Ω/3

18.4 $\int_{t_0}^{x} \frac{dt}{\sin^4 t \cdot \cos^4 t} = \left| \begin{matrix} y = \operatorname{tg} t & dt = \frac{dy}{1+y^2} \\ \sin t = \frac{y}{\sqrt{1+y^2}} & \cos t = \frac{1}{\sqrt{1+y^2}} \end{matrix} \right| =$

$$= \int_{\operatorname{tg} t_0}^{\operatorname{tg} t} \frac{dy}{\cancel{(1+y^2)} \left(\frac{y}{\sqrt{1+y^2}}\right)^{\frac{4}{2}} \left(\frac{1}{\sqrt{1+y^2}}\right)} = \int_{\operatorname{tg} t_0}^{\operatorname{tg} t} \frac{dy \cdot (1+y^2)^2}{y^4} =$$

$$= \int_{\operatorname{tg} t_0}^{\operatorname{tg} t} \frac{1+2y^2+y^4}{y^4} dy = \int \frac{1}{y^4} dy + 2\int \frac{1}{y^2} + \int 1 \cdot dy =$$

$$= -\frac{1}{3y^3}\Big|_{\operatorname{tg} x_0}^{\operatorname{tg} x} - \frac{2}{y}\Big|_{\operatorname{tg} x_0}^{\operatorname{tg} x} + y\Big|_{\operatorname{tg} x_0}^{\operatorname{tg} x} = -\frac{1}{3\operatorname{tg}^3 x} - \frac{2}{\operatorname{tg} x} + \operatorname{tg} x =$$

$$= -\frac{1}{3}\operatorname{ctg}^3 x - 2\operatorname{ctg} x + \operatorname{tg} x$$

18.5 $\int\limits_{x_0}^{x} \frac{dt}{\sin 2t} = \int\limits_{x_0}^{x} \frac{\sin 2t\,dt}{1-\cos^2 2t} = \left| \begin{matrix} y = \cos 2t \\ dy = -2\sin 2t\,dt \end{matrix} \right| =$

$= \int\limits_{\cos 2t_0}^{\cos 2t} -\frac{1}{2} \cdot \frac{dy}{1-y^2} = \frac{1}{2} \int\limits_{\cos 2t_0}^{\cos 2t} \frac{dy}{y^2-1} = \frac{1}{4} \ln \left| \frac{y-1}{y+1} \right| \Big|_{\cos 2t_0}^{\cos 2t} =$

$= \frac{1}{4} \ln \left| \frac{\cos 2t - 1}{\cos 2t + 1} \right| \Big|_{x_0}^{x} = \frac{1}{4} \ln |tg^2 x| = \frac{1}{2} \ln |tg\, x|$

18.7 $\int\limits_{x_0}^{x} \frac{dt}{3-2\cos 3t} = \left| \begin{matrix} tg \frac{t}{2} = y \\ \cos t = \frac{1-y^2}{1+y^2} \\ dt = \frac{2dy}{1+y^2} \end{matrix} \right| = \int\limits_{tg \frac{t_0}{2}}^{tg \frac{t}{2}} \frac{2dy}{(1+y^2)(3-2\frac{1-y^2}{1+y^2})} =$

$= \int\limits_{tg \frac{t_0}{2}}^{tg \frac{t}{2}} \frac{2dy}{(1+y^2)\frac{3+3y^2-2+2y^2}{1+y^2}} = \int\limits_{tg \frac{t_0}{2}}^{tg \frac{t}{2}} \frac{2dy}{\cancel{(1+y^2)}\frac{3+3y^2-1+2y^2}{\cancel{1+y^2}}} =$

$= \int\limits_{tg \frac{t_0}{2}}^{tg \frac{t}{2}} \frac{2dy}{1+5y^2} = 2 \int\limits_{\sqrt{5}y_0}^{\sqrt{5}y} \frac{\frac{1}{\sqrt{5}}\,dz}{1+z^2} = \frac{2}{\sqrt{5}} \operatorname{arctg} \sqrt{5}y = \frac{2}{\sqrt{5}} \operatorname{arctg} \sqrt{5}\, tg \frac{x}{2}$

18.11 $\int\limits_{x_0}^{x} \frac{8\cos^2 t \sin t}{\sin t + \cos t}\,dt = \left| \begin{matrix} y = tg\, t & dt = \frac{dy}{1+y^2} \\ \sin t = \frac{y}{\sqrt{1+y^2}} & \cos t = \frac{1}{\sqrt{1+y^2}} \end{matrix} \right| =$

$= 8 \int \frac{\frac{1}{1+y^2} \cdot \frac{y}{\sqrt{1+y^2}}}{\frac{y}{\sqrt{1+y^2}} + \frac{1}{\sqrt{1+y^2}}} \cdot \frac{dy}{1+y^2} = 8 \int \frac{y\,dy}{(1+y^2)^2(y+1)}$

$\frac{8y}{(1+y^2)^2(y+1)} = \frac{A}{1+y} + \frac{By+C}{1+y^2} + \frac{Dy+E}{(1+y^2)^2}$

$8y = A + 2Ay^2 + Ay^4 + By^4 + By^3 + By^2 + Cy^3 + Cy^2 + Cy + C + Dy^2 +$

$y^4: \; A+B=0 \qquad A=-2$

$y^3: \; B+C=0 \qquad B=2$

$y^2: \; 2A+B+C+D=0 \qquad C=-2$

$\qquad D=4$

$y: \; B+C+D+E=8 \qquad E=4$

$y^0: \; A+C+E=0$

$= \int \frac{-2}{y+1}\,dy + \int \frac{2y-2}{(y^2+1)}\,dy + \int \frac{4y+4}{(1+y^2)^2}\,dy = -2\ln|tg\,x+1| +$

$+ \int \frac{dy^2}{y^2+1} - \int \frac{2}{y^2+1}\,dy + \int \frac{4(y+1)}{(1+y^2)^2}\,dy$ (2)

$= 4\int \frac{y+1}{(1+y^2)^2}\,dy = \left| \begin{matrix} y = tg\,z \\ dy = \frac{dz}{\cos^2 z} \end{matrix} \right| = 4\int \frac{(tg\,z+1)\,dz}{(1+tg^2 z)^2\cos^2 z} =$

$= 4\int \cos^2 z\,(tg\,z+1) = 4\int (1-\sin^2 z)(tg\,z+1)\,dz =$

$= 4\int (-\sin^2 z \cdot tg\,z + \sin^2 z)(-tg\,z+1)\,dz = -4\int \sin^2 z\, tg\,z\,dz +$

$+ 4\int tg\,z\,dz - 4\sin^2 dz + 4\int dz = 4\int \sin z\cos z\,dz -$

$- 2\int 1\,dt + 2\int \cos 2z\,dt + 4\int 1\,dt = 2\sin^2 z + 2z + \sin 2z$

(2) $-2\ln|tg\,x+1| + \ln|\cos^2 x| - 2\,arctg(tg\,x) + 2\sin^2(arctg(tg\,x)) + 4x +$

$+ \sin 2x = -2\ln|tg\,x+1| + \ln|\cos^2 x| + 2\sin^2 x + \sin 2x$

12.13 $I_n = \int_{x_0}^{x} \cos^n t \, dt, \ n \in N$

$n=1$: $I_1 = \int_{x_0}^{x} \cos t \, dt = \sin x$

$n=2$

$I_2 = \int_{x_0}^{x} \cos^2 t \, dt = \int_{x_0}^{x} \frac{1+\cos 2t}{2} dt = \frac{1}{2}x + \frac{1}{4}\sin 2x$

$n \geq 3$

$I_n = \int_{x_0}^{x} \cos^n t \, dt = \left| \begin{matrix} u = \cos^{n-1} t & du = -(n-1)\cos^{n-2} t \sin t \, dt \\ dv = \cos t \, dt & v = \sin t \end{matrix} \right| =$

$= \cos^{n-1} t \sin t + (n-1) \cdot \int \sin t \cdot \cos^{n-2} t \cdot \sin t \, dt =$

$= \cos^{n-1} t \sin t + (n-1) \int (1-\cos^2 t) \cos^{n-2} t \, dt = \cos^{n-1} t \sin t +$

$+ (n-1) \int \cos^{n-2} t \, dt - (n-1) \int \cos^n t \, dt$

$I_n = \cos^{n-1} t \sin t + (n-1) I_{n-2} - (n-1) I_n = \frac{1}{n} \cos^{n-1} t \sin t + \frac{n-1}{n} I_{n-2}$

12.14 $I_n = \int_{x_0}^{x} \frac{dt}{\sin^n t}, \ n \in N$

$n=1$:

$I_1 = \int_{x_0}^{x} \frac{dt}{\sin t} = \left| \begin{matrix} \operatorname{tg}\frac{t}{2} = y & dt = \frac{2dy}{1+y^2} \\ \sin t = \frac{2y}{1+y^2} & \end{matrix} \right| = \int_{x_0}^{x} \frac{2dy}{2y} = \ln\left(\operatorname{tg}\frac{t}{2}\right)$

$n=2$

$I_2 = \int_{x_0}^{x} \frac{1}{\sin^2 t} dt = -\operatorname{ctg} x$

$n \geq 3$

$$I_n = \int_{x_0}^{x} \frac{dt}{\sin^n t} = \left| \begin{matrix} v = \cos t & dv = -\sin t\,dt \\ u = \frac{1}{\sin^{n-1} t} & du = -\frac{(n-1)\cos t\,dt}{\sin^n t} \end{matrix} \right| =$$

$$= \frac{\cos t}{\sin^{n-1} t} + \int \frac{(n-1)\cos^2 t\,dt}{\sin^n t} = \frac{\cos t}{\sin^{n-1} t} +$$

$$+ (n-1)\int \frac{dt}{\sin^n t} - (n-1)\int \frac{dt}{\sin^{n-2} t} = I_n = \frac{\cos t}{\sin^{n-1} t} +$$

$$= (n-1) I_n - (n-1) I_{n-2}$$

$$(n-2) I_n = -\frac{\cos t}{(n-2)\sin^{n-1} t} + \frac{n-1}{n-2} I_{n-2}$$

18.15 $f(x) = \operatorname{sgn} x$

$$f(x) = \begin{cases} 1, & x > 0 \\ -1, & x < 0 \\ 0, & x = 0 \end{cases}$$

$$F(x) = \begin{cases} x + C_1, & x > 0 \\ -x + C_2, & x < 0 \\ 0 + C_3, & x = 0 \end{cases}$$

$F(0+0) = F(0-0)$

$x + C_1 = -x + C_2$

$0 + C_1 = 0 + C_2$

$C_1 = C_2 = C_3$

$F(x) = |x| + C.$

18.16 $f(x) = \operatorname{sgn}(x^2 + x - 2)$

$\operatorname{sgn}(x^2 + x - 2) = \operatorname{sgn}(x-1)(x+2)$

$$f(x) = \begin{cases} x + C_1, & x \leq -2 \\ -x + C_2, & -2 \leq x \leq 2 \\ x + C_3, & x > 1 \end{cases}$$

$F(-2-0) = F(-2+0) \Rightarrow -2+C_1 = 2+C_2$

$C_1 = 4+C_2$

$C_2 = C$

$f(x) = x+4+C \quad (x \le -2)$

$F(1-0) = F(1+0) \Rightarrow -1+C_2 = 1+C_3$

$C_2 = C = 2+C_3$

$C_3 = C-2$

$f(x) = -x+C, \quad -2 \le x \le 1$

$f(x) = x-2+C, \quad x > 1$

N