Mathematical Methods in Linguistics

Prerequisite

- functions (basics)

Relations

Relations are similar to functions in that they establish connections between objects. But whereas a function associates only one output with every input, a relation is more flexible and allows connections to arbitrarily many elements.

Example 1.
The question *Is x a biological child of y* is a function because it maps any two x and y to either true or false, but never both. But if we slightly change the question to just *Name a biological child of y*, we are no longer dealing with a function because multiple answers are possible if y has more than one child. Instead, we can talk about the *biological child* relation R such that x is related to y via R iff x is a biological child of y.

1 Basic notation

Given some relation R, we write $x \ R \ y$ to indicate that R relates x to y. Note that $x \ R \ y$ does not imply that $y \ R \ x$ also holds, nor that it doesn't hold.

Example 2.
One relation you know very well is the "less than" relation $<$ over numbers. When we write $5 < 7$, we are saying that the relation $<$ relates 5 to 7. However, it is not the case that 7 is related to 5 via $<$, since $7 < 5$ does not hold.

Example 3.
Suppose *John* has exactly two siblings, *Mary* and *Sue*. Then the sibling relation establishes two connections for John: *John-Mary*, and *John-Sue*. Using S for the sibling relation, we have John S Mary and John S Sue.

In contrast to $<$, the sibling relation is **symmetric**. That is to say, if Mary is a sibling of John, then John is also a sibling of Mary. Therefore we also have Mary S John and Sue S John.

The relation is also **transitive**. If John is a sibling of Mary, and Sue is a sibling of John, then Sue is a sibling of Mary. So is also holds that Mary S Sue, and via symmetry we also get Sue S Mary.

Overall, we have $x \ S \ y$ where $x, y \in \{\text{John}, \text{Mary}, \text{Sue}\}$ and $x \neq y$.

Example 4.
The **substring relation** $\sqsubseteq$ holds between two strings u and v iff u is a substring of v. That is to say, $u \subseteq v$ iff there are $w, w' \in \Sigma^*$ such that $w \cdot u \cdot w' = v$. Note that for any given string u, there are infinitely many v that u is a substring of. Even if the alphabet contains only a, the string aa is a substring of aaa, $aaaa$, $aaaaa$, and so on, ad infinitum. It is also a substring of itself (in this case, $w = w' = \varepsilon$).

Example 5.
Relations can be defined over more complex objects like sets. An example of this is the subset relation $\subseteq$. The set $\{1\}$ is a subset of infinitely many other sets: $\{0, 1\}$, $\{1, 2\}$, $\{0, 1, 2\}$, $\{1, 3\}$, and so on.

Example 6.
Just like a function can take multiple arguments to return a single output, a relation can connect multiple elements. In the real world, the "jointly conceived" relation J would connect two individuals to their offspring. So the expression

$$\text{John}, \text{Mary}\ R\ \text{Sue}$$

encodes that John and Mary are the biological parents of Sue (let's just hope that those are not the same people as in the first example).

Example 7.
Here is an example of a very abstract relation. Consider the space of all possible functions from real numbers to real numbers — that's a lot of functions. Now let's define a boundedness relation B which relates function f to function g iff $f(x) \leq g(x)$ for every natural number x. Suppose, for instance, that $f(x) = x$ and $g(x) = x^2$. Then $f\ B\ g$, but not $g\ B\ f$.

Many functions aren't related via B at all. One example of this is $f(x) = -x + 1$ and $g(x) = x^2$. It is the case that $f(x) \leq g(x)$ for all $x \geq 1$, but $f(0) = 1 > 0 = g(0)$.

We will mostly be dealing with the special case of **binary relations** where exactly one element is related to some other element.

Example 8.
Almost all relations above are binary relations. The only exception is the "jointly conceived" relation J, which is a ternary relation as it relates three elements.

Given a binary relation R, $a\ R$ is the set of objects that a is related to. Similarly, $R\ b$ is the set of objects that are related to b.

$$a\ R := \{b \mid a\ R\ b\}$$

$$R\ b := \{a \mid a\ R\ b\}$$

Example 9.
Suppose that the parent-of relation P establishes the following relations between elements: John P Sue and Mary P Sue. Then John P= {Sue} and P Sue = {John, Mary}.

Exercise 1.
Let R be the relation that connects words to their parts of speech (N for nouns, V for verbs, A for adjectives, P for prepositions, D for determiners, and so on). List the following for English:

- export R
- apple R
- R P

2 Relations versus functions

Every function can be regarded as a relation.

Example 10.
Consider the function $f : \mathbb{N} \to \mathbb{N}$ with $x \mapsto 2x$. It is identical to the relation R with $x\ R\ y$ iff $y = 2x$.

The crucial difference between functions and relations is that functions are **right-unique relations**. That's a fancy way of saying that a function cannot provide more than one output, whereas a relation can unless it is right-unique. When we view a function f as a relation R, then it must hold for every a that $a\ R$ is either empty or contains exactly one element. Hence the term right-unique: if we look at the expression $a\ R\ b$, a is the left side and b the right side. If there cannot be more than one choice for b, then the right side of $a\ R\ b$ is uniquely determined.

The bottom line: every function is a relation, but not every relation is a function. If a is related to two elements or more (i.e. $|a\ R| \geq 2$), then R cannot be a function.

Exercise 2.
For each one of the following, say whether it is a function or just a relation.

- the parent-of relation (e.g. $j\ P\ m$ for "John is a parent of Mary")
- the parent-of relation in a world where the one-child policy is enforced globally
- the relation between a car's license plate and its owners
- the prefix relation, where u is a prefix of v iff there is some $w \in \Sigma^*$ such

that $v = u \cdot w$

Exercise 3.
Is the following statement true or false? Justify your answer.
Every relation R can be regarded as a function that maps x to $x\ R$.

3 Recap

- Relations are a generalization of functions in that a single input can be related to many distinct outputs.
- Given some relation R, we write $x\ R\ y$ to indicate that R relates x to y.
- We mostly focus on binary relations.
- For binary relations, we have the following notation:

$$a\ R := \{b \mid a\ R\ b\}$$

$$R\ b := \{a \mid a\ R\ b\}$$