

Comparing sets

PREREQUISITES

- sets (notation, operations, cardinality)

Two sets can stand in several distinct relations to each other:

1. subset
2. superset
3. identity
4. proper subset
5. proper superset
6. disjoint
7. incomparable

1 Subset and superset

Given two sets A and B , A is a **subset** of B iff every element of A is also an element of B . In this case, one writes $A \subseteq B$. For example, $\{a, b\} \subseteq \{a, b, c, d\}$. Alternatively, one also says in this case that B is a **superset** of A (written $B \supseteq A$).

EXAMPLE 1.

A transitive verb is a verb that occurs with a subject and an object: *devour*, *contradict*, *wager*, *flummox*, and many more. Not all verbs are transitive, e.g. *sleep* or *give*. Suppose T is the set of all English transitive verbs, whereas V is the set of all English verbs. Since every transitive verb is a verb, but not the other way round, we have $T \subseteq V$.

By the definition of subset, every set S is a subset of itself. The reasoning is simple. If $S \subseteq S$, then every member of S must be a member of S , which is obviously true (how could it be otherwise?).

In addition, the empty set is a subset of every set, including itself. This is because the empty set contains no elements at all, so it trivially holds that every member of the empty set is a member of every set.

EXERCISE 1.

Complete the table below.

| A | B | $A \subseteq B?$ | $A \supseteq B?$ |
|------------|------------------|------------------|------------------|
| $\{a, b\}$ | $\{a, a, b, c\}$ | | |

| | |
|------------|------------------|
| $\{a\}$ | $\{b\}$ |
| $\{\}$ | $\{a\}$ |
| $\{a, b\}$ | $\{a, a, b, b\}$ |

EXERCISE 2.

Say whether the following statement is true or false and justify your answer: for any two sets A and B , $A \subseteq B$ iff $A \cap B = A$.

2 Identity

Two sets are *identical* iff each one is a subset of the other. In formal terms, $A = B$ iff both $A \subseteq B$ and $B \subseteq A$ hold. The reason for this is again simple:

1. If two sets A and B are identical, then they must contain exactly the same elements. But then every member of A is a member of B , which implies $A \subseteq B$. And it's also the case that every member of B is a member of A , so that we have $B \subseteq A$, too.
2. In the other direction, if $A \subseteq B$ and $B \subseteq A$, then every member of A is a member of B , and every member of B is a member of A . But that can only happen if the sets are identical.

3 Proper subset and superset

We call A a **proper subset** of B ($A \subsetneq B$) iff A is a subset of B but A and B are not identical. In other words, every element of A is a member of B , but not every element of B is a member of A . We also say that B is a **proper superset** of A ($B \supsetneq A$).

EXAMPLE 2.

Given our previous discussion, the set T of transitive verbs is proper subset of the set V of verbs because it is a subset but not every verb is a transitive verb. In other words, $T \subseteq V$ yet $T \neq V$. Hence $T \subsetneq V$.

EXERCISE 3.

Fill in $=$, \subsetneq , or \supsetneq as appropriate.

- $\{a, b\} _ \{a\}$
- $\{a, a, b, c\} _ \{b, b, a, c\}$
- $\{1, 2, 3\} _ \{n + 5 \mid n \in \{-4, -3\}\}$
- $\emptyset _ \{a\}$

- $\emptyset _ \{\emptyset\}$

4 Disjoint and incomparable sets

If there are two sets A and B such that neither $A \subseteq B$ nor $B \subseteq A$, then there can be only two scenarios. One option is that A and B are **disjoint**, which means that there is no x such that both $x \in A$ and $x \in B$ — the two sets have absolutely no overlap. In mathematical terms, $A \cap B = \emptyset$. Alternatively, A and B might be **incomparable**. In this case the two sets have a limited overlap such that there is at least one x with both $x \in A$ and $x \in B$, but there are also $a \in A$ and $b \in B$ such that $a \notin B$ and $b \notin A$.

EXAMPLE 3.

The set of English prepositions (*on, to, at, ...*) and the set of English determiners (*a, the, this, ...*) have not a single word in common and thus are disjoint. The set of English verbs and the set of English nouns, on the other hand, are incomparable. Many words like *water, cut, fall, love, try, judge, beat, or cross* can be used as nouns or verbs, but many other words are used only as nouns (*tree, waterfall, idea, Ferrari*) or only as verbs (*write, convince, admonish*).

Remember that it is possible for both $A \subseteq B$ and $B \subseteq A$ to be true — in this case, $A = B$. But there can be no A and B such that $A \subsetneq B$ and $B \subsetneq A$.

EXERCISE 4.

For each line in the table, say whether the sets are disjoint, incomparable, identical, or stand in a proper subset/superset relation.

| A | B |
|---------------|--|
| $\{2, 5, 8\}$ | the set of all odd numbers |
| $\{a, b, c\}$ | $\{a, b\} \cup (\{a, c\} - \{b, d\})$ |
| \emptyset | $\{a, b\} \cap (\{a, c\} - \{b, d\})$ |
| \emptyset | $\{a, b\} \cap (\{a, c\} \cap \{b, d\})$ |

5 Remarks on notation

Similarity to \leq and \geq

Students sometimes confuse the symbols \subseteq and \supseteq . To avoid that, just keep in mind that these symbols are modeled after \leq and \geq for numbers. Just like $x \leq y$ means that x is at most as large as y , $x \subseteq y$ tells us that x contains at most all the elements of y , and nothing else.

A note on \subset

You may occasionally come across the symbol \subset in other math texts. Some authors use \subset instead of \subseteq , while others use it for \subsetneq . As you might imagine, this can be very confusing for the reader, so it's best to avoid \subset and use \subseteq and \subsetneq instead.

And then there's $\not\subseteq$

Sometimes we might just want to say that A is not a subset of B . We could paraphrase this, as in “it is not the case that $A \subseteq B$ ”. But mathematicians like to use symbols for common phrases, so there's a dedicated symbol for this: $\not\subseteq$. Careful, do not confuse $\not\subseteq$ with \subsetneq .

Here's an overview of all the relevant notation:

| Formula | means... |
|---------------------|--|
| $A \subseteq B$ | A is a subset of B (holds even if $A = B$) |
| $A \subsetneq B$ | A is a proper subset of B ($A \subseteq B$ and $A \neq B$) |
| $A \not\subseteq B$ | A is not a subset of B ($A \ni a \notin B$ for some a) |

As you might have expected, there's corresponding counterparts for superset: \supseteq , \supsetneq , $\not\supseteq$. But there is no standardized symbol for sets being incomparable, although some authors like to use \sim for this purpose.

6 Recap

DEFINITION 1. Let A and B be arbitrary sets. Then A is a **subset** of B ($A \subseteq B$) iff every member of A is a member of B . In this case, B is a **superset** of A ($B \supseteq A$).

DEFINITION 2. For A and B arbitrary sets, A is a **proper subset** of B ($A \subsetneq B$) iff $A \subseteq B$ and there is a $b \in B$ such that $b \notin A$. Similarly, B is a **proper superset** of A ($B \supsetneq A$).

DEFINITION 3. Let A and B be arbitrary sets. Then A and B are:

- **identical** iff $A \subseteq B$ and $B \subseteq A$ both hold,
- **disjoint** iff $A \cap B = \emptyset$,
- **incomparable** iff $A \not\subseteq B$ and $B \not\subseteq A$ and $A \cap B \neq \emptyset$.