Comparing sets

PREREQUISITES

- sets (notation, operations, cardinality)

Two sets can stand in several distinct relations to each other:

1. subset
2. superset
3. identity
4. proper subset
5. proper superset
6. disjoint
7. incomparable

1 Subset and superset

Given two sets A and B, A is a **subset** of B iff every element of A is also an element of B. In this case, one writes $A \subseteq B$. For example, $\{a, b\} \subseteq \{a, b, c, d\}$. Alternatively, one also says in this case that B is a **superset** of A (written $B \supseteq A$).

EXAMPLE 1.
A transitive verb is a verb that occurs with a subject and an object: *devour*, *contradict*, *wager*, *flummox*, and many more. Not all verbs are transitive, e.g. *sleep* or *give*. Suppose T is the set of all English transitive verbs, whereas V is the set of all English verbs. Since every transitive verb is a verb, but no the other way round, we have $T \subseteq V$.

By the definition of subset, every set S is a subset of itself. The reasoning is simple. If $S \subseteq S$, then every member of S must be a member of S, which is obviously true (how could it be otherwise?).

In addition, the empty set is a subset of every set, including itself. This is because the empty set contains no elements at all, so it trivially holds that every member of the empty set is a member of every set.

EXERCISE 1.
Complete the table below.

A	B	$A \subseteq B$?	$A \supseteq B$?
$\{a, b\}$	$\{a, a, b, c\}$		
$\{a\}$	$\{b\}$		

$\{\}$	$\{a\}$
$\{a, b\}$	$\{a, a, b, b\}$

Exercise 2.
Say whether the following statement is true or false and justify your answer: for any two sets A and B, $A \subseteq B$ iff $A \cap B = A$.

2 Identity

Two sets are *identical* iff each one is a subset of the other. In formal terms, $A = B$ iff both $A \subseteq B$ and $B \subseteq A$ hold. The reason for this is again simple:

1. If two sets A and B are identical, then they must contain exactly the same elements. But then every member of A is a member of B, which implies $A \subseteq B$. And it's also the case that every member of B is a member of A, so that we have $B \subseteq A$, too.

2. In the other direction, if $A \subseteq B$ and $B \subseteq A$, then every member of A is a member of B, and every member of B is a member of A. But that can only happen if the sets are identical.

3 Proper subset and superset

We call A a **proper subset** of B ($A \subsetneq B$) iff A is a subset of B but A and B are not identical. In other words, every element of A is a member of B, but not every element of B is a member of A. We also say that B is a **proper superset** of A ($B \supsetneq A$).

Example 2.
Given our previous discussion, the set T of transitive verbs is proper subset of the set V of verbs because it is a subset but not every verb is a transitive verb. In other words, $T \subseteq V$ yet $T \neq V$. Hence $T \subsetneq V$.

Exercise 3.
Fill in $=$, $\subsetneq$, or $\supsetneq$ as appropriate.

- $\{a, b\}\_\{a\}$
- $\{a, a, b, c\}\_\{b, b, a, c\}$
- $\{1, 2, 3\}\_\{n + 5 \mid n \in \{-4, -3\}\}$
- $\emptyset\_\{a\}$
- $\emptyset\_\{\emptyset\}$

4 Disjoint and incomparable sets

If there are two sets A and B such that neither $A \subseteq B$ nor $B \subseteq A$, then there can be only two scenarios. One option is that A and B are **disjoint**, which means that there is no x such that both $x \in A$ and $x \in B$ — the two sets have absolutely no overlap. In mathematical terms, $A \cap B = \emptyset$. Alternatively, A and B might be **incomparable**. In this case the two sets have a limited overlap such that there is at least one x with both $x \in A$ and $x \in B$, but there are also $a \in A$ and $b \in B$ such that $a \notin B$ and $b \notin A$.

Example 3.
The set of English prepositions (*on*, *to*, *at*, ...) and the set of English determiners (*a*, *the*, *this*, ...) have not a single word in common and thus are disjoint. The set of English verbs and the set of English nouns, on the other hand, are incomparable. Many words like *water*, *cut*, *fall*, *love*, *try*, *judge*, *beat*, or *cross* can be used as nouns or verbs, but many other words are used only as nouns (*tree*, *waterfall*, *idea*, *Ferrari*) or only as verbs (*write*, *convince*, *admonish*).

Remember that it is possible for both $A \subseteq B$ and $B \subseteq A$ to be true — in this case, $A = B$. But there can be no A and B such that $A \subsetneq B$ and $B \subsetneq A$.

Exercise 4.
For each line in the table, say whether the sets are disjoint, incomparable, identical, or stand in a proper subset/superset relation.

A	B
$\{2,5,8\}$	the set of all odd numbers
$\{a,b,c\}$	$\{a,b\} \cup (\{a,c\} - \{b,d\})$
$\emptyset$	$\{a,b\} \cap (\{a,c\} - \{b,d\})$
$\emptyset$	$\{a,b\} \cap (\{a,c\} \cap \{b,d\})$

5 Remarks on notation

Similarity to ≤ and ≥

Students sometimes confuse the symbols $\subseteq$ and $\supseteq$. To avoid that, just keep in mind that these symbols are modeled after $\leq$ and $\geq$ for numbers. Just like $x \leq y$ means that x is at most as large as y, $x \subseteq y$ tells us that x contains at most all the elements of y, and nothing else.

A note on $\subset$

You may occasionally come across the symbol $\subset$ in other math texts. Some authors use $\subset$ instead of $\subseteq$, while others use it for $\subsetneq$. As you might imagine, this can be very confusing for the reader, so it's best to avoid $\subset$ and use $\subseteq$ and $\subsetneq$ instead.

And then there's $\nsubseteq$

Sometimes we might just want to say that A is not a subset of B. We could paraphrase this, as in "it is not the case that $A \subseteq B$". But mathematicians like to use symbols for common phrases, so there's a dedicated symbol for this: $\nsubseteq$. Careful, do not confuse $\nsubseteq$ with $\subsetneq$.

Here's an overview of all the relevant notation:

Formula	means...
$A \subseteq B$	A is a subset of B (holds even if $A = B$)
$A \subsetneq B$	A is a proper subset of B ($A \subseteq B$ and $A \neq B$)
$A \nsubseteq B$	A is not a subset of B ($A \ni a \notin B$ for some a)

As you might have expected, there's corresponding counterparts for superset: $\supseteq$, $\supsetneq$, $\nsupseteq$. But there is no standardized symbol for sets being incomparable, although some authors like to use $\sim$ for this purpose.

6 Recap

Definition 1. Let A and B be arbitrary sets. Then A is a **subset** of B ($A \subseteq B$) iff every member of A is a member of B. In this case, B is a **superset** of A ($B \supseteq A$).

Definition 2. For A and B arbitrary sets, A is a **proper subset** of B ($A \subsetneq B$) iff $A \subseteq B$ and there is a $b \in B$ such that $b \notin A$. Similarly, B is a **proper superset** of A ($B \supsetneq A$).

Definition 3. Let A and B be arbitrary sets. Then A and B are:

- **identical** iff $A \subseteq B$ and $B \subseteq A$ both hold,
- **disjoint** iff $A \cap B = \emptyset$,
- **incomparable** iff $A \nsubseteq B$ and $B \nsubseteq A$ and $A \cap B \neq \emptyset$.