Optimization and Operations Research Assignment 4

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Question 1. Solve the following problem using Branch and Bound.

$$\begin{array}{llccccccl} \max & z = & 3x_1 & + & 7x_2 & + & 5x_3 & & \\ & \text{s.t.} & 2x_1 & + & 5x_2 & + & 4x_3 & \leqslant 20 \\ & & 2x_1 & + & 8x_2 & + & 5x_3 & \leqslant 13 \\ & & 3x_1 & + & 3x_2 & + & 10x_3 & \leqslant 14 \end{array}$$

with x_i are positive integers, $i = 1, 2, 3$.

(a) Draw a tree diagram using the branching strategy described in the course notes.

(b) Include documented MATLAB code to solve each relaxed LP.

(c) For each fathomed nodes of the tree clearly explain why the node is fathomed. Present this information as a list of fathomed nodes, along with the reason they are fathomed.

(d) State the optimal solution to the ILP.

Solution 1. To complete this problem, we will first go over all the steps, then finish each step's details.

The fisrt relaxed LP (namely LP1) is the original problem without the integer constraints, it has the following solution:

$\mathbf{x}_R \approx (4.06, 0.61, 0)$, $z_R \approx 16.44$.

So the branched integer is 0.61. Then we have two LPs, namely LP2 and LP3.

$$\begin{array}{lllcccccl} \text{LP2} & \max & z = & 3x_1 & + & 7x_2 & + & 5x_3 & \\ & & \text{s.t.} & 2x_1 & + & 5x_2 & + & 4x_3 & \leqslant 20 \\ & & & 2x_1 & + & 8x_2 & + & 5x_3 & \leqslant 13 \\ & & & 3x_1 & + & 3x_2 & + & 10x_3 & \leqslant 14 \\ & & & & & x_2 & & & \leqslant 0 \end{array}$$

$$\begin{array}{lllcccccl} \text{LP3} & \max & z = & 3x_1 & + & 7x_2 & + & 5x_3 & \\ & & \text{s.t.} & 2x_1 & + & 5x_2 & + & 4x_3 & \leqslant 20 \\ & & & 2x_1 & + & 8x_2 & + & 5x_3 & \leqslant 13 \\ & & & 3x_1 & + & 3x_2 & + & 10x_3 & \leqslant 14 \\ & & & & & -x_2 & & & \leqslant -1 \end{array}$$

Then we solve LP2 and LP3, and get the following results:

LP2: $\mathbf{x}_R \approx (4.67, 0, 0)$, $z_R = 14$.

LP3: $\mathbf{x}_R = (2.5, 1, 0)$, $z_R = 14.5$.

Similarly, we would branch LP2 and LP3 with LP2 first.

$$
\begin{array}{llcccccc}
\text{LP4} & \max & z = & 3x_1 & + & 7x_2 & + & 5x_3 \\
& \text{s.t.} & 2x_1 & + & 5x_2 & + & 4x_3 & \leqslant 20 \\
& & 2x_1 & + & 8x_2 & + & 5x_3 & \leqslant 13 \\
& & 3x_1 & + & 3x_2 & + & 10x_3 & \leqslant 14 \\
& & & & x_2 & & & \leqslant 0 \\
& & x_1 & & & & & \leqslant 4
\end{array}
$$

$$
\begin{array}{llcccccc}
\text{LP5} & \max & z = & 3x_1 & + & 7x_2 & + & 5x_3 \\
& \text{s.t.} & 2x_1 & + & 5x_2 & + & 4x_3 & \leqslant 20 \\
& & 2x_1 & + & 8x_2 & + & 5x_3 & \leqslant 13 \\
& & 3x_1 & + & 3x_2 & + & 10x_3 & \leqslant 14 \\
& & & & x_2 & & & \leqslant 0 \\
& & -x_1 & & & & & \leqslant -5
\end{array}
$$

Then we solve LP4 and LP5, and get the following results:

LP4: $\mathbf{x}_R = (4, 0, 0.2)$, $z_R = 13$.

LP5: infeasible.

For LP3, the branched problem is LP6 and LP7 with the following solutions.

$$
\begin{array}{llcccccc}
\text{LP6} & \max & z = & 3x_1 & + & 7x_2 & + & 5x_3 \\
& \text{s.t.} & 2x_1 & + & 5x_2 & + & 4x_3 & \leqslant 20 \\
& & 2x_1 & + & 8x_2 & + & 5x_3 & \leqslant 13 \\
& & 3x_1 & + & 3x_2 & + & 10x_3 & \leqslant 14 \\
& & & & -x_2 & & & \leqslant -1 \\
& & x_1 & & & & & \leqslant 2
\end{array}
$$

$$
\begin{array}{llcccccc}
\text{LP7} & \max & z = & 3x_1 & + & 7x_2 & + & 5x_3 \\
& \text{s.t.} & 2x_1 & + & 5x_2 & + & 4x_3 & \leqslant 20 \\
& & 2x_1 & + & 8x_2 & + & 5x_3 & \leqslant 13 \\
& & 3x_1 & + & 3x_2 & + & 10x_3 & \leqslant 14 \\
& & & & -x_2 & & & \leqslant -1 \\
& & -x_1 & & & & & \leqslant -3
\end{array}
$$

Then we solve LP6 and LP7, and get the following results:

LP6: $\mathbf{x}_R = (2, 1, 0.2)$, $z_R = 14$.

LP7: infeasible.

Since LP5 and LP7 are infeasible, thus fathomed. Now consider the left two LPs, we would choose LP4 to branch first.

$$\begin{array}{llcccccl} \text{LP8} & \max & z = & 3x_1 & + & 7x_2 & + & 5x_3 \\ & \text{s.t.} & 2x_1 & + & 5x_2 & + & 4x_3 & \leqslant 20 \\ & & 2x_1 & + & 8x_2 & + & 5x_3 & \leqslant 13 \\ & & 3x_1 & + & 3x_2 & + & 10x_3 & \leqslant 14 \\ & & & & x_2 & & & \leqslant 0 \\ & & x_1 & & & & & \leqslant 4 \\ & & & & & x_3 & & \leqslant 0 \end{array}$$

$$\begin{array}{llcccccl} \text{LP9} & \max & z = & 3x_1 & + & 7x_2 & + & 5x_3 \\ & \text{s.t.} & 2x_1 & + & 5x_2 & + & 4x_3 & \leqslant 20 \\ & & 2x_1 & + & 8x_2 & + & 5x_3 & \leqslant 13 \\ & & 3x_1 & + & 3x_2 & + & 10x_3 & \leqslant 14 \\ & & & & x_2 & & & \leqslant 0 \\ & & x_1 & & & & & \leqslant 4 \\ & & & & & -x_3 & & \leqslant -1 \end{array}$$

Then we solve LP8 and LP9, and get the following results:

LP8: $\mathbf{x}_R = (4, 0, 0)$, $z_R = 12$.

LP9: $\mathbf{x}_R = (1.33, 0, 1)$, $z_R = 9$.

Now consider LP6 to branch.

$$\begin{array}{llcccccl} \text{LP10} & \max & z = & 3x_1 & + & 7x_2 & + & 5x_3 \\ & \text{s.t.} & 2x_1 & + & 5x_2 & + & 4x_3 & \leqslant 20 \\ & & 2x_1 & + & 8x_2 & + & 5x_3 & \leqslant 13 \\ & & 3x_1 & + & 3x_2 & + & 10x_3 & \leqslant 14 \\ & & & & -x_2 & & & \leqslant -1 \\ & & x_1 & & & & & \leqslant 2 \\ & & & & & x_3 & & \leqslant 0 \end{array}$$

$$\begin{array}{llcccccl} \text{LP11} & \max & z = & 3x_1 & + & 7x_2 & + & 5x_3 \\ & \text{s.t.} & 2x_1 & + & 5x_2 & + & 4x_3 & \leqslant 20 \\ & & 2x_1 & + & 8x_2 & + & 5x_3 & \leqslant 13 \\ & & 3x_1 & + & 3x_2 & + & 10x_3 & \leqslant 14 \\ & & & & -x_2 & & & \leqslant -1 \\ & & x_1 & & & & & \leqslant 2 \\ & & & & & -x_3 & & \leqslant -1 \end{array}$$

Then we solve LP10 and LP11, and get the following results:

LP10: $\mathbf{x}_R = (2, 1.125, 0)$, $z_R = 13.875$.

LP11: $\mathbf{x}_R \approx (0, 1, 1)$, $z_R = 12$.

Now consider LP10 to branch since x_2 is still not an integer.

$$
\begin{array}{llrcrcrcl}
\text{LP12} & \max & z = & 3x_1 & + & 7x_2 & + & 5x_3 & \\
 & \text{s.t.} & 2x_1 & + & 5x_2 & + & 4x_3 & \leqslant 20 \\
 & & 2x_1 & + & 8x_2 & + & 5x_3 & \leqslant 13 \\
 & & 3x_1 & + & 3x_2 & + & 10x_3 & \leqslant 14 \\
 & & & & -x_2 & & & \leqslant -1 \\
 & & x_1 & & & & & \leqslant 2 \\
 & & & & & & x_3 & \leqslant 0 \\
 & & & & x_2 & & & \leqslant 1
\end{array}
$$

$$
\begin{array}{llrcrcrcl}
\text{LP13} & \max & z = & 3x_1 & + & 7x_2 & + & 5x_3 & \\
 & \text{s.t.} & 2x_1 & + & 5x_2 & + & 4x_3 & \leqslant 20 \\
 & & 2x_1 & + & 8x_2 & + & 5x_3 & \leqslant 13 \\
 & & 3x_1 & + & 3x_2 & + & 10x_3 & \leqslant 14 \\
 & & & & -x_2 & & & \leqslant -1 \\
 & & x_1 & & & & & \leqslant 2 \\
 & & & & & & x_3 & \leqslant 0 \\
 & & & & -x_2 & & & \leqslant -2
\end{array}
$$

Then we solve LP12 and LP13, and get the following results:

LP12: $\mathbf{x}_R = (2, 1, 0)$, $z_R = 13$.

LP13: infeasible.

So here we have finished all the LPs, and we can see that the optimal solution is LP12.

(a) The tree diagram is shown in the following figure.

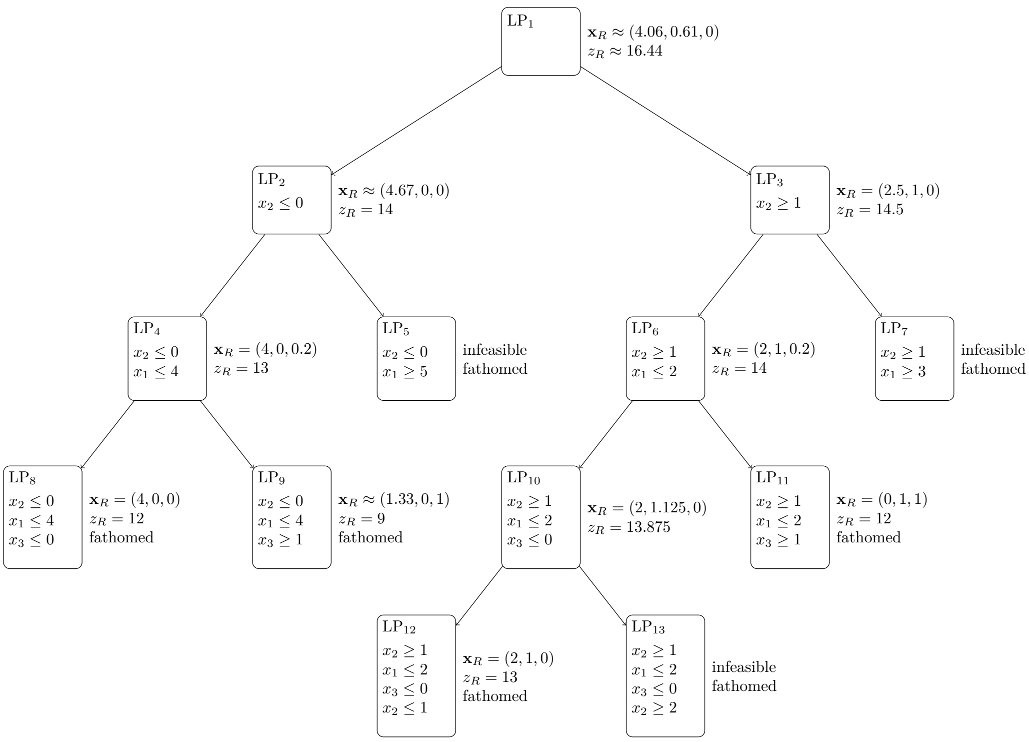

Figure 1: Question 1 Branch and Bound Tree Diagram

(b) Here are the MATLAB codes for each LP.

```
% Initial matrices and values
A = [2 5 4;
    2 8 5;
    3 3 10];
b = [20; 13; 14];
c = [3; 7; 5];

% LP1: No Additional Constraints
A1 = A;
b1 = b;
[x1, fval] = linprog(-c, A1, b1, [], [], zeros(size(c)));
z1 = -fval
x1

% LP2: Add x2 <= 0 from LP1
A2 = [A1; [0 1 0]];
b2 = [b1; 0];
[x2,fval]   = linprog(-c, A2, b2, [], [], zeros(size(c)));
z2 = - fval
x2

% LP3: Add x2 >= 1 from LP1
A3 = [A1; [0 -1 0]];
b3 = [b1; -1];
[x3,fval] = linprog(-c, A3, b3, [], [], zeros(size(c)));
z3 = - fval
x3

% LP4: Add x1 <= 4 from LP2
A4 = [A2;[1 0 0]];
b4 = [b2; 4];
[x4,fval] = linprog(-c, A4, b4, [], [], zeros(size(c)));
z4 = - fval
x4

% LP5: Add x1 >= 5 from LP2
A5 = [A2;[-1 0 0]];
b5 = [b2; -5];
[x5,fval] = linprog(-c, A5, b5, [], [], zeros(size(c)));
z5 = - fval
```

```
x5

% LP6: Add x1 <= 2 from LP3
A6 = [A3;[1 0 0]];
b6 = [b3; 2];
[x6,fval] = linprog(-c, A6, b6, [], [], zeros(size(c)));
z6 = - fval
x6

% LP7: Add x1 >= 3 from LP3
A7 = [A3;[-1 0 0]];
b7 = [b3; -3];
[x7,fval] = linprog(-c, A7, b7, [], [], zeros(size(c)));
z7 = - fval
x7

% LP8: Add x3 <= 0 to LP4
A8 = [A4;[0 0 1]];
b8 = [b4; 0];
[x8,fval] = linprog(-c, A8, b8, [], [], zeros(size(c)));
z8 = - fval
x8

% LP9: Add x3 >= 1 to LP4
A9 = [A4;[0 0 -1]];
b9 = [b4; -1];
[x9,fval] = linprog(-c, A9, b9, [], [], zeros(size(c)));
z9 = - fval
x9

% LP10: Add x3 <= 0 from LP6
A10 = [A6;[0 0 1]];
b10 = [b6; 0];
[x10,fval] = linprog(-c, A10, b10, [], [], zeros(size(c)));
z10 = - fval
x10

% LP11: Add x3 >= 1 from LP6
A11 = [A6;[0 0 -1]];
b11 = [b6; -1];
```

```
[x11,fval] = linprog(-c, A11, b11, [], [], zeros(size(c)));
z11 = - fval
x11

% LP12: Add x2 <= 1 to LP10
A12 = [A10;[0 1 0]];
b12 = [b10; 1];
[x12,fval] = linprog(-c, A12, b12, [], [], zeros(size(c)));
z12 = - fval
x12

% LP13: Add x2 >= 2 to LP10
A13 = [A10;[0 -1 0]];
b13 = [b10; -2];
[x13,fval] = linprog(-c, A13, b13, [], [], zeros(size(c)));
z13 = - fval
x13
```

(c) The list of fathomed nodes is as follows:

Node	Reason
LP5	After adding the constraint, the LP is infeasible.
LP7	After adding the constraint, the LP is infeasible.
LP8	After adding the constraint, the LP has an integer feasible solution.
LP9	After adding the constraint, the LP has a solution that is bounded by LP8.
LP11	After adding the constraint, the LP has an integer feasible solution.
LP12	After adding the constraint, the LP has an integer feasible solution.
LP13	After adding the constraint, the LP is infeasible.

(d) The optimal solution to the ILP is $\mathbf{x}_R = (2, 1, 0)$ with the optimal value $z_R = 13$.

Question 2. Consider the following linear program representing a knapsack problem for six items with total value z and maximum volume 35.

$$
\begin{array}{llcccccccccccl}
\max & z = & 19x_1 & + & 22x_2 & + & 30x_3 & + & 37x_4 & + & 11x_5 & + & 42x_6 & \\
& \text{s.t.} & 7x_1 & + & 6x_2 & + & 11x_3 & + & 13x_4 & + & 4x_5 & + & 13x_6 & \leqslant 35
\end{array}
$$

with $x_i = 0, 1$, for $i = 1, \cdots, 6$.

(a) Relax the integral constraints (to give the constraint that all the x_i are $0 \leqslant x_i \leqslant 1$) and solve the above problem using the problem based approach in MATLAB. Upload your code to MATLAB Grader for checking, you do not need to include it in your PDF submission.

(b) Find the optimal solution to the problem using the branch and bound as described in the course notes.

Since the tree for this problem is a large, so you should draw this diagram to show only the structure of the tree and the labels of the problems.

The detail of the relaxed problems and branching should then be provided in a table like the one below. Make sure you include reasons for fathoming.

It is not required that you submit any MATLAB code for this question, you just need to summarise results in your table.

(c) Finally, briefly comment on how effective branch and bound was for this problem. For example, would a brute force search have been more efficient?

Solution 2. (b) The tree diagram is shown in the following figure.

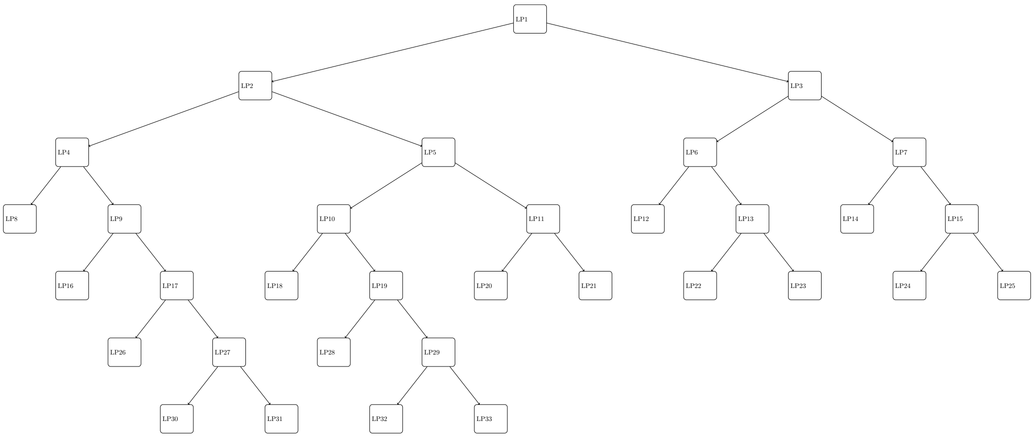

Figure 2: Question 2 Branch and Bound Tree Diagram

And here is the table summarising the information for all relaxed problems and branching status.

IP	Parent	New Constraints	z_R	x_R	Status	Reason
1	-	-	109.2500	(0,1,0,1,0.7500,1)	Branched	-
2	1	$x_5 \leqslant 0$	109.1818	(0,1,0.2727,1,0,1)	Branched	-
3	1	$x_5 \geqslant 1$	109.1538	(0,1,0,0.9231,1,1)	Branched	-
4	2	$x_3 \leqslant 0$	109.1429	(0.4286,1,0,1,0,1)	Branched	-
5	2	$x_3 \geqslant 1$	108.2308	(0,1,1,0.3846,0,1)	Branched	-
6	3	$x_4 \leqslant 0$	107.7143	(0.1429,1,1,0,1,1)	Branched	-
7	3	$x_4 \geqslant 1$	108.7692	(0,1,0,1,1,0.9231)	Branched	-
8	4	$x_1 \leqslant 0$	101	(0,1,0,1,0,1)	Fathomed	Integral Feasible
9	4	$x_1 \geqslant 1$	108.6154	(1,1,0,0.6923,0,1)	Branched	-
10	5	$x_4 \leqslant 0$	107.5714	(0.7143,1,1,0,0,1)	Branched	-
11	5	$x_4 \geqslant 1$	105.1538	(0,1,1,1,0,0.3846)	Branched	-
12	6	$x_1 \leqslant 0$	105	(0,1,1,0,1,1)	Fathomed	Integral Feasible
13	6	$x_1 \geqslant 1$	107.6364	(1,1,0.4545,0,1,1)	Branched	-
14	7	$x_6 \leqslant 0$	102.7143	(0.1429,1,1,1,0,0)	Fathomed	$z < z_{ip}$(IP12)
15	7	$x_6 \geqslant 1$	108.3333	(0,0.8333,0,1,1,1)	Branched	-
16	9	$x_4 \leqslant 0$	83	(1,1,0,0,0,1)	Fathomed	Integral Feasible
17	9	$x_4 \geqslant 1$	107.0769	(1,1,0,1,0,0.6923)	Branched	-
18	10	$x_1 \leqslant 0$	94	(0,1,1,0,0,1)	Fathomed	Integral Feasible
19	10	$x_1 \geqslant 1$	106.5385	(1,1,0,0,0,0.8462)	Branched	-

IP	Parent	New Constraints	z_R	x_R	Status	Reason
20	11	$x_6 \leqslant 0$	102.5714	(0.7143,1,1,1,0,0)	Fathomed	$z < z_{ip}$(IP12)
21	11	$x_6 \geqslant 1$	-	-	Branched	Infeasible
22	13	$x_3 \leqslant 0$	94	(1,1,0,0,1,1)	Fathomed	Integral Feasible
23	13	$x_3 \geqslant 1$	104.6154	(1,1,1,0,1,0.5385)	Fathomed	$z < z_{ip}$(IP12)
24	15	$x_2 \leqslant 0$	103.6364	(0,0,0.4545,1,1,1)	Fathomed	$z < z_{ip}$(IP12)
25	15	$x_2 \geqslant 1$	-	-	Fathomed	Infeasible
26	17	$x_6 \leqslant 0$	78	(1,1,0,1,0,0)	Fathomed	Integral Feasible
27	17	$x_6 \geqslant 1$	105.3333	(1,0.3333,0,1,0,1)	Branched	-
28	19	$x_6 \leqslant 0$	71	(1,1,1,0,0,0)	Fathomed	Integral Feasible
29	19	$x_6 \geqslant 1$	105.6667	(1,0.6667,1,0,0,1)	Branched	-
30	27	$x_2 \leqslant 0$	98	(1,0,0,1,0,1)	Fathomed	Integral Feasible
31	27	$x_2 \geqslant 1$	-	-	Fathomed	Infeasible
32	29	$x_2 \leqslant 0$	91	(1,0,1,0,0,1)	Fathomed	Integral Feasible
33	29	$x_2 \geqslant 1$	-	-	Fathomed	Infeasible

(c) By looking at the table, we can conclude that the branch and bound method is very effective for this problem since if one subproblem is infeasible, then all the subproblems that are branched from it are also infeasible so a lot of works are reduced.

However, a brute force search would be much less efficient, as in this case it need to check all possible solutions with one thing has two possibles, that is, $2^6 = 64$ solutions for checking, which would be computationally expensive for larger problems.