Proof: Equivalence of positive and negative grammars

Prerequisites

- general(abbreviations[w.l.o.g.])
- sets(notation, operations)
- strings(notation)

This section defines both negative and positive versions of n-gram grammars and shows that they are expressively equivalent. Like in the section on the equivalence of mixed and fixed n-gram grammars, this is accomplished by a **constructive** proof. A proof is constructive if it doesn't just derive the existence of some object, but gives a concrete procedure for constructing this object. In the case at hand, the proof shows how to construct a positive grammar from a negative one, and the other way around.

Definition 1. Let Σ be some alphabet, and Σ_E its extension with edge marker symbols $\rtimes, \ltimes \notin \Sigma$. An n-gram over Σ_E is an element of Σ_E^n ($n \geq 1$). An n-gram grammar G over alphabet Σ is a finite set of n-grams over Σ_E. Every n-gram grammar has a **polarity**:

- If G is negative (also denoted ${}^{-}G$), then a string s over Σ is well-formed with respect to ${}^{-}G$ iff there are no u, v over Σ_E and no $g \in^{-} G$ such that $\rtimes^{n-1} \cdot s \cdot \ltimes^{n-1} = u \cdot g \cdot v$.
- If G is positive (also denoted ${}^{+}G$), then a string s over Σ is well-formed with respect to ${}^{+}G$ iff for all u, v over Σ_E and $g \in \Sigma_E^n$ such that $\rtimes^{n-1} \cdot s \cdot \ltimes^{n-1} = u \cdot g \cdot v$, it holds that $g \in^{+} G$.

The **language of** G, denoted $L(G)$, contains all strings that are well-formed with respect to G, and only those.

Theorem 2. For every n-gram grammar G there exists a grammar G' of opposite polarity such that $L(G) = L(G')$.

Proof. We assume w.l.o.g. that G is a positive grammar and denote it by P. We define a negative counterpart N as $\Sigma_E^n - P$ and show that $L(P) = L(N)$.

First, every $s \in L(P)$ is necessarily a member of $L(N)$. Assume towards a contradiction that $s \notin L(N)$. Then there must be some $g \in N$ such that $\rtimes^{n-1} s \ltimes^{n-1} = u \cdot g \cdot v$ ($u, v, \in \Sigma_E^*$). But since $N := \Sigma_E^n - P$, $g \in N$ implies $g \notin P$, wherefore $s \notin L(P)$. As this contradicts our initial assumption that $s \in L(P)$, it cannot be the case that $s \notin L(N)$. So $s \in L(N)$ after all.

In the other direction, suppose that $s \notin L(P)$. Then by definition there are $u, v \in \Sigma_E^*$ and $\Sigma_E^n \ni g \notin P$ such that $\rtimes^{n-1} s \ltimes^{n-1} = u \cdot g \cdot v$. But then $g \in N$, which entails $s \notin L(N)$.

This shows that $s \in L(P)$ iff $s \in L(N)$. As s was arbitrary, this holds for all strings and establishes $L(P) = L(N)$, which concludes our proof.