Дз 20.23 $f = \int\limits_{t^2\sin 5t}^{x^2 t} \frac{e^y}{y}\,dy$

а) $\frac{df}{dx} = \frac{d}{dx}\int\limits_{t^2\sin 5t}^{x^2 t}\frac{e^y}{y}\,dy = \frac{e^{x^2 t}}{x^2 t}\cdot 2xt - \frac{e^{t^2\sin 5t}}{t^2\sin 5t}\left(t^2\sin 5t\right)' = 2\frac{e^{x^2 t}}{x}$

б) $\frac{df}{dt} = \frac{d}{dt}\int\limits_{t^2\sin 5t}^{x^2 t}\frac{e^y}{y}\,dy = \frac{e^{x^2 t}}{x^2 t}\cdot x^2 - \frac{e^{t^2\sin 5t}}{t^2\sin 5t}\left(2t\sin 5t + t^2\cos 5t\cdot\frac{1}{2 5t}\right) =$

$= \frac{e^{x^2 t}}{t} - \frac{2}{t}e^{t^2\sin 5t} - \frac{1}{25t}e^{t^2\sin 5t}\cdot \operatorname{ctg} 5t$

20.26 $\lim\limits_{x\to 0}\frac{\int_0^{x^2}\operatorname{arctg}(t^2)\,dt}{\int_{x^3}^{x^5}\operatorname{arctg} t\,dt} = \left[\frac{0}{0}\right] = \lim\limits_{x\to 0}\frac{f'}{g'} =$

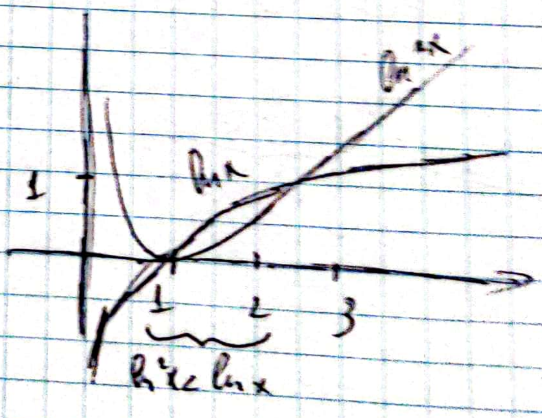
$= \lim\limits_{x\to 0}\frac{2x\operatorname{arctg}x^4}{5x^4\operatorname{arctg}x^5 - 3x^2\operatorname{arctg}x^3} = \lim\limits_{x\to 0}\frac{2\operatorname{arctg}x^4}{5x^3\operatorname{arctg}x^5 - 3x\operatorname{arctg}x^3} =$

$= \lim\limits_{x\to 0}\frac{2(x^4+o(x^5))}{5x^3(x^5+o(x^5)) - 3x(x^3+o(x^5))} = \lim\limits_{x\to 0}\frac{2x^4+o(x^5)}{5x^8-3x^4+o(x^5)} =$

$= \lim\limits_{x\to 0}\frac{2+o(x)}{5x^4-3+o(x)} = -\frac{2}{3}$

20.29 $\int\limits_1^2 \ln^2 x\,dx < \int\limits_1^2 \ln x\,dx$.

$\ln^2 x < \ln x$



20.30 $\int\limits_0^1 2^{x^2}\,dx > \int\limits_0^1 2^{x^3}\,dx$

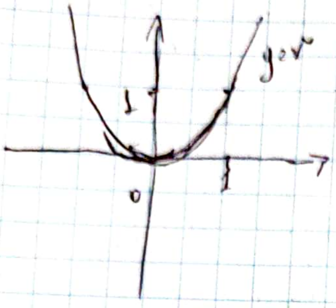
$2^{x^2} > 2^{x^3}$ при $x\in[0;1]$.

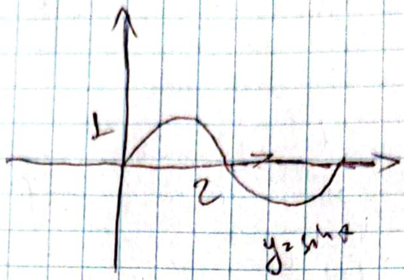
20.38 $\int_{\pi}^{2\pi} \sin x^2 dx$ $\sin x^2$ — неперервна

1 — константа

$$I = \int_{\pi}^{2\pi} \sin x^2 dx = \sin \xi^2 \int_{\pi}^{2\pi} 1 dx = \sin \xi^2 x \Big|_{\pi}^{2\pi} = \sin \xi^2 (2\pi - \pi) = \pi \sin \xi^2$$

$|\sin x^2| \le 1$





$\sin x^2$ періодична, $\sup(\sin x^2) = 1$, $\inf(\sin x^2) = -1$

$x = \sqrt{\frac{\pi}{2} + 2\pi n}$

$x = \sqrt{-\frac{\pi}{2} + 2\pi n}$

Нижня межа $\sqrt{\pi n} \ge \pi$; $\pi n \ge \pi^2$; $n \ge \pi$; $n \ge 3$

Верхня: $\sqrt{\pi n} \le 2\pi$; $\pi n \le 4\pi^2$; $n \le 4\pi$; $n \le 12$

Тому такі точки існують.

$-\pi \le I \le \pi$

20.40 $\int_{10}^{20} \frac{x^2 dx}{x^4+x+1}$

$f_1 = \frac{x^2}{4x^3+1}$ — монотонно спадає

$f_2 = \frac{4x^3+1}{x^4+x+1}$ — монотонно спадає

$$I = \int_{10}^{20} \frac{x^2 dx}{x^4+x+1} = f_1(10) \int_{10}^{\xi} f_2(x) dx = \frac{100}{4000+1} \int_{10}^{\xi} \frac{4x^3+1}{x^4+x+1} =$$

$$= \frac{100}{4001} \ln|x^4+x+1| \Big|_{10}^{\xi} = \frac{100}{4001} \left(\ln|\xi^4+\xi+1| - \ln 10011\right)$$

$\xi \in [0, 10] \Rightarrow 0 \le I \le \underset{\approx \frac{1}{40}}{\frac{100}{4001}} \ln \underset{\approx 5}{\frac{160021}{1011}} \approx \frac{1}{8}$

$$I = \int_{10}^{20} \frac{x^2\,dx}{x^4+x+1} = f_2(10)\int_{10}^{\xi} f_1(x)\,dx = \frac{4001}{1011}\int_{10}^{\xi} \frac{x^2}{4x^3+1}\,dx =$$

$$= \frac{1001}{1011}\int_{10}^{\xi} \frac{x^2}{4x^3+1}\,\frac{d(4x^3)}{12x^2} = \frac{4001}{1011}\ln|4x^3+1|\Big|_{10}^{\xi} = \frac{4001}{1011}\left(\ln|4\xi^3+1| - \ln 1001\right).$$

$\xi \in [10; 20] \Rightarrow 0 \le I \le \underset{\approx 4}{\frac{4001}{1001}} \ln \underset{\approx 2}{\frac{8001}{1001}} \approx 8$

Перша оцінка більш точна $\Rightarrow I \in [0; \frac{1}{8}]$

21.4 $y = \frac{16}{x^2}$, $y = 17 - x^2$, $x > 0$

$\frac{16}{x^2} = 17 - x^2$

$16 = 17x^2 - x^4$

$x^4 - 17x^2 + 16 = 0$

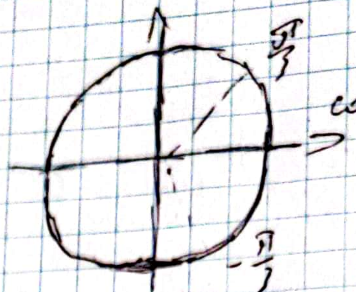
$x_1 = 4$

$x_2 = 1$ Точки перетину $(1; 16)$, $(4, 1)$

$$S = \int_1^4 \left(17 - x^2 - \frac{16}{x^2}\right)dx = 17x\Big|_1^4 - \frac{x^3}{3}\Big|_1^4 + \frac{16}{x}\Big|_1^4 = 68 - 17 - \frac{64}{3} + \frac{1}{3} + 4 - 16$$

$$= 39 - \frac{63}{3} = 39 - 21 = 18$$

4 п. $\rho = 2\cos\varphi, \ \rho \geq 1$



$2\cos\varphi \geq 1$

$\cos\varphi \geq \frac{1}{2}$

$\varphi \in [-\frac{\pi}{3} + 2\pi n, \ \frac{\pi}{3} + 2\pi n]$

$$S = \frac{1}{2}\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} 4\cos^2\varphi \, d\varphi = \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} 2\cos^2\varphi \, d\varphi = \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} (1+\cos 2\varphi) \, d\varphi =$$

$$= \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} 1 \, d\varphi + \frac{1}{2}\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \cos 2\varphi \, d2\varphi = \frac{\pi}{3} + \frac{\pi}{3} + \frac{1}{2}\sin 2\varphi \Big|_{-\frac{\pi}{3}}^{\frac{\pi}{3}} =$$

$$= \frac{2\pi}{3} + \frac{1}{2}\left(\sin\frac{2\pi}{3} + \sin\frac{2\pi}{3}\right) = \frac{2\pi}{3} + \frac{\sqrt{3}}{2}$$

5 п. $y = \ln x, \ x \in [\sqrt{3}, \sqrt{8}]$ монотонно зростає

$$L = \int_{\sqrt{3}}^{\sqrt{8}} \sqrt{1 + ((\ln x)')^2} \, dx = \int_{\sqrt{3}}^{\sqrt{8}} \sqrt{1 + \frac{1}{x^2}} \, dx = \int_{\sqrt{3}}^{\sqrt{8}} \frac{\sqrt{1+x^2}}{x} \, dx =$$

$$= \frac{1}{2}\int_{\sqrt{3}}^{\sqrt{8}} \sqrt{\frac{1+v}{v}} \, dv = \left| \begin{array}{l} u = \sqrt{1+v} \\ du = \frac{dv}{2\sqrt{1+v}} \\ v = u^2 - 1 \end{array} \right| = \frac{1}{2}\int_{\sqrt{3}}^{\sqrt{8}} \frac{2u^2 \, du}{u^2 - 1} = \int_{\sqrt{3}}^{\sqrt{8}} \frac{u^2 \, du}{u^2 - 1} =$$

$$= \int_{\sqrt{3}}^{\sqrt{8}} \left(1 + \frac{1}{(u-1)(u+1)}\right) du \quad (2)$$

$$\frac{1}{(u-1)(u+1)} = \frac{A}{u-1} + \frac{B}{u+1}$$

$1 = Au + A + Bu - B$

$$\begin{array}{l} u^1: \\ u^0: \end{array} \begin{cases} Au + Bu = 0 \\ A - B = 1 \end{cases} \quad \begin{cases} A = \frac{1}{2} \\ B = -\frac{1}{2} \end{cases}$$

1 $$\int_{\sqrt{3}}^{\sqrt{8}}\left(1+\frac{1}{2(u-1)}-\frac{1}{2(u+1)}\right)du = \int_{\sqrt{3}}^{\sqrt{8}} 1\,du+\frac{1}{2}\int_{\sqrt{3}}^{\sqrt{8}}\frac{1}{u-1}\,du-$$

$$-\frac{1}{2}\int_{\sqrt{3}}^{\sqrt{8}}\frac{1}{u+1}\,du = u\Big|_{\sqrt{3}}^{\sqrt{8}}+\frac{1}{2}\ln|u-1|\Big|_{\sqrt{3}}^{\sqrt{8}}-\frac{1}{2}\ln|u+1|\Big|_{\sqrt{3}}^{\sqrt{8}}=$$

$$=\sqrt{1+x^2}\Big|_{\sqrt{3}}^{\sqrt{8}}+\frac{1}{2}\ln|\sqrt{1+x^2}-1|\Big|_{\sqrt{3}}^{\sqrt{8}}-\frac{1}{2}\ln|\sqrt{1+x^2}+1|\Big|_{\sqrt{3}}^{\sqrt{8}}=$$

$$=3-2+\frac{1}{2}\ln 2-\frac{1}{2}\ln 1-\frac{1}{2}\ln 4+\frac{1}{2}\ln 3 = 1+\frac{1}{2}\ln\frac{3\cdot 2}{4} = 1+\frac{1}{2}\ln\frac{3}{2}$$