Comparing sets (Solutions)

Exercise 1.
Complete the table below.

A	B	$A \subseteq B$?	$A \supseteq B$?
$\{a, b\}$	$\{a, a, b, c\}$		
$\{a\}$	$\{b\}$		
$\{\}$	$\{a\}$		
$\{a, b\}$	$\{a, a, b, b\}$		

Solution

A	B	$A \subseteq B$?	$A \supseteq B$?
$\{a, b\}$	$\{a, a, b, c\}$	Y	N
$\{a\}$	$\{b\}$	N	N
$\{\}$	$\{a\}$	Y	N
$\{a, b\}$	$\{a, a, b, b\}$	Y	Y

Explanation

For $A \subseteq B$, we have to check that every member of A is a member of B. For $A \supseteq B$, the opposite has to hold: every membeer of B is a member of A. Keep in mind that the empty set $\{\}$ contains no elements at all, so it is a subset of every set A, no matter what A looks like. Finally, recall that $\{a, a, b, b\} = \{a, b\}$, so the last line is asking whether $\{a, b\}$ is a subset and/or superset of $\{a, b\}$, to which the answer is yes in both cases.

Exercise 2.
Say whether the following statement is true or false and justify your answer: for any two sets A and B, $A \subseteq B$ iff $A \cap B = A$.

Solution

This is true.

We have to show that $A \subseteq B$ implies $A \cap B = A$, and that $A \cap B = A$ implies that $A \subseteq B$. Suppose, then, that $A \subseteq B$. In this case, every element A is also an element of B. Since $A \cap B$ contains all elements that belong to both A and B, and every element of A is also an element of B, $A \cap B$ contains at least all elements of A. But by definition, $A \cap B$ cannot contain any elements that do not belong to A itself. Hence $A \cap B$ contains all elements of A, and nothing else, which means that $A \cap B = A$.

Now suppose that $A \cap B = A$. Since $A \cap B$ contains all elements that are members of A and members of B, $A \cap B = A$ entails that every member of A is a member of B. In other words, $A \subseteq B$.

Exercise 3.
Fill in $=$, $\subsetneq$, or $\supsetneq$ as appropriate.

1. $\{a, b\}\_\{a\}$
2. $\{a, a, b, c\}\_\{b, b, a, c\}$
3. $\{1, 2, 3\}\_\{n + 5 \mid n \in \{-4, -3\}\}$
4. $\emptyset\_\{a\}$
5. $\emptyset\_\{\emptyset\}$

Solution

1. $\{a, b\} \supsetneq \{a\}$
2. $\{a, a, b, c\} = \{b, b, a, c\}$
3. $\{1, 2, 3\} \supsetneq \{n + 5 \mid n \in \{-4, -3\}\}$
4. $\emptyset \subsetneq \{a\}$
5. $\emptyset \subsetneq \{\emptyset\}$

Explanation

1. Every member of $\{a\}$ is a member of $\{a, b\}$, but $\{a, b\}$ contains at least one element that is not a member of $\{a\}$, hence $\{a\}$ is a proper subset of $\{a, b\}$.
2. Just remember that sets cannot contain an element more than once, so $\{a, a, b, c\} = \{a, b, c\} = \{b, b, a, c\}$.
3. The set $\{n + 5 \mid n \in \{-4, -3\}\}$ is $\{1, 2\}$, which is a proper subset of $\{1, 2, 3\}$.
4. Remember that the empty set is a subset of every set, which means that it is a proper subset of every set that is distinct from the empty set.
5. The same reasoning applies here, it is just a bit more confusing because we are comparing the empty set to the set containing the empty set. Still, the empty set has to be a subset of $\{\emptyset\}$, and $\{\emptyset\}$ contains an element that is not contained in the empty set, namely the empty set itself.

Exercise 4.
For each line in the table, say whether the sets are disjoint, incomparable, identical, or stand in a proper subset/superset relation.

A	B
$\{2, 5, 8\}$	the set of all odd numbers
$\{a, b, c\}$	$\{a, b\} \cup (\{a, c\} - \{b, d\})$

$\emptyset$	$\{a,b\} \cap (\{a,c\} - \{b,d\})$
$\emptyset$	$\{a,b\} \cap (\{a,c\} \cap \{b,d\})$

Solution

A	B	relation
$\{2,5,8\}$	the set of all odd numbers	incomparable
$\{a,b,c\}$	$\{a,b\} \cup (\{a,c\} - \{b,d\})$	identical
$\emptyset$	$\{a,b\} \cap (\{a,c\} - \{b,d\})$	A is the proper subset of B
$\emptyset$	$\{a,b\} \cap (\{a,c\} \cap \{b,d\})$	identical

Explanation

1. The set of all odd numbers contains some numbers that occur in $\{2,5,8\}$, and many that do not. In the other direction, $\{2,5,8\}$ contains 8, which is not an element of the set of all numbers. Overall, the two sets have some overlap, but neither subsumes the other. Hence they are incomparable.
2. We have $\{a,c\} - \{b,d\} = \{a,c\}$, and $\{a,b\} \cup \{a,c\} = \{a,b,c\}$. That's clearly the same set as $\{a,b,c\}$.
3. We first have to compute $\{a,b\} \cap \{a,c\}$, which is $\{a\}$. Clearly that is distinct from the $\emptyset$, but the $\emptyset$ is a subste of every set, including $\{a\}$. Hence the empty set is a proper subset of $\{a\}$.
4. Again we perform some computations first: $\{a,c\} \cap \{b,d\} = \emptyset$, and $\{a,b\} \cap \emptyset = \emptyset$. Needless to say, $\emptyset = \emptyset$.