Linear Algebra: Homework #1

Due on August 28, 2019

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Problem 1 (1.1#1)

Solve the following system by using elementary row operations on the equations

$$x_1 + 5x_2 = 7$$
$$-2x_1 - 7x_2 = -5$$

Solution:

The system of equations can be represented by the following agumented matrices

$$\begin{aligned}
\begin{bmatrix} 1 & 5 & 7 \\ -2 & -7 & -5 \end{bmatrix} &\equiv \begin{bmatrix} 1 & 5 & 7 \\ 0 & 3 & 9 \end{bmatrix} && R2 = 2 * R1 + R2 \\
&\equiv \begin{bmatrix} 1 & 5 & 7 \\ 0 & 1 & 3 \end{bmatrix} && R2 = \frac{1}{3} * R2 \\
&\equiv \begin{bmatrix} 1 & 0 & -8 \\ 0 & 1 & 3 \end{bmatrix} && R1 = -5 * R2 + R1 \\
&\equiv \begin{Bmatrix} x_1 = -8 \\ x_2 = 3 \end{Bmatrix}
\end{aligned}$$

This is equivalent to the point $(-8, 3)$, which is the solution to the system of equations.

Problem 2 (1.1#3)

Find the point of intersection of the lines $x_1 - 5x_2 = 1$ and $3x_1 - 7x_2 = 5$ by using elementary row operations on the equations

Solution:

The system of equations can be represented by the following agumented matrices

$$\begin{aligned}
\begin{bmatrix} 1 & 5 & 7 \\ 1 & -2 & -2 \end{bmatrix} &\equiv \begin{bmatrix} 1 & 5 & 7 \\ 0 & -7 & -9 \end{bmatrix} && R2 = -1 * R1 + R2 \\
&\equiv \begin{bmatrix} 1 & 5 & 7 \\ 0 & 1 & \frac{9}{7} \end{bmatrix} && R2 = \frac{1}{7} * R2 \\
&\equiv \begin{bmatrix} 1 & 0 & \frac{4}{7} \\ 0 & 1 & \frac{9}{7} \end{bmatrix} && R1 = -5 * R2 + R1 \\
&\equiv \begin{Bmatrix} x_1 = \frac{4}{7} \\ x_2 = \frac{9}{7} \end{Bmatrix}
\end{aligned}$$

This is equivalent to the point $(\frac{4}{7}, \frac{9}{7})$, which is the point of intersection.

Problem 3 (1.1#7)

The agumented matrix of a linear system has been reduced by row operations to the form shown. In each case, continue the appropriate row operations and describe the solution set of the original system.

$$\begin{bmatrix} 1 & 7 & 3 & -4 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & -2 \end{bmatrix}$$

Solution:

There is no solution to the original system. Row 3 shows that this is a inconsistent system ($0 \neq 1$). This means that there is no solution to the system.

Problem 4 (1.1#8)

The agumented matrix of a linear system has been reduced by row operations to the form shown. In each case, continue the appropriate row operations and describe the solution set of the original system.

$$\begin{bmatrix} 1 & -4 & 9 & 0 \\ 0 & 1 & 7 & 0 \\ 0 & 0 & 2 & 0 \end{bmatrix}$$

Solution:

$$\begin{bmatrix} 1 & -4 & 9 & 0 \\ 0 & 1 & 7 & 0 \\ 0 & 0 & 2 & 0 \end{bmatrix} \equiv \begin{bmatrix} 1 & -4 & 9 & 0 \\ 0 & 1 & 7 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \qquad R3 = \frac{1}{2} * R3$$

$$\equiv \begin{bmatrix} 1 & -4 & 0 & 0 \\ 0 & 1 & 7 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \qquad R1 = -9 * R3 + R1$$

$$\equiv \begin{bmatrix} 1 & -4 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \qquad R2 = -7 * R3 + R2$$

$$\equiv \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \qquad R1 = 4 * R2 + R1$$

$$\equiv \left\{ \begin{matrix} x_1 = 0 \\ x_2 = 0 \\ x_3 = 0 \end{matrix} \right\}$$

The soltuion to the original system is the point $(0, 0, 0)$.

Problem 5 (1.1#11)

Solve the following system

$$\begin{aligned} x_2 + 4x_3 &= -5 \\ x_1 + 3x_2 + 5x_3 &= -2 \\ 3x_1 + 7x_2 + 7x_2 &= 6 \end{aligned}$$

Solution:

The system can be represented by the following agumented matrices

$$\begin{aligned} \begin{bmatrix} 0 & 1 & 4 & 5 \\ 1 & 3 & 5 & -2 \\ 3 & 7 & 7 & 6 \end{bmatrix} &\equiv \begin{bmatrix} 1 & 3 & 5 & -2 \\ 0 & 1 & 4 & -5 \\ 3 & 7 & 7 & 6 \end{bmatrix} && \text{Swap } R2 \text{ and } R1 \\ &\equiv \begin{bmatrix} 1 & 3 & 5 & -2 \\ 0 & 1 & 4 & -5 \\ 0 & -2 & -8 & 12 \end{bmatrix} && R3 = -3 * R1 + R3 \\ &\equiv \begin{bmatrix} 1 & 3 & 5 & -2 \\ 0 & 1 & 4 & -5 \\ 0 & 0 & 0 & 2 \end{bmatrix} && R3 = 2 * R2 + R3 \end{aligned}$$

There is no solution to this system, as evident by row 3, which shows that the system is inconsistent ($0 \neq 2$).

Problem 6 (1.1#18)

Do the three planes $x_1 + 2x_2 + x_3 = 4$, $x_2 - x_3 = 1$, and $x_1 + 3x_2 = 0$ have at least one common point of intersection? Explain.

Solution:

The equations of the three planes can be represented by the following augmented matrices

$$\begin{aligned} \begin{bmatrix} 1 & 2 & 1 & 4 \\ 0 & 1 & -1 & 1 \\ 1 & 3 & 0 & 0 \end{bmatrix} &\equiv \begin{bmatrix} 1 & 2 & 1 & 4 \\ 0 & 1 & -1 & 1 \\ 0 & 1 & -1 & -4 \end{bmatrix} && R3 = -1 * R1 + R3 \\ &\equiv \begin{bmatrix} 1 & 2 & 1 & 4 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & -5 \end{bmatrix} && R3 = -1 * R2 + R3 \end{aligned}$$

There are no common points of intersection between the three planes because they produce an inconsistent system, as evident by row 3 of the augmented matrix ($0 \neq -5$).

Problem 7 (1.1#20)

Determine the value(s) of h such that the matrix is the agumented matrix of a consistent linear system.

$$\begin{bmatrix} 1 & h & -3 \\ -2 & 4 & 6 \end{bmatrix}$$

Solution:

$$\begin{bmatrix} 1 & h & -3 \\ -2 & 4 & 6 \end{bmatrix} \equiv \begin{bmatrix} 1 & h & -3 \\ 0 & 4+2h & 0 \end{bmatrix} \qquad R2 = -1 * R1 + R2$$

The system is consistent for all possible values of h.

Problem 8 (1.1#23)

For each statement determine if it is True or False, and *justify* your answer. (If true, give the approximate location where a similar statement appears, or refer to a definition or theorem. If false, give the location of a statement that has been quoted or used incorrectly, or cite an example that shows the statement is not true in all cases.)

(a) Every elementary row operation is reversible.

(b) A 5×6 matrix has six rows.

(c) The solution set of a linear system involving variables $x_1, \ldots, x_n$ is a list of numbers $(s_1, \ldots, s_n)$ that make each equation in the system a true statement when the values $s_1, \ldots, s_n$ are substituded for $x_1, \ldots, x_n$, respectively.

(d) Two fundamental questions about a linear system involve existence and uniqueness.

Solution:

(a) This statemet is <u>true</u>, as stated on pg. 6 "It is important to note that row operations are *reversible*".

(b) This statement is <u>false</u>, a 5×6 matrix has 5 rows and 6 columns. According to pg. 4 "an $m \times n$ matrix is a rectangular array of numbers with m rows and $ncomulmns$."

(c) This statement is <u>true</u>, according to pg. 3, where it is stated "A **solution** of the system is a list $(s_1, \ldots, s_n)$ of numbers that makes each equation a true statement when the values $s_1, \ldots, s_n$ are substituded for $x_1, \ldots, x_n$, respectively."

(d) This statement is <u>true</u>, according to pg. 7, the two fundamental questions about a linear system are "is the system consistent" and "is the system unique".

Problem 9 (1.1#27)

Suppose the system below is consistent for all possible values of f and g. What can you say about the coefficients c and d? Justify your answer.

$$x_1 + 3x_2 = f$$
$$cx_1 + dx_2 = g$$

Solution:

The system can be represented by the following augmented matrices

$$\begin{bmatrix} 1 & 3 & f \\ c & d & g \end{bmatrix} \equiv \begin{bmatrix} 1 & 3 & f \\ 0 & d-3c & g-fc \end{bmatrix} \qquad R2 = -c * R1 + R2$$

In order for the system to be consistent for all possible values of f and g, $d - 3c \neq 0$. In other words, so long as the values of c and d satisfy $d - 3c \neq 0$, the system will be consistent.

Problem 10 (1.2#1)

Determine which matrices are in reduced echelon form and which others are only in echelon form.

(a) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

(d) $\begin{bmatrix} 1 & 1 & 0 & 1 & 1 \\ 0 & 2 & 0 & 2 & 2 \\ 0 & 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 0 & 4 \end{bmatrix}$

Solution:

(a) RREF

(b) RREF

(c) Nothing

(d) REF

Problem 11 (1.2#3)

Row reduce the matrix to reduced row echelon form. Circle the pivot positions in the final matrix and in the original matrix, and list the pivot columns.

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 4 & 5 & 6 & 7 \\ 6 & 7 & 8 & 9 \end{bmatrix}$$

Solution:

$$\begin{aligned}
\begin{bmatrix} 1 & 2 & 3 & 4 \\ 4 & 5 & 6 & 7 \\ 6 & 7 & 8 & 9 \end{bmatrix} &\equiv \begin{bmatrix} 1 & 2 & 3 & 4 \\ 4 & 5 & 6 & 7 \\ 0 & -5 & -10 & -15 \end{bmatrix} && R3 = -6 * R1 + R3 \\
&\equiv \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -3 & -6 & -9 \\ 0 & -5 & -10 & -15 \end{bmatrix} && R2 = -4 * R1 + R2 \\
&\equiv \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & -5 & -10 & -15 \end{bmatrix} && R2 = -\frac{1}{3} R2 \\
&\equiv \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} && R3 = 5 * R2 + R3 \\
&\equiv \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix} && R3 = \frac{1}{2} R3 \\
&\equiv \begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}
\end{aligned}$$

The pivot columns are C1, C2, and C4.

Problem 12 (1.2#7)

Find the general solution to the system whose augmented matrix is given below

$$\begin{bmatrix} 1 & 3 & 4 & 7 \\ 3 & 9 & 7 & 6 \end{bmatrix}$$

Solution:

$$\begin{aligned}
\begin{bmatrix} 1 & 3 & 4 & 7 \\ 3 & 9 & 7 & 6 \end{bmatrix} &\equiv \begin{bmatrix} 1 & 3 & 4 & 7 \\ 0 & 0 & -5 & -15 \end{bmatrix} && R2 = -3 * R1 + R2 \\
&\equiv \begin{bmatrix} 1 & 3 & 4 & 7 \\ 0 & 0 & 1 & 3 \end{bmatrix} && R2 = -\frac{1}{3} R2 \\
&\equiv \begin{bmatrix} 1 & 3 & 0 & -5 \\ 0 & 0 & 1 & 3 \end{bmatrix} && R1 = -4 * R2 + R1 \\
&\equiv \begin{Bmatrix} x_1 = -5 - 3x_2 \\ x_3 = 3 \end{Bmatrix}
\end{aligned}$$

The general solution to is all points that satisfy the system $\begin{Bmatrix} x_1 = -5 - 3x_2 \\ x_3 = 3 \end{Bmatrix}$, in other words all the points $(-5 - 3u, u, 3)$ for all possible values of u.

Problem 13 (1.2#11)

Find the general solution to the system whose augmented matrix is given below

$$\begin{bmatrix} 3 & -4 & 2 & 0 \\ -9 & 12 & -6 & 0 \\ -6 & 8 & -4 & 0 \end{bmatrix}$$

Solution:

$$\begin{aligned}
\begin{bmatrix} 3 & -4 & 2 & 0 \\ -9 & 12 & -6 & 0 \\ -6 & 8 & -4 & 0 \end{bmatrix} &\equiv \begin{bmatrix} 3 & -4 & 2 & 0 \\ -9 & 12 & -6 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} && R3 = 2 * R1 + R3 \\
&\equiv \begin{bmatrix} 3 & -4 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} && R2 = 3 * R1 + R2 \\
&\equiv \left\{ 3x_1 - 4x_2 + 2x_3 = 0 \right\} \\
&\equiv \left\{ x_1 = \tfrac{4}{3}x_2 - \tfrac{2}{3}x_3 \right\}
\end{aligned}$$

The general solution to is all points that satisfy the system $\left\{x_1 = \frac{4}{3}x_2 - \frac{2}{3}x_3\right\}$, in other words all the points $(\frac{4}{3}u - \frac{2}{3}v, u, v)$ for all possible values of u and v.

Problem 14 (1.2#19)

Choose h and k such that the system hase (a) no solution, (b) a unique solution, and (c) many solutions. Give separate answers for each part.

$$x_1 + hx_2 = 2$$
$$4x_1 + 8x_2 = k$$

Solution:

$$\begin{bmatrix} 1 & h & 2 \\ 4 & 8 & k \end{bmatrix} \equiv \begin{bmatrix} 1 & h & 2 \\ 0 & 8-4h & k-8 \end{bmatrix} \qquad R2 = -4 * R1 + R2$$

(a) If $h = 2$ and $k = 1$, then the system becomes inconsistent and there is no solution

(b) If $h = \frac{7}{4}$ and $k = 7$, then the system has the unique solution $(\frac{1}{4}, 1)$.

(c) If $h = 2$ and $k = 8$, then the system has infinite solutions of the form $x_1 = 2 - 2x_2$.

Problem 15 (1.2#24)

Suppose a system of linear equations has a 3×5 *augmented* matrix whose fifth column is a pivot column. Is the system consistent? Why or why not?

Solution:

If the fifth column is a pivot column, then there is a pivot position in the third row and fifth column. This means that the last row is $\begin{bmatrix} 0 & 0 & 0 & 0 & 1 \end{bmatrix}$. This augmented matrix represents a inconsistent systme.

Problem 16 (1.2#28)

What would you have to know about the pivot columns in an augmented matrix in order to know that the system is consistent and has a unique solution?

Solution:

If the last column of an augmented matrix is a pivot column, then the system is inconsistent. For example:

$$\begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

If every column except for the last is a pivot column then the system has a unique solution. For example:

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

Problem 17 (1.2#33)

Find the interpolating polynomial $p(t) = a_0 + a_1 t + a_2 t^2$ for the data $(1, 12)$, $(2, 15)$, $(3, 16)$. That is find a_0, a_1, and a_2 such that

$$\begin{aligned} a_0 + a_1(1) + a_2(1)^2 &= 12 \\ a_0 + a_1(2) + a_2(2)^2 &= 15 \\ a_0 + a_1(3) + a_2(3)^2 &= 16 \end{aligned}$$

Solution:

The system, $\left\{ \begin{aligned} a_0 + a_1 + a_2 &= 12 \\ a_0 + 2a_1 + 4a_2 &= 15 \\ a_0 + 3a_1 + 9a_2 &= 16 \end{aligned} \right\}$ can be represented by the following augmented matrices

$$\begin{bmatrix}1 & 1 & 1 & 12\\1 & 2 & 4 & 15\\1 & 3 & 9 & 16\end{bmatrix} \equiv \begin{bmatrix}1 & 1 & 1 & 12\\1 & 2 & 4 & 15\\0 & 2 & 8 & 4\end{bmatrix} \qquad R3 = -1 * R1 + R3$$

$$\equiv \begin{bmatrix}1 & 1 & 1 & 12\\0 & 1 & 3 & 3\\0 & 2 & 8 & 4\end{bmatrix} \qquad R2 = -1 * R1 + R2$$

$$\equiv \begin{bmatrix}1 & 1 & 1 & 12\\0 & 1 & 3 & 3\\0 & 0 & 2 & -2\end{bmatrix} \qquad R3 = -2 * R2 + R3$$

$$\equiv \begin{bmatrix}1 & 1 & 1 & 12\\0 & 1 & 3 & 3\\0 & 0 & 1 & -1\end{bmatrix} \qquad R3 = \frac{1}{2} * R3$$

$$\equiv \begin{bmatrix}1 & 1 & 1 & 12\\0 & 1 & 0 & 6\\0 & 0 & 1 & -1\end{bmatrix} \qquad R2 = -3 * R3 + R2$$

$$\equiv \begin{bmatrix}1 & 1 & 0 & 13\\0 & 1 & 0 & 6\\0 & 0 & 1 & -1\end{bmatrix} \qquad R1 = -1 * R3 + R1$$

$$\equiv \begin{bmatrix}1 & 0 & 0 & 7\\0 & 1 & 0 & 6\\0 & 0 & 1 & -1\end{bmatrix} \qquad R1 = -1 * R2 + R1$$

$$\equiv \begin{Bmatrix}a_0 = 7\\a_1 = 6\\a_2 = -1\end{Bmatrix}$$

The interpolating polynomial $p(t) = a_0 + a_1 t + a_2 t^2$ for the data $(1, 12)$, $(2, 15)$, $(3, 16)$, is $p(t) = 7 + 6t - t^2$.

Problem 18 (1.3#1)

Compute $\boldsymbol{u} + \boldsymbol{v}$ and $\boldsymbol{u} - 2\boldsymbol{v}$

$$\boldsymbol{u} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}, \boldsymbol{u} = \begin{bmatrix} -3 \\ -1 \end{bmatrix}$$

Solution:

$$\boldsymbol{u} + \boldsymbol{v} = \begin{bmatrix} -1 \\ 2 \end{bmatrix} + \begin{bmatrix} -3 \\ -1 \end{bmatrix} = \begin{bmatrix} -4 \\ 1 \end{bmatrix}$$

$$\boldsymbol{u} - 2\boldsymbol{v} = \begin{bmatrix} -1 \\ 2 \end{bmatrix} - 2\begin{bmatrix} -3 \\ -1 \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \end{bmatrix}$$

Problem 19 (1.3#5)

Write a system of equations that is equivalent to the given vector equation.

$$x_1 \begin{bmatrix} 6 \\ -1 \\ 5 \end{bmatrix} + x_2 \begin{bmatrix} -3 \\ 4 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -7 \\ -5 \end{bmatrix}$$

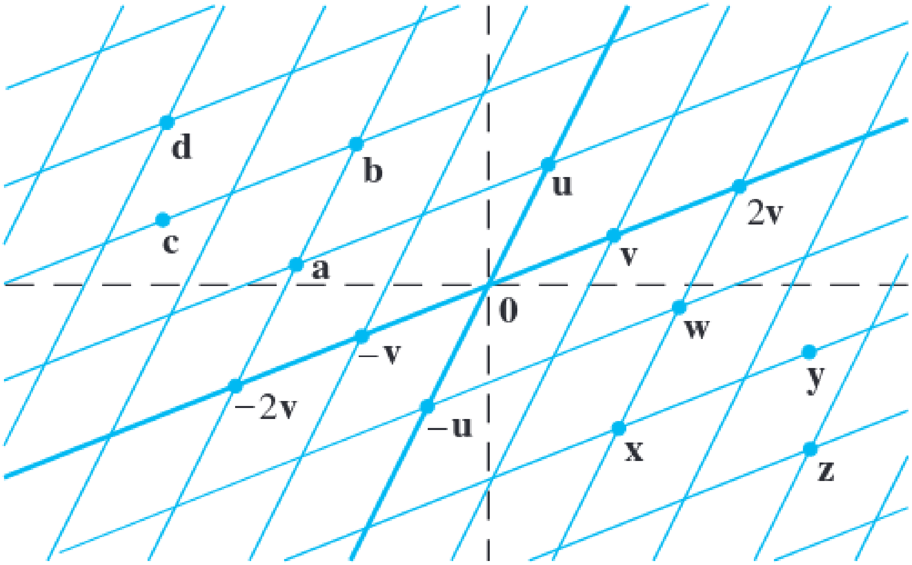
Solution:

The system of equations that is equivalent to the given vector equation is

$$\begin{Bmatrix} 6x_1 - 3x_2 = 1 \\ -x_1 + 4x_2 = -7 \\ 5x_1 = -5 \end{Bmatrix}$$

Problem 20 (1.3#7)

Use the accompanying figure to write vectors a,b,c and d as a linear combination of $\boldsymbol{u}$ and $\boldsymbol{v}$.



Solution:

$$\boldsymbol{a} = \boldsymbol{u} - 2\boldsymbol{v}$$

$$\boldsymbol{b} = 2\boldsymbol{u} - 2\boldsymbol{v}$$

$$\boldsymbol{c} = 2\boldsymbol{u} - \frac{7}{2}\boldsymbol{v}$$

$$\boldsymbol{d} = 3\boldsymbol{u} - 4\boldsymbol{v}$$

Problem 21 (1.3#9)

Write a vector equation that is equivalent to the given system of equations.

$$\begin{aligned} x_2 + 5x_3 &= 0 \\ 4x_1 + 6x_2 - x_3 &= 0 \\ -x_1 + 3x_2 - 8x_3 &= 0 \end{aligned}$$

Solution:

The vector equation that is equivalent to the given system of equations is

$$x_1 \begin{bmatrix} 0 \\ 4 \\ -1 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 6 \\ 3 \end{bmatrix} + x_3 \begin{bmatrix} 5 \\ -1 \\ -8 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Problem 22 (1.3#13)

Determine if $\boldsymbol{b}$ is a linear combination of the vectors formed from the columns of the matrix A.

$$A = \begin{bmatrix} 1 & -4 & 2 \\ 0 & 3 & 5 \\ -2 & 8 & -4 \end{bmatrix}, \boldsymbol{b} = \begin{bmatrix} 3 \\ -7 \\ -3 \end{bmatrix}$$

Solution:

$$\begin{bmatrix} 1 & -4 & 2 & 3 \\ 0 & 3 & 5 & 7 \\ -2 & 8 & -4 & -3 \end{bmatrix} \equiv \begin{bmatrix} 1 & -4 & 2 & 3 \\ 0 & 3 & 5 & 7 \\ 0 & 0 & 0 & 3 \end{bmatrix} \qquad R3 = 2 * R1 + R3$$

$\boldsymbol{b}$ is not a linear combination of the vectors formed from the columns of the matrix A.

Problem 23 (1.3#17)

Let $\boldsymbol{a_1} = \begin{bmatrix} 1 \\ 4 \\ -2 \end{bmatrix}$, $\boldsymbol{a_2} = \begin{bmatrix} -2 \\ -3 \\ 7 \end{bmatrix}$, and $\boldsymbol{b} = \begin{bmatrix} 4 \\ 1 \\ h \end{bmatrix}$. For what value(s) of h is $\boldsymbol{b}$ in the plane spanned by $\boldsymbol{a_1}$ and $\boldsymbol{a_2}$?

Solution:

$$\begin{bmatrix} 1 & -2 & 4 \\ 4 & -3 & 1 \\ -2 & 7 & h \end{bmatrix} \equiv \begin{bmatrix} 1 & -2 & 4 \\ 4 & -3 & 1 \\ 0 & 3 & h+8 \end{bmatrix} \qquad R3 = 2 * R1 + R3$$

$$\equiv \begin{bmatrix} 1 & -2 & 4 \\ 0 & 5 & -15 \\ 0 & 3 & h+8 \end{bmatrix} \qquad R2 = -4 * R1 + R2$$

$$\equiv \begin{bmatrix} 1 & -2 & 4 \\ 0 & 1 & -3 \\ 0 & 3 & h+8 \end{bmatrix} \qquad R2 = \frac{1}{5} R2$$

$$\equiv \begin{bmatrix} 1 & -2 & 4 \\ 0 & 1 & -3 \\ 0 & 0 & h+17 \end{bmatrix} \qquad R3 = -3 * R2 + R3$$

$$\equiv \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & -3 \\ 0 & 0 & h+17 \end{bmatrix} \qquad R1 = 2 * R2 + R1$$

In order for $\boldsymbol{b}$ to be in the splane spanned by $\boldsymbol{a_1}$ and $\boldsymbol{a_2}$, $h + 17 = 0$, so $h = -17$.

Problem 24 (1.3#19)

Give a geometric description of the Span $\{\boldsymbol{v_1}, \boldsymbol{v_2}\}$ for vectors $\boldsymbol{v_1} = \begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix}$, $\boldsymbol{v_2} = \begin{bmatrix} -2 \\ 0 \\ 3 \end{bmatrix}$

Solution:

The geometric description of the Span $\{\boldsymbol{v_1}, \boldsymbol{v_2}\}$ is the plane that contains the vectors $\begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} -2 \\ 0 \\ 3 \end{bmatrix}$ and passes through the origin.

Problem 25 (1.3#24)

Determine whether each statement is True or False. Justify your answer.

(a) Any list of five real numbers is a vector in $\mathbb{R}^5$.

(b) The vector $\boldsymbol{u}$ results when a vector $\boldsymbol{u} - \boldsymbol{v}$ is added to the vector $\boldsymbol{v}$.

(c) The weights $c_1, \ldots, c_p$ in a linear combination $c_1\boldsymbol{v_1}, \ldots, c_p\boldsymbol{v_p}$ cannot be all zero.

(d) When $\boldsymbol{u}$ and $\boldsymbol{v}$ are nonzero vectors, Span $\{\boldsymbol{u}, \boldsymbol{v}\}$ contains the line through $\boldsymbol{u}$ and the origin.

(e) Asking whether the linear system corresponding to an augmented matrix $\begin{bmatrix} \boldsymbol{a_1} & \boldsymbol{a_2} & \boldsymbol{a_3} & \boldsymbol{b} \end{bmatrix}$ has a solution amounts to asking if $\boldsymbol{b}$ is in Span $\{\boldsymbol{a_1}, \boldsymbol{a_2}, \boldsymbol{a_3}\}$

Solution:

(a) <u>True</u>, $\mathbb{R}^n$ is the set of all lists or real numbers of length n.

(b) <u>True</u>, $(\boldsymbol{u} - \boldsymbol{v}) + \boldsymbol{v} = \boldsymbol{u} + (\boldsymbol{v} - \boldsymbol{v}) = \boldsymbol{u}$.

(c) <u>False</u>, the zero matrix is a linear combination of any set of vectors with zero weights.

(d) <u>True</u>, the Span $\{\boldsymbol{u}, \boldsymbol{v}\}$ contains all points that are a linear combination of $\boldsymbol{u}$ and $\boldsymbol{v}$ and all the points on the line through $\boldsymbol{u}$ and the origin are a linear combination of $\boldsymbol{u}$ and $\boldsymbol{v}$.

(e) <u>True</u>, if $\boldsymbol{b}$ is in Span $\{\boldsymbol{a_1}, \boldsymbol{a_2}, \boldsymbol{a_3}\}$, then there is some linear combination of $\boldsymbol{a_1}, \boldsymbol{a_2}, \boldsymbol{a_3}$ that is equal to $\boldsymbol{b}$, and in order for a linear combination to exist there must be a solution to the linear system.

Problem 26 (1.3#25)

Let $A = \begin{bmatrix} 1 & 0 & -4 \\ 0 & 3 & -2 \\ -2 & 6 & 3 \end{bmatrix}$ and $\boldsymbol{b} = \begin{bmatrix} 4 \\ 1 \\ -4 \end{bmatrix}$. Denote the columns of A by $\boldsymbol{a_1}$, $\boldsymbol{a_2}$, $\boldsymbol{a_3}$, and let $W = \text{Span}\,\{\boldsymbol{a_1}, \boldsymbol{a_2}, \boldsymbol{a_3}\}$.

(a) is $\boldsymbol{b}$ in $\{\boldsymbol{a_1}, \boldsymbol{a_2}, \boldsymbol{a_3}\}$? How many vectors are in $\{\boldsymbol{a_1}, \boldsymbol{a_2}, \boldsymbol{a_3}\}$?

(b) is $\boldsymbol{b}$ in W? How many vectors are in W?

(c) Show that $\boldsymbol{a_1}$ is in W. [Hint: Row operations are unnecessary.]

Solution:

(a) No, there are three vectors in the set $\{\boldsymbol{a_1}, \boldsymbol{a_2}, \boldsymbol{a_3}\}$

(b) Yes, there are an infinite number of vectors in W.

(c) The vector $\boldsymbol{a_1}$ can be written as a linear combination of $\boldsymbol{a_1}$, $\boldsymbol{a_2}$, $\boldsymbol{a_3}$, so it is in W ($\boldsymbol{a_1} = 1\boldsymbol{a_1} + 0\boldsymbol{a_2} + 0\boldsymbol{a_3}$).

Problem 27 (1.3#29)

Let $v_1, \ldots, v_k$ be points in $\mathbb{R}^3$ and supposed that for $j = 1, \ldots, k$ an object with mass m_j is located at point v_j. Physicists call such objects *point masses*. The total mass of the system of point masses is

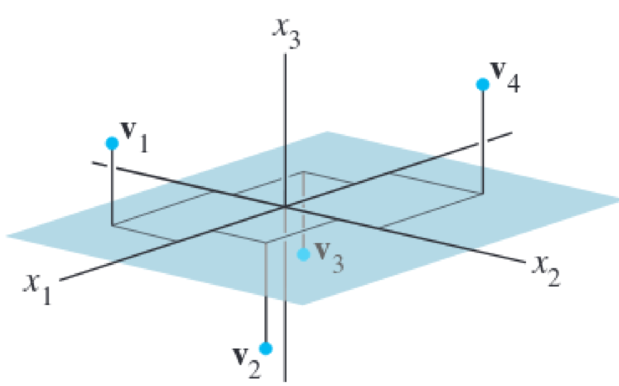
$$m = m_1 + \ldots + m_k$$

The *center of mass* of the system is

$$\boldsymbol{v} = \frac{1}{m}(m_1 v_1 + m_2 v_2 + \ldots + m_k v_k)$$

Compute the center of gtavity of the system consisting of the following point masses

Point	Mass
$\mathbf{v}_1 = (5, -4, 3)$	2 g
$\mathbf{v}_2 = (4, 3, -2)$	5 g
$\mathbf{v}_3 = (-4, -3, -1)$	2 g
$\mathbf{v}_4 = (-9, 8, 6)$	1 g



Solution:

$$\boldsymbol{v} = \frac{1}{10}(2\begin{bmatrix} 5 \\ -4 \\ 3 \end{bmatrix} + 5\begin{bmatrix} 4 \\ 3 \\ -1 \end{bmatrix} + 2\begin{bmatrix} -4 \\ -3 \\ -1 \end{bmatrix} + \begin{bmatrix} -9 \\ 8 \\ 6 \end{bmatrix}) = \begin{bmatrix} \frac{13}{10} \\ \frac{9}{10} \\ 0 \end{bmatrix}$$

The center of mass of the system is at $(\frac{13}{10}, \frac{9}{10}, 0)$.

Problem 28 (1.3#34)

Use the vector $\boldsymbol{u} = (u_1, \ldots, u_n)$ to verify the following algebraic properties of $\mathbb{R}^n$.

(a) $\boldsymbol{u} + (-\boldsymbol{u}) = (-\boldsymbol{u}) + \boldsymbol{u} = 0$

(b) $c(d\boldsymbol{u}) = (cd)\boldsymbol{u}$ for all scalars c and d

Solution:

Proof. $\boldsymbol{u} + (-\boldsymbol{u}) = (-\boldsymbol{u}) + \boldsymbol{u}$

$$\begin{aligned}
\boldsymbol{u} + (-\boldsymbol{u}) &= (u_1, \ldots, u_n) - (u_1, \ldots, u_n) && \text{(by substution)} \\
&= (u_1 - u_1, \ldots, u_n - u_n) && \text{(by vector addition)} \\
&= ((-u_1) + u_1, \ldots, (-u_n) + u_n) && \text{(by associative property of addition)} \\
&= (-\boldsymbol{u}) + \boldsymbol{u} && \text{(by vector addition)}
\end{aligned}$$

□

Proof. $\boldsymbol{u} + (-\boldsymbol{u}) = 0$

$$\begin{aligned}
\boldsymbol{u} + (-\boldsymbol{u}) &= (u_1, \ldots, u_n) - (u_1, \ldots, u_n) && \text{(by substution)} \\
&= (u_1 - u_1, \ldots, u_n - u_n) && \text{(by vector addition)} \\
&= (0, \ldots, 0) && \text{(by addition)} \\
&= 0 && \text{(by def. of zero vector)}
\end{aligned}$$

□

Proof. $c(d\boldsymbol{u}) = (cd)\boldsymbol{u}$ for all scalars c and d

$$\begin{aligned}
c(d\boldsymbol{u}) &= c(du_1, \ldots, cu_n) && \text{(by scalar multiplication)} \\
&= (c(du_1), \ldots, c(cu_n)) && \text{(by scalar multiplication of vector)} \\
&= ((cd)u_1, \ldots, (cd)u_n) && \text{(by communative property of multiplication)} \\
&= (cd)\boldsymbol{u} && \text{(by scalar multiplication of vector)}
\end{aligned}$$

□