Kepler's Equation

$$\boxed{M = E - e\sin(E)}$$

where M and E are in rad

Problem: Given a mean anomaly M and eccentricity e, find eccentric anomaly E

Kepler's Equation

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Problem: Given a mean anomaly M and eccentricity e, find eccentric anomaly E

To solve this, we use a numerical method (finding the solution by using numerical procedure/algorithm) called the **Newton-Raphson method**.

Newton-Raphson method

A numerical method to solve the following mathematical problem:

Given a function $f(x)$. *Find such* x^* *that makes the function equals to zero;* $f(x^*) = 0$

Basically, the method is just guessing a number and checking whether it solves the equation

Make a guess. Check the solution.

Not the answer?

Make another guess. Check the solution.

Not the answer?

Make another guess. Check the solution.

…and so on and so on

Ex: Given $f(x)=x^2-1$. What x is $f(x)=0$?

Guess $x=2 \rightarrow f(2) = 2^2-1 = 3$

Not zero.

Guess $x=0.5 \rightarrow f(0.5) = 0.5^2-1 = -0.75$.

Not zero.

Guess $x=1 \rightarrow f(1) = 1^2-1 = 0$.

Zero! $x=1$ is the solution!

But how do we do this 'intelligently' so that we don't have to do this guessing and checking forever?

Newton-Raphson method

A numerical method to solve the following mathematical problem:

Given a function $f(x)$. Find such x^* that makes the function equals to zero; $f(x^*) = 0$

How Newton-Raphson solve this? Use this algorithm

1. Make a guess for the value x. Let's give this value an index i: x_i

2. evaluate $f(x_i)$

3. evaluate $f'(x_i)$ (the first derivative of $f(x_i)$)

4. Solve for the new guess $x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$

5. Repeat Step 1 until $\frac{f(x_i)}{f'(x_i)} < \epsilon$, where ϵ is called a 'tolerance' which is typically a very small number (e.g. $\epsilon = 10^{-8}$)

OK now let's apply the **Newton-Raphson algorithm** ($find\ such\ x\ where\ f(x)\ is\ zero$) to our **Kepler's Equation** (Given a M and e, find E)

Newton-Raphson method applied to Kepler's Equation

First, let's do some algebraic manipulation, bringing

$$M = E - e\sin E$$
$$E - e\sin E - M = 0$$

Now let's set the function, $f(E)$, as

$$f(E) = E - e\sin E - M$$

So that we can modify the problem that NR is trying to solve…

A numerical method to solve the following mathematical problem:

Given a function $\cancel{f(x)}$ $f(E)$. *Find such* $\cancel{x^*}$ E *that makes the function equals to zero;* $\cancel{f(x^*)}$ $f(E)$ $= 0$

See?

Newton-Raphson method applied to Kepler's Equation

Use this algorithm

1. *Make a guess* for the value* E. *Let's give this value an index i:* E_i
2. *evaluate* $f(E_i) = E_i - e\sin E_i - M$
3. *evaluate* $f'(E_i) = 1 - e\cos E_i$
4. *Solve for the updated guess* $E_{i+1} = E_i - \frac{f(E_i)}{f'(E_i)}$

   in other words $E_{new} = E_{old} - \frac{f(E_{old})}{f'(E_{old})}$
5. *Repeat Step 1 until* $\frac{f(E_i)}{f'(E_i)} < \epsilon$. *pick a small tolerance, like* $\epsilon = 10^{-8}$

We start the process with a good **first guess. If* $M < \pi$, then $E = M + e/2$. If $M > \pi$, then $E = M - e/2$.

Newton-Raphson method applied to **Kepler's Equation**

example from Curtis 2020 (pp 150)

Redo this example

EXAMPLE 3.2

In the previous example, find the true anomaly at 3 h after perigee passage.

Solution

Since the time (10,800 s) is greater than one-half the period, the true anomaly must be greater than 180°.

First, we use Eq. (3.8) to calculate the mean anomaly for $t = 10{,}800\,\mathrm{s}$.

$$M_e = 2\pi\frac{t}{T} = 2\pi\frac{10{,}800}{18{,}830} = 3.6029\,\mathrm{rad} \tag{a}$$

Kepler's equation, $E - e\sin(E) = M_e$ (with all angles in radians) is then employed to find the eccentric anomaly. This transcendental equation will be solved using Algorithm 3.1 with an error tolerance of 10^{-6}. Since $M_e > \pi$, a good starting value for the iteration is $E_0 = M_e - e/2 = 3.4166$. Executing the algorithm yields the following steps:

Step 1:

$$\begin{aligned}
E_0 &= 3.4166 \\
f(E_0) &= -0.085124 \\
f'(E_0) &= 1.3585 \\
\text{ratio} &= \frac{-0.085124}{1.3585} = -0.062658 \\
|\text{ratio}| &> 10^{-6} \qquad \therefore \text{repeat}
\end{aligned}$$

Step 2:

$$\begin{aligned}
E_1 &= 3.4166 - (-0.062658) = 3.4793 \\
f(E_1) &= -0.0002134 \\
f'(E_1) &= 1.3515 \\
\text{ratio} &= \frac{-0.0002134}{1.3515} = -1.5778\times10^{-4} \\
|\text{ratio}| &> 10^{-6} \qquad \therefore \text{repeat}
\end{aligned}$$

Step 3:

$$\begin{aligned}
E_2 &= 3.4793 - \left(-1.5778\times10^{-4}\right) = 3.4794 \\
f(E_2) &= -1.5366\times10^{-9} \\
f'(E_2) &= 1.3515 \\
\text{ratio} &= \frac{-1.5366\times10^{-9}}{1.3515} = -1.137\times10^{-9} \\
|\text{ratio}| &< 10^{-6} \qquad \therefore \text{stop}
\end{aligned}$$

Convergence to even more than the desired accuracy occurred after just two iterations. With $E = 3.4794$, the true anomaly is found from Eq. (3.13a) to be

$$\tan\frac{\theta}{2} = \sqrt{\frac{1+e}{1-e}}\tan\frac{E}{2} = \sqrt{\frac{1+0.37255}{1-0.37255}}\tan\frac{3.4794}{2} = -8.6721 \Rightarrow \boxed{\theta = 193.2^\circ}$$

Practice/Validate your algorithm

3.9 An earth-orbiting satellite has a perigee radius of 7000 km and an apogee radius of 10,000 km.
(a) What true anomaly $\Delta\theta$ is swept out between $t = 0.5$ h and $t = 1.5$ h after perigee passage?
(b) [redacted]

{Ans.: (a) 128.7°; (b) [redacted]}

3.11 A satellite in earth orbit has perigee and apogee radii of $r_p = 7500$ km and $r_a = 16{,}000$ km, respectively. Find its true anomaly 40 min after passing the true anomaly of 80°.

{Ans.: 174.7°}

Barker's Equation (Parabolic)

Section 3.5

3.14 Calculate the time required for a spacecraft launched into a parabolic trajectory at a perigee altitude of 200 km to leave the earth's sphere of influence (see Table A.2).

{Ans.: 7.77 days}

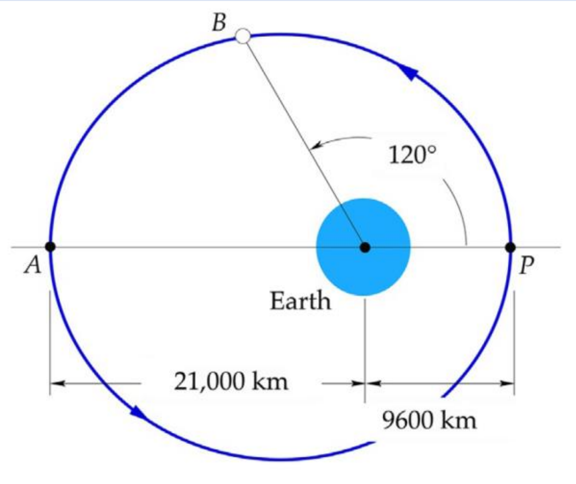
3.15 A spacecraft on a parabolic trajectory around the earth has a perigee radius of 6600 km.
(a) How long does it take to coast from $\theta = -90°$ to $\theta = +90°$?
(b) How far is the spacecraft from the center of the earth 36 h after passing through perigee?

{Ans.: (a) 0.8897 h; (b) 304,700 km}

In Space Mechanics, most analytical calculations are **long**, **difficult** and **tedious**.

Pen-and-paper calculation is not enough.

Be smart and use **computational tools**
e.g. spreadsheet, Octave or Matlab, Python



Elliptic orbit

$$M = E - e \sin E$$

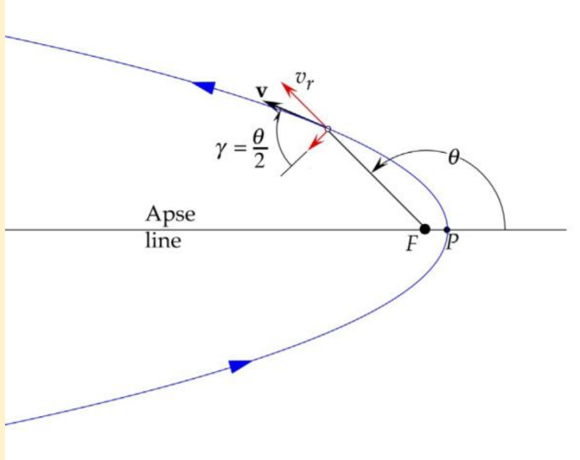
Given true anomaly θ, find time-of-flight t

$$\theta \rightarrow E \rightarrow M \rightarrow t$$

Given time-of-flight t, find true anomaly θ

$$t \rightarrow M \rightarrow E \rightarrow \theta$$

Use Newton-Raphson method



Parabolic orbit

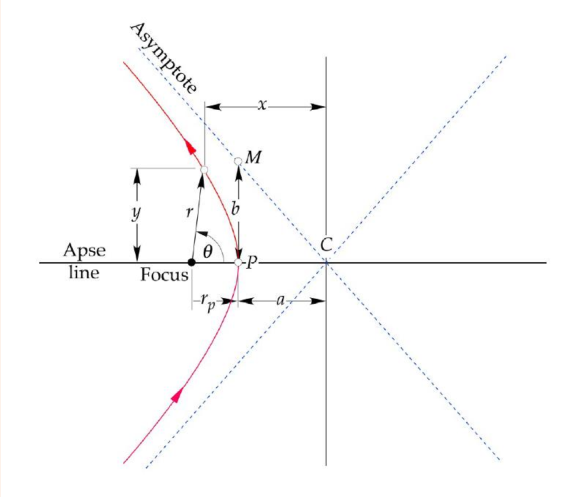
Use the Barker's Equation for both problems

Given true anomaly θ, find time-of-flight t

$$\theta \rightarrow t$$

Given time-of-flight t, find true anomaly θ

$$t \rightarrow \theta$$



Hyperbolic orbit

$$M = e \sinh F - F$$

Given true anomaly θ, find time-of-flight t

$$\theta \rightarrow F \rightarrow M \rightarrow t$$

Given time-of-flight t, find true anomaly θ

$$t \rightarrow M \rightarrow F \rightarrow \theta$$

Use Newton-Raphson method

TASK 1

Create a **Kepler's Equation Solver (for elliptical orbit)** using spreadsheet e.g. Microsoft Excel/Google Sheet

Some guide: https://youtu.be/uyJnU5qgz3s?t=187

Make sure your solver is reliable, you are going to use it for later assignments.

TASK 2

Create a **Barker's Equation Solver (for parabolic trajectory)** using spreadsheet e.g. Microsoft Excel/Google Sheet

Some guide: https://en.wikipedia.org/wiki/Parabolic_trajectory#:~:text=Barker's%20equation,-Barker's%20equation%20relates&text=D%20%3D%20tan(%CE%BD%2F2,is%20the%20standard%20gravitational%20parameter

Make sure your solver is reliable, you are going to use it for later assignments.

TASK 3

Create a **Kepler’s Equation Solver (for hyperbolic trajectory)** using spreadsheet e.g. Microsoft Excel/Google Sheet

Some guide: Curtis (2020) page 163, Algorithm 3.2 or Appendix D.12

All quantities in this formula are dimensionless (radians, not degrees).

ALGORITHM 3.2

Solve Kepler’s equation for the hyperbola for the hyperbolic eccentric anomaly F given the eccentricity e and the hyperbolic mean anomaly M_h. See Appendix D.12 for the implementation of this algorithm in MATLAB.

1. Choose an initial estimate of the eccentric anomaly F.
   a. For hand computations, read a rough value of F_0 (no more than two significant figures) from Fig. 3.17 to keep the number of iterations to a minimum.
   b. In computer software, let $F_0 = M_h$, an inelegant choice that may result in many iterations but will nevertheless rapidly converge on today’s high-speed desktops and laptops.
2. At any given step, having obtained F_i from the previous step, calculate $f(F_i) = e \sinh F_i - F_i - M_h$ and $f'(F_i) = e \cosh F_i - 1$.
3. Calculate $\text{ratio}_i = f(F_i)/f'(F_i)$.
4. If $|\text{ratio}_i|$ exceeds the chosen tolerance (e.g., 10^{-8}), then calculate an updated value of F_i. Return to Step 2.
5. If $|\text{ratio}_i|$ is less than the tolerance, then accept F_i as the solution to within the desired accuracy.

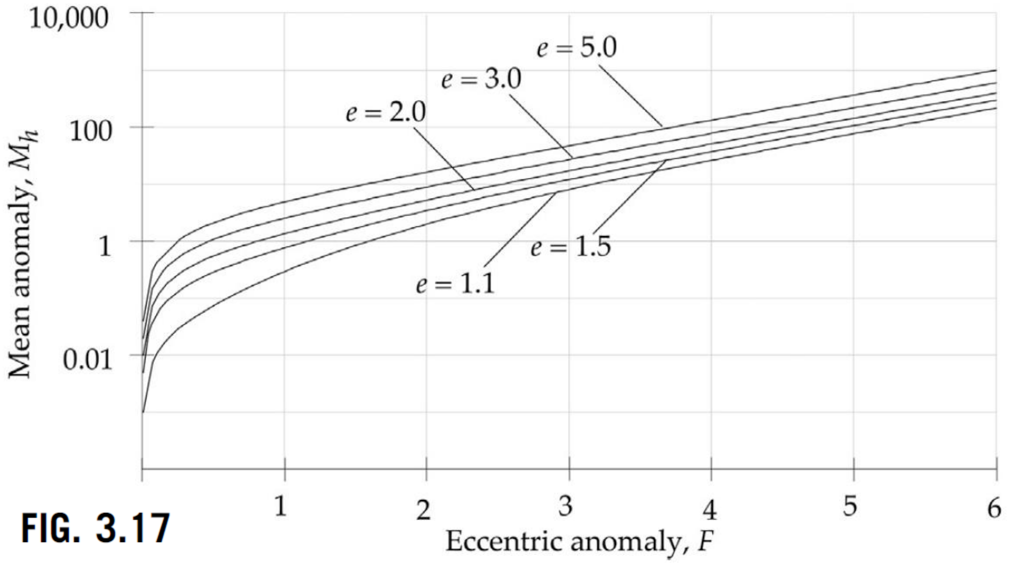


FIG. 3.17

Semester Project 1: TASK 1, 2, and 3

Create these solvers in a single spreadsheet file which include three separate sheets. Name them:

1. **Kepler's Solver** (elliptic) – in Sheet 1
2. **Barker's Solver** (parabolic) – in Sheet 2
3. **Kepler's Solver** (hyperbolic) – in Sheet 3
4. Self-regulated Learning Journal – Plan (before), Monitor (during), Evaluate (after)

Name your file `keplerssolvers_[`*`matric no`*`]_[`*`firstnames`*`].xlsx` and `learningjournal_[`*`matric no`*`]_[`*`firstnames`*`].docx`

You'll be submitting these in PutraBlast (an assignment box will be created)

Due date: Before Test 1?

Self-Regulated Learning (Semester Project 1)

Planning • What is the instructor's goal in having me do this task? • What are all the things I need to do to successfully accomplish this task? • What resources do I need to complete the task? How will I make sure I have them? • How much time do I need to complete the task? • If I have done something like this before, how could I do a better job this time?	
Monitoring • What strategies am I using that are working well or not working well to help me learn? • What other resources could I be using to complete this task? What action should I take to get these? • What is most challenging for me about this task? Most confusing? • What could I do differently mid-assignment to address these challenges and confusions?	
Evaluating • To what extent did I successfully accomplish the goals of the task? • To what extent did I use resources available to me? • If I were the instructor, what would I identify as strengths of my work and flaws in my work? • When I do an assignment or task like this again, what do I want to remember to do differently? What worked well for me that I should use next time?	