

## Taking the closure of a relation

Let  $D$  be some fixed domain over which we define a relation  $R \subseteq D \times D$ . The  $P$ -closure of  $R$  is the smallest  $R'$  such that  $R \subseteq R' \subseteq D \times D$  and  $R$  has property  $P$ . For example, the reflexive, symmetric, transitive closure of  $R$  is the smallest superset  $R'$  that is reflexive, symmetric, and transitive.

### EXAMPLE 1.

Consider the relation  $R := \{\langle 1, 2 \rangle, \langle 2, 3 \rangle, \langle 3, 4 \rangle\}$ . We want to compute its transitive closure. We first add  $\langle 1, 3 \rangle$  and  $\langle 2, 4 \rangle$ , which we can construct from the existing pairs via transitivity. But the result is still not transitive. The relation now contains both  $\langle 1, 3 \rangle$  and  $\langle 3, 4 \rangle$ , so we also have to add  $\langle 1, 4 \rangle$ . At this point, the relation is transitive, so we do not add any more edges.

### EXAMPLE 2.

Consider the relation  $R := \{\langle 1, 2 \rangle, \langle 3, 2 \rangle\}$ . Its transitive closure is just  $R$ .

### EXAMPLE 3.

Consider once more the relation  $R := \{\langle 1, 2 \rangle, \langle 3, 2 \rangle\}$ . Now we want to compute its symmetric, transitive closure. As  $R$  is not symmetric, we have to add additional pairs, in this case  $\langle 2, 1 \rangle$  and  $\langle 2, 3 \rangle$ . But  $R' := \{\langle 1, 2 \rangle, \langle 2, 1 \rangle, \langle 2, 3 \rangle, \langle 3, 2 \rangle\}$  is not transitive. In order to make  $R'$  transitive, we have to add  $\langle 1, 3 \rangle$  and  $\langle 3, 1 \rangle$ , but also  $\langle 1, 1 \rangle$ ,  $\langle 2, 2 \rangle$ , and  $\langle 3, 3 \rangle$ . The resulting  $R'' := \{1, 2, 3\} \times \{1, 2, 3\}$  is transitive and symmetric, and even reflexive.

### EXERCISE 1.

Calculate all of the following, assuming that the relation's domain is  $D := \{a, b, c, d\}$ :

- the reflexive closure of  $\{\langle a, b \rangle, \langle b, a \rangle\}$
- the transitive closure of  $\{\langle a, b \rangle, \langle b, a \rangle\}$
- the transitive closure of  $\{\langle a, b \rangle, \langle b, c \rangle, \langle c, d \rangle, \langle d, a \rangle\}$
- the reflexive, symmetric, transitive closure of  $\{\langle a, b \rangle, \langle a, c \rangle, \langle d, c \rangle\}$
- the reflexive, symmetric, transitive closure of  $\emptyset$