th} x}-\frac{1}{\operatorname{tg} x}}{x} = \lim_{x\to 0}\frac{\frac{\operatorname{ch} x}{\operatorname{sh} x}-\frac{\cos x}{\sin x}}{x} = \lim_{x\to 0}\frac{\frac{1+\frac{x^2}{2}}{1+\frac{x^3}{6}}-\frac{1-\frac{x^2}{2}}{x-\frac{x^3}{6}}+o(x)}{x}$$

$$= \lim_{x\to 0}\frac{x-\frac{x^3}{6}+\frac{x^3}{2}-x-\frac{x^3}{6}+\frac{x^3}{2}}{x^3} = \lim_{x\to 0}\frac{2x^3}{3x^3} = \frac{2}{3}$$

13.23 $\lim\limits_{x\to 0} \frac{\cos\sin x - \cos x}{2x^4}$

$\sin x = x - \frac{x^3}{3!}$

$\cos\sin x = 1 - \frac{\left(x-\frac{x^3}{3!}\right)^2}{2!} + \frac{\left(x-\frac{x^3}{3!}\right)^4}{4!} + o(x^4)$

$\lim\limits_{x\to 0} \frac{1 + \frac{2x^4}{6} - \frac{x^2}{2!} + \frac{x^4}{4!} - 1 + \frac{x^2}{2!} - \frac{x^4}{4!}}{2x^4} = \lim\limits_{x\to 0} \frac{x^4}{6x^4} = \frac{1}{6}$

1.7 $py^{p-1}(x-y) \le x^p - y^p \le px^{p-1}(x-y)$, $0<y<x$, $p>1$

$0<y<\varepsilon<x$.

$y^{p-1} < \varepsilon^{p-1} < x^{p-1}$

$x^p - y^p = p\varepsilon^{p-1}(x-y)$

$py^{p-1}(x-y) \le x^p - y^p \le px^{p-1}(x-y)$

Доведено.

14.15 $x^3 + 3x + 6x\ln x + 2 > 6x^2$, $x>1$

$\varphi(1) = \psi(1) = 6$

$\varphi'(x) = 3x^2 + 3 + 6\ln x + 6$ | $\psi'(x) = 12x$

$\varphi'(1) = 12$ | $\psi'(1) = 12$

$\varphi''(x) = 6x + \frac{6}{x}$ | $\psi''(x) = 12$

$\varphi''(1) = \psi''(1) = 12$

$\varphi^{(3)}(x) = 6 - \frac{6}{x^2}$ | $\psi^{(3)}(x) = 0$

$\varphi^{(3)} > \psi^{(3)}(x)$ при $x>1$. $\Rightarrow \varphi(x) > \psi(x)$

Доведено

(14.19) $x\ln x + y\ln y \geq (x+y)\ln\frac{x+y}{2}$, $x>0, y>0$

$f(t) = t\ln t$

$f'(t) = 1+\ln t$

$f''(t) = \frac{1}{t} > 0$ при $t>0$ => функція опукла вниз

$\frac{x}{2}\ln x + \frac{y}{2}\ln y \geq \frac{x+y}{2}\ln\frac{x+y}{2}$ за теор. Іенсена

(14.20) $\sqrt{\sin\frac{x+y}{2}} \geq \frac{1}{2}(\sqrt{\sin x} + \sqrt{\sin y})$, $\{x,y\} \subset [0,\pi]$

$f(t) = \sqrt{\sin t}$

$f'(t) = \frac{\cos t}{2\sqrt{\sin t}}$

$f''(t) = \frac{-2\sin t\sqrt{\sin t} - \frac{\cos^2 t}{\sqrt{\sin t}}}{4\sin t} = \frac{-2\sin^2 t - \cos^2 t}{4\sin t\sqrt{\sin t}} = \frac{-2\sin^2 t - 1 + \sin^2 t}{4\sin t\sqrt{\sin t}} =$

$= \frac{-1-\sin^2 t}{4\sin t\sqrt{\sin t}} < 0$, Функція опукла ~~вниз~~ вверх

$\sqrt{\sin\frac{x+y}{2}} \geq \frac{1}{2}(\sqrt{\sin x} + \sqrt{\sin y})$ за нерівністю Іенсена

(14.21) $\cos\left(\frac{x+y}{2}\right)^2 \geq \frac{1}{2}(\cos x^2 + \cos y^2)$, $\{x,y\} \subset [0, \sqrt{\frac{\pi}{2}}]$

$f(t) = \cos t^2$

$f'(t) = -\sin t^2 \cdot 2t$

$f''(t) = -2(2t^2\cos t^2 + \sin t^2) = -4t^2\cos t^2 - 2\sin t^2 \leq 0.$

Функція опукла вверх

$\cos\left(\frac{x+y}{2}\right)^2 \geq \frac{1}{2}\cos x^2 + \frac{1}{2}\sin y^2$ за нерівн. Іенсена

14.22 $\dfrac{x^n+y^n+z^n}{3} > \left(\dfrac{x+y+z}{3}\right)^n$, $x>0$, $y>0$, $z>0$, $n>1$, $x\neq y$, $y\neq z$, $z\neq x$

$f(t) = t^n$

$f'(t) = n \cdot t^{n-1}$

$f''(t) = n t^{n-1}(\log t) + t^{n-1} > 0$. Функція опукла вниз.

$\dfrac{x^n+y^n+z^n}{3} > \left(\dfrac{x+y+z}{3}\right)^n$ за нерівністю Єнсена