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Discrete Mathematics 293 (2005) 313–320
DISCRETE MATHEMATICS
www.elsevier.com/locate/disc

Research problems from the 19th British Combinatorial Conference

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The Research Problems section presents unsolved problems in discrete mathematics. In issues devoted to particular problems, these typically are problems collected by the guest editors. In regular issues, the Research Problems generally consist of problems submitted on an individual basis.

Older problems are acceptable if they are not as widely known as they should be or if the exposition features a new partial result. Concise definitions and commentary (such as motivation and known partial results) should be provided to make the problems accessible and interesting to a broad cross-section of the readership. Problems are solicited from all readers. Ideally, they should be presented in the style below, occupy at most one journal page, and be sent to

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Most of the problems in this issue were presented at the problem session of the 19th British Combinatorial Conference at the University College of North Wales, Bangor, 29 June–4 July, 2001. Some problems submitted later have been added. As is traditional for this conference, the problems are circularly ordered. These problems have been edited by Peter J. Cameron.

Comments and questions of a technical nature about a particular problem should be sent to the correspondent for that problem. Other comments and information about partial or full

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0012-365X/$ - see front matter
doi:10.1016/j.disc.2005.02.006

solutions should be sent to Professor Cameron (for possible later updates in BCC problems lists).

Several problems have been removed for editorial reasons or because they have been solved. For full details, conference participants should see the original list.

PROBLEM 417 (BCC19.1). Tight single-change covering designs

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A *tight single-change covering design* is an array such that

(a) each pair of its entries occurs in at least one column (block), so all pairs are "covered",
(b) when each block is formed from the previous one, only a single change is made, and
(c) when each new block is formed, the new entry occurs only with entries with which it has not previously been paired, so the arrangement is "tight".

The array below is such an array.

1	1	1	1	1	3	3	3	2	2
2	2	5	6	7	7	7	7	7	6
3	4	4	4	4	4	6	5	5	5

Tight single-change covering designs with block sizes 2 (trivial), 3, 4 and 5 are known and in Refs. [16,15].

Question. Do there exist tight single-change covering designs with block size greater than 5? (No such designs are currently known.)

PROBLEM 418 (BCC19.2). Three problems on partial Latin squares

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A *partial Latin square* of order n is an $n \times n$ array using at most n symbols such that each symbol appears at most once in each row and each column. An *intercalate* is a partial Latin square having four entries in two rows and two columns and with two symbols.

A partial Latin square is *completable* to a Latin square if entries can be added so that each symbol appears exactly once in each row and column. A *critical set* in a Latin square L is a partial Latin square contained in L and in no other Latin square of the same size.

There has long been interest in which partial Latin squares are completable (Lindner [13] is an early reference); here we raise questions related to the number of completions.

Question 1. Let P be a completable partial Latin square P of order n with $n \geqslant 6$. If P has fewer than $\lfloor n^2/4 \rfloor$ entries, then must there be an intercalate I such that adding I to P yields a completable partial Latin square?

Question 2. Let P be a partial Latin square of order n with fewer than $\lfloor n^2/4 \rfloor$ entries. Must the number of completions of P to Latin squares be even?

Question 3. Let $\ell(n)$ be the maximum size of a critical set over all Latin squares of order n. Does ℓ satisfy

$$\max\{A(n), B(n), C(n), D(n)\} \leqslant \ell(n) \leqslant E(n),$$

where

$$\begin{aligned}
&A(n) = (n^2 - n)/2,\\
&B(n) = \max\{(3(pq)^2 - pq^2 - qp^2 - pq)/4\colon\ p, q \in \mathbb{N},\ pq = n\},\\
&C(n) = |\{(i, j) : 0 \leqslant i, j < n, i\&j \neq 0\}| \text{ definition of } i\&j \text{ below}),\\
&D(n) = \max\{\ell(p)q^2 + \ell(q)p^2 - \ell(p)\ell(q)\colon\ p, q \in \mathbb{N},\ pq = n\},\\
&E(n) = n^2 - n^{\log 3/\log 2}?
\end{aligned}$$

Comments. Note that $B(n) \leqslant D(n)$ for all n; B appears in the lower bound in case D fails to be a lower bound. The function & in the definition of $C(n)$ is "bitwise and": write the arguments i and j in base 2, then multiply the digits in each position and interpret the result as the base 2 representation of $i\&j$. The conjectured lower bound $C(n)$ is Sequence A080572 in the *Online Encyclopedia of Integer Sequences* [19]; see also [10].

PROBLEM 419 (BCC19.3). Designs with three eigenvalues

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A *binary equireplicate proper incomplete-block design* with v treatments is a set of blocks such that each block consists of a fixed number k of distinct treatments (with $k < v$) and each treatment occurs in a fixed number r of blocks. The *Levi graph* of such a design has as vertices the treatments and blocks, a block being adjacent to all the blocks it contains. Such a design is *connected* if its Levi graph is connected. It is *balanced* if any two treatments lie in a constant number of blocks. Its *concurrence matrix* is the $v \times v$ matrix with rows and columns indexed by treatments, whose (i, j) entry is the number of blocks in which i and j both occur.

Let D be a connected binary equireplicate proper incomplete-block design, and let Λ be its concurrence matrix. Connectivity ensures that the maximum eigenvalue rk has multiplicity one. If D is balanced, then Λ has two distinct eigenvalues: rk and one "interesting" eigenvalue. We therefore seek designs with two "interesting" eigenvalues.

Question. When does the concurrence matrix of a connected binary equireplicate proper incomplete block design have exactly three distinct eigenvalues? Is there a nice characterization of designs with this property, for example by a finite number of infinite families and a finite number of exceptions?

Comments. The case where one of the nontrivial eigenvalues is zero has long been recognized as important: see [4] for the statistical motivation, [3] for the combinatorial motivation, and [8] for the link with optimal designs. Updates from these three points of view are in [5, Section 7.3; 9,1], respectively. Some information on the case when both interesting eigenvalues are non-zero is given in [5, Section 8.3].

The matrix Λ has exactly three distinct eigenvalues at least when D lies in one of the following cases (see [5] for the definitions of these classes and proof of the statement): (1) duals of non-symmetric BIBDs, (2) partially balanced IBDs with two associate classes, (3) duals of symmetric partially balanced IBDs with two associate classes, (4) duals of non-symmetric partially balanced IBDs with two associate classes and having an efficiency factor 1 (if not balanced).

The original form of the question asked whether there are any such designs not in the four classes listed. Both Caliński and Kageyama [5] and van Dam and Spence [9] answered this in the affirmative. Given a design with three eigenvalues one of which is zero, another can be made by replacing each treatment by a constant number of new treatments. Other examples given in both references are partially balanced incomplete-block designs with three associate classes in which two of the four eigenvalues happen to coincide.

PROBLEM 420 (BCC19.4). Forcing domination number of products of cycles

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A *dominating set* in a graph G is a set S of vertices such that every vertex not in S is adjacent to a vertex in S. A *forcing domination set* is a set of vertices which is contained in a unique dominating set of minimum cardinality. See [7].

Conjecture. If G is the Cartesian product of k copies of the odd cycle C_{2k+1}, then the smallest forcing domination set in G has cardinality k.

Comments. The conjecture is known to be true if $k = 2$ (see [7]). For $k = 3$, a forcing domination set of size 3 was found by Ms. Jahanbakht, while R. Bean claims to have one of size 2.

A prize of 1 000 000 Rials (about Ł300 as of July 2004) is offered for settling the problem.

PROBLEM 421 (BCC19.6). Cospectral graphs

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The *characteristic polynomial* of a graph is the characteristic polynomial of its adjacency matrix, and the *spectrum* of the graph is the multiset of zeros of this polynomial.

Question. Does almost every graph (a proportion tending to 1 as the number of vertices tends to ∞) have a cospectral mate (another graph with the same characteristic polynomial?) The same question can be asked for other graph polynomials such as the chromatic polynomial.

Comments. (by the editors) A recent paper by Haemers and Spence [11] suggests that the answer to the problem may be negative. It is shown that the proportion of graphs having a cospectral mate increases for $n \leqslant 10$ but is smaller for $n = 11$ than for $n = 10$. On the other hand, Schwenk [18] showed that almost every tree has a spectral mate.

PROBLEM 422 (BCC19.7). An arithmetic graph

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Let m be an integer greater than 1. Let G_m be the graph whose vertex set is the set $\mathbb{Z}$ of integers and whose edges are the pairs $\{x, y\}$ such that x and y have difference or ratio m. A *one-way Hamiltonian path* in an infinite graph is a sequence $(x_i\colon i \geqslant 0)$ that contains each vertex exactly once and has the property that $x_i x_{i+1}$ is an edge for all $i \geqslant 0$). A *two-way Hamiltonian path* is a sequence $(x_i\colon i \in \mathbb{Z})$ that contains each vertex exactly once and has the property that x_i, x_{i+1} is an edge for all $i \in \mathbb{Z}$.

The question was raised of whether G_m contains such paths. Russell [17] showed that G_m, in general, does not have a one-way Hamiltonian path, but it does have a two-way Hamiltonian path. The following question remains.

Question. Can the vertices of G_m be covered by disjoint finite cycles?

PROBLEM 423 (BCC19.8). Directed paths in the cube

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Let Q_n denote the graph of the n-cube with vertex set $\{0, 1\}^n$ and edge set consisting of all pairs of vertices differing in one coordinate. Bollobás and Leader [2] showed that, if A and B are disjoint vertex sets in Q_n with $|A| = |B| = 2^k$, then there exist $2^k(n - k)$ edge-disjoint paths from A to B. This is clearly best possible, since if A is a subcube, then there are just $2^k(n - k)$ edges from A to its complement.

Problem. Regarding Q_n as a poset (orienting each edge toward the endpoint with more 1s), suppose that A is a down-set and B is an up-set. Must there be $2^k(n-k)$ edge-disjoint paths directed upwards from A to B?

PROBLEM 424 (BCC19.10). Sum-free sets in the square

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A *sum-free set* in the plane is a set of points not containing two points and their vector sum. One can ask for the largest sum-free set contained in a given region.

Conjecture. The size of the largest sum-free set in the square $\{1, \dots, n\} \times \{1, \dots, n\}$ is $cn^2 + \mathrm{O}(n)$. If so, find the constant c.

Comment. Upper and lower bounds for c are $\mathrm{e}^{-1/2}$ and $3/5$, respectively, see [6]. These are fairly close, since $\mathrm{e}^{-1/2} = 0.6065\dots$ and $3/5 = 0.6$.

PROBLEM 425 (BCC19.11). The second largest maximal k-arc

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Let $\mathrm{PG}(2, q)$ be the Desarguesian projective plane over a finite field of *square* order q. A *k-arc* in a projective plane is a set of k points with no three lying in a single line. Let $m(2, q)$ be the largest size of a (maximal) k-arc in $\mathrm{PG}(2, q)$, and let $m'(2, q)$ be the second largest size of a maximal k-arc. Background motivation and the result below for even q can be found in [12, Chapter 10].

Conjecture. If $q > 9$, then $m'(2, q) = q - \sqrt{q} + 1$ for $q > 9$.

Comments. It is known that $m(2, q) = q + 1$ when q is odd and $m(2, q) = q + 2$ when q is even. It is known also that $m'(2, q) = q - \sqrt{q} + 1$ when q is an even power of 2. The problem is to prove this also for odd prime power squares.

PROBLEM 426 (BCC19.13). Disjoint intersections in intersecting families

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Let $X = \{1, 2, \ldots, n\}$, and let $\binom{X}{r}$ be the family of all r-element subsets of X. An *intersecting family* is a family of sets such that any two members have a common element. Given an intersecting family $\mathscr{A}$ contained in $\binom{X}{r}$, define

$$\mathscr{A}\langle 1\rangle = \{x \in X\colon \exists A, B \in \mathscr{A} \text{ with } A \cap B = \{x\}\}.$$

Furthermore, let

$$\alpha_r = \max_{n \geqslant r} \left\{ |\mathscr{A}\langle 1\rangle|\colon \mathscr{A} \text{ is an intersecting family in } \binom{X}{r} \right\}.$$

Lovász [14] proved for $r \geqslant 1$ that α_r in fact exists. The best known bounds for α_r are due to Tuza [20].

For $1 \leqslant k \leqslant r \leqslant n$, let

$$\mathscr{A}\langle k\rangle = \left\{ C \in \binom{X}{r}\colon \exists A, B \in \mathscr{A} \text{ with } A \cap B = C \right\}.$$

Conjecture. If $A_1, A_2, \ldots, A_m \in \mathscr{A}\langle k\rangle$ are pairwise disjoint, then $m \leqslant \alpha_{r-k+1}$.

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