Дз №10

$$f(x,\theta,h)=\begin{cases}\frac{1}{2h}, & x\in[\theta-h;\ \theta+h]\\ 0, & x\notin[\theta-h;\ \theta+h]\end{cases}$$

$$\hat{\theta}_1=\frac{1}{2}\left(\max\{\xi_i\}-\min\{\xi_i\}\right)$$

$$\hat{\theta}_2=\frac{1}{2}\left(\min\{\xi_i\}+\max\{\xi_i\}\right)$$

$$\hat{\theta}_3=\min\{\xi_i\}-\frac{1}{n-1}\left(\max\{\xi_i\}-\min\{\xi_i\}\right)$$

$$\hat{\theta}_4=\max\{\xi_i\}+\frac{1}{n-1}\left(\max\{\xi_i\}-\min\{\xi_i\}\right)$$

$$F_{\max\{\xi_i\}}=P(\max\xi_i\le t)=\prod_{i=1}^{n}P(\xi\le t)=\left(\int_{\theta-h}^{t}\frac{1}{2h}\,dx\right)^n=$$
$$=\left(\frac{t-\theta+h}{2h}\right)^n$$

$$f_{\max\{\xi_i\}}(t)=F'_{\max\{\xi_i\}}(t)=\frac{n(t-\theta+h)^{n-1}}{(2h)^n}$$

$$M(\max\{\xi\})=\int_{\theta-h}^{\theta+h}\frac{tn(t-\theta+h)^{n-1}}{(2h)^n}\,dt=\frac{t(t-\theta+h)^n}{(2h)^n}\Bigg|_{\theta-h}^{\theta+h}-$$
$$-\int_{\theta-h}^{\theta+h}\frac{(t-\theta+h)^n}{(2h)^n}\,dt=\theta+h-\frac{(t-\theta+h)^{n+1}}{(2h)^n(n+1)}\Bigg|_{\theta-h}^{\theta+h}=$$
$$=\theta+h-\frac{2h}{n+1}=\theta+\frac{h(n-1)}{n+1}\xrightarrow{n\to\infty}\theta+h$$

$$F_{\min\{\xi_i\}}=P(\min\xi_i\le t)=1-P(\min\xi_i\ge t)=1-\prod_{i=1}^{n}P(\xi_i>t)=$$
$$=1-\left(\int_{t}^{\theta+h}\frac{1}{2h}\right)^n=1-\left(\frac{\theta+h-t}{2h}\right)^n$$

$$f_{\min\{\xi_i\}} = n\left(\frac{\frac{\theta+h-t}{2h}}{2h}\right)^{n-1} = \frac{n(\theta+h-t)^{n-1}}{(2h)^n}$$

$$M\min\{\xi_i\} = \int_{\theta-h}^{\theta+h} t\, n \cdot \frac{(\theta+h-t)^{n-1}}{(2h)^n}\,dt =$$

$$= -t\left(\frac{\theta+h-t}{2h}\right)^n \Bigg|_{\theta-h}^{\theta+h} + \int_{\theta-h}^{\theta+h} \frac{(\theta+h-t)^n}{(2h)^n}\,dt =$$

$$= \theta - h + \frac{-(\theta+h-t)^{(n+1)}}{(n+1)(2h)^n}\Bigg|_{\theta-h}^{\theta+h} =$$

$$= \theta - h + \frac{2h}{n+1} = \theta - \frac{h(n-1)}{n+1}$$

$$\hat{\theta}_1 : M\left(\frac{1}{2}(\max\{\xi_i\} - \min\{\xi_i\})\right) = \frac{1}{2}\big(M(\max\{\xi_i\}) -$$

$$- M(\min\{\xi_i\})\big) = \frac{1}{2}\left(\theta + \frac{h(n-1)}{n+1} - \theta + \frac{h(n-1)}{n+1}\right) =$$

$= \frac{h(n-1)}{n+1} \xrightarrow{n\to\infty} h \to$ асимптотично незсунута оцінка

$$\hat{\theta}_2 : M\left(\frac{1}{2}(\max\{\xi_i\} + \min\{\xi_i\})\right) = \frac{1}{2}\big(M(\max\{\xi_i\}) +$$

$$+ M(\min\{\xi_i\})\big) = \frac{1}{2}\left(\theta + \frac{h(n-1)}{(n+1)} + \theta - \frac{h(n-1)}{n+1}\right) =$$

$= \theta$ – незсунута оцінка

$$\hat{\theta}_3 : M\left(\min\{\xi_i\} - \frac{1}{n-1}(\max\{\xi_i\} - \min\{\xi_i\})\right) =$$

$$= M(\min\{\xi_i\}) - \frac{2}{n-1} M\left(\frac{\max\{\xi_i\} - \min\{\xi_i\}}{2}\right) =$$

$$= \Theta - \frac{h(n-1)}{n+1} - \frac{2}{n-1}\left(\frac{h(n-1)}{n+1}\right) = \Theta - \frac{h(n-1)}{n+1} - \frac{2h}{n+1} =$$

$$= \Theta - \frac{h(n+1)}{n+1} = \Theta - h$$ — незсунута оцінка

$$\hat{\Theta}_4 = \max\{\xi_i\} + \frac{1}{n-1}(\max\{\xi_i\} - \min\{\xi_i\}) =$$

$$= \Theta + \frac{h(n-1)}{(n+1)} + \frac{2}{n-1} \cdot \frac{h(n-1)}{n+1} = \Theta + h$$

незсунута оцінка

$$M(\max\{\xi_i\})^2 = \int_{\Theta-h}^{\Theta+h} \frac{t^2 n(t-\Theta+h)^{n-1}}{(2h)^n} dt =$$

$$= \frac{t^2(h-\Theta+t)^n}{(2h)^n}\Bigg|_{\Theta-h}^{\Theta+h} - 2\int_{\Theta-h}^{\Theta+h} \frac{t(t-\Theta+h)\,dt}{(2h)^n} =$$

$$= (\Theta+h)^2 - 2\left(\int_{\Theta-h}^{\Theta+h} \frac{t(t-\Theta+h)^n}{(2h)^n} dt\right) = (\Theta+h)^2 - 2 \cdot$$

$$\cdot \left(\frac{t(t-\Theta+h)^{n+1}}{(n+1)(2h)^n}\Bigg|_{\Theta-h}^{\Theta+h} - \int_{\Theta-h}^{\Theta+h} \frac{(t-\Theta+h)^{n+1}}{(n+1)(2h)^n}\right) =$$

$$= (\Theta+h)^2 + \frac{(-\Theta+h)(2h)}{n+1} + 2\frac{(t-\Theta+h)^{n+2}}{(n+1)(2h)^n(n+2)}\Bigg|_{\Theta-h}^{\Theta+h} =$$

$$= (\Theta+h)^2 + \frac{-(\Theta+h)(2h)}{n+1} + \frac{2 \cdot 4h^2}{(n+1)(n+2)} =$$

$$= \frac{n \cdot 4(\Theta^2 + 2h\Theta + h^2) - (\Theta+h)2h(n+2) + 8h^2}{(n+1)(n+2)} =$$

$$= \frac{\Theta^2(n+2)(n+1) + 2h\Theta(n^2+3n+2-2n-4) + h^2(n^2+3n+2-4n)}{(n+1)(n+2)} =$$

$$= \frac{\Theta^2(n+2)(n+1) + 2h\Theta(n^2+n-2) + h^2(n^2-n+2)}{(n+1)(n+2)}$$

$$D_{\max\{\xi_i\}} = M(\max\{\xi_i\})^2 - (M\max\{\xi\})^2 =$$

$$= \frac{\Theta^2(n+2)(n+1) + 2h\Theta(n^2+n-2) + h^2(n^2-n+2)}{(n+1)(n+2)} - \frac{(\Theta(n+1)+h(n-1))^2}{(n+1)^2} =$$

$$= \frac{\Theta^2(n+2)(n+1) + 2h(n^2+n-2)\Theta + h^2(n^2-n+2)}{(n+1)(n+2)} -$$

$$- \frac{\Theta(n+2)(n+1) - 2h\Theta(n-1)(n+2)}{(n+1)(n+2)} - \frac{h(n-1)^2}{(n+1)^2} =$$

$$= \frac{h^2(n^2-n+2)}{(n+1)(n+2)} - \frac{h^2(n-1)^2}{(n+1)^2}$$

$$M(\min\{\xi_i\})^2 = \int_{\Theta-h}^{\Theta+h} \frac{t^2 n(\Theta+h-t)^{n-1}}{(2h)^n}\, dt =$$

$$= \frac{(n^2+3n+2)\Theta^2 - 2h\Theta(n^2+n-2) + h^2(n^2-n+2)}{(n+1)(n+2)}$$

$$D_{\min\{\xi\}} = M(\min\{\xi\})^2 - (M(\min\{\xi\}))^2 =$$

$$= \frac{(n^2+3n+2)\Theta^2 - 2h\Theta(n^2+n-2) + h^2(n^2-n+2)}{(n+1)(n+2)} - \left(\Theta - h\frac{(n-1)}{(n+1)}\right)^2 =$$

$$= \frac{(n^2+3n+2)\Theta^2 - 2h\Theta(n^2+n-2) + h^2(n^2-n+2)}{(n+1)(n+2)} -$$

$$- \frac{\Theta^2(n+1)(n+2) - 2h\Theta(n+2)(n-1)}{(n+1)(n+2)} - \frac{h^2(n-1)^2}{(n+1)^2} = \frac{h^2(n^2-n+2)}{(n+1)(n+2)} -$$

$$- \frac{h^2(n-1)^2}{(n+1)^2} = \frac{h^2(n^2-n+2)(n+1) - h^2(n^2-2n+1)(n+2)}{(n+1)^2(n+2)} =$$

$$= \frac{h^2(n^3+n^2-n^2-n+2n+2-(n^3+2n^2-2n^2-4n+n+2)}{(n+1)^2(n+2)} =$$

$$= \frac{h^2(n+3n)}{(n+1)^2(n+2)} = \frac{4n\,h^2}{(n+1)^2(n+2)} \xrightarrow{n\to\infty} 0$$

$\hat{\theta}_1$: $D\left(\frac{1}{2}(\max\{\xi\} - \min\{\xi\})\right) = \frac{1}{4}\left(D\max\{\xi\} -\right.$

$- 2\,\mathrm{cov}(\max\{\xi_i\}, \min\{\xi\}) + D\min\{\xi\} =$

$= \frac{1}{4}(D\max\{\xi_i\} + D\min\{\xi_i\} = \frac{1}{4}\left(\frac{5h^2}{(n+1)^2(n+2)}\right) \to 0$

консистентная

$\hat{\theta}_2$: $D\left(\frac{1}{2}(\max\{\xi\} + \min\{\xi_i\})\right) = \frac{1}{4}\left(\frac{5h^2 n}{(n+1)^2(n+2)}\right) \to 0$

неконзистентна

$\hat{\theta}_3$: $D\left(\min\{\xi_i\} - \frac{1}{n-1}(\max\{\xi_i\} - \min\{\xi_i\})\right) =$

$= D\left(\left(1+\frac{1}{n-1}\right)\min\{\xi_i\} - \frac{1}{n-1}\max\{\xi_i\}\right) = \left(1+\frac{1}{n-1}\right)^2 D(\min\{\xi_i\}) +$

$+ \left(\frac{1}{n-1}\right)^2 D(\max\{\xi_i\}) = \left(\frac{n}{n-1}\right)^2 \frac{4h^2 n}{(n+1)^2(n+2)} + \left(\frac{1}{n-1}\right)^2 \frac{4h^2 n}{(n+1)^2(n+2)} \to 0$

конзистентна

$\hat{\theta}_4$ $D\left(\max\{\xi_i\} + \frac{1}{n-1}(\max\{\xi_i\} - \min\{\xi_i\})\right) = \left(\frac{n}{n-1}\right)^2 D(\max\{\xi_i\}) -$

$-2\left(\frac{n}{n-1}\max\{\xi_i\}, \frac{1}{n-1}\min\{\xi\}\right) = \left(\frac{1}{n-1}\right)^2 D(\min\{\xi_i\}) \to 0$

конзистентна

2) $f(x, \alpha) = \begin{cases} 0, \text{ якщо } x<0 \\ \frac{1}{\alpha} e^{-\frac{1}{\alpha}x}, & x \geq 0 \end{cases}$

$\hat{\Theta}_n = \bar{\xi} = \frac{1}{n}\sum_{i=1}^{n} \xi_i$

$M\hat{\Theta}_1 = \frac{1}{n}\sum_{i=1}^{n} M(\xi_i) = \frac{1}{n}\sum_{i=1}^{n} \alpha = \frac{n\alpha}{n} = \alpha$

Оцінка незсунута

$M\xi_i = \int_0^{+\infty} t \frac{1}{\alpha} e^{-\frac{1}{\alpha}t} dt = \frac{1}{\alpha}\int_0^{+\infty} t e^{-\frac{1}{\alpha}} dt = \frac{1}{\alpha}\left(-t\alpha e^{-\frac{1}{\alpha}t}\Big|_0^{+\infty} =\right.$

$\left.+\int_0^{+\infty} \alpha e^{-\frac{1}{\alpha}t} dt\right) = \frac{1}{\alpha}\cdot\alpha^2 = \alpha$

$D\hat{\Theta} = D\left(\frac{1}{n}\sum_{k=1}^{n}\xi_k\right) = \frac{1}{n}\sum_{k=1}^{n} D(\xi_k)\cdot\frac{1}{n^2} = \sum_{k=1}^{n}\alpha^2 = \frac{n\alpha^2}{n^2} \to 0$

Оцінка конзистентна

$\hat{\Theta}_2 = \frac{1}{2}(\xi_{n-1} + \xi_n)$

$M\hat{\Theta} = \frac{1}{2}(M\xi_{n-1} + M\xi_n) = \frac{1}{2}(\alpha+\alpha) = \alpha$

Оцінка незсунута

$P\left(\frac{1}{2}(\xi_{n+1} + \xi_n) - \alpha\right) > \varepsilon = P|\hat{\Theta}_2 - \alpha) > \varepsilon| = \int f(x) dx =$

$= \int \frac{1}{\alpha} e^{-\frac{1}{\alpha}} \neq 0$ не конзистентна

3) $P(k, \Theta) = C_N^k \Theta^k (1-\Theta)^{n-k}, \ k = 0, 1 \dots N$

$$\hat{\Theta} = \frac{1}{Nn} \sum_{k=1}^{n} \xi_k$$

$$M\hat{\Theta} = M\left(\frac{1}{Nn} \sum_{i=1}^{n} \xi_i\right) = \frac{1}{Nn} \sum_{i=1}^{n} M(\xi_i) = \frac{1}{Nn} \sum_{i=1}^{n} N\Theta =$$

$$= \frac{nN}{nN} \Theta = \Theta$$

Незсунута оцінка

$$D\hat{\Theta} = D\left(\frac{1}{Nn} \sum_{i=1}^{n} \xi_i\right) = \frac{1}{N^2 n^2} \sum_{i=1}^{n} D(\xi_i) = \frac{1}{N^2 n^2} \sum_{i=1}^{n} N\Theta(1-\Theta) =$$

$$= \frac{1}{n} \cdot \frac{N\Theta(1-\Theta)}{N^2} \to 0$$ конзистентна.