Cardinality

PREREQUISITES

- sets (notation, operations)
- functions (basic notation, domain terminology)

Sets can be compared based on how many elements they contain.

EXAMPLE 1.
The set $A := \{a, b, c\}$ contains exactly as many elements as the set $X := \{x, y, z\}$, namely 3. Each set has fewer members than $A \cup X$, which contains 6 elements, whereas $A \cap X$ has 0 members.

This is commonly called the **size** of a set, but the more accurate term is **cardinality**. The cardinality of a set A is denoted $|A|$. We say that $|A| \leq B$ iff there is a function f such that every element of A is mapped to some element of B and every element of B has at most one element of A mapped to it.

EXAMPLE 2.
Suppose that $A := \{a, b, c\}$ and $D := \{d, e, f, g\}$. Then $|A| \leq |D|$, as every element of A can be mapped to some distinct element of D. For instance, we could have a function with $a \mapsto f$, $b \mapsto d$, $c \mapsto g$.

In the other direction, $|D| \nleq |A|$. No matter how one maps the elements of D to members of A, at least two members of D will have to be mapped to the same element in A.

Clearly $|A| = |B|$ iff $|A| \leq |B|$ and $|B| \leq |A|$ are both true. But there is a more direct definition: $|A| = |B|$ iff there is a function f such that f maps every element of A to some element of B and every element of B has exactly one element of A mapped to it. We also say that there is a **bijection** between A and B, which is a technical term for a 1-to-1 correspondence between the elements of A and B.

EXAMPLE 3.
We already saw that $|A| = |X|$ in the previous example. A possible choice of f would be $a \mapsto x$, $b \mapsto y$, $c \mapsto z$.

EXAMPLE 4.
The sets $A := \{0, 1, 2\}$ and $B := \{2, 3\}$ obviously have distinct cardinality. The set A contains 3 elements, the set B only 2. But let us see how we get the same result via our mathematical definition.

Suppose we have some arbitrary function $f : A \to B$. If f is a bijection, then it must map every element of A to some element of B. But since there are three elements in A and only two in B, some element of B must be the output for at least two elements of A. But then f is not a bijection.

In the other direction, consider some arbitrary function $g : B \to A$. Since a function maps each input to at most one output, the two elements of B are mapped to at most two elements of A. But A has three elements, so one element of A cannot be an output for any element of B. Again we find that g cannot be bijection.

This exhausts all cases we need to consider, and we may conclude that no function from A to B, or the other way round, can be a bijection. Hence A and B must have distinct cardinality.

Exercise 1.
Show that $\left|\{0 \leq n < 10 \mid n \text{ is odd}\} = |\{0 \leq n < 10 \mid n \text{ is even}\}|\right|$.

For finite sets, our intuitive notion of size closely matches the technical term of cardinality. However, size and cardinality diverge once we look at infinite sets.

Example 5.
Consider the set $\mathbb{N} := \{0, 1, 2, ...\}$ of all natural numbers and the set $\mathbb{N}_+ := \{1, 2, ...\}$ of all positive natural numbers. Intuitively, $\mathbb{N}$ is larger than $\mathbb{N}_+$ because it contains all members of $\mathbb{N}_+$ as well as 0, which is not in $\mathbb{N}_+$. But the function $f : \mathbb{N} \to \mathbb{N}_+$ with $n \mapsto n + 1$ is a bijection. Hence $|\mathbb{N}| = |\mathbb{N}_+|$ even though intuitively the two sets have distinct size.

Exercise 2.
Show that the set of natural numbers has the same cardinality as the set of all even natural numbers.

In a later unit, we will see that our definition of cardinality entails that there are different "sizes" of infinity, and that we want one specific infinity size to talk about language.