

## Domains and co-domains

Every function has a **domain** and a **co-domain**. The domain is the set of objects from which its arguments can be drawn, and the **co-domain** is the set of objects from which outputs can be drawn. A function is undefined on any arguments that do not belong to its domain. One commonly writes  $f : D \rightarrow C$  to indicate that  $f$  is a function from domain  $D$  to co-domain  $C$ .

### EXAMPLE 1.

Consider the function  $f(x) = x + 1$ . This actually represents multiple functions depending on how one picks the domain and co-domain.

Suppose  $f : \mathbb{N} \rightarrow \mathbb{N}$ . Then  $f$  is a function from natural numbers  $(0, 1, 2, \dots)$  to natural numbers. In this case we have, for instance,  $f(0) = 1$  and  $f(500) = 501$ . However,  $f(-1)$  or  $f(2.5)$  would be undefined because  $-1$  and  $2.5$  are not natural numbers.

### EXAMPLE 2.

Now suppose that we have  $f(x) = x + 1$  with  $f : \mathbb{R} \rightarrow \mathbb{R}$ , i.e.  $f$  is a function from real numbers to real numbers. (We haven't encountered real numbers yet, just assume that  $\mathbb{R}$  includes pretty much number you have encountered in high school, e.g.  $1$ ,  $1.38702$ ,  $-\frac{5}{17}$ ,  $\sqrt{2}$ , and so on.) Now  $f(-1) = 0$  and  $f(2.5) = 3.5$ .

### EXERCISE 1.

Suppose that  $f$  is still defined by  $f(x) = x + 1$ , but we have  $f : \mathbb{R} \rightarrow \mathbb{N}$ . For each one of the following, say whether it is defined or undefined.

1.  $f(0)$
2.  $f(-1)$
3.  $f(-2)$
4.  $f(2.5)$

### EXAMPLE 3.

When a car wash is viewed as a function, its domain is the set of all cars (both dirty and clean), whereas the co-domain only contains clean cars.

### EXERCISE 2.

What would be the domain and co-domain of a broken car wash that fails to remove even the tiniest speck of dirt?

Since it is so important to know the domain and co-domain of a function, those are usually specified before the precise mapping from inputs to outputs is given.

**EXAMPLE 4.**

Let  $E$  be the set of English first names. Then the function  $f : E \rightarrow \{0, 1\}$  maps  $n$  to 1 iff  $n$  contains at least three syllables.

The mapping from arguments to outputs can be defined in various ways, e.g. in plain English, or as a formula like  $f(x) = \frac{(x+x^2+5)^{x+1}}{1000^x}$ . For very simple functions whose name was already mentioned, one often writes  $x \mapsto y$  instead of  $f(x) = y$ .

**EXAMPLE 5.**

Instead of  $f(x) = 5 \times x - 3$ , we may simply write  $x \mapsto 5 \times x - 3$ .

**Caution:** Notice the difference between  $\rightarrow$  and  $\mapsto$ . The first is used when specifying the domain and co-domain, whereas the latter indicates the concrete mapping from an argument to an output.