

# The Powerset

## PREREQUISITES

- sets (notation, operations, cardinality)

Sometimes it is useful to take a set and consider all sets one could build from its elements. For example, each one of the following sets can be built from the elements of  $\{1, 2, 3\}$ :

1.  $\emptyset$
2.  $\{1\}$
3.  $\{2\}$
4.  $\{3\}$
5.  $\{1, 2\}$
6.  $\{1, 3\}$
7.  $\{2, 3\}$
8.  $\{1, 2, 3\}$

Note that each one of the sets in this list is a subset of  $\{1, 2, 3\}$ , and every subset of  $\{1, 2, 3\}$  is on this list. So the above is the list of all subsets of  $\{1, 2, 3\}$ . The set of all these sets is called the **powerset** of  $\{1, 2, 3\}$ .

**DEFINITION 1.** For  $A$  a set, the **powerset** of  $A$  is  $\wp(A) := \{S \mid S \subseteq A\}$ .

### EXAMPLE 1.

Suppose we have the set  $\{a\}$  and want to compute  $\wp(\{a\})$ . This can be done in many ways, but here is one that's easy for beginners.

1. First, we write down the set itself:  $\{a\}$
2. Next, we write down all proper subsets of the set. In this case, there's only one:  $\emptyset$
3. Finally, we put set brackets around the list of sets we wrote down:  $\wp(\{a\}, \emptyset)$ .

And that's it. As long as the set in question isn't too large, you can always follow this mechanical procedure when you aren't sure how to compute the set's powerset.

### EXERCISE 1.

For each one of the following sets, compute its powerset.

1.  $\{a, b\}$
2.  $\{a, b, c, d\}$

3.  $\{\{a\}\}$
4.  $\emptyset$
5.  $\{\emptyset\}$
6.  $\wp(\{\{a\}\})$
7.  $\wp(\wp(\emptyset))$

## 1 Powerset notation

There are many alternative notations for the powerset. A particularly common one is  $2^A$  as it highlights two interesting aspects of the powerset. Remember that the cardinality of a set  $A$  measures the number elements it contains, and we denote it by  $|A|$ . For example,  $|\{1, 2, 3\}| = 3$ . Now we can state a universal truth for the cardinality of powersets.

**THEOREM 2.** For every set  $A$  with  $|A| = n$ , it holds that  $|\wp(A)| = |2^A| = 2^{|A|} = 2^n$ .

This is witnessed by our example set  $\{1, 2, 3\}$ , the powerset of which has 8 members (see the list at the beginning of this unit).

### EXERCISE 2.

For each set  $A$  in the previous exercise, verify that  $|\wp(A)| = 2^{|A|}$ .

## 2 Recap

- The powerset of a set  $A$  is the set of all subsets of  $A$ , including  $A$  itself.
- For every finite set  $A$ , it holds that  $|\wp(A)| = 2^{|A|}$ .