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Research Problems from the BCC21

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ABSTRACT

A collection of open problems, mostly presented at the problem session of the 21st British Combinatorial Conference.

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The Research Problems section presents unsolved problems in discrete mathematics. In special issues from conferences, most problems come from the meeting and are collected by the guest editors. In regular issues, the Research Problems collect problems submitted individually.

Older problems are acceptable if they are not widely known and the exposition features a new partial result. Concise definitions and commentary (such as motivation or known partial results) should be provided to make the problems accessible and interesting to a broad cross-section of the readership. Problems are solicited from all readers; they should be presented in the style below, occupy at most one journal page, and be sent to

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Most problems below were presented at the problem session of the 21st British Combinatorial Conference. Some problems contributed after the session were added. The problems are ordered according to subject matter.

Several of the problems presented at the meeting have been solved and hence have been removed, resulting in some gaps in the BCC numbering. One solved problem remains: it is a problem on on-line sorting proposed by Nicolas Lichiardopol whose solution by Adam Philpotts and Rob Waters appears in this volume.

These problems were collected and edited by:

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Comments and questions of a technical nature about a particular problem should be sent to the correspondent for that problem. Other comments and information about partial or full solutions should be sent to Professor Cameron (for potential later updates).

PROBLEM 950. (BCC21.1) A generalization of Erdős–Ko–Rado

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Let X be a set of cardinality n, and let r be a divisor of n. Partition X into n/r subsets $S_1, \ldots, S_{n/r}$, each of cardinality r.

Let $N(n, r, k)$ be the largest cardinality of a family $\mathcal{A}$ of k-subsets of X with the property that for all $A, B \in \mathcal{A}$, there exists $i \in \{1, \ldots, n/r\}$ such that $A \cap S_i \neq \emptyset$ and $B \cap S_i \neq \emptyset$.

Conjecture 1: Let $\mathcal{A}^*$ denote the family of all k-sets A such that $A \cap S_1 \neq \emptyset$. If $k < d_r n(1 - \epsilon_r(n))$, where $d_r = 1 - 2^{-1/r}$ and $\epsilon_r(n) = o(1)$, then

$$N(n, r, k) = |\mathcal{A}^*| = \binom{n}{k} - \binom{n-r}{k}.$$

Comment: For $r = 1$, the truth of the conjecture is the well-known Erdős–Ko–Rado Theorem [1]. The conjecture has also been proved for $r = 2$ (Johnson–Talbot, [2]). If n/r is odd and $k > d_r n(1+\delta_r(n))$, where $\delta_r(n) = o(1)$, then $N(n, r, k) = |\mathcal{A}_m|$, where

$$\mathcal{A}_m = \{A\colon\ A \text{ meets more than half of the sets } S_i\}.$$

A similar construction holds when n/r is even and k is in this range.

A more speculative question is open even for $r = 2$:

Question 2: Is it true that $N(n, r, k) = \max\{|\mathcal{A}^*|, |\mathcal{A}_m|\}$ for all k?

References

[1] P. Erdős, C. Ko, and R. Rado, Intersection theorems for systems of finite sets, *Quart. J. Math. Oxford* (2) **12** (1961), 313–320.
[2] J. R. Johnson and J. Talbot, *G*-intersection theorems for matchings and other graphs, *Combin. Probab. Comput.* **17** (2008), 559–575.

PROBLEM 951. (BCC21.2) A-optimality of graphs

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Given positive integers v and b such that $b \geq v - 1$, a connected graph with v vertices and b edges is *A-optimal* if it minimizes the total variance among all graphs with v vertices and b edges. Regarding the graph as an electrical network with 1 Ω resistors on every edge, A-optimality is equivalent to minimizing the sum of the resistances of the network between all pairs of terminals.

Concerning leaves in A-optimal graphs, the following properties are known (see [1]):
(a) if $b \geq v(v-1)/2$ then A-optimal graphs have no leaves;
(b) for each $c \geq 0$, there is a threshold $N(c)$ such that if $b = v + c$ and $v \geq N(c)$, then A-optimal graphs have many leaves.

Question: Does there exist a threshold function f such that

1. if $b > f(v)$, then A-optimal graphs have no leaves, and
2. if $b \leq f(v)$, then A-optimal graphs have (many) leaves?

If such a function exists, find it (explicitly or asymptotically).

Reference

[1] R. A. Bailey, Designs for two-colour microarray experiments, *J. Roy. Statist. Soc. Ser. C* **56** (2007), 365–394.

PROBLEM 952. (BCC21.3) The row space of an adjacency matrix

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Let G be a graph with at least one edge, and let A be the adjacency matrix of G.

Question: Is it always true that there is a nonzero $\{0, 1\}$-vector in the row space of A (over the real numbers) that is not a row of A?

Comment: The answer is yes for graphs with at most 9 vertices and also for line graphs.

Reference

[1] S. Akbari, P. J. Cameron, and G.B. Khosrovshahi, The rank and signature of adjacency matrices (in preparation).

PROBLEM 953. (BCC21.4) Eulerian transversal to a triangle partition

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An *even graph* is a graph in which every vertex has even degree (it may contain isolated vertices). A 2-factor of a graph is a spanning 2-regular subgraph.

Problem: Let (G, F) be a graph with a given 2-factor F that consists only of triangles $\Delta_1, \ldots, \Delta_k$. Find a "natural" condition X such that, if G satisfies X, then G contains an induced even subgraph H that contains exactly one vertex of each triangle.

Comment: A possibility for X is the condition that for $i, j \in \{1, \ldots, k\}$, every vertex in Δ_i is adjacent to an even number of vertices in Δ_j. If this condition suffices, then every cubic graph with a dominating cycle has a nowhere-zero 6-flow.

Reference

[1] A. Hoffmann-Ostenhof, A counterexample to the bipartizing matching conjecture, *Discrete Math.* **307** (2007), 2723–2733.

PROBLEM 954. (BCC21.5) Saturated k-fold colorings

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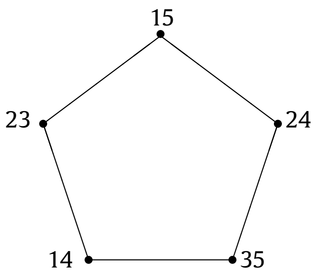
A *k-fold coloring* of a graph G is an assignment of k distinct colors to each vertex such that adjacent vertices are assigned disjoint sets of colors. The *k-fold chromatic number* $\chi^{(k)}(G)$ is the minimum number of colors in a k-fold coloring. Note that $\chi^{(1)}(G)$ equals $\chi(G)$, the chromatic number of G.

A k-fold coloring of G is *saturated* if every color occurs at $\alpha(G)$ vertices, where $\alpha(G)$ is the independence number of G. The figure shows a saturated 2-fold coloring of the 5-cycle C_5. Note that $\chi^{(2)}(C_5) = 5$ and $\chi^{(3)}(C_5) = 8$.



Question: Is it true that, if every k-fold coloring of G is saturated, then the following equation holds?

$$\chi^{(k+1)}(G) = \chi^{(k)}(G) + \chi(G).$$

Comment: The question is inspired by [1] but has not been published before.

Reference

[1] S. Stahl, *n*-tuple colorings and associated graphs, *J. Combin. Theory B* **20** (1976), 185–203.

PROBLEM 955. (BCC21.6) A list-colouring problem

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For a graph G, a k-list assignment L is a function that assigns to each vertex v of G a set L_v of k colors. An L-coloring of G is a proper coloring of G such that the color on each vertex comes from its list.

Question: Does there exist a graph G and a 2-list assignment L for G such that the lists all contain a common element and G has exactly three L-colorings?

Comment: It is shown in [1] that a graph G has a 2-list assignment L with exactly three L-colorings if and only if at least one of the blocks of G is not a complete graph with more than two vertices.

Reference

[1] S. Akbari, D. Kiani, F. Mohammadi, S. Moradi, Classification of $M_3(2)$-graphs (submitted).

PROBLEM 956. (BCC21.7) 3-cordial colourings of a graph

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This problem is an old conjecture of Hovey [1] on labellings, restated as a colouring problem.

Let G be a simple graph. Given a (not necessarily proper) 3-colouring of the vertices of G, induce a colouring of the edges as follows. If the endpoints of e have been given the same colour, then give this colour also to e; if they have different colours, then give the third colour to e. The colouring is 3-*cordial* if the numbers of vertices of any two colours differ by at most 1 and the numbers of edges of any two colours differ by at most 1.

Question: Is it true that every connected simple graph has a 3-cordial colouring?

Reference

[1] M. Hovey, A-cordial graphs, *Discrete Math.*, **93** (1991) 183–194.

PROBLEM 957. (BCC21.8) Switching edge-colourings

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An edge-colouring of a multigraph G is *optimal* if it is a proper edge-colouring using the fewest colours.

Conjecture (Vizing [1])**:** Given any proper edge-colouring of a multigraph G, it is possible to obtain an optimal edge-colouring of G by a sequence of colour switches along maximal bicoloured paths or bicoloured (even) cycles.

Reference

[1] V. G. Vizing, The chromatic class of a multigraph, *Kybernetika (Kiev)* **3** (1965), 29–39.

PROBLEM 958. (BCC21.9) Covering with matchings

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We define a decision problem COVER(m) for a fixed positive integer m. The input is a graph G and an integer k. The question is whether $E(G)$ can be expressed as the union of at most k matchings of size exactly m.

Question (R. Rizzi)**:** What is the least integer m (if one exists) such that COVER(m) is NP-hard?

Comment: The optimization version of the question seeks the least k such that $E(G)$ is a union of k matchings of size m, if this is possible. For $m = 1$, this number is $|E(G)|$. For $m = 2$, it can be checked in polynomial time whether such a union exists; if so, then it is easy to show (see [1]) that the minimum number of 2-matchings needed is

$$\max\{\chi'(G), \lceil |E(G)|/2 \rceil\},$$

where $\chi'(G)$ is the chromatic index of G.

Reference

[1] D. Cariolaro and H.-L. Fu, Covering graphs with matchings of fixed size, *Discrete Math.* (this issue).

PROBLEM 959. (BCC21.11) Counting toroidal graphs

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Let a_n be the number of simple graphs on the vertex set $\{1, \ldots, n\}$ which are embeddable on the fixed surface S. It is known [2] that $(a_n/n!)^{1/n} \to \gamma$ as $n \to \infty$, where γ is the planar growth constant, which was determined by Giménez and Noy to be 27.2268 . . . [1]. (It is the same constant for each surface S.)

Question: Is it true that na_{n-1}/a_n tends to a limit as $n \to \infty$?

Comment: If the limit exists, then it must be $1/\gamma$. The answer is "yes" for the sphere but is not known for any other surface. In particular, is it true for the torus?

References

[1] O. Giménez and M. Noy, Estimating the growth constant of labelled planar graphs, Mathematics and computer science. III, 133–139, *Trends Math.*, Birkhäuser, Basel, 2004.
[2] C. McDiarmid, Random graphs on surfaces, preprint 722, CRM, Barcelona, http://www.crm.es/, 2006.

PROBLEM 960. (BCC21.12) Orbital chromatic roots

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Let Γ be a graph, and let G be a group of automorphisms of Γ. The *orbital chromatic polynomial* $P_{\Gamma,G}(x)$ is the polynomial whose value at the positive integer k is the number of orbits of G on proper vertex colourings of Γ with k colours (see [1] for background). Note that $P_{\Gamma,1}(x)$ is the chromatic polynomial $\chi_\Gamma(x)$ of Γ, where 1 denotes the trivial group. Roots of orbital chromatic polynomials are investigated in [2].

Question 1: Is it true that the real roots of $P_{\Gamma,G}(x)$ are bounded above by the largest real root of the chromatic polynomial $\chi_\Gamma(x)$?

Question 2: Is it true that real roots of $P_{\Gamma,G}(x)$, in cases where G consists of even permutations of the vertex set of Γ, are dense in $[1, \infty)$?

Comment: There are no roots less than 1 except zero. Without the parity condition, real roots are dense in the real line. A result of Thomassen [3] shows that real roots are dense in $[\frac{32}{27}, \infty)$.

References

[1] P. J. Cameron, B. Jackson and J. Rudd, Orbit-counting polynomials for graphs and codes, *Discrete Math.* **308** (2008), 920–930.
[2] P. J. Cameron and K. K. Kayibi, Orbital chromatic and flow roots, *Combin. Probab. Comput.* **16** (2007), 401–407.
[3] C. Thomassen, The zero-free intervals for chromatic polynomials of graphs, *Combin. Probab. Comput.* **6** (1997), 497–506.

PROBLEM 961. (BCC21.13) Covers of the symmetric group

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This problem arises in studying the automorphism group of the countable "n-coloured random graph" [1]. An *involution* in a group is an element of order 2.

Question: Let n be a multiple of 8. Do there exist G and ϕ, where G is a finite group and ϕ is a homomorphism from G onto the symmetric group $\mathbb{S}_n$, such that no involution in G maps under ϕ to a fixed-point-free involution in H? If so, what is the order of the smallest such group?

Comment: If n is odd, then $\mathbb{S}_n$ has no fixed-point-free involutions, and so the smallest group is $\mathbb{S}_n$ itself. Hoffman and Humphreys [2] proved that if n is even but not divisible by 8, then there is such a group whose order is just twice that of $\mathbb{S}_n$.

References

[1] P. J. Cameron and S. Tarzi, On the automorphism group of the m-coloured random graph, (in preparation).
[2] P. N. Hoffman and J. F. Humphreys, *Projective Representations of the Symmetric Groups*, Clarendon Press, Oxford, 1992.

PROBLEM 962. (BCC21.14) Sudoku groups

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Consider a filled $n^2 \times n^2$ Sudoku puzzle. Choose a symbol, say 1, and delete all occurrences of the other symbols. Now for each collection of n of the $n \times n$ subsquares forming a row or column of the puzzle, form an $n \times n$ matrix with n ones by superimposing these matrices. It may happen that all these matrices are permutation matrices; further, it may happen that the distinct elements among the $2n$ permutation matrices obtained in this way form a group.

Example: Consider the following Sudoku puzzle:

1					2		3	5
				1			8	
			8		4	2		1
				7	3		1	
		1	6					
4	8		1		9			
3					1	6	2	
						1		3
9	1		7					

The positions of the 1s in the boxes in the rows give the three permutation matrices

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix};$$

the columns similarly give

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Together these make up the symmetric group $\mathbb{S}_3$. Readers may wish to complete the Sudoku puzzle; in the completed grid, every symbol gives rise to the group S_3 in this way.

Question: Which groups of order at most $2n$ can arise in this way?

Comment: This problem is connected with the concept of *symmetric Sudoku* devised by Robert Connelly and investigated in [1]. Using coding theory, when n is an odd prime power one can construct an $n^2 \times n^2$ symmetric Sudoku for which the row and column permutations comprise the group

$$\{x \mapsto \pm x + c : c \in F\},$$

where F is the finite field with n elements.

Reference

[1] R. A. Bailey, P. J. Cameron and R. Connelly, Sudoku, gerechte designs, resolutions, affine space, spreads, reguli, and Hamming codes, *American Math. Monthly* **115** (2008), 383–404.

PROBLEM 963. (BCC21.15) Quasi-magic Sudoku squares

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A 9×9 Sudoku square is said to be *quasi-magic* if each 3×3 subsquare has the property that all its rows, columns, and diagonals have sums in $\{13, 14, 15, 16, 17\}$. Several facts have been proved computationally about quasi-magic Sudoku squares.

Problem: Give direct proofs of the following computationally-verified statements.

1. The entry 5 cannot occur in a "box edge" (a cell in one of the positions (1, 2), (2, 1), (2, 3) or (3, 2) of a subsquare).
2. The entries 3 and 7 cannot occur more than once each in a "box centre" (a cell in position (2, 2) of a subsquare).
3. If 3 and 7 both occur in box centres, then their boxes must be in the same row or column.

For readers' interest, the proposer has supplied a quasi-magic Sudoku puzzle:

								7
								8
		8	1	3	9	6		
6								
1								

PROBLEM 964. (BCC21.18) An arithmetic permutation

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Let p be a prime congruent to 1 or 3 modulo 8. Let S and N be the sets of quadratic residues and non-residues modulo p. Consider the permutation π whose two-row form is the following array L (all entries modulo p):

$$\begin{aligned} L_{1,i} &= \begin{cases} 0 & \text{if } i = 0, \\ -i & \text{if } i \in S, \\ 2i & \text{if } i \in N, \end{cases} \\ L_{2,i} &= L_{1,i-1} + 1. \end{aligned}$$

For $p = 11$,

$$L = \begin{pmatrix} 0 & 10 & 4 & 8 & 7 & 6 & 1 & 3 & 5 & 2 & 9 \\ 10 & 1 & 0 & 5 & 9 & 8 & 7 & 2 & 4 & 6 & 3 \end{pmatrix},$$

and π is the single cycle $(0, 10, 1, 7, 9, 3, 2, 6, 8, 5, 4)$.

Conjecture: Always π is a single cycle of order p.

Comment: The conjecture can be formulated in terms of permutation polynomials. It has been verified for $p < 10^7$.

PROBLEM 965. (BCC21.19) Arranging the first $n + 1$ natural numbers

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Question: Let k and n be positive integers with $n/2 \leq k < n$. Must there exist a permutation $(a_0, \ldots, a_n)$ of $\{0, \ldots, n\}$ starting with $a_0 = 0$ such that the multiset of unsigned differences $|a_i - a_{i-1}|$ for $i \in \{1, \ldots, n\}$ has two copies of each integer from k down to $2k - n + 1$ and one copy of each positive integer below that?

Example: For $n = 7$ and $k = 5$, take $\mathbf{a} = (0, 5, 1, 6, 4, 7, 3, 2)$. Successive values of $|a_i - a_{i-1}|$ are 5, 4, 5, 2, 3, 4, 1, which when ordered become 5, 5, 4, 4, 3, 2, 1.

PROBLEM 966. (BCC21.21) On-line sorting

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Two integers m and n are given, with $1 \leq m \leq n$. A row of r empty boxes is available. There will be m distinct random integers from $\{1, \ldots, n\}$ announced. The task is to place each number in a box as it is announced so that, at the end, the numbers are stored in increasing order (with gaps allowed).

Question: What is the minimum value of r for which this task can be accomplished? What is the minimum such value when the m numbers announced are not necessarily distinct and must be put in non-decreasing order.

Comment: This problem has been solved by Adam Philpotts and Rob Waters. The solution appears in this volume [1].

Reference

[1] A. R. Philpotts and R. J. Waters, Solution to a problem of Nicolas Lichiardopol, *Discrete Math.* (this issue).