ДЗ 8.

№2. a) $F(x) = \frac{3}{4} + \frac{1}{2\pi} \operatorname{arctg} x$

$F(+\infty) = \frac{3}{4} + \frac{1}{2\pi} \cdot \frac{\pi}{2} = 1$

$F(-\infty) = \frac{3}{4} - \frac{1}{2\pi} \cdot \frac{\pi}{2} = \frac{1}{2} \neq 0$

не є.

б) $F(x) = e^{-e^{-x}}$

$F(+\infty) = e^{0} = 1$

$F(-\infty) = e^{-\infty} = 0$

$F'(x) = e^{-e^{-x}} \cdot (-e^{-x})' = e^{-e^{-x}} \cdot e^{-x} \geq 0$

Функція неспадна, не має точок розриву, може бути функ. розподілу

с) $F(x) = \begin{cases} \frac{x}{1+x}, & x \geq 0 \\ 0, & x < 0 \end{cases}$

$F(-\infty) = 0$

$F(+\infty) = \lim\limits_{x \to +\infty} \frac{x}{1+x} = 1$

$F'(x) = \left(\frac{x}{1+x}\right)' = \frac{(1+x)-x}{(1+x)^2} = \frac{1}{(1+x)^2} \geq 0$ => неспадна

$F(0) = \frac{0}{1+0} = 0$ => $F(x)$ неперервна, може бути функ. розподілу

d) $F(x) = \begin{cases} 1 - \frac{1-e^{-x}}{x}, & x \geq 0 \\ 0, & x < 0 \end{cases}$

$F(-\infty) = 0$

$F(+\infty) = 1 - \frac{1-0}{+\infty} \to 1$

$F'(x) = \frac{xe^{-x} - (x)'(1-e^{-x})}{x^2} = \frac{xe^{-x} + e^{-x} - 1}{x^2} = \frac{1 - e^{x} - xe^{-x}}{x^2} \geq 0$

$F(0) = 1 - \frac{1-e^{0}}{0} = 1 - \frac{1-1-x-o(x)}{x} = 1-1 = 0$

F-неперервна, функ. розподілу

e) $F(x) = \begin{cases} 1, & x \geq 2 \\ \frac{[x]}{2}, & 0 \leq x < 2 \\ 0, & x < 0 \end{cases}$

$F(-\infty) = 0$, $F(+\infty) = 1$

Функція має розрив 1 роду в т. $x = 1$, $x = 2$

$F(1+) = F(1) = \frac{1}{2}$, $F(2+) = F(2) = 1$

Функція неперервна справа, є функ розподілу

3. $MX = 0$

$[a; 4]$, $f(x) = Ax^2$

$$\int_a^4 x f(x) dx = 0$$

$$\int_0^4 x^3 A\,dx = 0$$

$$\frac{x^4}{4} A \Big|_a^4 = 0$$

$$A\left(\frac{4^4}{4} - \frac{a^4}{4}\right) = 0$$

$A \neq 0, \; a < 4$

$a = -4$

$$F(x) = \int_{-4}^{x} f(u)\,du = \int_{-4}^{x} Au^2\,du = A\int_{-4}^{x} u^2\,du = A\frac{u^3}{3}\Big|_{-4}^{x}$$

$$A\left(\frac{x^3}{3} + \frac{64}{3}\right) = 1$$

$F(4) = 1$

$$A\left(\frac{4^3}{3} + \frac{64}{3}\right) = 1$$

$$A = \frac{1}{\frac{128}{3}} = \frac{3}{128}$$

$$F(2) = \frac{3}{128} \cdot \frac{72}{3} = \frac{9}{16}$$

$a = -4, \quad A = \frac{3}{128}, \quad F(2) = \frac{9}{16}$

4. $f_\xi(x) = \begin{cases} 0, & x \notin [0;2] \\ \frac{1}{2}, & 0 < x \le 1 \\ \frac{x}{3}, & 1 < x \le 2 \end{cases}$

$Y = X^2$

$F_Y = P(x^2 \le t) = P(x \le \sqrt{t}), \; t \ge 0$

$= 0, \; t < 0$

$$\int_{-\sqrt{t}}^{\sqrt{t}} f_x\, x\, dx = \int_0^{\sqrt{t}} f(x)\,dx$$

$$F_Y = \begin{cases} t<1 & : \int_0^{t} \frac{1}{2}dx = \frac{5t}{2} \\ 1<t<4 & \frac{1}{2} + \int_1^{t} \frac{x}{3}dx = \frac{1}{2} + \frac{x^2}{6}\Big|_1^{t} = \frac{1}{2} + \frac{t}{6} - \frac{1}{6} = \frac{t}{6} + \frac{1}{3} \\ 0, & t<0 \\ 1, & t \geq 4 \end{cases}$$

$$f_Y(t) = \begin{cases} 0, & t \notin [0;4] \\ \frac{1}{4\sqrt{t}}, & t \in [0;1) \\ \frac{1}{6}, & t \in [1;4] \end{cases}$$

$$MY = \int_{-\infty}^{\infty} x f_Y(x)\,dx = \int_0^4 x f_x\,dx = \int_0^1 \frac{\sqrt{x}}{4}dx + \int_1^4 \frac{x}{6}dx =$$

$$= \frac{x^{\frac{3}{2}}}{6}\Big|_0^1 + \frac{x^2}{12}\Big|_1^4 = \frac{1}{6} + \frac{16}{12} - \frac{1}{12} = \frac{17}{12}$$

$$MY^2 = \int_{-\infty}^{+\infty} x^2 f(x)\,dx = \int_0^4 x^2 f(x) = \int_0^1 \frac{x^{\frac{3}{2}}}{4} + \int_1^4 \frac{x^2}{6}dx =$$

$$= \frac{x^{\frac{5}{2}}}{10}\Big|_0^1 + \frac{x^3}{18}\Big|_1^4 = \frac{1}{10} + \frac{64}{18} - \frac{1}{18} = \frac{1}{10} + \frac{7}{2} = \frac{36}{10} = 3{,}6$$

5. $Y = [X]$

$F_X(t) = P(X \leq t) = 1 - e^{-\lambda t}$

$f(x)(1) = \lambda e^{-\lambda t}$

$$F_Y = P(X < [t+1]) = \int_0^{[t+1]} f(a)\,da = \int_0^{[t+1]} f(a)\,da = \int_0^{[t+1]} \lambda e^{-\lambda a}\,da =$$

$$= \lambda \int_0^{[t+1]} e^{-\lambda a}\,da = \lambda\left(-\frac{e^{-\lambda a}}{\lambda}\right)\Big|_0^{[t+1]} = e^{-\lambda a}\Big|_0^{[t+1]} = 1 - e^{-\lambda[t+1]}$$

$$P(\xi = t) = 1 - e^{-\lambda(t+1)} - (1 - e^{-\lambda t}) = 1 - 1 + e^{-\lambda t} - e^{-\lambda t - \lambda} =$$

$$= e^{-\lambda t}(1 - e^{-\lambda})$$

$$MY = \sum_{t=0}^{\infty} e^{-\lambda t}(1 - e^{-\lambda}) = (1 - e^{-\lambda}) \sum_{t=0}^{\infty} e^{-\lambda t} = (1 - e^{-\lambda}) \cdot \frac{1}{1-e^{\lambda}}$$