Optimsation & Operations Research

Haide College, Spring Semester

Tutorial 2

These questions are on Topic 2: Simplex Algorithm

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1. **Interpretation:**

   Imagine we start with a LP

$$
\begin{array}{rcl}
\max\ z & = & 6x_1 + 14x_2 + 13x_3 \\
\text{subject to} & & \\
& & \frac{1}{2}x_1 + 2x_2 + x_3 \leq 24 \\
& & x_1 + 2x_2 + 4x_3 \leq 60 \\
& & x_i \geq 0
\end{array}
$$

   which we put into standard equality form (adding slack variables), and then into the tableau

x_1	x_2	x_3	x_4	x_5	z	b	basic variable
1/2	2	1	1	0	0	24	x_4
1	2	4	0	1	0	60	x_5
−6	−14	−13	0	0	1	0	

   We perform Simplex, and end up with the Tableau

x_1	x_2	x_3	x_4	x_5	z	b	basic variable
1	6	0	4	−1	0	36	x_1
0	−1	1	−1	1/2	0	6	x_3
0	9	0	11	1/2	1	294	

   The optimal solution is therefore $\mathbf{x}^* = (36, 0, 6)$, with $z^* = 294$

a) How close to equality are the original constraints at this solution?

> **Solution:** Both are tight, *i.e.*, they LHS = the RHS.

b) Intepret that "closeness" in the light of the value of the slack variables.

> **Solution:** The slack variables are both 0, indicating that these constraints are tight.

c) If we were to increase one of the constraint values, say 60 → 61, we could increase one of the slack variables – which one and by how much?

> **Solution:** If we were to increase one of the constraint values, say 60 → 61, we could increase x_5 by 1.

d) Use the final row of the tableau to estimate the potential affect of this on the value of z^*

> **Solution:** the new Tableau corresponds to a set of equations, and a new objective function where the coefficient of x_5 is $1/2$. Thus, increasing x_5 by 1, would increase the objective funtion by $1/2$. This might not be the new optimal point (given other constraints), but certainly indicates that we can do better than the old solution. The value $1/2$ we found here is a *shadow price*.

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2. Consider the following Linear Program

$$
\begin{array}{llrcrcrcl}
(P) & \max z & = & -x_1 & + & x_2 & + & 2x_3 & - \; 12 \\
& \text{s.t.} & & -2x_1 & + & 2x_2 & + & x_3 & \leq \; 6 \\
& & & 3x_1 & + & x_2 & - & x_3 & \leq \; 9 \\
& & & \multicolumn{6}{c}{x_1 \geq 0,\; x_2 \geq 0,\; x_3 \geq 0.}
\end{array}
$$

Find the optimal solution to this linear program using the Simplex algorithm.

Explain each step along the way. Make sure to identify pivot locations, and how they were chosen, and include values of slack variables in your solution.

Solution: We introduce a slack variables $s_4, x_s \geq 0$ for the inequalities, which puts it into feasible canonical form, and could proceed with Phase II directly.

1. Starting Tableau, and first pivot location, chosen by Bland's rule to be the left-most column with $-c_j < 0$, and the row with the minimum ratio b_i/a_{ij} for $a_{ij} > 0$

x_1	x_2	x_3	s_4	s_5	z	b
-2	**2**	1	1	0	0	6
3	1	-1	0	1	0	9
1	-1	-2	0	0	1	-12

2. Tableau resulting from the pivot, and second pivot location, chosen as the only column with $-c_j < 0$, and the only row with $a_{ij} > 0$

x_1	x_2	x_3	s_4	s_5	z	b
-1	1	**1/2**	$1/2$	0	0	3
4	0	$-3/2$	$-1/2$	1	0	6
0	0	$-3/2$	$1/2$	0	1	-9

3. Tableau resulting from the pivot, and second pivot location, chosen as the only column with $-c_j < 0$, and the only row with $a_{ij} > 0$

x_1	x_2	x_3	s_4	s_5	z	b
-2	2	1	1	0	0	6
1	3	0	1	1	0	15
-3	3	0	2	0	1	0

4. Fourth Tableau, and we stop because there are no remaining columns with $-c_j < 0$.

x_1	x_2	x_3	s_4	s_5	z	b
0	8	1	3	2	0	36
1	3	0	1	1	0	15
0	12	0	5	3	1	45

Phase II is now complete since all $-c_j \geq 0$, so that reading off the optimal solution we get

$$\mathbf{x}^* = \begin{pmatrix} 15 \\ 0 \\ 36 \\ 0 \\ 0 \end{pmatrix}$$

and with optimal solution $z^* = 45$.
NB: not using Bland's rule, you might get different Tableaus, but should get the same result.

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3. *Translate the following scenario into a linear program.*

A food stall in the OUC canteen is creating a recipe for a new dish.

It is to be a delicious beef noodle soup made up of the following ingredients:

- Noodles
- Beef
- Vegetables
- Broth

The nutritional and other information of the four ingredients is given in the below table.

	Noodles	**Beef**	**Vegetables**	**Broth**
Energy (per 100g serve)	560 kJ	1046 kJ	54 kJ	13 kJ
Protein (per 100g serve)	4.1 g	35 g	2.5g	1 g
Vitamin C (per 100g)	0 mg	0 mg	19.7 mg	0 mg
Volume (per 100g)	156.2 mL	104.6 mL	94.8 mL	100 mL
Cost (per 100g)	¥0.8	¥2.6	¥2.1	¥0.3

The soup as a whole should meet the following nutritional requirements:

- Energy: must have at least 2000 kJ
- Protein: must have at least 30 g of protein
- Vitamin C: must have at least 30 mg

The soup should exactly fill a standard 1000 mL bowl. It should have no more than 400 mL of broth to leave plenty of room for the other ingredients.

To maximise profit, the noodle soup should be produced as cheaply as possible.

a) State the variables (including units), the objective and the constraints.

Solution:

- Variables

$$\begin{aligned} x_1 &= \text{amount noodles } (100\,\text{g}) \\ x_2 &= \text{amount beef } (100\,\text{g}) \\ x_3 &= \text{amount vegetables } (100\,\text{g}) \\ x_4 &= \text{amount broth } (100\,\text{g}) \end{aligned}$$

- Objective: minimise $z = 0.8x_1 + 2.6x_2 + 2.1x_3 + 0.3x_4$
- Constraints: such that

$$\begin{aligned} 560x_1 + 1046x_2 + 54x_3 + 13x_4 &\geq 2000 \text{ (energy)} \\ 4.1x_1 + 35x_2 + 2.5x_3 + x_4 &\geq 40 \text{ (protein)} \\ 19.7x_3 &\geq 30 \text{ (vitamin c)} \\ x_4 &\leq 4 \text{ (max broth)} \\ 156.2x_1 + 104.6x_2 + 94.8x_3 + 100x_4 &= 1000 \\ x_1, x_2, x3, x_4 &\geq 0 \text{ (non-negativity)} \end{aligned}$$

b) Solve your linear program from part (a) using `linprog`.

Solution:

```
% Tutorial 2, Q3(b)
% Mike Chen, Feb 2025

% Use linprog to solve the noodle soup problem

% Noodle soup LP as a minimisation problem (minimise cost).
%f = [1.5; 5.1; 4.3; 0.5]/2;
f = [0.8; 2.6; 2.1; 0.3];

A = -[560 1046 54 13;
    4.1 35 2.5 1;
    0 0 19.7 0;
    0 0 0 -1];

b = [-2000; -40; -30; 4];

Aeq = [156.2 104.6 94.8 100];
beq = 1000;

lb = zeros(1,4);
ub = [];
% Could use ub = [Inf(1,3) 4] here instead of first two
% constraints above.
```

```
% Call linprog:
[x,fval] = linprog(f,A,b,Aeq,beq,lb,ub)

nutritional_info = -A(1:3,:)*x
```

c) Using the output from `linprog`, state the optimal solution. This should include the value of the objective and the values of the variables. Make sure you include units for all quantities.

Solution: The optimal solution is $x_1 \approx 2.50$, $x_2 \approx 0.63$, $x_3 \approx 1.52$, $x_4 = 4$ with $z = 8.03$.
To minimise cost ("make as cheaply as possible") the food stall's recipe should be

- 250 g noodle
- 63 g beef
- 152 g vegetables
- 400 g broth (or equivalently 400 mL)

This soup will cost ¥8.03 to produce.

d) State the nutritional information for this noodle soup. This should include energy (kJ), protein (g) and vitamin C (mg).

Solution: The complete noodle soup has the following nutritional information:

- energy: 2119 kJ
- protein: 40 g
- vitatmin C: 30 mg

e) Advise the food stall on a price for the noodle soup and state the resulting profit per bowl. (**Hint:** There is no "right" answer here, but you should explain your reasoning.)

Solution: To make a profit, the price should be greater than the cost (in this case ¥8.03).
A typical price for this sort of dish might be ¥12, which would give

$$\text{profit} = \text{price} - \text{cost} = 12 - 8.03 = 3.97.$$

If the food stall charges ¥12 for this dish they will make a profit of ¥3.97 per bowl.

Bonus questions

1. **Calculations:** Consider the LP with the Simplex Tableau:

x_1	x_2	x_3	x_4	x_5	x_6	z	b
0	1	−2	0	3	1	0	3
1	2	4	0	1	0	0	4
4	−2	1	1	1	0	0	2
−3	−1	−5	0	0	7	1	10

a) Explain why each of the following positions would not be a suitable choice for the next pivot position, if the Simplex Method were to be applied to the above tableau.

i. Row 1, Column 1

Solution: $a_{11} = 0$; only pivot on $a_{ij} > 0$.

ii. Row 1, Column 3

Solution: $a_{13} < 0$; only pivot on $a_{ij} > 0$.

iii. Row 3, Column 4

Solution: Column 4 is a basic column already, so there is no pivoting to be done.

iv. Row 2, Column 1

Solution: $a_{21} > 0$ and $-c_1 < 0$, but $\frac{b_2}{a_{21}} = 4 > \frac{b_3}{a_{31}} = \frac{1}{2}$ so a_{21} does not produce the smallest ratio.

v. Row 2, Column 8

Solution: Never choose a pivot position from the $b-$ column.

vi. Row 4, Column 3

Solution: Never choose a pivot position from the $z-$row.

b) Nominate two distinct entries that *could* be selected as suitable pivot positions for the Simplex Method.

Solution: Two of the following:

- Row 3, Column 1 (a_{31})
- Row 2, Column 2(a_{22})
- Row 2, Column 3 (a_{23}).

c) What happens to the value of the objective function if you pivot in Column 5?

Solution: The value of the objective function would not change, because $-c_5 = 0$. (If you had a final tableau, with all $-c_j \geq 0$, then pivoting in a column k with $-c_k = 0$ would allow you to check if there are alternative $\mathbf{x}$ to give the same optimal z^*.)