ALGORITHM 4.2

Obtain orbital elements from the state vector. A MATLAB version of this procedure appears in Appendix D.18. Applying this algorithm to orbits around other planets or the sun amounts to defining the frame of reference and substituting the appropriate gravitational parameter μ.

1. Calculate the distance, $r = \sqrt{\mathbf{r}\cdot\mathbf{r}} = \sqrt{X^2+Y^2+Z^2}$.
2. Calculate the speed, $v = \sqrt{\mathbf{v}\cdot\mathbf{v}} = \sqrt{v_X^2+v_Y^2+v_Z^2}$.
3. Calculate the radial velocity, $v_r = \mathbf{r}\cdot\mathbf{v}/r = (Xv_X + Yv_Y + Zv_Z)/r$.
   Note that if $v_r > 0$, the spacecraft is flying away from perigee. If $v_r < 0$, it is flying toward perigee.
4. Calculate the specific angular momentum,

$$\mathbf{h} = \mathbf{r}\times\mathbf{v} = \begin{vmatrix} \hat{\mathbf{I}} & \hat{\mathbf{J}} & \hat{\mathbf{K}} \\ X & Y & Z \\ v_X & v_Y & v_Z \end{vmatrix}$$

5. Calculate the magnitude of the specific angular momentum, $h = \sqrt{\mathbf{h}\cdot\mathbf{h}}$.
   This is the first orbital element.
6. Calculate the inclination,

$$i = \cos^{-1}(h_Z/h) \tag{4.7}$$

   This is the second orbital element. Recall that i must lie between 0° and 180°, which is precisely the range (principal values) of the arccosine function. Hence, there is no quadrant ambiguity to contend with here. If $90° < i \le 180°$, the angular momentum **h** points in a southerly direction. In that case, the orbit is retrograde, which means that the motion of the satellite around the earth is opposite to earth's rotation.
7. Calculate

$$\mathbf{N} = \hat{\mathbf{K}}\times\mathbf{h} = \begin{vmatrix} \hat{\mathbf{I}} & \hat{\mathbf{J}} & \hat{\mathbf{K}} \\ 0 & 0 & 1 \\ h_X & h_Y & h_Z \end{vmatrix} \tag{4.8}$$

   This vector defines the node line.
8. Calculate the magnitude of **N**, $N = \sqrt{\mathbf{N}\cdot\mathbf{N}}$.
9. Calculate the right ascension of the ascending node, $\Omega = \cos^{-1}(N_X/N)$. This is the third orbital element. If $N_X > 0$, then Ω lies in either the first or fourth quadrant. If $N_X < 0$, then Ω lies in either the second or third quadrant. To place Ω in the proper quadrant, observe that the ascending node lies on the positive side of the vertical XZ plane ($0 \le \Omega < 180°$) if $N_Y > 0$. On the other hand, the ascending node lies on the negative side of the XZ plane ($180° \le \Omega < 360°$) if $N_Y < 0$. Therefore, $N_Y > 0$ implies that $0 \le \Omega < 180°$, whereas $N_Y < 0$ implies that $180° \le \Omega < 360°$. In summary,

$$\Omega = \begin{cases} \cos^{-1}\left(\dfrac{N_X}{N}\right) & (N_Y \ge 0) \\ 360° - \cos^{-1}\left(\dfrac{N_X}{N}\right) & (N_Y < 0) \end{cases} \tag{4.9}$$

10. Calculate the eccentricity vector. Starting with Eq. (2.40),

$$\mathbf{e} = \frac{1}{\mu}\left[\mathbf{v}\times\mathbf{h} - \mu\frac{\mathbf{r}}{r}\right] = \frac{1}{\mu}\left[\mathbf{v}\times(\mathbf{r}\times\mathbf{v}) - \mu\frac{\mathbf{r}}{r}\right] = \frac{1}{\mu}\left[\overbrace{\mathbf{r}v^2 - \mathbf{v}(\mathbf{r}\cdot\mathbf{v})}^{bac-cab\ rule} - \mu\frac{\mathbf{r}}{r}\right]$$

so that

$$\mathbf{e} = \frac{1}{\mu}\left[\left(v^2 - \frac{\mu}{r}\right)\mathbf{r} - rv_r\mathbf{v}\right] \tag{4.10}$$

11. Calculate the eccentricity, $e = \sqrt{\mathbf{e}\cdot\mathbf{e}}$, which is the fourth orbital element. Substituting Eq. (4.10) leads to a form depending only on the scalars obtained thus far,

$$e = \sqrt{1 + \frac{h^2}{\mu^2}\left(v^2 - \frac{2\mu}{r}\right)} \tag{4.11}$$

12. Calculate the argument of perigee,

$$\omega = \cos^{-1}\left(\frac{\mathbf{N}}{N}\cdot\frac{\mathbf{e}}{e}\right)$$

This is the fifth orbital element. If N·$\mathbf{e} > 0$, then ω lies in either the first or fourth quadrant. If N·$\mathbf{e} < 0$, then ω lies in either the second or third quadrant. To place ω in the proper quadrant, observe that perigee lies above the equatorial plane ($0° \le \omega < 180°$) if $\mathbf{e}$ points up (in the positive Z direction) and that perigee lies below the plane ($180° \le \omega < 360°$) if $\mathbf{e}$ points down. Therefore, $e_Z \ge 0$ implies that $0° < \omega < 180°$, whereas $e_Z < 0$ implies that $180° < \omega < 360°$. To summarize,

$$\omega = \begin{cases} \cos^{-1}\left(\dfrac{\mathbf{N}\cdot\mathbf{e}}{Ne}\right) & (e_Z \ge 0) \\ 360° - \cos^{-1}\left(\dfrac{\mathbf{N}\cdot\mathbf{e}}{Ne}\right) & (e_Z < 0) \end{cases} \tag{4.12}$$

13. Calculate the true anomaly,

$$\theta = \cos^{-1}\left(\frac{\mathbf{e}}{e}\cdot\frac{\mathbf{r}}{r}\right)$$

This is the sixth and final orbital element. If $\mathbf{e}\cdot\mathbf{r} > 0$, then θ lies in the first or fourth quadrant. If $\mathbf{e}\cdot\mathbf{r} < 0$, then θ lies in the second or third quadrant. To place θ in the proper quadrant, note that if the satellite is flying away from perigee ($\mathbf{r}\cdot\mathbf{v} \ge 0$), then $0 \le \theta < 180°$, whereas if the satellite is flying toward perigee ($\mathbf{r}\cdot\mathbf{v} < 0$), then $180° \le \theta < 360°$. Therefore, using the results of Step 3 above

$$\theta = \begin{cases} \cos^{-1}\left(\dfrac{\mathbf{e}}{e} \cdot \dfrac{\mathbf{r}}{r}\right) & (v_r \geq 0) \\ 360^\circ - \cos^{-1}\left(\dfrac{\mathbf{e}}{e} \cdot \dfrac{\mathbf{r}}{r}\right) & (v_r < 0) \end{cases} \tag{4.13a}$$

Substituting Eq. (4.10) yields an alternative form of this expression,

$$\theta = \begin{cases} \cos^{-1}\left[\dfrac{1}{e}\left(\dfrac{h^2}{\mu r} - 1\right)\right] & (v_r \geq 0) \\ 360^\circ - \cos^{-1}\left[\dfrac{1}{e}\left(\dfrac{h^2}{\mu r} - 1\right)\right] & (v_r < 0) \end{cases} \tag{4.13b}$$

The procedure described above for calculating the orbital elements is not unique.

EXAMPLE 4.3

Given the state vector,

$$\mathbf{r} = -6045\hat{\mathbf{I}} - 3490\hat{\mathbf{J}} + 2500\hat{\mathbf{K}}\,(\text{km})$$
$$\mathbf{v} = -3.457\hat{\mathbf{I}} + 6.618\hat{\mathbf{J}} + 2.533\hat{\mathbf{K}}\,(\text{km/s})$$

find the orbital elements h, i, Ω, e, ω, and θ using Algorithm 4.2.

Solution

Step 1:

$$r = \sqrt{\mathbf{r}\cdot\mathbf{r}} = \sqrt{(-6045)^2 + (-3490)^2 + 2500^2} = 7414\,\text{km}$$

Step 2:

$$v = \sqrt{\mathbf{v}\cdot\mathbf{v}} = \sqrt{(-3.457)^2 + 6.618^2 + 2.533^2} = 7.884\,\text{km/s}$$

Step 3:

$$v_r = \frac{\mathbf{v}\cdot\mathbf{r}}{r} = \frac{(-3.457)\cdot(-6045) + 6.618\cdot(-3490) + 2.533\cdot 2500}{7414} = 0.5575\,\text{km/s} \tag{a}$$

Since $v_r > 0$, the satellite is flying away from perigee.

Step 4:

$$\mathbf{h} = \mathbf{r}\times\mathbf{v} = \begin{vmatrix} \hat{\mathbf{I}} & \hat{\mathbf{J}} & \hat{\mathbf{K}} \\ -6045 & -3490 & 2500 \\ -3.457 & 6.618 & 2.533 \end{vmatrix} = -25{,}380\hat{\mathbf{I}} + 6670\hat{\mathbf{J}} - 52{,}070\hat{\mathbf{K}}\left(\text{km}^2/\text{s}\right)$$

Step 5:

$$h = \sqrt{\mathbf{h}\cdot\mathbf{h}} = \sqrt{(-25{,}380)^2 + 6670^2 + (-52{,}070)^2} \Rightarrow \boxed{h = 58{,}310\,\text{km}^2/\text{s}}$$

Step 6:

$$i = \cos^{-1}\frac{h_Z}{h} = \cos^{-1}\left(\frac{-52{,}070}{58{,}310}\right) \Rightarrow \boxed{i = 153.2^\circ}$$

Since i is greater than 90°, this is a retrograde orbit.

Step 7:

$$\mathbf{N} = \hat{\mathbf{K}} \times \mathbf{h} = \begin{vmatrix} \hat{\mathbf{I}} & \hat{\mathbf{J}} & \hat{\mathbf{K}} \\ 0 & 0 & 1 \\ -25{,}380 & 6670 & -52{,}070 \end{vmatrix} = -6670\hat{\mathbf{I}} - 25{,}380\hat{\mathbf{J}}\left(\text{km}^2/\text{s}\right) \tag{b}$$

Step 8:

$$N = \sqrt{\mathbf{N}\cdot\mathbf{N}} = \sqrt{(-6670)^2 + (-25,380)^2} = 26,250\text{km}^2/\text{s}$$

Step 9:

$$\Omega = \cos^{-1}\frac{N_X}{N} = \cos^{-1}\left(\frac{-6670}{26{,}250}\right) = 104.7° \quad \text{or} \quad 255.3°$$

From Eq. (b) we know that $N_Y < 0$; therefore, Ω must lie in the third quadrant,

$$\boxed{\Omega = 255.3°}$$

Step 10:

$$\begin{aligned} \mathbf{e} &= \frac{1}{\mu}\left[\left(v^2 - \frac{\mu}{r}\right)\mathbf{r} - rv_r\mathbf{v}\right] \\ &= \frac{1}{398{,}600}\left[\left(7.884^2 - \frac{398{,}600}{7414}\right)\left(-6045\hat{\mathbf{I}} - 3490\hat{\mathbf{J}} + 2500\hat{\mathbf{K}}\right)\right. \\ &\quad \left. -(7414)(0.5575)\left(-3.457\hat{\mathbf{I}} + 6.618\hat{\mathbf{J}} + 2.533\hat{\mathbf{K}}\right)\right] \\ \mathbf{e} &= -0.09160\hat{\mathbf{I}} - 0.1422\hat{\mathbf{J}} + 0.02644\hat{\mathbf{K}} \end{aligned} \tag{c}$$

Step 11:

$$e = \sqrt{\mathbf{e}\cdot\mathbf{e}} = \sqrt{(-0.09160)^2 + (-0.1422)^2 + (0.02644)^2} \quad \Rightarrow \quad \boxed{e = 0.1712}$$

Clearly, the orbit is an ellipse.

Step 12:

$$\omega = \cos^{-1}\frac{\mathbf{N}\cdot\mathbf{e}}{Ne} = \cos^{-1}\left[\frac{(-6670)(-0.09160) + (-25,380)(-0.1422) + (0)(0.02644)}{(26,250)(0.1712)}\right] = 20.07° \quad \text{or} \quad 339.9°$$

ω lies in the first quadrant if $e_Z > 0$, which is true in this case, as we see from Eq. (c). Therefore,

$$\boxed{\omega = 20.07°}$$

Step 13:

$$\theta = \cos^{-1}\left(\frac{\mathbf{e}\cdot\mathbf{r}}{er}\right) = \cos^{-1}\left[\frac{(-0.09160)(-6045) + (-0.1422)\cdot(-3490) + (0.02644)(2500)}{(0.1712)(7414)}\right] = 28.45° \quad \text{or} \quad 331.6°$$

From Eq. (a) we know that $v_r > 0$, which means $0° \le \theta < 180°$. Therefore,

$$\boxed{\theta = 28.45°}$$

Having found the six orbital elements, we can go on to compute other parameters. The perigee and apogee radii are

$$\begin{aligned} r_p &= \frac{h^2}{\mu}\frac{1}{1 + e\cos(0)} = \frac{58{,}310^2}{398{,}600}\frac{1}{1 + 0.1712} = 7284\text{km} \\ r_a &= \frac{h^2}{\mu}\frac{1}{1 + e\cos(180°)} = \frac{58{,}310^2}{398{,}600}\frac{1}{1 - 0.1712} = 10{,}290\text{km} \end{aligned}$$

From these it follows that the semimajor axis of the ellipse is

$$a = \frac{1}{2}\left(r_p + r_a\right) = 8788\text{km}$$

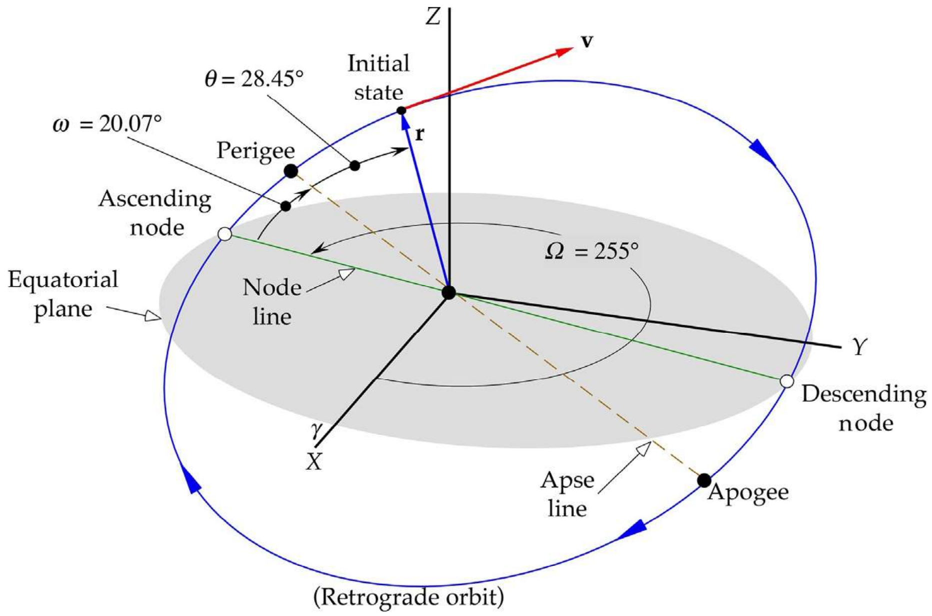

FIG. 4.8

A plot of the orbit identified in Example 4.3.

This leads to the period,

$$T = \frac{2\pi}{\sqrt{\mu}} a^{3/2} = 2.278\,\mathrm{h}$$

The orbit is illustrated in Fig. 4.8.

We have seen how to obtain the orbital elements from the state vector. To arrive at the state vector, given the orbital elements, requires performing coordinate transformations, which are discussed in the next section.

4.5 COORDINATE TRANSFORMATION

The Cartesian coordinate system was introduced in Section 1.2. Fig. 4.9 shows two such coordinate systems: the unprimed system with axes xyz, and the primed system with axes $x'y'z'$. The orthogonal unit basis vectors for the unprimed system are $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$, and $\hat{\mathbf{k}}$. The fact they are unit vectors means

$$\hat{\mathbf{i}} \cdot \hat{\mathbf{i}} = \hat{\mathbf{j}} \cdot \hat{\mathbf{j}} = \hat{\mathbf{k}} \cdot \hat{\mathbf{k}} = 1 \tag{4.14}$$

Since they are orthogonal,

$$\hat{\mathbf{i}} \cdot \hat{\mathbf{j}} = \hat{\mathbf{i}} \cdot \hat{\mathbf{k}} = \hat{\mathbf{j}} \cdot \hat{\mathbf{k}} = 0 \tag{4.15}$$