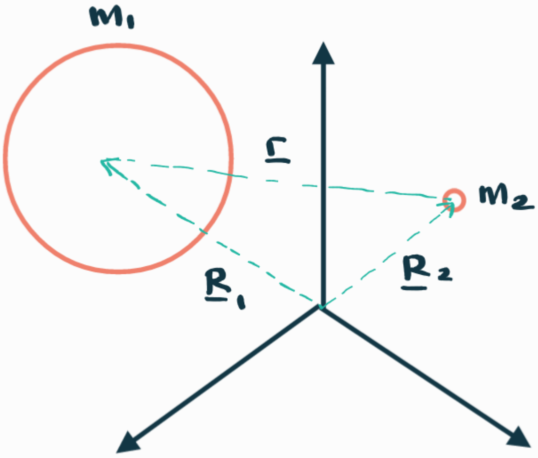
The Two-Body Problem

How do we mathematically describe the motion of an object (e.g. satellite) around a large gravitating body (e.g. Earth)?

A: Let's limit our physical system to just <u>two bodies</u> and then use N.L.U.G.



$$\underline{r} = \underline{R}_2 - \underline{R}_1 \longrightarrow 1$$

Two masses m_1 and m_2 with position vectors $\underline{R}_1$ and $\underline{R}_2$

(assumption: $m_1 >> m_2$)

Force from m_2 to m_1

$$m_1 \ddot{\underline{R}}_1 \leftsquigarrow \underline{F}_{12} = \frac{G m_1 m_2}{r^2}\hat{r} \rightarrow \cancel{m_1}\ddot{\underline{R}}_1 = \frac{G \cancel{m_1} m_2}{r^2}\hat{r} \rightarrow \ddot{\underline{R}}_1 = \frac{G m_2}{r^2}\hat{r} \quad 2$$

Force from m_1 to m_2

$$m_2 \ddot{\underline{R}}_2 \leftsquigarrow \underline{F}_{21} = -\frac{G m_1 m_2}{r^2}\hat{r} \rightarrow \cancel{m_2}\ddot{\underline{R}}_2 = -\frac{G m_1 \cancel{m_2}}{r^2}\hat{r} \rightarrow \ddot{\underline{R}}_2 = -\frac{G m_1}{r^2}\hat{r} \quad 3$$

$$\underline{r} = \underline{R}_2 - \underline{R}_1 \longrightarrow \text{from } 1$$

$$\ddot{\underline{r}} = \ddot{\underline{R}}_2 - \ddot{\underline{R}}_1$$

$$\ddot{\underline{r}} = -\frac{G m_1}{r^2} - \frac{G m_2}{r^2} \longrightarrow \text{from } 2 \;\&\; 3$$

$$\ddot{\underline{r}} = -\frac{G(m_1 + m_2)}{r^2} \longrightarrow \text{since } m_1 >> m_2, \quad G(m_1 + m_2) \approx G m_1$$

$$\boxed{G m_1 \equiv \mu = 398{,}600 \ \frac{km^3}{s^2}}$$

gravitational parameter of Earth

$$\boxed{\ddot{\underline{r}} = -\frac{\mu}{r^3}\underline{r}}$$

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Two results here:

By simplifying our system to just <u>one gravitating body</u> we obtain the governing equation for a <u>two-body system</u>

1. $\ddot{\underline{r}} = -\frac{\mu}{r^3}\underline{r}$

2. $\mu = GM_{earth} = 398{,}600 \ km^3/s^2$

$R_E = 6378$ km