# Домашня робота

**1.1** $$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

При $n=1$ маємо:

$$1^2 = \frac{1\cdot 2\cdot 3}{6}$$

$$1 = 1 \quad (+)$$

$$A_{n+1}: \sum_{k=1}^{n+1} k^2 = \frac{(n+1)(n+2)(2n+3)}{6}$$

$$\sum_{k=1}^{n+1} k^2 = \sum_{k=1}^{n} k^2 + (n+1)^2 = \frac{n(n+1)(2n+1)}{6} + \cancel{\ldots} =$$

$$\cancel{= \frac{(n^2+n)(2n+1)}{6}} = \frac{n(n+1)(2n+1)}{6} + (n+1)(n+1) = (n+1)\left(\frac{n\cdot(2n+1)}{6} + (n+1)\right) =$$

$$= (n+1)\left(\frac{2n^2+7n+6}{6}\right) = (n+1)\left(\frac{(n+2)(2n+3)}{6}\right) = \frac{(n+1)(n+2)(2n+3)}{6}$$

Що і треба було довести.

**1.3.** $$\sum_{k=1}^{n-1} (-1)^{k-1} k^2 = (-1)^n \frac{(n-1)n}{2}; \quad n \geq 2$$

При $n=2$ маємо:

$$1^2 = (-1)^2 \frac{2}{2} \quad (+)$$

$$A_{n+1}: \sum_{k=1}^{n} (-1)^{k-1} k^2 = (-1)^{n+1} \cdot \frac{n\cdot(n+1)}{2} = (-1)^{n+1} \cdot \frac{n^2+n}{2}$$

$$\sum_{k=1}^{n} (-1)^{k-1} k^2 = \sum_{k=1}^{n-1} (-1)^{k-1} k^2 + (-1)^{n-1} n^2 = (-1)^n \frac{(n-1)n}{2} + (-1)^{n-1} \cdot n^2 =$$

$$= (-1)^n \left(\frac{(n-1)n}{2} - n^2\right) = (-1)^n \left(\frac{n^2-n-2n^2}{2}\right) = (-1)^n \left(\frac{-n^2-n}{2}\right) = (-1)^n \cdot (-1) \cdot \left(\frac{n^2+n}{2}\right) =$$

$= (-1)^{n+1} \cdot \frac{n^2+n}{2}$ ⊕

Що і треба було довести

**1.4** $\sum_{k=1}^{n} k \cdot k! = (n+1)! - 1$

При $n=1$ $1 \cdot 1 = 2! - 1$ ⊕

$A_{n+1}: \sum_{k=1}^{n+1} k \cdot k! = (n+2)! - 1$

$$\sum_{k=1}^{n+1} k \cdot k! = \sum_{k=1}^{n} k \cdot k! + (n+1)(n+1)! = (n+1)! - 1 + (n+1)(n+1)! =$$

$$= (n+1)!(n+1+1) - 1 = (n+2)! - 1.$$ ⊕

Що і треба було довести.

**1.5** $\sum_{k=1}^{n} k(k+1) = \frac{n(n+1)(n+2)}{3}$

При $n=1$ $1 \cdot 2 = \frac{1 \cdot 2 \cdot \cancel{3}}{\cancel{3}}$ ⊕

$A_{n+1}: \sum_{k=1}^{n+1} k(k+1) = \frac{(n+1)(n+2)(n+3)}{3}$

$$\sum_{k=1}^{n+1} k(k+1) = \sum_{k=1}^{n} k(k+1) + (n+1)(n+2) = \frac{n(n+1)(n+2)}{3} + (n+1)(n+2) =$$

$$= \frac{n(n+1)(n+2) + 3(n+1)(n+2)}{3} = \frac{(n+1)(n+2)(n+3)}{3}$$ ⊕

Що і треба було довести.

**1.2** $\sum_{k=1}^{n} k^3 = \left(\sum_{k=1}^{n} k\right)^2$

При $n=1$

$1^3 = 1^2$ ⊕

$A_{n+1}: \sum_{k=1}^{n+1} k^3 = \left(\sum_{k=1}^{n+1} k\right)^2$

$$\sum_{k=1}^{n+1} k^3 = \sum_{k=1}^{n} k^3 + (n+1)^3 = \left(\sum_{k=1}^{n} k\right)^2 + (n+1)^3 = \left(\frac{n(n+1)}{2}\right)^2 + (n+1)^3 =$$

$$= \frac{(n+1)^2\left(n^2+4(n+1)^3\right)}{4} = \frac{(n+1)^2(n^2+4n+4)}{4} = \frac{(n+1)^2(n+2)^2}{4} = \left(\frac{(n+1)(n+2)}{2}\right)^2 =$$

$$= \left(\sum_{k=1}^{n+1} k\right)^2$$ ⊕

Що і треба було довести

**1.6** $\sum_{k=1}^{n} \operatorname{arctg}\frac{1}{2k^2} = \operatorname{arctg}\frac{n}{n+1}$

При $n=1$:

$\operatorname{arctg}\frac{1}{2} = \operatorname{arctg}\frac{1}{2}$ ⊕

$A_{n+1}: \sum_{k=1}^{n+1} = \operatorname{arctg}\frac{(n+1)}{(n+2)}$

$$\sum_{k=1}^{n+1} \operatorname{arctg}\frac{1}{2k^2} = \sum_{k=1}^{n} \operatorname{arctg}\frac{1}{2k^2} + \operatorname{arctg}\frac{1}{2(n+1)^2} = \operatorname{arctg}\frac{n}{n+1} + \operatorname{arctg}\frac{1}{2(n+1)^2} =$$

$$= \operatorname{arctg}\left(\frac{\frac{n}{n+1} + \frac{1}{2(n+1)^2}}{1 - \frac{n}{2(n+1)^3}}\right) = \operatorname{arctg}\left(\frac{\frac{2n^2+2n+1}{2(n+1)^2}}{1-\frac{n}{2(n+1)^3}}\right) =$$

$$= \operatorname{arctg}\left(\frac{\frac{2n^2+2n+1}{2(n+1)^2}}{\frac{2(n+1)^3-n}{2(n+1)^3}}\right) = \operatorname{arctg}\left(\frac{(2n^2+2n+1)(n+1)}{2(n+1)^3-n}\right) =$$

$$= \operatorname{arctg}\left(\frac{2n^3+2n^2+2n^2+2n+n+1}{2(n^3+3n^2+3n+1)-n}\right) = \operatorname{arctg}\left(\frac{2n^3+4n^2+3n+1}{2n^3+6n^2+6n+2+n}\right) =$$

$$= \operatorname{arctg}\left(\frac{2n^3+2n^2+2n^2+2n+n+1}{2n^3+4n^2+2n^2+5n+2}\right) = \operatorname{arctg}\left(\frac{2n^2(n+1)+2n(n+1)+1(n+1)}{2n^2(n+2)+2n(n+2)+1(n+2)}\right) =$$

$$= \operatorname{arctg}\left(\frac{(n+1)\cancel{(2n^2+2n+1)}}{(n+2)\cancel{(2n^2+2n+1)}}\right) = \operatorname{arctg}\frac{(n+1)}{(n+2)} \quad \oplus$$

Що і треба було довести.

1.17 $n^{n+1} > (n+1)^n, \; n \geq 3$

При $n=3 \quad 3^4 > 4^3 \quad \oplus$

$A_{n+1}: \quad (n+1)^{n+2} > (n+2)^{n+1}$

$$\frac{(n+2)^{n+1}}{(n+1)^{n+2}} < 1$$

$$\frac{n+2}{(n+1)^2} \cdot \left(\frac{n+2}{n+1}\right)^n < 1$$

$$\frac{n+2}{(n+1)^2} \cdot \left(1+\frac{1}{n+1}\right)^n < \frac{n+2}{(n+1)^2} \cdot \left(1+\frac{1}{n}\right)^n$$

$$\frac{(n+2)n}{(n+1)^2} \cdot \boxed{\frac{(n+1)^n}{n^{n+1}}} < \cancel{1}\ \frac{(n+2)n}{(n+1)^2}.$$

$< 1$

$$\frac{(n+2)n}{(n+1)^2} = \frac{n^2+2n}{n^2+2n+1} < 1$$

Що і треба було довести.

**1.10** $\sum\limits_{k=1}^{n} \frac{1}{n+k} > \frac{13}{24},\ n \geq 2$

При $n=2$

$\frac{1}{2+1} + \frac{1}{2+2} > \frac{13}{24}$

$\frac{14}{24} > \frac{13}{24}$ $\oplus$

$A_{n+1}:\ \sum\limits_{k=1}^{n+1} \frac{1}{n+1+k} > \frac{13}{24}$

$\sum\limits_{k=1}^{n-1} \frac{1}{n+1+k} + \frac{1}{2n+1} + \frac{1}{2n+2} > \frac{13}{24}$

$\sum\limits_{k=1}^{n} \frac{1}{n+k} + \left(\frac{1}{2n+1} + \frac{1}{2n+2} - \frac{1}{n+1}\right) > \frac{13}{24}$

$\sum\limits_{k=1}^{n} \frac{1}{n+k} + \frac{2n+2+2n+1-4n-2}{2(2n+1)(n+1)} > \frac{13}{24}$

$\underbrace{\sum\limits_{k=1}^{n} \frac{1}{n+k}}_{> \frac{13}{24}} + \underbrace{\frac{1}{2(2n+1)(n+1)}}_{>0} > \frac{13}{24}$

Що і треба було довести

**1.11** $\sqrt{n} < \sum\limits_{k=1}^{n} \frac{1}{\sqrt{k}} < 2\sqrt{n},\ n \geq 2$

При $n=2$

$\sqrt{2} < \frac{1}{1} + \frac{1}{\sqrt{2}} < 2\sqrt{2}$

$\sqrt{2} < \frac{2+\sqrt{2}}{2} < 2\sqrt{2}$ $\oplus$

$A_{n+1}:\ \sqrt{n+1} < \sum\limits_{k=1}^{n+1} \frac{1}{\sqrt{k}} < 2\sqrt{n+1}$

$$\sum_{k=1}^{n+1} \frac{1}{\sqrt{k}} = 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \frac{1}{4} + \dots \frac{1}{\sqrt{n}} + \frac{1}{\sqrt{n+1}} > \frac{1}{\sqrt{n+1}} + \frac{1}{\sqrt{n+1}} + \dots \frac{1}{\sqrt{n+1}} = \frac{n+1}{\sqrt{n+1}} = \sqrt{n+1}$$

$$1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} \dots + \frac{1}{\sqrt{n}} + \frac{1}{\sqrt{n+1}} < \frac{2}{\sqrt{n+1}} + \frac{2}{\sqrt{n+1}} + \dots + \frac{2}{\sqrt{n+1}} =$$

$$= \frac{2(n+1)}{\sqrt{n+1}} = 2\sqrt{n+1}$$

$$\frac{1}{\sqrt{n}} < \frac{2}{\sqrt{n+1}}$$

$$\frac{1}{\sqrt{n}} - \frac{2}{\sqrt{n+1}} < 0;$$

$$\frac{\sqrt{n+1} - 2\sqrt{n}}{\sqrt{n(n+1)}} < 0$$

$$\sqrt{n+1} < 2\sqrt{n}$$

$$n+1 < 4n$$

$$3n > 1$$