# Modeling misspecification as a parameter in Bayesian structural equation models

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May 4, 2023

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# Outline

# Introduction

# Study context

**Misspecification is active research area for Bayesian SEMs**

- Levy (2011): Posterior predictive checking using the likelihood ratio test; Bayesian SRMR
- Hoofs, van de Schoot, Jansen, and Kant (2018): Bayesian RMSEA
- Garnier-Villarreal and Jorgensen (2020): Several translations of frequentist fit indices: RMSEA, CFI, TLI, ...
- Cain and Zhang (2019): Deviance information criterion
- Fit hypthesized model $\xmapsto{\textit{then}}$ compute misspecification

# Current approach

**Fit hypthesized model and estimate misspecification simultaneously**

- Wu and Browne (2015) in frequentist context:
  - Parameter estimates have greater uncertainty about them reflecting degree of model misspecification
  - Estimates the RMSEA as a model parameter
- We could simply recreate Wu and Browne (2015) in a Bayesian context, but:
  - The RMSEA is not so easy to interpret (e.g. Chen, Curran, Bollen, Kirby, & Paxton, 2008; Savalei, 2012)
  - The SRMR has clearer interpretations.
  - Model misspecification so parameter estimates reflect degree of model incorrectness, while estimating an SRMR-type index.

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# Model

# CFA as an example

Begin with Muthén and Asparouhov (2012):

$$\mathbf{\Sigma} = \underbrace{\mathbf{\Lambda}\mathbf{\Phi}\mathbf{\Lambda}^{\mathsf{T}} + \mathbf{\Delta}}_{\text{Standard}} + \mathbf{\Psi} \tag{1}$$

$\mathbf{\Lambda}$: Loading matrix, $\mathbf{\Phi}$: Interfactor correlation matrix, $\mathbf{\Delta}$: Standard residual covariance matrix (often diagonal)

$\mathbf{\Psi}$: Residual covariance matrix, with all off-diagonal elements estimated.

- Theoretically: Assumed to reflect the influence of minor factors (MacCallum & Tucker, 1991)
- Practically: $\mathbf{\Psi}$ is not identified. Muthén and Asparouhov (2012) used an inverse-Wishart prior with known parameters to shrink elements in $\mathbf{\Psi}$ to zero.

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# Goal is to model misspecification

$$\mathbf{\Sigma} = \underbrace{\mathbf{\Lambda\Phi\Lambda}^{\mathsf{T}} + \mathbf{\Delta}}_{\text{Standard}} + \mathbf{\Psi}$$

Let $\psi_{ij}$ ($i \neq j$) be off diagonal elements in $\mathbf{\Psi}$, reflecting misspecification / minor factor influences.

To model minor factor influences:

$$\underbrace{\frac{\psi_{ij}}{\sqrt{\sigma_{jj}\sigma_{ii}}}}_{\text{SRCs}} \sim \mathcal{N}(0, \tau), \ \tau \sim \mathcal{N}^{+}(0, 1)$$

$\sigma_{ii/jj}$: indicator variances i.e. SRCs: standardized residual covariances.

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# A hierarchical model for SRCs

$$\underbrace{\frac{\psi_{ij}}{\sqrt{\sigma_{jj}\sigma_{ii}}}}_{\text{SRCs}} \sim \mathcal{N}(0, \tau),\ \tau \sim \mathcal{N}^{+}(0, 1)$$

## Implications

- SRCs are assumed to be zero on average
- SRC scale parameter ($\tau$) is the root mean square error of SRCs with location constrained to be zero
- $\tau$ is akin to the correlation root mean square residual (CRMR)
- Smaller values of $\tau$ reflect a model with smaller SRCs, lower misspecification OR smaller influence of minor factors
- $\tau < 0.05 \implies$ most SRCs $< 0.10$

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# Simulation studies

# Three simulation studies

## List of simulation studies

1. Can the proposed approach recover model parameters?
    - Yes, upcoming
2. Can differences in $\tau$ between competing models be used for model selection?
    - Lacks power, more promising to use approximate leave-one-out cross validation to select between competing models.
3. Can the $\boldsymbol{\Psi}$ matrix be used to detect specific residual covariances that are too large?
    - Yes, especially for larger samples.

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# Study 1: Recovery of structural parameters

Simulate data with minor factor influences ($\Psi$)

**5 modeling approaches**

- Two **ignore** minor factor influences:
  1. *Freq*: Standard frequentist
  2. *Baseline*: Standard Bayesian
- One **fixes** the size of minor factor influences:
  3. *AZ*: Muthén and Asparouhov (2012)
- Two **estimate** the size of minor factor influences:
  4. *LKJ*: another proposed method (in paper)
  5. *NRM*: hierarchical normal method (focus here)

**Results**: Bias adequate; Coverage poor for methods that ignore or fix the size of the influence of minor factors

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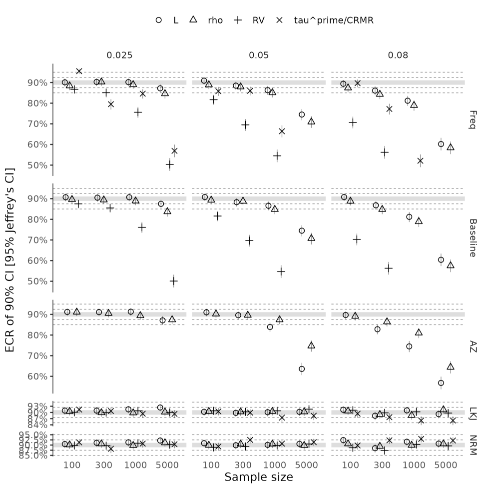
# Study 1: Empirical coverage rate of 90% CI

Coverage of structural parameters and $\tau'$



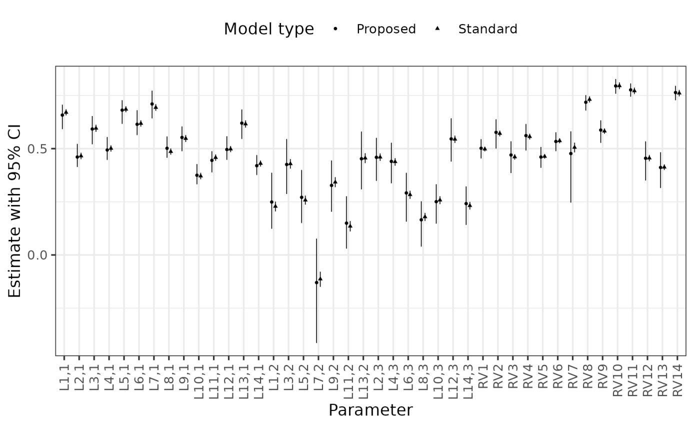

L: loading; RV: residual variance; rho: interfactor correlation

**Findings:**

- *Freq* and *Baseline* models fail when sample size or $\tau$ is large
- Performance of *AZ* model deteriorates at large $\tau$
- Methods that estimate the size of minor factor influences perform as expected

# An empirical example: Hospital Anxiety and Depression scale

Bifactor model parameters, $n = 21820, \tau = 0.028$. Note narrow intervals for the *Standard* approach.




LX.Y are loadings of factor Y on item X; RV: residual variances. Standard estimates are from a model without $\Psi$.

# Closing

# Discussion

## Discussion points / future work

- Approach works for fitting hypothesized models simultaneously with misspecification
- There is an R package for this: minorbsem
- Package also includes:
  - Implementation of Wu and Browne (2015) method, other priors for standardized residual covariances e.g. lasso, generalized double Pareto (global-local)
  - Global-local prior on all cross-loadings, relaxing simple structure
- Extensions to other data (e.g. binary)
  - Additional issues/extensions are on GitHub: `https://github.com/jamesuanhoro/minorbsem/issues/3`

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