7 marks 1. Consider the following linear program.

$$\begin{aligned} \max \quad & z = 4x_1 + 3x_2 + 2x_3 + x_4, \\ \text{subject to} \quad & 10x_1 + 7x_2 - x_3 + x_4 \leq 18, \\ & 3x_2 + 8x_3 - 6x_4 \leq 16, \\ & 15x_1 - 20x_2 - 40x_3 - 50x_4 \leq 27, \end{aligned}$$

with x_1, x_2, x_3 and $x_4 \geq 0$.

6 a) Solve the above linear program using Simplex.
At each step explain why a pivot has been chosen, clearly stating the mandatory and (if applicable) discretionary rules which have been applied.
When you reach the final step, clearly explain why the algorithm has stopped.
State your optimal solution (values of the variables and objective).

Solution:
Step 1. Introduce slack variables and form the Simplex tableau as usual.

x_1	x_2	x_3	x_4	x_5	x_6	x_7	z	b
10	7	−1	1	1	0	0	0	18
0	3	8	−6	0	1	0	0	16
15	−20	−40	−50	0	0	5	0	27
−4	−2	−3	−1	0	0	0	1	0

Pivot at (1,1) identified as follows.

- *Column choice (mandatory).* Columns 1, 2, 3 and 4 have $-c_j < 0$.
- *Column choice (discretionary).* Column 1 is chosen because it is the left-most, ie. has smallest index, of these options.
- *Row choice (mandatory).* Rows 1 and 3 have $a_{ij} > 0$ and equal smallest b_i/a_{ij} $(18/10 = 27/15)$.
- *Row choice (discretionary).* Row 1 is chosen because it has the smallest index of these options.

Step 2. The resulting tableau is:

x_1	x_2	x_3	x_4	x_5	x_6	x_7	z	b
1	7/10	−1/10	1/10	1/10	0	0	0	9/5
0	3	8	−6	0	1	0	0	16
0	−61/2	−77/2	−103/2	−3/2	0	5	0	0
0	4/5	−17/5	−3/5	2/5	0	0	1	36/5

Pivot at (2,3) identified as follows.

- *Column choice (mandatory).* Columns 3 and 4 have $-c_j < 0$.
- *Column choice (discretionary).* Column 3 is chosen because it has smallest index of these choices.
- *Row choice (mandatory).* Row 2 is chosen because it is the only $a_{ij} > 0$.
- *Row choice (discretionary).* Not required.

Step 3. The resulting tableau is:

x_1	x_2	x_3	x_4	x_5	x_6	x_7	z	b
1	59/80	0	1/40	1/10	1/80	0	0	2
0	3/8	1	−3/4	0	1/8	0	0	2
0	−257/16	0	−643/8	−3/2	77/16	5	0	77
0	83/40	0	−63/20	2/5	17/40	0	1	14

Pivot at (1,4) identified as follows.

- *Column choice (mandatory).* Columns 4 is chosen because it has the only $-c_j < 0$.
- *Column choice (discretionary).* Not required.
- *Row choice (mandatory).* Row 1 is chosen because it has the only $a_{ij} > 0$.
- *Row choice (discretionary).* Not required.

Step 4. The resulting tableau is:

x_1	x_2	x_3	x_4	x_5	x_6	x_7	z	b
40	59/2	0	1	4	1/2	0	0	80
30	45/2	1	0	3	1/2	0	0	62
3215	2355	0	0	320	45	5	0	6507
126	95	0	0	13	2	0	1	266

Stop because there are no remaining columns with $-c_j < 0$.

The optimal solution is $(x_1, x_2, x_3, x_4) = (0, 0, 62, 80)$ with $z = 266$.

1 b) **MATLAB Grader.** Check your answer with MATLAB Grader.

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6 marks 2. Consider the following primal linear program (P),

$$
\begin{array}{llrcrcrcl}
(P) & \max z = & 3x_1 & - & x_2 & + & x_3 & & \\
& \text{subject to} & 2x_1 & - & 5x_2 & + & 2x_3 & \leq & 8 \\
& & 5x_1 & + & 4x_2 & - & x_3 & \leq & 14 \\
& & \multicolumn{7}{l}{x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0}
\end{array}
$$

2 a) State the dual problem (D) of (P). As usual, your should include the objective and all constraints.

Solution: The dual (D) of (P) is

$$\begin{array}{lllrcrcr}
(D) & \min w & = & 8y_1 + 14y_2 & & & & \\
& & & 2y_1 & + & 5y_2 & \geq & 3 \\
& & & -5y_1 & + & 4y_2 & \geq & -1 \\
& & & 2y_1 & - & y_2 & \geq & 1 \\
& & y_1, y_2 \geq 0. & & & & &
\end{array}$$

where non-negativity arises as usual (introduction of slack variable etc).

1 b) Find the complementary slackness relations for the above primal problem (P).

Solution: The Complementary Slackness Relations, for the Primal (and for completeness the Dual) are

$$\begin{aligned}
y_1(2x_1 - 5x_2 + 2x_3 - 8) &= 0 \quad (i) \\
y_2(5x_1 + 4x_2 - x_3 - 14) &= 0 \quad (ii) \\
x_1(2y_1 + 5y_2 - 3) &= 0 \quad (iii) \\
x_2(-5y_1 + 4y_2 + 1) &= 0 \quad (iv) \\
x_3(2y_1 - y_2 - 1) &= 0 \quad (v)
\end{aligned}$$

3 c) You are given that (P) has the optimal solution

$$x_1^* = 0, x_2^* = 12, x_3^* = 34.$$

Use this optimal solution and the complementary slackness relations from Question 2(b) to find the optimal solution of the dual (D).

Note: no marks will given for a Simplex Method solution.

Solution:

- The solution $\vec{x}^T = (0, 12, 34)$ satisfies (i) and (ii).
- The variable $x_1 = 0$, and hence CSR (iii) is automatically satisfied.
- The basic variables x_2 and x_3 are non-zero, so CSRs (iv) and (v) become

$$\begin{aligned}
-5y_1 + 4y_2 + 1 &= 0 \quad (iv) \\
2y_1 - y_2 - 1 &= 0 \quad (v)
\end{aligned}$$

Solving simultaneously gives $\vec{y}^* = (1, 1)$ and $w^* = 22$.

For $\vec{x}^* = (0, 12, 34)$ have $z^* = 22 = w^*$, so by strong duality this is the optimal solution.

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7 marks

3. Consider the following linear problem

$$\begin{aligned} \max \quad z &= 4x_1 + 2x_2 + x_3, \\ \text{subject to} \quad x_1 &\leq 5, \\ 4x_1 + x_2 &\leq 25, \\ 8x_1 + 4x_2 + x_3 &\leq 125, \end{aligned}$$

with $x_1, x_2, x_3 \geq 0$. This is an example of Klee-Minty problems, which is a class of linear programs often used to test the performance of algorithms.

2 a) Solve the above problem using Simplex with Bland's rules for column/row choice. In addition to the usual mandatory rules, these rules are:

- column choice: if more than one $-c_j < 0$ choose the option with smallest j.
- row choice: if more than one row has equal smallest b_i/a_{ij}, choose the row with smallest i.

In your submission, rather than writing out each tableau, record the following information about each step:

- The choice of pivot location (i, j)
- The value of the objective
- The value of the variables (x_1, x_2, x_3)
- The non-basic variables.

A table or list would be an appropriate way to present this information.

Solution: Here is a table summarising Simplex Phase II with the given discretionary rules. Note here x_4, x_5 and x_6 are slack variables.

Step	Pivot	z	(x_1, x_2, x_3)	Non-basic variables
Initial	-	0	$(0, 0, 0)$	x_1, x_2, x_3
1	(1,1)	20	$(5, 0, 0)$	x_2, x_3, x_4
2	(2,2)	30	$(5, 5, 0)$	x_3, x_4, x_5
3	(3,3)	95	$(5, 5, 65)$	x_4, x_5, x_6
4	(2,5)	105	$(5, 0, 85)$	x_2, x_4, x_6
5	(1,4)	125	$(0, 0, 125)$	x_1, x_2, x_6

2 b) Now solve the problem again, but this time with different discretionary rules. In addition to the usual mandatory rules apply the following

- column choice: if more than one $-c_j < 0$ choose the minimum value of $-c_j$ (the most strongly negative). Where two values are equal, choose the option with the smallest j.
- row choice: if more than one row has equal smallest b_i/a_{ij}, choose the row with largest a_{ij}. Where two values are equal, choose the option with the smallest i.

In your submission, include the details of each step as in Question 3(a) presented as a table or list.

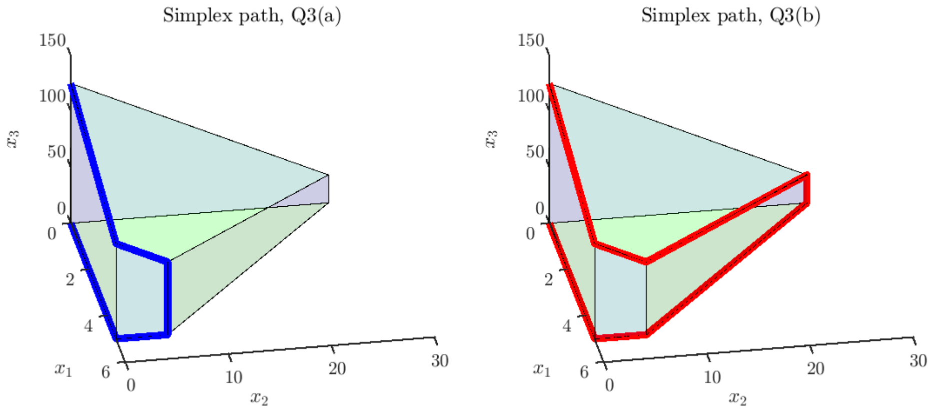
Solution: Here is a table summarising Simplex Phase II with the given discretionary rules. Note here x_4, x_5 and x_6 are slack variables.

Step	Pivot	z	(x_1, x_2, x_3)	Non-basic variables
Initial	-	0	$(0, 0, 0)$	x_1, x_2, x_3
1	(1,1)	20	$(5, 0, 0)$	x_2, x_3, x_4
2	(2,2)	30	$(5, 5, 0)$	x_3, x_4, x_5
3	(1,4)	50	$(0, 25, 0)$	x_1, x_3, x_5
4	(3,3)	75	$(0, 25, 25)$	x_1, x_5, x_6
5	(1,1)	95	$(5, 5, 65)$	x_4, x_5, x_6
6	(2,5)	105	$(5, 0, 85)$	x_2, x_4, x_6
7	(1,4)	125	$(0, 0, 125)$	x_1, x_2, x_6

2 c) Visualise the trajectories of the solutions from Questions 3(a) and 3(b). To do this make a 3D plot of the (x_1, x_2, x_3) co-ordinates of each step of the Simplex.

Hint: One way to make such a plot is with MATLAB's `plot3` function (see function documentation for examples). You may find it helpful to indicate the feasible region as part of your plot(s).

Solution: The trajectory for the solution from Q3(a) is shown on the left with a blue line, and the trajectory for the solution from Q3(b) is shown on the right with a red line. Both plots show the feaseible region (a distored 3D cube).



Code not required, but here's how the above plot was produced for those who are interested.

```
% Define paths
path_3a = [0 0 0; 5 0 0; 5 5 0; 5 5 65; 5 0 85; 0 0 125];
path_3b = [0 0 0; 5 0 0; 5 5 0; 0 25 0;
    0 25 25; 5 5 65; 5 0 85; 0 0 125];

% Use the Q3(b) data to define the faces of the Klee-Minty cube
km_faces_1 = [1 2 7 8; 3 4 5 6];
km_faces_2 = [1 2 3 4; 5 6 7 8];
km_faces_3 = [1 4 5 8; 2 3 6 7];

% Plot paths with plot3, and add the feasible region with patch
subplot(1,2,1)
plot3(path_3a(:,1),path_3a(:,2),path_3a(:,3),'b','LineWidth',4)
hold on
patch('Faces',km_faces_1,'Vertices',path_3b,'FaceColor','black','FaceAlpha',0.1)
patch('Faces',km_faces_2,'Vertices',path_3b,'FaceColor','green','FaceAlpha',0.1)
patch('Faces',km_faces_3,'Vertices',path_3b,'FaceColor','blue','FaceAlpha',0.1)
```

```
hold off
xlabel('$x_1$'); ylabel('$x_2$'); zlabel('$x_3$');
title('Simplex path, Q3(a)')
view([440 40])

subplot(1,2,2)
plot3(path_3b(:,1),path_3b(:,2),path_3b(:,3),'r','LineWidth',4)
hold on
patch('Faces',km_faces_1,'Vertices',path_3b,'FaceColor','black','FaceAlpha',0.1)
patch('Faces',km_faces_2,'Vertices',path_3b,'FaceColor','green','FaceAlpha',0.1)
patch('Faces',km_faces_3,'Vertices',path_3b,'FaceColor','blue','FaceAlpha',0.1)
hold off
xlabel('$x_1$'); ylabel('$x_2$'); zlabel('$x_3$');
title('Simplex path, Q3(b)')
view([440 40])
set(gcf,'Position',[0 0 800 300])
```

d) Briefly discuss how Simplex proceeds with the different discretionary choice rules from Question 3(a) and 3(b). Refer to your plots from Question 3(c) as part of your answer.

Solution: The trajectories both start at $(x_1, x_2, x_3) = (0, 0, 0)$ and finish at the optimal solution at (0,0,125).
The discretionary rules in Q3(a) result in a trajectory with 5 steps to get to the optimal solution, while the rules in Q3(b) result in a slightly longer trajectory with 7 steps.
Some additional things to notice:

- the trajectory from Q3(b) visits every feasible node, meaning that (without doubling back) this is as inefficient a path as possible, equivalent to a brute force search.
- neither set of rules find the optimal solution as quickly as possible - this would be a direct trajectory up the x_3 axis from (0,0,0) to (0,0,125) with a single pivot at (3,3). An example of a discretioinary rule that would have done is to have the discretionary column rule choose the option with largest j.