THE UNIVERSITY *of* ADELAIDE

Optimsation & Operations Research

Haide College, Spring Semester

Tutorial 3

These questions are (mostly) on Topic 2: Simplex Algorithm

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1. **Translation:** The following is called the *Transportation problem.*

There are three warehouses at different cities: Sydney, Melbourne and Adelaide. They have 250, 130 and 235 tonnes of paper accordingly. There are four publishers, in Sydney, Melbourne, Brisbane and Hobart. They ordered 75, 230, 240 and 70 tonnes of paper to publish new books. There are the following costs (in tens of dollars) of transportation of one tonne of paper:

From / To	Sydney	Brisbane	Melbourne	Hobart
Sydney	15	20	16	21
Melbourne	25	13	5	11
Adelaide	15	15	7	17

The idea is to find a *transportation plan* such that all orders will be met and the transportation costs will be minimized.

a) Formulate the problem as a Linear Program.

Solution: Let x_{ij} = number of tonnes of paper shipped from Warehouse i ($i = 1$: Sydney; $i = 2$: Melbourne; $i = 3$: Adelaide), to publisher in City j ($j = 1, 2, 3, 4$ for Sydney, Brisbane, Melbourne and Hobart, respectively). Then constraints are equations, such as $x_{11} + x_{12} + x_{13} + x_{14} = 250$ (amount that can be shipped from Sydney to the group of publishers). Similarly for deliveries from Melbourne and Adelaide. Also, $x_{11} + x_{21} + x_{31} = 75$ (amount that is delivered to the Sydney publisher from the group of three warehouses). Similarly for deliveries to Brisbane, Melbourne and Hobart.
We can write these constraints concisely in the form

$$\begin{aligned} \sum_{i=1}^{n} x_{ij} &= \mathbf{w}, \\ \sum_{j=1}^{m} x_{ij} &= \mathbf{p}, \end{aligned}$$

where $\mathbf{w}$ and $\mathbf{p}$ are the demand and supply coefficients, and n and m are the numbers of warehouses and publishers, respectively. The objective function is

$$\min \sum_{i,j} c_{i,j} x_{i,j}.$$

where c_{ij} is the vector of costs of shipping from i to j.
However, to write it in standard form (suitable for input to MATLAB) we need to construct A, $\mathbf{b}$ and $\mathbf{c}$ as follows.

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$\mathbf{b} = [250, 130, 235, 75, 240, 230, 70]^T$, and
$\mathbf{c} = [15, 20, 16, 21, 25, 13, 5, 11, 15, 15, 7, 17]^T$.

Here, $\mathbf{c} = (c_{ij})$ is the vectorised version of the costs.

The problem is then: min (cost) $z = \mathbf{c}^T\mathbf{x}$ s.t. $A\mathbf{x} = \mathbf{b}$, $\mathbf{x} \geq \mathbf{0}$, all x_{ij} integral.

b) How could you change this if the total paper ordered was less than the total supply?

Solution: If the total paper ordered was less than the total supply then the constraints on supply become inequalities. That is, the first three equations expressed above become inequalities.
Alternatively, introduce a fictitious publisher $j = 5$, with a supply equal to s, the shortfall between total demand and the amount available. Put $c_{i5} = t$, a constant, *e.g.*, $t = 10$. The value doesn't matter, since $\sum_{i=1}^{3} t.x_{i5} = t \sum_{i=1}^{3} x_{i5} = t.s$, a constant.
The second approach avoids adding three extra slack variables.

c) What if only whole tonnes of paper can be delivered?

Solution: If only whole tonnes can be delivered, then we add the constraints that the x_{ij} must be integers.

Hints: remember to look for three things:

- the variables: here we could use a 2D array of variables x_{ij} (remember to define what these mean);
- the objective (the thing you want to maximise or minimise); and
- the constraints (here there will be a constraint for each warehouse and each publisher).

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2. Consider the following primal Linear Program

$$\begin{array}{llrcrcrcrcr} (P) & \max z & = & & -x_1 & + & x_2 & + & 2x_3 & - & 12 \\ & & \text{s.t.} & & -2x_1 & + & 2x_2 & + & x_3 & \leq & 6 \\ & & & & 3x_1 & + & x_2 & - & x_3 & \leq & 9 \\ & & & & \multicolumn{7}{l}{x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0.} \end{array}$$

a) Write down the dual (D) of (P).

Solution:

$$\begin{array}{llrcrcr}
(D) & \min w = & 6y_1 & + & 9y_2 & - & 12 \\
 & & -2y_1 & + & 3y_2 & \geq & -1 \\
 & & 2y_1 & + & y_2 & \geq & 1 \\
 & & y_1 & - & y_2 & \geq & 2 \\
 & y_1, y_2 \geq 0. & & & & &
\end{array}$$

b) Write down the Complementary Slackness Relations (CSRs) for (P) and (D).

Solution: The complementary slackness relationships are

$$\begin{aligned}
y_1(-2x_1 + 2x_2 + x_3 - 6) &= 0 \quad (i) \\
y_2(3x_1 + x_2 - x_3 - 9) &= 0 \quad (ii) \\
x_1(-2y_1 + 3y_2 + 1) &= 0 \quad (iii) \\
x_2(2y_1 + y_2 - 1) &= 0 \quad (iv) \\
x_3(y_1 - y_2 - 2) &= 0 \quad (v)
\end{aligned}$$

c) Use $\mathbf{x}^T = (15, 0, 36)$, and your CSRs to find the optimal solution(s) to the dual (D), verifying all of the CSRs in the process. What does that tell you about $\mathbf{x}$?

Solution:

- The solution $\mathbf{x}^T = (15, 0, 36)$ satisfies (i) and (ii).
- The variables $x_2 = 0$, and hence CSR (iv) is automatically satisfied.
- The basic variables x_1 and x_3 are non-zero, so CSRs (iii) and (v) becomes

$$\begin{aligned}
-2y_1 + 3y_2 + 1 &= 0 \quad (iii) \\
y_1 - y_2 - 2 &= 0 \quad (v)
\end{aligned}$$

  Solving simultaneously gives $\mathbf{y}^* = (5, 3)$ and $w = 45$ (as expected).

  The solution must satisfy the CSRs because apart from the ones already satisfied, this is built using the CSRs. Hence $\mathbf{x}$ is the optimal solution to the primal.

Bonus questions

1. **Calculations:** Graph the constraints of the following problem and its corresponding dual.

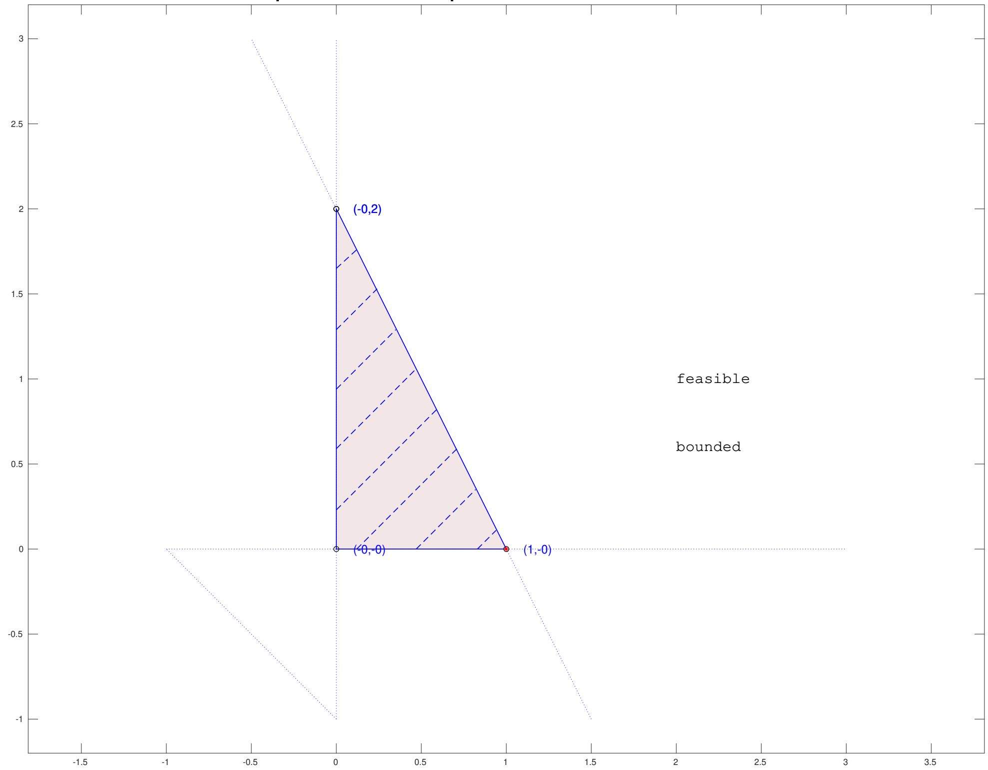
$$\begin{array}{llrrl}
(P) & \max \quad z = & x_1 & -x_2 & \\
 & \text{s.t.} & 2x_1 & +x_2 & \leq 2 \\
 & & -x_1 & -x_2 & \leq 1 \\
 & & x_1, & x_2 \geq 0 &
\end{array}$$

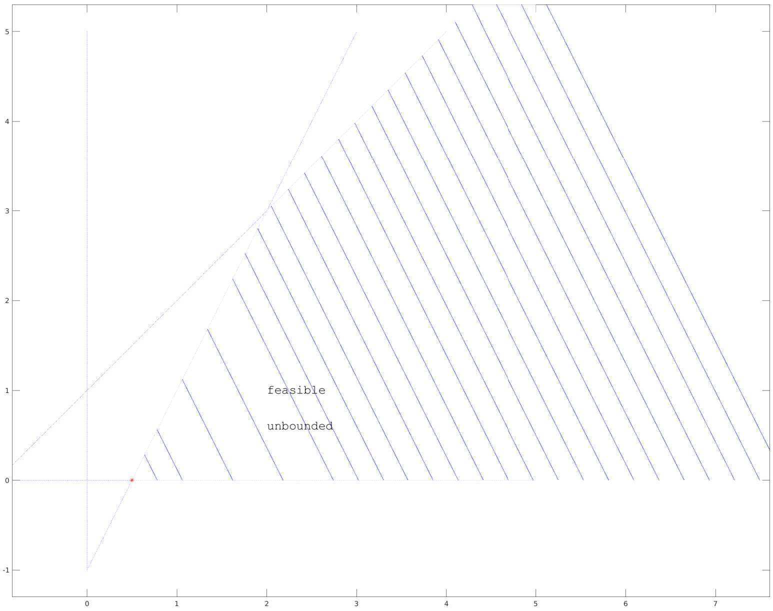
Intepret the sketches and their relationship.

Solution: The dual is

$$
\begin{array}{llrrcr}
\text{(D)} & \min \quad w = & 2y_1 & +y_2 & & \\
 & \text{s.t.} & 2y_1 & -y_2 & \geq & 1 \\
 & & y_1 & -y_2 & \geq & -1 \\
 & & y_1, & y_2 \geq 0 & &
\end{array}
$$

Sketches of the two problems are provided below.





The primal is a pretty standard feasible, bounded region, with a maximum at $(1/2, 0)$. The dual is an unbounded region, but it is bounded in the direction of the objective, and has a minimum at $(0, -1)$.
Note that in both cases one of the constraints plays no role in defining the region of interest.

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2. **Proof of the week:** There are more general duality conditions than presented in the course notes. One example is given below.

   Assuming that we had the primal problem

$$\begin{array}{rrcl} \max & z & = & \mathbf{c}^T\mathbf{x} + z_0 \\ \text{such that} & A\mathbf{x} & \leq & \mathbf{b} \\ \text{and} & \mathbf{x} & \geq & 0 \end{array}$$

   its dual is

$$\begin{array}{rrcl} \min & w & = & \mathbf{b}^T\mathbf{y} + z_0 \\ \text{such that} & A^T\mathbf{y} & \geq & \mathbf{c} \\ \text{and} & \mathbf{y} & \geq & 0 \end{array}$$

   Prove that $w \geq z$ for any feasible points $\mathbf{x}$ and $\mathbf{y}$ of the two respective problems.

Solution:

$$w = \sum_{i=1}^{m} y_i b_i + z_0.$$

From the Primal $b_i \geq \sum_{j=1}^{n} a_{ij} x_j$ so

$$\begin{aligned} w &\geq \sum_{i=1}^{m} y_i \left(\sum_{j=1}^{n} a_{ij} x_j \right) + z_0 \\ &= \sum_{j=1}^{n} \left(\sum_{i=1}^{m} y_i a_{ij} \right) x_j + z_0. \end{aligned}$$

From the Dual $\sum_{i=1}^{m} y_i a_{ij} \geq c_j$

$$w \geq \sum_{j=1}^{n} c_j x_j + z_0$$

Hence $w \geq z$ for any feasible solutions $\mathbf{x}$ and $\mathbf{y}$.