THE UNIVERSITY *of* ADELAIDE

Optimsation & Operations Research

Haide College, Spring Semester

Tutorial 1

These questions are on Topic 1: Optimisation & linear programming

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1. **Translation:** A farmer has three crops he can grow: wheat, rice and cotton. Each results in revenue (per acre) of \$100, \$300, \$200. However, the rice and cotton must be irrigated taking 110 kilolitres and 100 kilolitres of water per acre per day, respectively, and the farmer has only 3000 kilolitres of water available per day for the whole farm. All must be fertilised, with respective costs of \$40, \$30 and \$20 per acre.

   Interrogate the problem and formulate an optimisation problem to tell the farmer how much of each crop to grow on his/her 50 acre farm to maximise his/her profits.

   Hints: remember to look for three things:

   1. the variables (the things you can control);
   2. the objective (the thing you want to maximise or minimise); and
   3. the constraints (there are 2 main constraints here, but don't forget non-negativity).

   Tabulate the data, and then construct the optimisation in standard form.

   Solution:

   1. The variables are x_1, x_2 and x_3, the number of acres the crops wheat, rice and cotton.
   2. The objective is the profit in \$ (for the whole farm)

   $$z = (100-40)x_1 + (300-30)x_2 + (200-20)x_3 = 60x_1 + 270x_2 + 180x_3.$$

   3. The constraints are

   $$\begin{array}{rcrcrcrl} & & 110x_2 & + & 100x_3 & \leq & 3000, & \text{water constraint in kilolitres} \\ x_1 & + & x_2 & + & x_3 & \leq & 50, & \text{land area constraint in acres} \end{array}$$

   with $x_i \geq 0$, because you can't farm a negative area.

   So the problem can be written in standard inequality form as

   $$\begin{array}{rrcl} \max & z & = & \mathbf{c}^T\mathbf{x} \\ \text{such that} & A\mathbf{x} & \leq & \mathbf{b} \\ \text{and} & \mathbf{x} & \geq & 0 \end{array}$$

   where

   $$A = \begin{pmatrix} 0 & 110 & 100 \\ 1 & 1 & 1 \end{pmatrix}, \quad \mathbf{b}^T = (3000, 50), \quad \text{and } \mathbf{c}^T = (60, 270, 180).$$

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2. **Translation:** Consider the following *portfolio management* problem. A bank has \$1 million to invest in variety of bonds offered by the government and other agencies. Assume that there are 4 bonds considered here, with quantities related to each denoted with $i = 1, 2, 3, 4$.

   Each bond has a *rated quality*, q_i, an *after-tax yield*, y_i and a *years to maturity*, m_i (how long the investment is committed). The portfolio manager must try to maximise the return on investment, but must also meet other criteria:

   1. the average quality of the portfolio of bonds cannot be worse than 1.5 (note that for quality, a low number corresponds to high-quality)
   2. the average years to maturity of the portfolio of bonds should not exceed 4 years.

a) What are the variables? *Hint: define variables x_1, x_2, x_3 and x_4.*

Solution: Each x_i is the number of bond i we purchase for $i = 1, 2, 3, 4$.

b) What is the objective?

Solution: Maximise after-tax yield. Denote yield of bond i by y_i, we want to

$$\max \sum_{i=1}^{4} y_i x_i.$$

c) Write a series of linear constraints. *Hint: there should be three.*

Solution: Each bond i has a quality q_i, a years to maturity m_i and a cost c_i.

$$\begin{aligned}
\frac{\sum_{i=1}^{4} q_i x_i}{\sum_{i=1}^{4} x_i} &\leq Q, \\
\frac{\sum_{i=1}^{4} m_i x_i}{\sum_{i=1}^{4} x_i} &\leq M, \\
\sum_{i=1}^{4} c_i x_i &\leq C,
\end{aligned}$$

where $C = 1,000,000$ is the available investment fund, $M = 4$ is the required average of years to maturity, and $Q = 1.5$ is the required average quality.
To create a linear program, we have to rearrange the average constraints by multiplying both sides by the denominator, and then moving terms involving x_i to the left, so we get

$$\begin{aligned}
\sum_{i=1}^{4} q_i x_i - Q \sum_{i=1}^{4} x_i &\leq 0, \\
\sum_{i=1}^{4} m_i x_i - M \sum_{i=1}^{4} x_i &\leq 0, \\
\sum_{i=1}^{4} c_i x_i &\leq C,
\end{aligned}$$

We can make the problem's structure more obvious by writing it as

$$\begin{aligned}
\sum_{i=1}^{4} (q_i - Q) x_i &\leq 0, \\
\sum_{i=1}^{4} (m_i - M) x_i &\leq 0, \\
\sum_{i=1}^{4} c_i x_i &\leq C,
\end{aligned}$$

Note that

- The last constraint is *implicit* in the problem: we didn't state it, but it should be clear from the context that we can't spend more money than we have.
- Instead of averages on the LHS, the sum, and this accounts for the extra factor of 4 on the right. This approach can reduce computation just a little bit.

d) What are the bounds on the variables?

Solution: We require $x_i \geq 0$ for all i, though in real investment funds it is sometimes possible to purchase "derivatives" that might look like a negative investment.

Note that, although this problem is specified for a particular set of numbers, e.g., the number of choices is 4, we could extend it to an arbitrary number of bonds, and change the various coefficients as needed. We will investigate how to practically solve such problems in the next practical.

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3. **Interpretation:** Imagine that we took a problem expressed in the form

$$\begin{aligned}
\max \quad & z = \mathbf{c}^T \mathbf{x} \\
\text{subject to} \quad & A' \mathbf{x} \leq \mathbf{b} \\
& \mathbf{x} \geq \mathbf{0}
\end{aligned}$$

and when we converted it into standard inequality form, we arrived at a Tableau for the equalities, on which we performed a series of row operations to obtain

$$\begin{array}{|cccccc|c|}
\hline
1 & 1 & 0 & 1 & 0 & 0 & 5 \\
0 & 3 & 1 & 0 & -1 & 0 & 7 \\
0 & 3 & 0 & 1 & -1 & 1 & 4 \\
\hline
\end{array}$$

a) Write down a solution to this problem with three basic, and three non-basic variables (Hint: it should be possible to do so immediately).

Solution: The basic variables correspond to the unit columns, and the non-basic variables are set to zero, so the solution is

$$\mathbf{x} = (5, 0, 7, 0, 0, 4)$$

b) Interpret this solution in the light of the optimisation problem specified in terms on inequalities above.

Solution:

- The solution is the intersection of 3 planes in the original space, corresponding to the zeros, *i.e.*, two of the slack variables x_4 and $x_5 = 0$, meaning two of the original boundary planes are involved, and the third comes from $x_2 = 0$.
- The solution is feasible (as all values are non-negative), and hence the solution is a vertex of the feasible region originally specified.

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4. **Calculations:** Translate the following problem into standard equality form.

$$
\begin{array}{lcrcrcrcr}
\min \; z & = & 2x_1 & - & 2x_2 & + & 3x_3 & & \\
\text{subject to} & & & & & & & & \\
 & & -x_1 & + & 2x_2 & + & x_3 & \leq & 4 \\
 & & 2x_1 & - & x_2 & + & 2x_3 & \geq & -2
\end{array}
$$

with $x_1 \geq 0$, $x_2 \leq 0$, and x_3 free.

Hints: some tricks you will need:

1. You need to convert it into a maximisation problem.
2. You need to convert a $\geq$ constraint into a $\leq$
3. You need to swap a non-positive variable with a non-negative one.
4. You need to replace a free variable with two non-negative variables.
5. You need to add slack variables to convert the constraints into equalities.

Solution: Steps:

- multiply the objective by -1, to make it a maximisation.
- multiply the second equation by -1, so that it is a $\leq$.
- replace x_2 with $x_2' = -x_2$, and hence $x_2' \geq 0$
- convert $x_3 = x_3^+ - x_3^-$, so that we have non-negative variables; and
- add two slack variables to convert the inequatilities to equalities.

$$
\begin{array}{lcrcrcrcrcrcrcr}
\max \; z & = & -2x_1 & - & 2x_2' & - & 3x_3^+ & + & 3x_3^- & & & & & & \\
\text{subject to} & & & & & & & & & & & & & & \\
 & & -x_1 & - & 2x_2' & + & x_3^+ & - & x_3^- & + & x_4 & & & = & 4 \\
 & & -2x_1 & - & x_2' & - & 2x_3^+ & + & 2x_3^- & & & + & x_5 & = & 2
\end{array}
$$

with $x_i \geq 0$ for all i.
