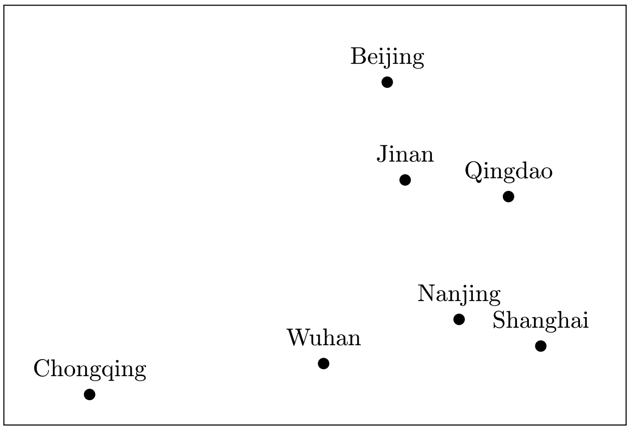
Optimsation & Operations Research Haide College, Spring Semester

Tutorial 4

These questions are on Topic 4: Integer programming problems.

Consider the seven Chinese cities shown in the map below.



1. A tourist would like the visit all of the cities shown above exactly once, starting and finishing in Qingdao. They would like to do this in the shortest distance possible.

   The distances between each pair of cities in kilometres are given in the table below.

	Qingdao	Shanghai	Beijing	Wuhan	Jinan	Chongqing	Nanjing
Qingdao	0	560	550	840	310	1480	480
Shanghai	560	0	1070	690	740	1440	270
Beijing	550	1070	0	1050	360	1460	900
Wuhan	840	690	1050	0	720	750	460
Jinan	310	740	360	720	0	1250	540
Chongqing	1480	1440	1460	750	1250	0	1200
Nanjing	480	270	900	460	540	1200	0

   What route around the above cities should the tourist take?

a) This is an example of a type of problem covered in this course. State the name of this type of problem and describe a greedy heuristic to find a solution.

> **Solution:** This is an example of the travelling salesperson problem.
> The greedy heuristic is that the next city in the route is always is the closest city that has not already been visited.

b) Apply the greedy heuristic to find a route around the seven cities **starting and ending in Qingdao**. State the order of your route and the total distance travelled.

> **Solution:**
> Starting in Qingdao, the order of cities is:

City	Distance from previous
Qingdao	-
Jinan	310
Beijing	360
Nanjing	900
Shanghai	270
Wuhan	690
Chongqing	750
Qingdao	1480
Total	4760

c) Is the route you found with the greedy heuristic the shortest route possible? Suggest what could be improved about the route you found.

Solution: The Beijing-Nanjing, Qingdao-Jinan, Chongqing-Qingdao links all cross which seems non-optimal. A more circular route might be preferred. Eg. Qingdao-Beijing-Jinan-Chongqing-Wuhan-Nanjing-Shanghai-Qingdao.

City	Distance from previous
Qingdao	-
Beijing	550
Jinan	360
Chongqing	1250
Wuhan	750
Nanjing	460
Shanghai	270
Qingdao	560
Total	4200

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2. A student at Haide College enjoys travelling to other cities by train. They would like to travel as many kilometres as possible, but not spend more than ¥2400.

The cost of a return train ticket from Qingdao to the other six cities is:

From Qingdao to ...	Shanghai	Beijing	Wuhan	Jinan	Chongqing	Nanjing
Return ticket (¥)	600	700	1000	300	1700	600

The distance from Qingdao to each city is given in the table for part (a).

Which cities should the student visit?

a) This is an example of a type of problem covered in this course. State the name of this type of problem and describe a greedy heuristic to find a solution.

Solution: This is a knapsack problem.
The greedy heuristic starts by calculating the "value-to-weight" ratio (here kilometres per ¥). Items are included in the order largest to smallest ratio, provided they do not exceed the "weight" constraint (here the budget constraint).

b) Apply the greedy heuristic to find which cities the student should visit. State the total number of kilometres travelled and the total cost of the tickets.

Solution:

From Qingdao to ...	Shanghai	Beijing	Wuhan	Jinan	Chongqing	Nanjing
Return ticket (¥)	600	700	1000	300	1700	600
Distance (km)	560	550	840	310	1480	480
km/¥	0.93	0.79	0.84	1.03	0.87	0.8

NB. should really double distance for return trip, but result is the same.
Include these in order (provided they do not exceed budget):

City	Ratio	Included	Cost cumulative
Jinan	1.03	Y	300
Shanghai	0.93	Y	900
Chongqing	0.87	N	900
Wuhan	0.84	Y	1900
Nanjing	0.8	N	1900
Beijing	0.79	N	1900

The cities visited are Jinan, Shanghai and Wuhan. The total cost is ¥1900, and the distance travelled is $310 + 560 + 840 = 1710$ km.

c) The greedy solution to this problem is not the optimal solution. With reference to your answer to 2(a), are there any aspects of the greedy solution that are particularly non-optimal? In view of this, can you suggest a better solution?

Solution: The greedy solution leaves ¥500 unspent, which could have been used to travel.
A better solution that gets closer to budget is Chongqing & Shanghai (¥2300 for 2040 km).

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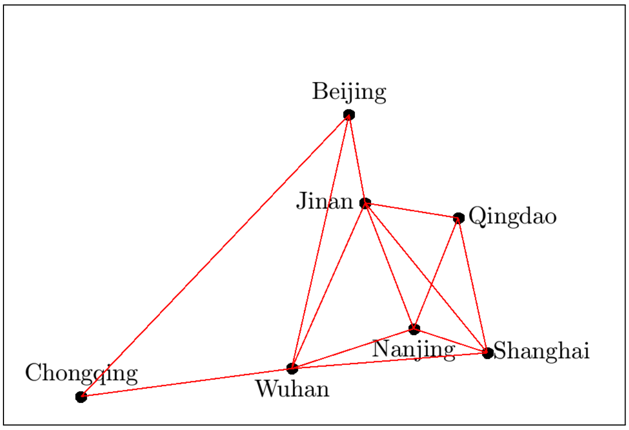
3. Another student from Haide College is a fan of high-speed rail. Assume that our 7 cities are connected in a high-speed rail network as shown below (this is a simplified version of the real network). Also assume that the distacnce by rail is the same as that given in the table in Question 1.



Starting in Qingdao, what is the shortest path to visit each city using this high-speed rail network?

a) This is an example of a type of problem covered in this course. State the name of this type of problem and which (greedy) algorithm can be used to find a solution.

Solution: This is a shortest paths problem. The shortests paths to each city can be found with Djikstra's algorithm.

b) Apply the greedy heuristic to find a routes around the seven cities **starting in Qingdao**. State your solution both in the usual notation of the algorithm, as well as writing out the shortest routes from Qingdao to each of the other cities.
Hint: in applying this algorithm, it might be easier to denote each city with a number (Qingdao = 1, Shanghai = 2 and so on).

Solution: Apply Djikstra's algorithm as follows.
Denote each city with a number in the order they appear in the table:
Here is the table of distances again, just show the cities linked in this network.

	1 Qingdao	**2 Shanghai**	**3 Beijing**	**4 Wuhan**	**5 Jinan**	**6 Chongqing**	**7 Nanjing**
1 Qingdao	0	560	-	-	310	-	480
2 Shanghai	560	0	-	690	740	-	270
3 Beijing	-	-	0	1050	360	1460	-
4 Wuhan	-	690	1050	0	720	750	460
5 Jinan	310	740	360	720	0	-	540
6 Chongqing	-	-	1460	750	-	0	-
7 Nanjing	480	270	-	460	540	-	0

- Initialise. $S = \{1\}$, $D = (0, \mathbf{560}, -, -, \mathbf{310}, -, \mathbf{480})$
- Iteration 1
  - Step 1. $S = \{1, 5\}$, $D = (0, 560, -, -, 310, -, 480)$
  - Step 2. $S = \{1, 5\}$,

$$D = (0, \min(560, 310 + 740), 310 + 360, 310 + 720, 310, -, \min(480, 310 + 540))$$
$$= (0, 560, \mathbf{670}, \mathbf{1030}, 310, -, 480)$$

- Iteration 2
  - Step 1. $S = \{1, 5, 7\}$, $D = (0, 560, 670, 1030, 310, -, 480)$
  - Step 2. $S = \{1, 5, 7\}$,

$$D = (0, \min(560, 480 + 270), 670, \min(1030, 480 + 460), 310, -, 480)$$
$$= (0, 560, 670, \mathbf{940}, 310, -, 480)$$

- Iteration 3
  - Step 1. $S = \{1, 5, 7, 2\}$, $D = (0, 560, 670, 940, 310, -, 480)$
  - Step 2. $S = \{1, 5, 7, 2\}$,

$$D = (0, 560, 670, \min(940, 560 + 690), 310, -, 480)$$
$$= (0, 560, 670, 940, 310, -, 480)$$

- Iteration 4
  - Step 1. $S = \{1, 5, 7, 2, 3\}$, $D = (0, 560, 670, 940, 310, -, 480)$
  - Step 2. $S = \{1, 5, 7, 2, 3\}$,

$$D = (0, 560, 670, \min(940, 670 + 1050), 310, 670 + 1460, 480)$$
$$D = (0, 560, 670, 940, 310, \mathbf{2130}, 480)$$

- Iteration 5
  - Step 1. $S = \{1, 5, 7, 2, 3, 4\}$, $D = (0, 560, 670, 940, 310, 2130, 480)$
  - Step 2. $S = \{1, 5, 7, 2, 3, 4\}$,

$$D = (0, 560, 670, 940, 310, \min(2130, 940 + 750), 480)$$
$$D = (0, 560, 670, 940, 310, \mathbf{1690}, 480)$$

- Iteration 6
  - Step 1. $S = \{1, 5, 7, 2, 3, 4, 6\}$, $D = (0, 560, 670, 940, 310, 1690, 480)$
  - Stop.

Looking back at our working we can spot that the predecessor nodes are

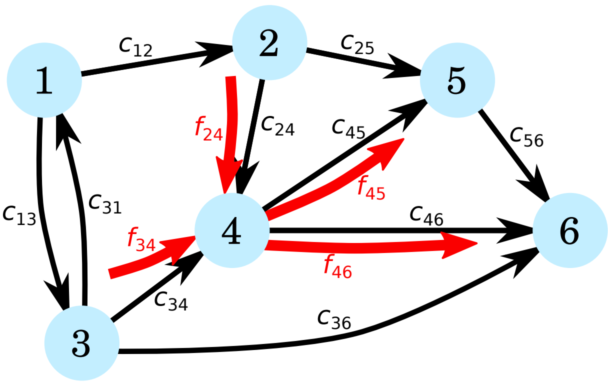
Node	**Predecessor**	**Iteration**
1 Qingdao	-	-
2 Shanghai	1 Qingdao	Initialisation
3 Beijing	5 Jinan	1
4 Wuhan	7 Nanjing	2
5 Jinan	1 Qingdao	Initialisation
6 Chongqing	4 Wuhan	5
7 Nanjing	1 Qingdao	Initialisation

Construct routes and summarise solution to give:

Destination	**Route**	**Distance (km)**
1 Qingdao	-	0
2 Shanghai	1 Qingdao - 2 Shanghai	560
3 Beijing	1 Qingdao - 5 Jinan - 3 Beijing	670
4 Wuhan	1 Qingdao - 7 Nanjing - 4 Wuhan	940
5 Jinan	1 Qingdao - 5 Jinan	310
6 Chongqing	1 Qingdao - 7 Nanjing - 4 Wuhan - 6 Chongqing	1690
7 Nanjing	1 Qingdao - 7 Nanjing	480

Bonus questions

1. **Translation:** A *flow network* is a directed graph $G = (V, E)$ where each edge (u, v) has a capacity $c_{u,v}$, and each edge could receive a flow $f_{u,v}$ of some commodity (*e.g.*, electricity or traffic). See the figure below for an example.



A flow network is constrained in various ways:

- Flows are non-negative.
- The flow on an edge may not exceed the capacity. For example, in the figure, we must choose $f_{24} \leq c_{24}$. In particular, if no edge exists (*e.g.*, between nodes 1 and 4), this flow must be zero.
- Flows are conserved, *i.e.*, everything that flows into a node must also flow out, except at a source or sink, For instance, in the figure

$$f_{34} + f_{24} = f_{45} + f_{46}.$$

- The total flow in at the source must be balanced by the flow out at the sink.

The *maximum flow* problem is the problem: given source node s and destination sink t, what is the maximum flow that can pass across the network between these two (any flow that enters at the source must exit at the sink).

a) Formulate this problem as a LP in general terms, *i.e.*, write the objective and constraints as a set of general equations or inequalities (you don't need to translate into standard form yet).

Solution: The max-flow primal LP has variables $f_{u,v} \geq 0$; its objective is to maximize

$$|f| = \sum_{v:(s,v)\in E} f_{s,v} = \sum_{u:(u,t)\in E} f_{u,t}$$

(the flow from i to j) and it has constraints

$$\begin{aligned} f_{u,v} &\leq c_{u,v}, \ \forall (u,v) \in E, \quad \text{capacity constraint} \\ \sum_{u:(u,v)\in E} f_{u,v} &= \sum_{u:(v,u)\in E} f_{v,u}, \ \forall v \in V \backslash \{s,t\}, \quad \text{flow conservation} \end{aligned}$$

and we have the special conditions for the source and sink that flow is conserved except for $|f|$ the total flow in/out/across the network.

b) Take the particular problem above (assuming all possible flows, not just those illustrated), and write it in a form suitable for input to MATLAB, *i.e.*, work out the matrices A and A_{eq} and vectors $\mathbf{b}$, $\mathbf{b}_{eq}$ and $\mathbf{c}$, assuming that node 1 is a source (*i.e.*, it can supply an arbitrary amount of flow) and 6 is a sink (it can absorb an arbitrary amount). You should have 1 variable for each edge, and 1 constraint for each node (conservation) and for each edge (capacity).

Solution: In the particular case list the variables in lexical order

$$\mathbf{x} = (f_{12}, f_{13}, f_{24}, f_{25}, f_{31}, f_{34}, f_{36}, f_{45}, f_{46}, f_{56})^T.$$

Then

$$\mathbf{c} = (1, 1, 0, 0, 0, 0, 0, 0, 0, 0)^T.$$

and write the equality and inequality constraints separately we get inequalities expressed by

$$\mathbf{b} = (c_{12}, c_{13}, c_{24}, c_{25}, c_{31}, c_{34}, c_{36}, c_{45}, c_{46}, c_{56})^T,$$

and $A = I$ for the capacity constraints, and equalities expressed through

$$\mathbf{b}_{eq} = (0, 0, 0, 0, 0)^T,$$

and

$$A_{eq} = \begin{array}{cc} \begin{array}{cccccccccc} 12 & 13 & 24 & 25 & 31 & 34 & 36 & 45 & 46 & 56 \end{array} & \\ \begin{pmatrix} 1 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & -1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & -1 \\ 1 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & -1 \end{pmatrix} & \begin{array}{l} \text{node 2} \\ \text{node 3} \\ \text{node 4} \\ \text{node 5} \\ \text{nodes 1,6} \end{array} \end{array}$$

Where the first 4 rows are conservation constraints at intermediate nodes (2-5), and the last constraint corresponds to the special source/sink constraint (flow into node 1 = flow out of node 6). That is, what flows in must flow out.

(NB: don't solve – there are better approaches than just solving directly as a LP, *e.g.*, the Ford-Fulkerson algorithm).

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2. **Calculations:** Consider the primal LP

$$\text{(P)} \quad \begin{array}{llrl} \max & z & = 3x_1 - x_2 + 7x_3 & \\ \text{subject to} & & -x_1 + x_2 + 2x_3 & \leq 8 \\ & & x_1, x_2, x_3 & \geq 0 \end{array}$$

a) Write down the dual.

Solution: The dual LP is

$$
\begin{array}{llrl}
\text{(D)} & \min & w & = 8y_1 \\
& \text{subject to} & y_1 & \geq 0 \\
& & -y_1 & \geq 3 \qquad \text{i.e., } y_1 \leq -3 \\
& & y_1 & \geq -1 \\
& & 2y_1 & \geq 7
\end{array}
$$

b) Show by inspection that the dual is infeasible.

Solution:
The constraints of (D) are contradictory, so (D) is infeasible.

c) What can you conclude about the solution to (P)?

Solution: This means (P) has no optimal solution.

$$\text{Thus (P) is infeasible, or unbounded.}$$

We can see by inspection (P) has feasible solutions, e.g. $(0, 0, 0)$, and so (P) must be unbounded.
Obviously, a solution such as

$$(t, 0, 0) \text{ is feasible for (P) for all } t > 0, \text{ and } z \to \infty \text{ as } t \to \infty.$$

d) Now describe the relationship between primal and dual again if we restricted the variables to be integers.

Solution: If we restrict the problem to the integers (D) is still infeasible, and (P) is still unbounded, so the primal-dual relationship still holds.