4.6 TRANSFORMATION BETWEEN GEOCENTRIC EQUATORIAL AND PERIFOCAL FRAMES

The perifocal frame of reference for a given orbit was introduced in Section 2.10. Fig. 4.16 illustrates the relationship between the perifocal and geocentric equatorial frames. Since the orbit lies in the $\bar{x}\bar{y}$ plane, the components of the state vector of a body relative to its perifocal reference are, according to Eqs. (2.119) and (2.125),

$$\mathbf{r} = \bar{x}\hat{\mathbf{p}} + \bar{y}\hat{\mathbf{q}} = \frac{h^2}{\mu}\frac{1}{1+e\cos\theta}(\cos\theta\hat{\mathbf{p}} + \sin\theta\hat{\mathbf{q}}) \tag{4.43}$$

$$\mathbf{v} = \dot{\bar{x}}\hat{\mathbf{p}} + \dot{\bar{y}}\hat{\mathbf{q}} = \frac{\mu}{h}[-\sin\theta\hat{\mathbf{p}} + (e+\cos\theta)\hat{\mathbf{q}}] \tag{4.44}$$

In matrix notation, these may be written

$$\{\mathbf{r}\}_{\bar{x}} = \frac{h^2}{\mu}\frac{1}{1+e\cos\theta}\begin{Bmatrix}\cos\theta\\ \sin\theta\\ 0\end{Bmatrix} \tag{4.45}$$

$$\{\mathbf{v}\}_{\bar{x}} = \frac{\mu}{h}\begin{Bmatrix}-\sin\theta\\ e+\cos\theta\\ 0\end{Bmatrix} \tag{4.46}$$

The subscript $\bar{x}$ is shorthand for "the $\bar{x}\bar{y}\bar{z}$ coordinate system" and is used to indicate that the components of these vectors are given in the perifocal frame, as opposed to, say, the geocentric equatorial frame (Eqs. 4.2 and 4.3).

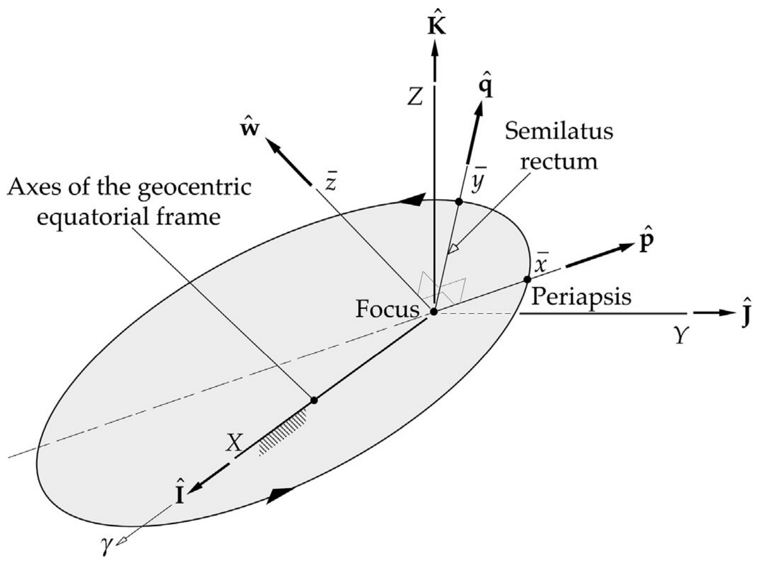

FIG. 4.16

Perifocal ($\bar{x}\bar{y}\bar{z}$) and geocenric equatorial (XYZ) frames.

The transformation from the geocentric equatorial frame into the perifocal frame may be accomplished by the classical Euler angle sequence $[\mathbf{R}_3(\gamma)][\mathbf{R}_1(\beta)][\mathbf{R}_3(\alpha)]$ in Eq. (4.37) (see Fig. 4.7). In this case, the first rotation angle is Ω, the right ascension of the ascending node. The second rotation is i, the orbital inclination angle, and the third rotation angle is ω, the argument of perigee. Ω is measured around the Z axis of the geocentric equatorial frame, i is measured around the node line, and ω is measured around the $\bar{z}$ axis of the perifocal frame. Therefore, the direct cosine matrix $[\mathbf{Q}]_{X\bar{x}}$ of the transformation from XYZ to $\bar{x}\,\bar{y}\,\bar{z}$ is

$$[\mathbf{Q}]_{X\bar{x}} = [\mathbf{R}_3(\omega)][\mathbf{R}_1(i)][\mathbf{R}_3(\Omega)] \tag{4.47}$$

From Eq. (4.38) we get

$$[\mathbf{Q}]_{X\bar{x}} = \begin{bmatrix} -\sin\Omega\cos i\sin\omega + \cos\Omega\cos\omega & \cos\Omega\cos i\sin\omega + \sin\Omega\cos\omega & \sin i\sin\omega \\ -\sin\Omega\cos i\cos\omega - \cos\Omega\sin\omega & \cos\Omega\cos i\cos\omega - \sin\Omega\sin\omega & \sin i\cos\omega \\ \sin\Omega\sin i & -\cos\Omega\sin i & \cos i \end{bmatrix} \tag{4.48}$$

Remember that this is an orthogonal matrix, which means that the inverse transformation $[\mathbf{Q}]_{\bar{x}X}$ from $\bar{x}\,\bar{y}\,\bar{z}$ to XYZ, is given by $[\mathbf{Q}]_{\bar{x}X} = \left([\mathbf{Q}]_{X\bar{x}}\right)^T$, or

$$[\mathbf{Q}]_{\bar{x}X} = \begin{bmatrix} -\sin\Omega\cos i\sin\omega + \cos\Omega\cos\omega & -\sin\Omega\cos i\cos\omega - \cos\Omega\sin\omega & \sin\Omega\sin i \\ \cos\Omega\cos i\sin\omega + \sin\Omega co\omega & \cos\,\cos i\cos\omega - \sin\Omega\sin\omega & -\cos\Omega\sin i \\ \sin i\sin\omega & \sin i\cos\omega & \cos i \end{bmatrix} \tag{4.49}$$

If the components of the state vector are given in the geocentric equatorial frame,

$$\{\mathbf{r}\}_X = \begin{Bmatrix} X \\ Y \\ Z \end{Bmatrix} \qquad \{\mathbf{v}\}_X = \begin{Bmatrix} v_X \\ v_Y \\ v_Z \end{Bmatrix}$$

then the components in the perifocal frame are found by carrying out the matrix multiplications

$$\{\mathbf{r}\}_{\bar{x}} = \begin{Bmatrix} \bar{x} \\ \bar{y} \\ 0 \end{Bmatrix} = [\mathbf{Q}]_{X\bar{x}}\{\mathbf{r}\}_X \qquad \{\mathbf{v}\}_{\bar{x}} = \begin{Bmatrix} \dot{\bar{x}} \\ \dot{\bar{y}} \\ 0 \end{Bmatrix} = [\mathbf{Q}]_{X\bar{x}}\{\mathbf{v}\}_X \tag{4.50}$$

Likewise, the transformation from perifocal to geocentric equatorial components is

$$\{\mathbf{r}\}_X = [\mathbf{Q}]_{\bar{x}X}\{\mathbf{r}\}_{\bar{x}} \qquad \{\mathbf{v}\}_X = [\mathbf{Q}]_{\bar{x}X}\{\mathbf{v}\}_{\bar{x}} \tag{4.51}$$

ALGORITHM 4.5

Given the orbital elements h, e, i, Ω, ω, and θ, compute the state vectors $\mathbf{r}$ and $\mathbf{v}$ in the geocentric equatorial frame of reference. A MATLAB implementation of this procedure is listed in Appendix D.22. This algorithm can be applied to orbits around other planets or the sun.

1. Calculate position vector $\{\mathbf{r}\}_{\bar{x}}$ in perifocal coordinates using Eq. (4.45).
2. Calculate velocity vector $\{\mathbf{v}\}_{\bar{x}}$ in perifocal coordinates using Eq. (4.46).
3. Calculate the matrix $[\mathbf{Q}]_{\bar{x}X}$ of the transformation from perifocal to geocentric equatorial coordinates using Eq. (4.49).
4. Transform $\{\mathbf{r}\}_{\bar{x}}$ and $\{\mathbf{v}\}_{\bar{x}}$ into the geocentric frame by means of Eq. (4.51).

EXAMPLE 4.7

For a given earth orbit, the elements are $h = 80,000\ \text{km}^2/\text{s}$, $e = 1.4$, $i = 30°$, $\Omega = 40°$, $\omega = 60°$, and $\theta = 30°$. Using Algorithm 4.5, find the state vectors **r** and **v** in the geocentric equatorial frame.

Solution

Step 1:

$$\{\mathbf{r}\}_{\bar{x}} = \frac{h^2}{\mu}\frac{1}{1+e\cos\theta}\begin{Bmatrix}\cos\theta\\ \sin\theta\\ 0\end{Bmatrix} = \frac{80{,}000^2}{398{,}600}\frac{1}{1+1.4\cos 30°}\begin{Bmatrix}\cos 30°\\ \sin 30°\\ 0\end{Bmatrix} = \begin{Bmatrix}6285.0\\ 3628.6\\ 0\end{Bmatrix}(\text{km})$$

Step 2:

$$\{\mathbf{v}\}_{\bar{x}} = \frac{\mu}{h}\begin{Bmatrix}-\sin\theta\\ e+\cos\theta\\ 0\end{Bmatrix} = \frac{398{,}600}{80{,}000}\begin{Bmatrix}-\sin 30°\\ 1.4+\cos 30°\\ 0\end{Bmatrix} = \begin{Bmatrix}-2.4913\\ 11.290\\ 0\end{Bmatrix}(\text{km/s})$$

Step 3:

$$[\mathbf{Q}]_{X\bar{x}} = \begin{bmatrix}\cos\omega & \sin\omega & 0\\ -\sin\omega & \cos\omega & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\ 0 & \cos i & \sin i\\ 0 & -\sin i & \cos i\end{bmatrix}\begin{bmatrix}\cos\Omega & \sin\Omega & 0\\ -\sin\Omega & \cos\Omega & 0\\ 0 & 0 & 1\end{bmatrix}$$

$$= \begin{bmatrix}\cos 60° & \sin 60° & 0\\ -\sin 60° & \cos 60° & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\ 0 & \cos 30° & \sin 30°\\ 0 & -\sin 30° & \cos 30°\end{bmatrix}\begin{bmatrix}\cos 40° & \sin 40° & 0\\ -\sin 40° & \cos 40° & 0\\ 0 & 0 & 1\end{bmatrix}$$

$$= \begin{bmatrix}-0.099068 & 0.89593 & 0.43301\\ -0.94175 & -0.22496 & 0.25000\\ 0.32139 & -0.38302 & 0.86603\end{bmatrix}$$

This is the direction cosine matrix for $XYZ \to \bar{x}\,\bar{y}\,\bar{z}$. The transformation matrix for $\bar{x}\,\bar{y}\,\bar{z} \to XYZ$ is the transpose,

$$[\mathbf{Q}]_{\bar{x}X} = \begin{bmatrix}-0.099068 & -0.94175 & 0.32139\\ 0.89593 & -0.22496 & -0.38302\\ 0.43301 & 0.25000 & 0.86603\end{bmatrix}$$

Step 4:
The geocentric equatorial position vector is

$$\{\mathbf{r}\}_X = [\mathbf{Q}]_{\bar{x}X}\{\mathbf{r}\}_{\bar{x}}$$

$$= \begin{bmatrix}-0.099068 & -0.94175 & 0.32139\\ 0.89593 & -0.22496 & -0.38302\\ 0.43301 & 0.25 & 0.86603\end{bmatrix}\begin{Bmatrix}6285.0\\ 3628.6\\ 0\end{Bmatrix} = \boxed{\begin{Bmatrix}-4040\\ 4815\\ 3629\end{Bmatrix}(\text{km})} \quad \text{(a)}$$

whereas the geocentric equatorial velocity vector is

$$\{\mathbf{v}\}_X = [\mathbf{Q}]_{\bar{x}X}\{\mathbf{v}\}_{\bar{x}}$$

$$= \begin{bmatrix}-0.099068 & -0.94175 & 0.32139\\ 0.89593 & -0.22496 & -0.38302\\ 0.43301 & 0.25 & 0.86603\end{bmatrix}\begin{Bmatrix}-2.4913\\ 11.290\\ 0\end{Bmatrix} = \boxed{\begin{Bmatrix}-10.39\\ -4.772\\ 1.744\end{Bmatrix}(\text{km/s})}$$

The state vectors **r** and **v** are shown in Fig. 4.17. By holding all the orbital parameters except the true anomaly fixed and allowing θ to take on a range of values, we generate a sequence of position vectors $\{\mathbf{r}\}_{\bar{x}}$ from Eq. (4.45). Each of these is

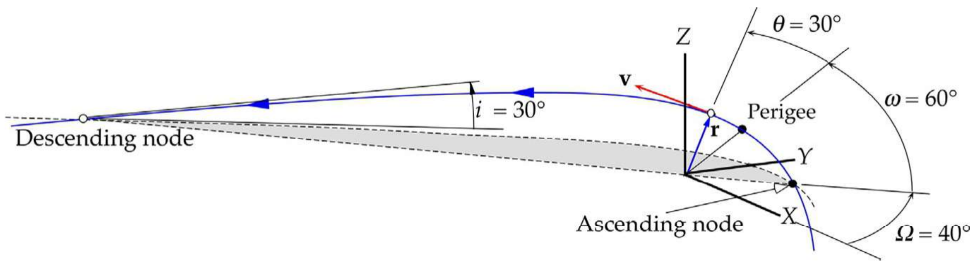

FIG. 4.17

A portion of the hyperbolic trajectory of Example 4.7.

projected into the geocentric equatorial frame as in Eq. (a), using repeatedly the same transformation matrix $[\mathbf{Q}]_{\bar{x}X}$. By connecting the end points of all the position vectors $\{\mathbf{r}\}_X$, we trace out the trajectory illustrated in Fig. 4.17.

4.7 EFFECTS OF THE EARTH'S OBLATENESS

The earth, like all planets with comparable or higher rotational rates, bulges out at the equator because of centrifugal force. The earth's equatorial radius is 21 km (13 miles) larger than the polar radius. This flattening at the poles is called oblateness, which is defined as follows:

$$\text{Oblateness} = \frac{\text{Equatorial radius} - \text{Polar radius}}{\text{Equatorial radius}}$$

The earth is an oblate spheroid, lacking the perfect symmetry of a sphere (a basketball can be made an oblate spheroid by sitting on it). This lack of symmetry means that the force of gravity on an orbiting body is not directed toward the center of the earth. Although the gravitational field of a perfectly spherical planet depends only on the distance from its center, oblateness causes a variation also with latitude (i.e., the angular distance from the equator (or pole)). This is called a zonal variation. The dimensionless parameter that quantifies the major effects of oblateness on orbits is J_2, the second zonal harmonic. J_2 is not a universal constant. Each planet has its own value, as illustrated in Table 4.3, which lists variations of J_2 as well as oblateness.

Table 4.3 Oblateness and second zonal harmonics

Planet	Oblateness	J_2
Mercury	0.000	$60(10^{-6})$
Venus	0.000	$4.458(10^{-6})$
Earth	0.003353	$1.08263(10^{-3})$
Mars	0.00648	$1.96045(10^{-3})$
Jupiter	0.06487	$14.736(10^{-3})$
Saturn	0.09796	$16.298(10^{-3})$
Uranus	0.02293	$3.34343(10^{-3})$
Neptune	0.01708	$3.411(10^{-3})$
(Moon)	0.0012	$202.7(10^{-6})$