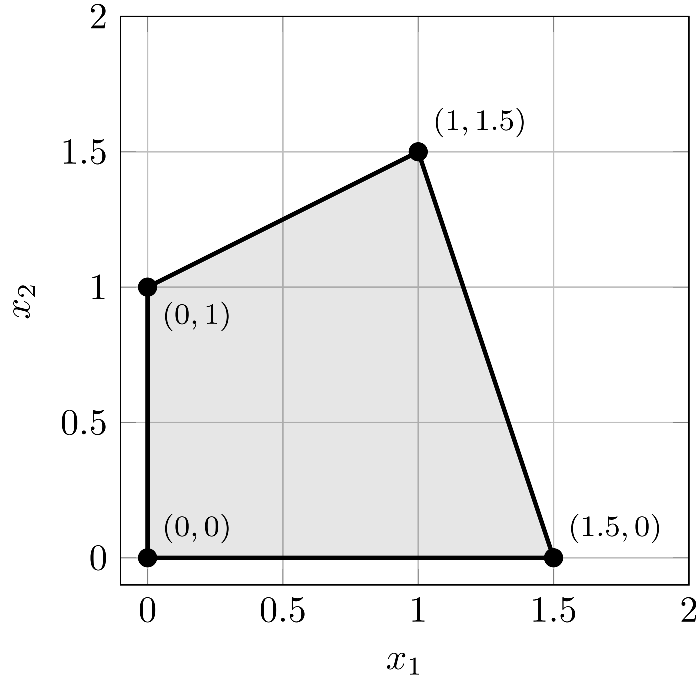
Question 1 [32 marks total]

(a) $\boxed{\text{6 marks}}$ Write the inequalities that describe the shaded feasible region specified by the thick lines. The (x_1, x_2) coordinates of each vertex are indicated.



Note that each line specifies one constraint. Ensure you include all constraints.

Solution: x_1 and x_2 non-negative and

$$\begin{array}{rcrcl} -x_1 & + & 2x_2 & \leq & 2 \\ 6x_1 & + & 2x_2 & \leq & 9 \end{array}$$

(b) $\boxed{\text{4 marks}}$ Fill in the missing parts of the Simplex Tableau corresponding to the following equations and optimisation objective:

$$\begin{array}{lrcrcr} \max\ z = & -2x_1 & - & x_2 & - & 2 \\ \text{subject to} & & & & & \\ - & x_1 & - & x_2 & \leq & 10 \\ & 2x_1 & - & 2x_2 & \geq & 1 \end{array}$$

for non-negative variables x_1 and x_2.

	z	b
$-1 \quad -1$	0	
	0	
	1	

Include the completed tableau on your answer sheet.

Solution:

x_1	x_2	x_3	x_4	z	b
-1	-1	1	0	0	10
-2	2	0	1	0	-1
2	1	0	0	1	-2

(c) $\boxed{\text{2 marks}}$ Explain the next step of the Simplex (phase II) algorithm for the following Tableau:

x_1	x_2	x_3	x_4	z	b
3	3	1	0	0	3
0	-1	0	1	0	2
-2	1	0	0	1	1

Solution: Pivot at the $(1,1)$ entry because the 1st column is the only column with a negative value in the final row, and the only a_{i1} entry that is positive is in the 1st row.

(d) $\boxed{\text{2 marks}}$ Explain the next step of the Simplex (phase II) algorithm.

x_1	x_2	x_3	x_4	x_5	x_6	x_7	z	b
0	$5/2$	1	$-1/2$	1	$-1/2$	0	0	$3/2$
1	$-1/2$	0	1	$-1/2$	$1/2$	0	0	$1/2$
0	$7/2$	0	0	$-1/2$	$1/2$	1	0	$7/2$
0	$11/2$	0	1	$3/2$	$-1/2$	0	1	$13/2$

Solution: Pivot at the $(2,6)$ entry because the 6th column is the only column with a negative value in the final row, and it has the a_{i6} entry that is positive and minimises b_i/a_{i6} in the 2nd row.

nclude explanation

(e) $\boxed{\text{2 marks}}$ Explain the next step of the Simplex (phase II) algorithm.

x_1	x_2	x_3	x_4	x_5	x_6	x_7	z	b
0	4	4	1	0	0	0	0	3
0	4	-1	0	1	0	0	0	5
1	$2/3$	1	0	0	$1/3$	0	0	$4/3$
0	$-4/3$	-1	0	0	$-2/3$	1	0	$7/3$
0	$7/3$	1	0	0	$2/3$	0	1	$2/3$

Solution: There are no negative values in the final row, so we cannot select a new column, and hence have reached the end, and in this case have achieved the **optimal solution**.

(f) $\boxed{\text{2 marks}}$ Explain the next step of the Simplex (phase II) algorithm.

x_1	x_2	x_3	x_4	z	b
2	0	1	0	0	1
3	−1	0	1	0	1
0	−1	0	0	1	−1

Solution: There is a negative entry in the last row so we choose column 2, but none of the corresponding a_{ij} values are positive, so we cannot select a row. Hence Simplex has terminated, and the solution is **unbounded**.

(g) 4 marks Is the following Tableau in feasible canonical form? **Explain your answer.**

x_1	x_2	x_3	x_4	x_5	x_6	x_7	z	b
1	2	1	1/2	1/2	0	0	0	2
2	−1	0	2	−1	1	0	0	1
−1	4	0	−1	0	0	1	0	3
1	5	0	2	1	0	0	1	−7

Solution: Yes, because it has a full set of basic columns and the **b** values are non-negative.

(h) $\boxed{\text{2 marks}}$ Complete the empty boxes for the next step of the Simplex algorithm.

x_1	x_2	x_3	s_4	s_5	z	b
2	0	1/2	1	−1/2	0	5/2
2	1	1/2	0	1/2	0	1/2
5	0	−1	0	1	1	3

x_1	x_2	x_3	s_4	s_5	z	b
		0			0	
		1			0	
		0			1	

Include the completed tableau on your answer sheet.

Solution: Pivot in the $(2,3)$ entry.

x_1	x_2	x_3	s_4	s_5	z	b
0	−1	0	1	−1	0	2
4	2	1	0	1	0	1
9	2	0	0	2	1	4

tableau as above

(i) $\boxed{\text{2 marks}}$ Complete the empty boxes for the next step of the Simplex algorithm.

x_1	x_2	x_3	x_4	x_5	x_6	z	b
4	4	2	1	1	0	0	3
0	3	−1	1	0	1	0	2
−1	−1	0	1	0	0	1	3

			1/4	1/4	0	0	3/4
			1	0	1	0	2
			5/4	1/4	0	1	

Include the completed tableau on your answer sheet.

Solution: Pivot in the $(1,1)$ entry, using Bland's rule.

1	1	1/2	1/4	1/4	0	0	3/4
0	3	−1	1	0	1	0	2
0	0	1/2	5/4	1/4	0	1	15/4

tableau as above

(j) $\boxed{\text{2 marks}}$ Given the following Simplex Tableau, which variables are basic, and which are non-basic?

x_1	x_2	x_3	x_4	x_5	z	b
0	$-1/4$	1	$1/4$	0	0	$1/4$
4	0	0	0	1	0	6
1	$-1/2$	0	$1/2$	0	1	$3/2$

Solution: The basic variables are x_3 and x_5.
The non-basic variables are x_1, x_2 and x_4.

(k) $\boxed{\text{4 marks}}$ Given the following Simplex Tableau, what is the optimal solution to the problem?

x_1	x_2	x_3	x_4	x_5	x_6	z	b
0	1	1	$1/4$	0	0	0	$1/4$
1	1	0	$1/4$	1	0	0	$5/4$
4	2	0	0	0	1	0	2
0	2	0	$1/2$	0	0	1	$-1/2$

Solution: The solution is $\mathbf{x}^T = (0,\ 0,\ 1/4,\ 0,\ 5/4,\ 2)$
and $z = -1/2$.

Question 2 [14 marks total]

Translate the following problem into a Linear Program. Make sure you define all terms. Put the program into **standard inequality form**.

Do not solve the problem.

A manufacturer produces drinks. It can produce four types of drink:

- Orange
- Diet Orange
- Raspberry
- Diet Raspberry

The revenue for each type of drink is fixed at \$1 per liter, but the cost for each is different:

- the Diet versions cost \$0.05 per litre more than the regular version,
- Raspberry flavour costs \$0.04 per litre more than Orange.

Additionally,

- At least 20% of production must be devoted to each type of drink.
- The company can bottle at most 10,000 litres per week.

If the regular Orange drink costs \$$M$ per litre to make, and presuming that each type of drink makes at least a small profit, determine how the company should maximise profit.

(a) $\boxed{\text{2 marks}}$ What are the variables of the problem?

Solution: Variables:

$$\begin{aligned} x_{O,S} &= \text{litres of Standard Orange pop,} \\ x_{R,S} &= \text{litres of Standard Raspberry pop,} \\ x_{O,D} &= \text{litres of Diet Orange pop,} \\ x_{R,D} &= \text{litres of Diet Raspberry pop.} \end{aligned}$$

(b) $\boxed{\text{6 marks}}$ What is the objective?

Solution: Objective: maximise profit z (measured in \$ per litre) assuming Standard Orange pop costs \$$M$ per litre to manufacture:

$$\begin{aligned} z &= (1-M)x_{O,S} + (1-M-0.04)x_{R,S} + (1-M-0.05)x_{O,D} + (1-M-0.09)x_{R,D} \\ &= 1x_{O,S} + 0.96x_{R,S} + 0.95x_{O,D} + 0.91x_{R,D} - M(x_{O,S} + x_{R,S} + x_{O,D} + x_{R,D}). \end{aligned}$$

(c) $\boxed{\text{4 marks}}$ What are the constraints of this problem?

Solution: Constraints: First note that (presuming a profit is made on each type of drink) the company will always ship the maximum total of pop each week so

$$x_{O,S} + x_{R,S} + x_{O,D} + x_{R,D} = 10000.$$

We can use this to simplify z, and also to write the 1/5 constraint as

$$x_{i,j} \geq 2000, \text{ for } i = O, R \text{ and } j = S, D.$$

(d) $\boxed{\text{2 marks}}$ State the complete problem written in standard inequality form.

Solution: Complete problem written in standard inequality form.

$$\begin{array}{lrcrcrcrcrcr}
\max\ z = & 1x_{O,S} & + & 0.96x_{R,S} & + & 0.95x_{O,D} & + & 0.91x_{R,D} & - & 10000M & & \\
\text{subject to} & & & & & & & & & & & \\
& x_{O,S} & + & x_{R,S} & + & x_{O,D} & + & x_{R,D} & \leq & 10000 & & \\
& -x_{O,S} & & & & & & & \leq & -2000 & & \\
& & & -x_{R,S} & & & & & \leq & -2000 & & \\
& & & & & -x_{O,D} & & & \leq & -2000 & & \\
& & & & & & & -x_{R,D} & \leq & -2000 & &
\end{array}$$

and all variables non-negative, noting the 1st constraint could be an equality.

Question 3 [12 marks total]

(a) $\boxed{\text{4 marks}}$ Write down the dual (D) of the following primal (P) linear program.

$$
\begin{array}{rrcrcrcrcl}
(P) & \max z & = & x_1 & + & 2x_2 & + & x_3 & + & 3 \\
 & & & 2x_1 & + & x_2 & & & \leq & 6 \\
 & & - & x_1 & + & x_2 & + & 2x_3 & = & 4 \\
 & & & \multicolumn{7}{l}{x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0.}
\end{array}
$$

Solution:

$$
\begin{array}{rrcrcrcl}
(D) & \min w & = & 6y_1 & + & 4y_2 & + & 3 \\
 & & & 2y_1 & - & y_2 & \geq & 1 \\
 & & & y_1 & + & y_2 & \geq & 2 \\
 & & & & & 2y_2 & \geq & 1 \\
 & & & \multicolumn{5}{l}{y_1 \geq 0, \quad \text{and} \quad y_2 \text{ free.}}
\end{array}
$$

(b) $\boxed{\text{4 marks}}$ Write down the Complementary Slackness Relations (CSRs) for (P) and (D).

Solution:

$$
\begin{array}{rcll}
y_1(2x_1 + x_2 - 6) & = & 0 & (i) \\
y_2(-x_1 + x_2 + 2x_3 - 4) & = & 0 & (ii) \\
x_1(2y_1 - y_2 - 1) & = & 0 & (iii) \\
x_2(y_1 + y_2 - 2) & = & 0 & (iv) \\
x_3(2y_2 - 1) & = & 0 & (v)
\end{array}
$$

(c) $\boxed{\text{4 marks}}$ The solution to the primal is $x_1^* = 2/3$, $x_2^* = 14/3$ and $x_3^* = x_4^* = 0$, with $z^* = 13$. Calculate the solution to the Dual (D), verifying all of the CSRs in the process.

Solution: From (iii) and (iv), as $x_1, x_2 \neq 0$, we need $2y_1 - y_2 = 1$ and $y_1 + y_2 = 2$, so that $y_1 = y_2 = 1$ and $w^* = 6(1) + 4(1) + 3 = 13$ as required and all the other CSRs are satisfied.

That is,

$$
\begin{array}{rcllr}
y_1(2x_1 + x_2 - 6) & = & 0 & (i) & 2(2/3) + (14/3) = 6 \\
y_2(-x_1 + x_2 + 2x_3 - 4) & = & 0 & (ii) & -(2/3) + (14/3) = 4 \\
x_3(2y_2 - 1) & = & 0 & (v) & x_3 = 0
\end{array}
$$

Question 4 [18 marks total]

Consider the following knapsack problem.

A hiker can choose from the following items when packing a knapsack:

Item	1 chocolate	2 raisins	3 camera	4 jumper	5 drink
w_i (kg)	0.5	0.4	0.8	1.6	0.6
v_i (value)	2.75	2.5	1	5	3.0

The hiker cannot carry more than 2.5 kg all together.

Use the **greedy heuristic** covered in the course to choose the number of each item to pack in order to maximise the total value of the goods packed, without violating the mass constraint.

Solution: See section 4.6 of the notes.

Question 5 [24 marks total]

Solve the following optimisation problem using Branch and Bound.

Maximise

$$z = 5x_1 + 8x_2,$$

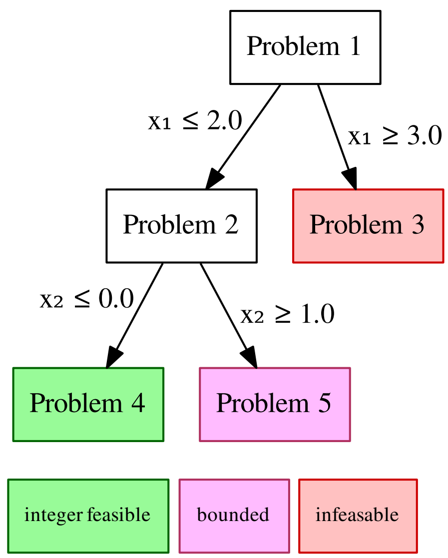
such that

$$4x_1 + 10x_2 \leq 11,$$

where $\mathbf{x}$ are non-negative and integer.

Illustrate your solution tree, and ensure you describe relaxed solutions to each along with the reasons to branch or bound at a particular point. You may find sketching the problem useful to understand the space.

Solution: The solution tree is as follows:

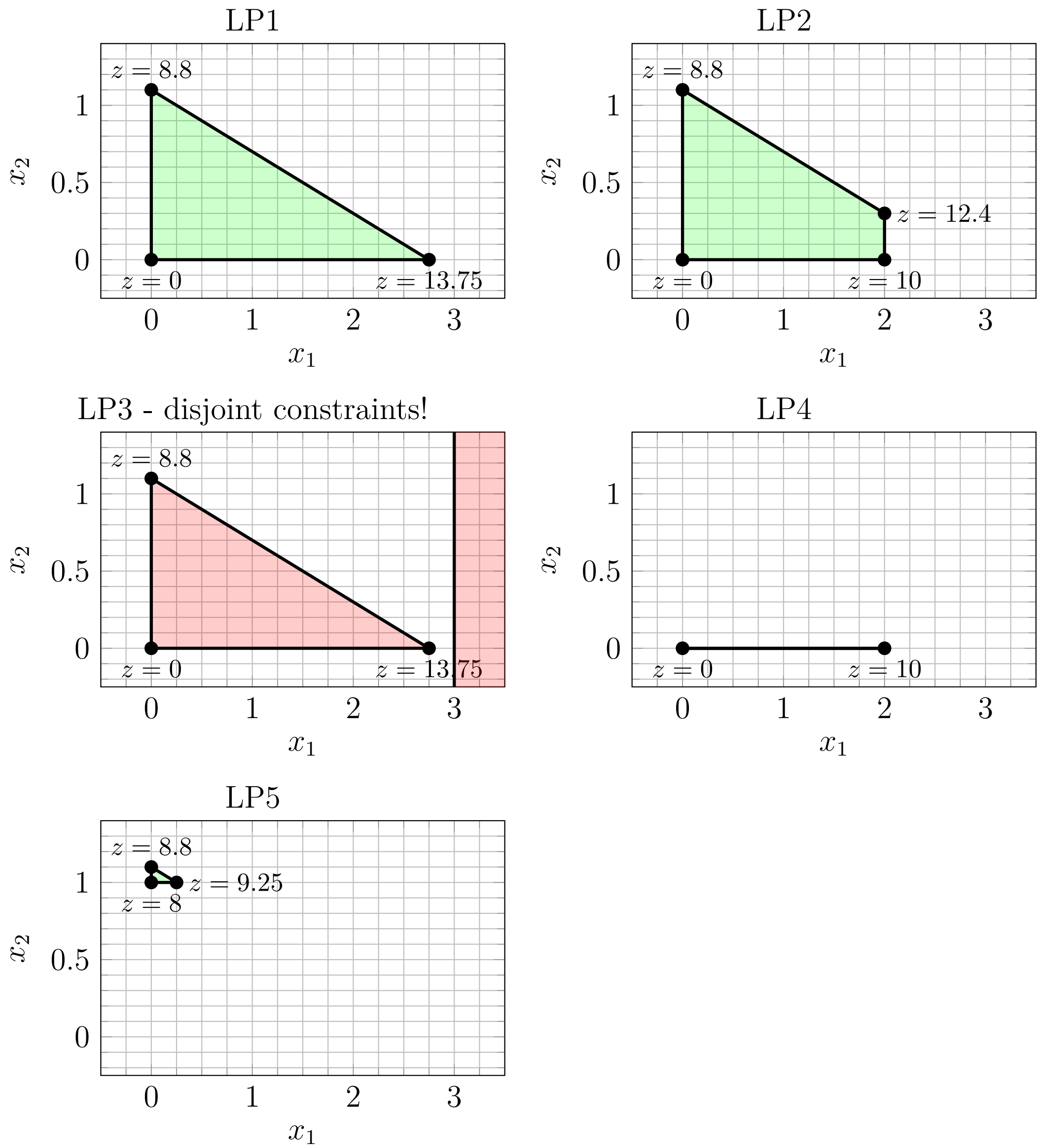


The details of the relaxed problems are

- LP1. $\mathbf{x}_R = (2.75, 0)$, $z_R = 13.75$. Branch on x_1.
- LP2. $\mathbf{x}_R = (2, 0.3)$, $z_R = 12.4$. Branch on x_2.
- LP3. Disjoint constraints. Infeasible (fathromed).

- LP4. $\mathbf{x}_R = (2, 0)$, $z_R = 10 = z_{ip}$. Integer feasible solution (fathomed).
- LP5. $\mathbf{x}_R = (0.25, 1)$, $z_R = 9.25 < z_{ip}$. Bounded by integer feasible solution LP4 (fathomed)

The five problems are illustrated graphically below.



End of examination questions.