Д/з

1. е) $a_n = \sqrt{n+2} - 2\sqrt{n+1} + \sqrt{n}$

$$S_n = \sum_{k=1}^{\infty}\left(\sqrt{k+2} - 2\sqrt{k+1} + \sqrt{k}\right) = \sum_{k=1}^{\infty}\sqrt{k+2} - 2\sum_{k=1}^{\infty}\sqrt{k+1} +$$

$$+ \sum_{k=1}^{\infty}\sqrt{k} = \sqrt{3} + \sqrt{4} + \dots + \sqrt{n} + \sqrt{n+1} + \sqrt{n+2} - 2\sqrt{2} - 2\sqrt{3} - \dots - 2\sqrt{n} -$$

$$- 2\sqrt{n+1} + \sqrt{1} + \sqrt{2} \dots + \sqrt{n} = \sqrt{n+1} - \sqrt{n+2} - 2\sqrt{2} - 2\sqrt{n+1} + 1 + \sqrt{2} =$$

$$= \sqrt{n+2} - \sqrt{n+1} - \sqrt{2} + 1$$

$$\lim_{n\to\infty} S_n \left(\underbrace{\sqrt{n+2} - \sqrt{n+1}}_{0} - \sqrt{2} + 1\right) = 1 - \sqrt{2}$$

1. ж) $a_n = q^n \cos n\alpha$, $|q| < 1$

Розгл. $\sum_{n=1}^{\infty} \cos nx$. $x \neq \pi k$, k – ціле число.

$\lim_{n\to\infty} \cos nx = 0$, $x \neq \pi k$

$\lim_{n\to\infty} \cos(n+1)x = 0 \Rightarrow \lim_{n\to\infty} (\sin nx \cos x + \cos nx \sin x) = 0$

$\lim_{n\to\infty} nx = 0$, $x \neq \pi k$

Тому $\lim_{n\to\infty}(\cos^2 nx+\sin^2 nx)=0$. Але $\sin^2 x+\cos^2 x=1$. Отримуємо суперечність.

При $x=\pi k$ ряд розбіжний, при $x\neq\pi k$ збіжний

$\sum_{n=1}^{\infty}\cos nx$ при $x=\pi k$ — сума $\pm 1\sum_{n=1}^{\infty}(-1)^n$; $\to 0$.

$\lim_{n\to\infty} S_n = 0$

1.12) $a_n=\frac{1}{(n+1)(n+2)(n+3)}$

$$S_n=\sum_{k=1}^{n}\frac{1}{(k+1)(k+2)(k+3)}=\sum_{k=1}^{n}\left(\frac{1}{2}\cdot\frac{1}{k+1}-\frac{1}{k+2}+\frac{1}{2}\cdot\frac{1}{k+3}\right)=$$

$$=\sum_{k=1}^{n}\left(\frac{1}{2}\left(\frac{1}{k+1}-\frac{1}{k+2}\right)+\frac{1}{2}\left(\frac{1}{k+3}-\frac{1}{k+2}\right)\right)=\frac{1}{2}-\frac{1}{2(n+1)}+$$

$$+\frac{1}{2(n+2)}-\frac{1}{4}$$

$\lim_{n\to\infty} S_n=\frac{1}{4}$

1.4) $a_n=\operatorname{arctg}\frac{1}{2n^2}$

$$S_n=\sum_{k=1}^{n}\operatorname{arctg}\frac{1}{2k^2}=\sum_{k=1}^{n}(\operatorname{arctg}(2k+1)-\operatorname{arctg}(2k-1))=\operatorname{arctg}3-\operatorname{arctg}1+$$

$$+\operatorname{arctg}5-\operatorname{arctg}5+\ldots+\operatorname{arctg}(2n-1)-\operatorname{arctg}(2n-1)=-\operatorname{arctg}1+$$

$$+\operatorname{arctg}(2n+1)$$

$$\lim_{n\to\infty} S_n=-\frac{\pi}{4}+\frac{\pi}{2}=\frac{\pi}{4}$$

2) к) $a_n = \frac{1}{\sqrt{(2n-1)(2n+3)}}$

$\lim\limits_{n\to\infty} a_n = 0$

$\frac{1}{\sqrt{(2n-1)(2n+3)}} = \frac{1}{\sqrt{4n^2+4n-3}} - a_n = O\left(\frac{1}{n}\right)$ - розбіжн.

2л) $a_n = tg\frac{\pi}{4n} \to 0 \quad n\to\infty$

$a_n \approx \frac{\pi}{4n} + \frac{1}{3}\left(\frac{\pi}{4n}\right)^3 + O\left(\frac{1}{n^3}\right) \sim O\left(\frac{1}{n}\right)$ - розбіжн.

2м) $a_n = \frac{1}{\ln(n+e)}$

$\lim\limits_{n\to\infty} a_n = 0$

$a_n = \frac{1}{\ln(n+e)} = \frac{1}{\ln\left(n\left(1+\frac{e}{n}\right)\right)} = \frac{1}{\ln n + \ln\left(1+\frac{e}{n}\right)} =$

$= \log_{n+e} e$

3б) $a_n = \frac{\cos x^n}{n^2}$

$|S_{n+p} - S_n| = \left|\frac{\cos x^{n+1}}{(n+1)^2} + \frac{\cos x^{n+2}}{(n+2)^2} + \dots + \frac{\cos x^{n+p}}{(n+p)^2}\right| \le$

$\le \frac{1}{(n+1)^2} + \frac{1}{(n+2)^2} + \dots + \frac{1}{(n+p)^2} < \frac{1}{n(n+1)} + \frac{1}{(n+1)(n+2)} + \dots +$

$+ \frac{1}{(n+p-1)(n+p)} = \frac{1}{n} - \frac{1}{n+p} < \frac{1}{n}$

При $\forall p \in N$, $\forall n > \frac{1}{\varepsilon}$ ряд збіжний

4у) $a_n = \frac{3^n \cdot n!}{n^n}$

$\lim\limits_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim\limits_{n\to\infty} \frac{3^{n+1}\cdot(n+1)!\cdot n^n}{3^n\cdot n!\cdot(n+1)^{n+1}} = \lim\limits_{n\to\infty} \frac{3\cdot n^n}{(n+1)^n} =$

$= \lim\limits_{n\to\infty} 3\cdot\left(\frac{1}{1+\frac{1}{n}}\right)^n = \frac{3}{e} > 1$ Ряд розбіжний

4с) $a_n = \frac{(n!)^2}{2^{n^2}}$

$\lim\limits_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim\limits_{n\to\infty} \frac{((n+1)!)^2\cdot 2^{n^2}}{(n!)^2\cdot 2^{(n+1)^2}} = \lim\limits_{n\to\infty} \frac{(n+1)^2\cdot 2^{n^2}}{2^{n^2+2n+1}} =$

$= \lim\limits_{n\to\infty} \frac{(n+1)^2}{2^{n+1}} \to \infty$. Ряд розбіжний.

4к) $a_n = n^2\cdot\left(2+\frac{1}{n}\right)^{-n-2}$

$\lim\limits_{n\to\infty} \sqrt[n]{n^2\cdot\left(2+\frac{1}{n}\right)^{-(n+2)}} = \lim\limits_{n\to\infty} \frac{1}{2+\frac{1}{n}}\cdot\sqrt[n]{\frac{n^2}{\left(2+\frac{1}{n}\right)^2}} = \frac{1}{2} < 1$. Ряд збіжний

5а) $a_n = \frac{n!\, e^n}{n^{n+p}}$

$\frac{a_n}{a_{n+1}} = \frac{n!\cdot e^n\,(n+1)^{n+p+1}}{n^{n+p}\,(n+1)!\,e^{n+1}} = \frac{1}{e}\left(1+\frac{1}{n}\right)^{n+p} = \frac{1}{e}\cdot e^{(n+p)\ln\left(1+\frac{1}{n}\right)} =$

$= e^{-1+(n+p)\left(\frac{1}{n}-\frac{1}{2n^2}+O\left(\frac{1}{n^2}\right)\right)} = e^{\frac{p-0,5}{n}+o(} \ldots = 1+\frac{p-0,5}{n}+o\left(\frac{1}{n}\right)$

при $p > \frac{3}{2}$ ряд збіжний

6в) $\ln^p(\sec\frac{\pi}{n})$, $\ln\left(\ln\frac{1}{\cos\frac{\pi}{n}}\right)^p = \left(-\ln\cos\frac{\pi}{n}\right)^p = \left(-\ln\left(1-\frac{\pi^2}{2n^2}+o\left(\frac{1}{n^2}\right)\right.\right.$

$= \left(\frac{\pi^2}{2n^2}+o\left(\frac{1}{n^2}\right)\right)^p = O\left(\frac{1}{n^2 p}\right)$

$2p > 1$

$p > \frac{1}{2}$

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6а) $a_n = \frac{1}{n^p} \sin\frac{\pi}{2}$

$a_n = \frac{1}{n^p} \cdot \left(\frac{\pi}{n} + o\left(\frac{1}{n}\right)\right) \sim O\left(\frac{1}{n^{p+1}}\right)$

$p+1 > 1$

$p > 0 \Rightarrow$ ряд розбіжн.

12а) $a_n = \frac{\ln^{100} n}{n} \cdot \sin\frac{\pi n}{4}$

$\left|\sum_{k=1}^{n} \sin\frac{k\pi}{4}\right| = \sin\left(\frac{\pi}{8}\right)^{-1} \cdot \left|\sin\frac{\pi n}{8} \cdot \sin\frac{n+1}{8}\pi\right| < \frac{1}{\sin\frac{\pi}{8}}$

$\frac{\ln^{100} n}{n} \to 0, \ n \to \infty$

ряд є збіжним

12г) $a_n = (-1)^n \cdot \sin\frac{1}{n}$

$\sin\left(\frac{1}{n+1}\right) < \sin\left(\frac{1}{n}\right), \quad \sin\left(\frac{1}{n}\right) \sim \frac{1}{n} \to 0, \ n \to \infty$

ряд збіжний

13а) $a_n = \frac{(-1)^n}{n^p}$

$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^p}$ - збіжний за Лейбніцом при $p > 0$

$\sum_{n=1}^{\infty} \left|\frac{(-1)^n}{n^p}\right| = \sum_{n=1}^{\infty} \frac{1}{n^p}$ - збіжний $(p > 1)$

$p \in (0, 1]$ - умовно збіжн.

$p \in (1; +\infty)$ - абсолютно збіжн.

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