

# Linear Algebra: Homework #2

Due on September 11, 2019

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## Problem 1 (1.6#4)

Suppose an economy has four sectors, Agriculture (A), Energy (E), Manufacturing (M), and Transportation (T). Sector A sells 10% of its output to E and 25% to M and retains the rest. Sector E sells 30% of its output to A, 35% to M, and 25% to T and retains the rest. Sector M sells 30% of its output to A, 15% to E, and 40% to T and retains the rest. Sector T sells 20% of its output to A, 10% to E, and 30% to M and retains the rest.

- Construct the exchange table for this economy.
- [M] Find a set of equilibrium prices for the economy.

### Part A:

A	E	M	T	Purchased By
0.65	0.30	0.30	0.20	A
0.10	0.10	0.15	0.10	E
0.25	0.35	0.15	0.30	M
0.00	0.25	0.40	0.40	T

### Part B:

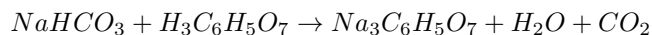
$$\begin{aligned}
 \begin{cases} p_a = 0.65p_a + 0.30p_e + 0.30p_m + 0.20p_t \\ p_e = 0.10p_a + 0.10p_e + 0.15p_m + 0.10p_t \\ p_m = 0.25p_a + 0.35p_e + 0.15p_m + 0.30p_t \\ p_t = 0.00p_a + 0.25p_e + 0.40p_m + 0.40p_t \end{cases} &\equiv \begin{cases} 0 = -0.35p_a + 0.30p_e + 0.30p_m + 0.20p_t \\ 0 = 0.10p_a - 0.90p_e + 0.15p_m + 0.10p_t \\ 0 = 0.25p_a + 0.35p_e - 0.85p_m + 0.30p_t \\ 0 = 0.00p_a + 0.25p_e + 0.40p_m - 0.60p_t \end{cases} \\
 &\equiv \begin{bmatrix} -0.35 & 0.30 & 0.30 & 0.20 & 0 \\ 0.10 & -0.90 & 0.15 & 0.10 & 0 \\ 0.25 & 0.35 & -0.85 & 0.30 & 0 \\ 0.00 & 0.25 & 0.40 & -0.60 & 0 \end{bmatrix} \\
 &\equiv \begin{bmatrix} 1 & 0 & 0 & -2.0279 & 0 \\ 0 & 1 & 0 & -0.5311 & 0 \\ 0 & 0 & 1 & -1.1681 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \\
 &\equiv \begin{cases} p_a = 2.0279p_t \\ p_e = 0.5311p_t \\ p_m = 1.1681p_t \\ p_t = p_t \end{cases}
 \end{aligned}$$

The set of all equilibrium prices are

$$\mathbf{p} = p_t \begin{bmatrix} 2.0279 \\ 0.5311 \\ 1.1681 \\ 1 \end{bmatrix}$$

**Problem 2 (1.6#7)**

Alka-Seltzer contains sodium bicarbonate ( $NaHCO_3$ ) and citric acid ( $H_3C_6H_5O_7$ ). When a tablet is dissolved in water, the following reaction produces sodium citrate, water, and carbon dioxide (gas):



Balance the chemical equation using the vector equation approach.

**Solution:**

$$NaHCO_3 : \begin{bmatrix} 1 \\ 1 \\ 1 \\ 3 \end{bmatrix}, H_3C_6H_5O_7 : \begin{bmatrix} 0 \\ 8 \\ 6 \\ 7 \end{bmatrix}, Na_3C_6H_5O_7 : \begin{bmatrix} 3 \\ 5 \\ 6 \\ 7 \end{bmatrix}, H_2O : \begin{bmatrix} 0 \\ 2 \\ 0 \\ 1 \end{bmatrix}, CO_2 : \begin{bmatrix} 0 \\ 0 \\ 1 \\ 2 \end{bmatrix}$$

So the vector equation that represents the reaction is

$$x_1 \begin{bmatrix} 1 \\ 1 \\ 1 \\ 3 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 8 \\ 6 \\ 7 \end{bmatrix} = x_3 \begin{bmatrix} 3 \\ 5 \\ 6 \\ 7 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 2 \\ 0 \\ 1 \end{bmatrix} + x_5 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 2 \end{bmatrix}$$

This is equivalent to the augmented matrices

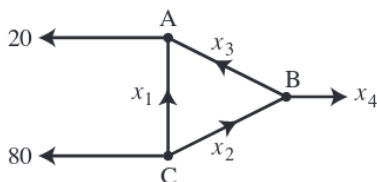
$$\begin{aligned}
 & \begin{bmatrix} 1 & 0 & -3 & 0 & 0 & 0 \\ 1 & 8 & -5 & -2 & 0 & 0 \\ 1 & 6 & -6 & 0 & -1 & 0 \\ 3 & 7 & -7 & -1 & -2 & 0 \end{bmatrix} \equiv \begin{bmatrix} 1 & 0 & -3 & 0 & 0 & 0 \\ 0 & 8 & -2 & -2 & 0 & 0 \\ 1 & 6 & -6 & 0 & -1 & 0 \\ 3 & 7 & -7 & -1 & -2 & 0 \end{bmatrix} \\
 & \quad R_2 = R_2 - R_1 \\
 & \equiv \begin{bmatrix} 1 & 0 & -3 & 0 & 0 & 0 \\ 0 & 8 & -2 & -2 & 0 & 0 \\ 0 & 6 & -3 & 0 & -1 & 0 \\ 3 & 7 & -7 & -1 & -2 & 0 \end{bmatrix} \equiv \begin{bmatrix} 1 & 0 & -3 & 0 & 0 & 0 \\ 0 & 8 & -2 & -2 & 0 & 0 \\ 0 & 6 & -3 & 0 & -1 & 0 \\ 0 & 7 & 2 & -1 & -2 & 0 \end{bmatrix} \\
 & \quad R_3 = R_3 - R_1 \quad R_4 = R_4 - 4R_1 \\
 & \equiv \begin{bmatrix} 1 & 0 & -3 & 0 & 0 & 0 \\ 0 & 1 & -\frac{1}{4} & -\frac{1}{4} & 0 & 0 \\ 0 & 6 & -3 & 0 & -1 & 0 \\ 0 & 7 & 2 & -1 & -2 & 0 \end{bmatrix} \equiv \begin{bmatrix} 1 & 0 & -3 & 0 & 0 & 0 \\ 0 & 1 & -\frac{1}{4} & -\frac{1}{4} & 0 & 0 \\ 0 & 6 & -3 & 0 & -1 & 0 \\ 0 & 0 & \frac{15}{4} & \frac{3}{4} & -2 & 0 \end{bmatrix} \\
 & \quad R_2 = R_2/8 \quad R_4 = R_4 - 7R_2 \\
 & \equiv \begin{bmatrix} 1 & 0 & -3 & 0 & 0 & 0 \\ 0 & 1 & -\frac{1}{4} & -\frac{1}{4} & 0 & 0 \\ 0 & 0 & -\frac{3}{2} & \frac{3}{2} & -1 & 0 \\ 0 & 0 & \frac{15}{4} & \frac{3}{4} & -2 & 0 \end{bmatrix} \equiv \begin{bmatrix} 1 & 0 & -3 & 0 & 0 & 0 \\ 0 & 1 & -\frac{1}{4} & -\frac{1}{4} & 0 & 0 \\ 0 & 0 & 1 & -1 & \frac{2}{3} & 0 \\ 0 & 0 & \frac{15}{4} & \frac{3}{4} & -2 & 0 \end{bmatrix} \\
 & \quad R_3 = R_3 - 6R_2 \quad R_3 = -2R_3/3 \\
 & \equiv \begin{bmatrix} 1 & 0 & -3 & 0 & 0 & 0 \\ 0 & 1 & -\frac{1}{4} & -\frac{1}{4} & 0 & 0 \\ 0 & 0 & 1 & -1 & \frac{2}{3} & 0 \\ 0 & 0 & 0 & \frac{9}{2} & -\frac{9}{2} & 0 \end{bmatrix} \equiv \begin{bmatrix} 1 & 0 & -3 & 0 & 0 & 0 \\ 0 & 1 & -\frac{1}{4} & -\frac{1}{4} & 0 & 0 \\ 0 & 0 & 1 & -1 & \frac{2}{3} & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 \end{bmatrix} \\
 & \quad R_4 = R_4 - \frac{15}{4}R_3 \quad R_4 = 2R_4/9 \\
 & \equiv \begin{bmatrix} 1 & 0 & -3 & 0 & 0 & 0 \\ 0 & 1 & -\frac{1}{4} & -\frac{1}{4} & 0 & 0 \\ 0 & 0 & 1 & 0 & -\frac{1}{3} & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 \end{bmatrix} \equiv \begin{bmatrix} 1 & 0 & -3 & 0 & 0 & 0 \\ 0 & 1 & -\frac{1}{4} & 0 & -\frac{1}{4} & 0 \\ 0 & 0 & 1 & 0 & -\frac{1}{3} & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 \end{bmatrix} \\
 & \quad R_3 = R_3 + R_4 \quad R_2 = R_2 + R_4/4 \\
 & \equiv \begin{bmatrix} 1 & 0 & -3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -\frac{1}{6} & 0 \\ 0 & 0 & 1 & 0 & -\frac{1}{3} & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 \end{bmatrix} \equiv \begin{bmatrix} 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & -\frac{1}{3} & 0 \\ 0 & 0 & 1 & 0 & -\frac{1}{3} & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 \end{bmatrix} \\
 & \quad R_2 = R_2 - R_3/4 \quad R_1 = R_1 + 3R_3 \\
 & \equiv \begin{bmatrix} x_1 = x_5 \\ x_2 = \frac{1}{3}x_5 \\ x_3 = \frac{1}{3}x_5 \\ x_4 = x_5 \end{bmatrix}
 \end{aligned}$$

So, the balanced reaction is



### Problem 3 (1.6#11)

Find the general flow pattern of the network shown in the figure. Assuming that the flows are all nonnegative, what is the largest possible value for  $x_3$ ?



#### Solution:

The inputs and outputs of each node in the network can be described as

$$A : x_3 + x_2 = 20$$

$$B : x_2 = x_3 + x_4$$

$$C : 80 = x_1 + x_2$$

This is equivalent to the following augmented matrices

$$\begin{aligned} \begin{bmatrix} 1 & 0 & 1 & 0 & 20 \\ 0 & -1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 80 \end{bmatrix} &\equiv \begin{bmatrix} 1 & 0 & 1 & 0 & 20 \\ 0 & -1 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 & 60 \end{bmatrix} & R_3 = R_3 - R_1 \\ &\equiv \begin{bmatrix} 1 & 0 & 1 & 0 & 20 \\ 0 & -1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 60 \end{bmatrix} & R_3 = R_3 + R_2 \\ &\equiv \begin{bmatrix} 1 & 0 & 1 & 0 & 20 \\ 0 & -1 & 1 & 0 & -60 \\ 0 & 0 & 0 & 1 & 60 \end{bmatrix} & R_2 = R_2 - R_3 \\ &\equiv \begin{bmatrix} 1 & 0 & 1 & 0 & 20 \\ 0 & 1 & -1 & 0 & 60 \\ 0 & 0 & 0 & 1 & 60 \end{bmatrix} & R_2 = -R_2 \end{aligned}$$

So the network can be described by the system

$$x_1 = 20 - x_3$$

$$x_2 = 60 + x_3$$

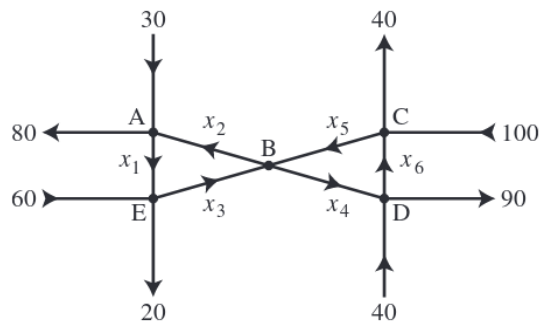
$$x_3 = x_3$$

$$x_4 = 60$$

The largest possible value of  $x_3$  for all nonnegative flows is 20.

### Problem 4 (1.6#13)

- Find the general flow pattern in the network shown in the figure.
- Assuming that the flow must be in the directions indicated, find the minimum flows in the branches denoted by  $x_2, x_3, x_4$  and  $x_5$ .



#### Part A:

The inputs and outputs of each node in the network can be described as

$$A : 30 + x_2 = 80 + x_1$$

$$B : x_3 + x_5 = x_2 + x_4$$

$$C : 100 + x_6 = x_5 + 40$$

$$D : x_4 + 40 = x_6 + 90$$

$$E : x_1 + 60 = 20 + x_3$$

This is equivalent to the following augmented matrices

$$\begin{aligned}
 & \begin{bmatrix} -1 & 1 & 0 & 0 & 0 & 0 & 50 \\ 0 & -1 & 1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 & -60 \\ 0 & 0 & 0 & 1 & 0 & -1 & 50 \\ 1 & 0 & -1 & 0 & 0 & 0 & -40 \end{bmatrix} \equiv \begin{bmatrix} -1 & 1 & 0 & 0 & 0 & 0 & 50 \\ 0 & -1 & 1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 & -60 \\ 0 & 0 & 0 & 1 & 0 & -1 & 50 \\ 0 & 1 & -1 & 0 & 0 & 0 & 10 \end{bmatrix} \\
 & \qquad \qquad \qquad R5 = R5 + R_1 \\
 & \begin{bmatrix} -1 & 1 & 0 & 0 & 0 & 0 & 50 \\ 0 & -1 & 1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 & -60 \\ 0 & 0 & 0 & 1 & 0 & -1 & 50 \\ 0 & 0 & 0 & -1 & 1 & 0 & 10 \end{bmatrix} \equiv \begin{bmatrix} -1 & 1 & 0 & 0 & 0 & 0 & 50 \\ 0 & -1 & 1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 & -60 \\ 0 & 0 & 0 & 1 & 0 & -1 & 50 \\ 0 & 0 & 0 & 0 & 1 & -1 & 60 \end{bmatrix} \\
 & \qquad \qquad \qquad R5 = R5 + R_2 \qquad R5 = R5 + R_4 \\
 & \begin{bmatrix} -1 & 1 & 0 & 0 & 0 & 0 & 50 \\ 0 & -1 & 1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 & -60 \\ 0 & 0 & 0 & 1 & 0 & -1 & 50 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \equiv \begin{bmatrix} -1 & 1 & 0 & 0 & 0 & 0 & 50 \\ 0 & -1 & 1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 & 50 \\ 0 & 0 & 0 & 0 & -1 & 1 & -60 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \\
 & \qquad \qquad \qquad R5 = R5 + R_3 \qquad \text{Swap } R_3 \text{ \& } R_4 \\
 & \begin{bmatrix} -1 & 1 & 0 & 0 & 0 & 0 & 50 \\ 0 & -1 & 1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 & 50 \\ 0 & 0 & 0 & 0 & 1 & -1 & 60 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \equiv \begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 & -50 \\ 0 & -1 & 1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 & 50 \\ 0 & 0 & 0 & 0 & 1 & -1 & 60 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \\
 & \qquad \qquad \qquad R_4 = -R_4 \qquad R_1 = -R_1 \\
 & \begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 & -50 \\ 0 & 1 & -1 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 & 50 \\ 0 & 0 & 0 & 0 & 1 & -1 & 60 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \equiv \begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 & -50 \\ 0 & 1 & -1 & 1 & 0 & -1 & 60 \\ 0 & 0 & 0 & 1 & 0 & -1 & 50 \\ 0 & 0 & 0 & 0 & 1 & -1 & 60 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \\
 & \qquad \qquad \qquad R_2 = -R_2 \qquad R_2 = R_2 + R_4 \\
 & \begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 & -50 \\ 0 & 1 & -1 & 0 & 0 & 0 & 10 \\ 0 & 0 & 0 & 1 & 0 & -1 & 50 \\ 0 & 0 & 0 & 0 & 1 & -1 & 60 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \equiv \begin{bmatrix} 1 & 0 & -1 & 0 & 0 & 0 & -40 \\ 0 & 1 & -1 & 0 & 0 & 0 & 10 \\ 0 & 0 & 0 & 1 & 0 & -1 & 50 \\ 0 & 0 & 0 & 0 & 1 & -1 & 60 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \\
 & \qquad \qquad \qquad R_2 = R_2 - R_3 \qquad R_1 = R_1 + R_2
 \end{aligned}$$

So the network can be described by the system

$$x_1 = x_3 - 40$$

$$x_2 = x_3 + 10$$

$$x_3 = x_3$$

$$x_4 = x_6 + 50$$

$$x_5 = x_6 + 60$$

$$x_6 = x_6$$

**Part B:**

The minimum flows for  $x_2, x_3, x_4$  and  $x_5$  are  $x_2 = 50, x_3 = 40, x_4 = 50, x_5 = 60$ .

**Problem 5 (1.7#1)**

Determine if the vectors are linearly independent. Justify answer.

$$\begin{bmatrix} 5 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 7 \\ 2 \\ -6 \end{bmatrix}, \begin{bmatrix} 9 \\ 4 \\ -8 \end{bmatrix}$$

**Solution:**

$$\begin{aligned} \begin{bmatrix} 5 & 7 & 9 \\ 0 & 2 & 4 \\ 0 & -6 & -8 \end{bmatrix} &\equiv \begin{bmatrix} 5 & 7 & 9 \\ 0 & 2 & 4 \\ 0 & 0 & 4 \end{bmatrix} \\ &R_3 = R_3 + 3R_2 \\ &\equiv \begin{bmatrix} 5 & 7 & 9 \\ 0 & 1 & 2 \\ 0 & 0 & 4 \end{bmatrix} \\ &R_2 = R_2/2 \quad R_3 = R_3/4 \\ &\equiv \begin{bmatrix} 5 & 7 & 9 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &R_2 = R_2 - R_3 \quad R_1 = R_1 - 9R_3 \\ &\equiv \begin{bmatrix} 5 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &R_1 = R_1 - 7R_2 \quad R_1 = R_1/5 \end{aligned}$$

The vectors are linearly independent, because their coefficient matrix has a pivot point in each row.



**Problem 6 (1.7#3)**

Determine if the vectors are linearly independent. Justify answer.

$$\begin{bmatrix} 1 \\ -3 \end{bmatrix}, \begin{bmatrix} -3 \\ 9 \end{bmatrix}$$

**Solution:**

The vectors are linearly dependent because  $\begin{bmatrix} -3 \\ 9 \end{bmatrix} = -3 \begin{bmatrix} 1 \\ -3 \end{bmatrix}$ .

**Problem 7 (1.7#5)**

Determine if the columns of the matrix form a linearly independent set. Justify your answer.

$$\begin{bmatrix} 0 & -8 & 5 \\ 3 & -7 & 4 \\ -1 & 5 & -4 \\ 1 & -3 & 2 \end{bmatrix}$$

**Solution:**

$$\begin{aligned} \begin{bmatrix} 0 & -8 & 5 \\ 3 & -7 & 4 \\ -1 & 5 & -4 \\ 1 & -3 & 2 \end{bmatrix} &\equiv \begin{bmatrix} 1 & -3 & 2 \\ 3 & -7 & 4 \\ -1 & 5 & -4 \\ 0 & -8 & 5 \end{bmatrix} \\ &\quad \text{Swap } R_1 \text{ \& } R_4 \\ &\equiv \begin{bmatrix} 1 & -3 & 2 \\ 3 & -7 & 4 \\ 0 & 2 & -2 \\ 0 & -8 & 5 \end{bmatrix} \equiv \begin{bmatrix} 1 & -3 & 2 \\ 0 & 2 & -2 \\ 0 & 2 & -2 \\ 0 & -8 & 5 \end{bmatrix} \\ &\quad R_3 = R_3 + R_1 \quad R_2 = R_2 - 3R_1 \\ &\equiv \begin{bmatrix} 1 & -3 & 2 \\ 0 & 2 & -2 \\ 0 & 0 & 0 \\ 0 & -8 & 5 \end{bmatrix} \equiv \begin{bmatrix} 1 & -3 & 2 \\ 0 & 2 & -2 \\ 0 & -8 & 5 \\ 0 & 0 & 0 \end{bmatrix} \\ &\quad R_3 = R_3 - R_2 \quad \text{Swap } R_3 \text{ \& } R_4 \\ &\equiv \begin{bmatrix} 1 & -3 & 2 \\ 0 & 2 & -2 \\ 0 & 0 & -3 \\ 0 & 0 & 0 \end{bmatrix} \equiv \begin{bmatrix} 1 & -3 & 2 \\ 0 & 2 & -2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \\ &\quad R_3 = R_3 + 4R_2 \quad R_3 = -R_3/3 \end{aligned}$$

The columns do form a linearly independent set because there is only the trivial solution to  $A\mathbf{x} = 0$ .

**Problem 8 (1.7#7)**

Determine if the columns of the matrix form a linearly independent set. Justify your answer.

$$\begin{bmatrix} 1 & 4 & -3 & 0 \\ -2 & -7 & 5 & 1 \\ -4 & -5 & 7 & 5 \end{bmatrix}$$

**Solution:**

The columns do not form a linearly independent set because the coefficient matrix is underdetermined, and will therefore have a free variable.

**Problem 9 (1.7#11)**

Find the value(s) of  $h$  for which the vectors are linearly dependent. Justify your answer.

$$\begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix}, \begin{bmatrix} 3 \\ -5 \\ 7 \end{bmatrix}, \begin{bmatrix} -1 \\ 5 \\ h \end{bmatrix}$$

**Solution:**

$$\begin{aligned} \begin{bmatrix} 1 & 3 & -1 \\ -1 & -5 & 5 \\ 4 & 7 & h \end{bmatrix} &\equiv \begin{bmatrix} 1 & 3 & -1 \\ 0 & -2 & 4 \\ 4 & 7 & h \end{bmatrix} && R_2 = R_2 + R_1 \\ &\equiv \begin{bmatrix} 1 & 3 & -1 \\ 0 & -2 & 4 \\ 0 & -5 & h+4 \end{bmatrix} && R_3 = R_3 - 4R_1 \\ &\equiv \begin{bmatrix} 1 & 3 & -1 \\ 0 & 1 & -2 \\ 0 & -5 & h+4 \end{bmatrix} && R_2 = -\frac{R_2}{2} \\ &\equiv \begin{bmatrix} 1 & 3 & -1 \\ 0 & 1 & -2 \\ 0 & 0 & h-6 \end{bmatrix} && R_3 = R_3 + 5R_2 \end{aligned}$$

The vectors will be linearly dependent if  $h = 6$ , because if  $h = 6$  then there is a free variable.

**Problem 10 (1.7#13)**

Find the value(s) of  $h$  for which the vectors are linearly dependent. Justify your answer.

$$\begin{bmatrix} 1 \\ 5 \\ -3 \end{bmatrix}, \begin{bmatrix} -2 \\ -9 \\ 6 \end{bmatrix}, \begin{bmatrix} 3 \\ h \\ -9 \end{bmatrix}$$

**Solution:**

$$\begin{bmatrix} 1 & -2 & 3 \\ 5 & -9 & h \\ -3 & 6 & -9 \end{bmatrix} \equiv \begin{bmatrix} 1 & -2 & 3 \\ 5 & -9 & h \\ 0 & 0 & 0 \end{bmatrix} \qquad R_3 = R_3 + 3R_1$$

The vectors will be linearly dependent for all possible value of  $h$ , because there is already a free variable regardless of the value of  $h$ .

**Problem 11 (1.7#15)**

Determine by inspection whether the vectors are linearly independent. Justify your answer.

$$\begin{bmatrix} 5 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 8 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ 7 \end{bmatrix}$$

**Solution:**

The vectors are linearly independent, because no vector is equal to the scalar multiple of another.

**Problem 12 (1.7#17)**

Determine by inspection whether the vectors are linearly independent. Justify your answer.

$$\begin{bmatrix} 3 \\ 5 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -6 \\ 5 \\ 4 \end{bmatrix}$$

**Solution:**

$$\begin{aligned} \begin{bmatrix} 3 & 0 & -6 \\ 5 & 0 & 5 \\ -1 & 0 & 4 \end{bmatrix} &\equiv \begin{bmatrix} 1 & 0 & -2 \\ 5 & 0 & 5 \\ -1 & 0 & 4 \end{bmatrix} \\ &R_1 = R_1/3 \\ &\equiv \begin{bmatrix} 1 & 0 & -2 \\ 1 & 0 & 1 \\ -1 & 0 & 4 \end{bmatrix} \equiv \begin{bmatrix} 1 & 0 & -2 \\ 1 & 0 & 1 \\ 0 & 0 & 2 \end{bmatrix} \\ &R_2 = R_2/5 \quad R_3 = R_3 + R_1 \\ &\equiv \begin{bmatrix} 1 & 0 & -2 \\ 0 & 0 & 3 \\ 0 & 0 & 2 \end{bmatrix} \equiv \begin{bmatrix} 1 & 0 & -2 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{bmatrix} \\ &R_2 = R_2 - R_1 \quad R_3 = R_3 - 2R_2/3 \end{aligned}$$

The vectors are linearly dependent because there is not a pivot point in each row of the coefficient matrix formed by the vectors.

**Problem 13 (1.7#19)**

Determine by inspection whether the vectors are linearly independent. Justify your answer.

$$\begin{bmatrix} -8 \\ 12 \\ -4 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ -1 \end{bmatrix}$$

**Solution:**

Yes, the vectors are linearly independent because they are not multiples of each other.

**Problem 14 (1.7#21)**

Mark each statment True or False. Justify each answer on the basis of a careful reading of the text.

- (a) The columns of a matrix  $A$  are linearly independent if the equation  $A\mathbf{x} = 0$  has the trivial solution
- (b) If  $S$  is a linearly dependent set, then each vector is a linear combination of the other vectors in  $S$ .
- (c) The columns of any  $4 \times 5$  matrix are linearly dependent.
- (d) If  $x$  and  $y$  are linearly independent, and if  $\{x, y, z\}$  is linearly dependent, then  $z$  is in  $\text{Span}\{x, y\}$

**Solution:**

- (a) False, the columns maybe linearly independent, but having the trivial solution does not imply that they are linearly independent.
- (b) False, for the vectors in  $S$  to be linearly independent, they cannot be linear combinations of the other vectors in the set.
- (c) True, a  $4 \times 5$  matrix is underdetermined, and therefore it's columns are a linearly dependent.
- (d) True, if  $\{x, y, z\}$  is linearly dependent, then  $z$  can be written as a linear combination of  $x$  and  $y$ , and therefore is in  $\text{Span}\{x, y\}$ .

**Problem 15 (1.7#23)**

Describe the possible echelon forms of  $A$ , a  $3 \times 3$  matrix with linearly independent columns. Use the notation of Example 1 in Section 1.2

**Solution:**

$$\begin{bmatrix} \blacksquare & * & * \\ 0 & \blacksquare & * \\ 0 & 0 & \blacksquare \end{bmatrix}$$

**Problem 16 (1.7#25)**

Describe the possible echelon forms of  $A = [\mathbf{a}_1, \mathbf{a}_2]$ , a  $4 \times 2$  matrix, such that  $\mathbf{a}_2$  is not a multiple of  $\mathbf{a}_1$  with linearly dependent columns. Use the notation of Example 1 in Section 1.2

**Solution:**

$$\begin{bmatrix} \blacksquare & * \\ 0 & \blacksquare \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

**Problem 17 (1.8#1)**

Let  $A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ , and define  $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  by  $T(\mathbf{x}) = A\mathbf{x}$ . Find the images under  $T$  of  $\mathbf{u} = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$ , and  $\mathbf{v} = \begin{bmatrix} a \\ b \end{bmatrix}$ .

**Solution:**

$$T(\mathbf{u}) = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ -3 \end{bmatrix} = \begin{bmatrix} 2 \\ -6 \end{bmatrix}$$

$$T(\mathbf{v}) = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 2a \\ 2b \end{bmatrix}$$

**Problem 18 (1.8#2)**

Let  $A = \begin{bmatrix} .5 & 0 & 0 \\ 0 & .5 & 0 \\ 0 & 0 & .5 \end{bmatrix}$ ,  $\mathbf{u} = \begin{bmatrix} 1 \\ 0 \\ -4 \end{bmatrix}$ , and  $\mathbf{v} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ . Define  $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$  by  $T(\mathbf{x}) = A\mathbf{x}$ . Find  $T(\mathbf{u})$  and  $T(\mathbf{v})$ .

**Solution:**

$$T(\mathbf{u}) = \begin{bmatrix} .5 & 0 & 0 \\ 0 & .5 & 0 \\ 0 & 0 & .5 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ -4 \end{bmatrix} = \begin{bmatrix} 0.5 \\ 0 \\ -2 \end{bmatrix}$$

$$T(\mathbf{v}) = \begin{bmatrix} .5 & 0 & 0 \\ 0 & .5 & 0 \\ 0 & 0 & .5 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0.5a \\ 0.5b \\ 0.5c \end{bmatrix}$$

**Problem 19 (1.8#3)**

Find a vector  $\mathbf{x}$  whose image under  $T$  is  $\mathbf{b}$  and determine whether  $\mathbf{x}$  is unique.  $T(\mathbf{x}) = A\mathbf{x}$ .

$$A = \begin{bmatrix} 1 & 0 & -2 \\ -2 & 1 & 6 \\ 3 & -2 & -5 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} -1 \\ 7 \\ -3 \end{bmatrix}$$

**Solution:**

$$\begin{bmatrix} 1 & 0 & -2 & -1 \\ -2 & 1 & 6 & 7 \\ 3 & -2 & -5 & -3 \end{bmatrix} \equiv \begin{bmatrix} 1 & 0 & -2 & -1 \\ 0 & 1 & 2 & 5 \\ 3 & -2 & -5 & -3 \end{bmatrix}$$

$$R_2 = R_2 + 2R_1$$

$$\equiv \begin{bmatrix} 1 & 0 & -2 & -1 \\ 0 & 1 & 2 & 5 \\ 0 & -2 & 1 & 0 \end{bmatrix} \equiv \begin{bmatrix} 1 & 0 & -2 & -1 \\ 0 & 1 & 2 & 5 \\ 0 & 0 & 5 & 10 \end{bmatrix}$$

$$R_3 = R_3 - 3R_1 \quad R_3 = R_3 + 2R_2$$

$$\equiv \begin{bmatrix} 1 & 0 & -2 & -1 \\ 0 & 1 & 2 & 5 \\ 0 & 0 & 1 & 2 \end{bmatrix} \equiv \begin{bmatrix} 1 & 0 & -2 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

$$R_3 = R_3/5 \quad R_2 = R_2 - R_3$$

$$\equiv \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

$$R_1 = R_1 + 2R_3$$

The vector  $\mathbf{x}$  whose image under  $T$  is  $\mathbf{b}$  is unique and is equal to  $\begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$ .

**Problem 20 (1.8#5)**

Find a vector  $\mathbf{x}$  whose image under  $T$  is  $\mathbf{b}$  and determine whether  $x$  is unique.  $T(\mathbf{x}) = A\mathbf{x}$ .

$$A = \begin{bmatrix} 1 & -5 & -7 \\ -3 & 7 & 5 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} -2 \\ -2 \end{bmatrix}$$

**Solution:**

$$\begin{aligned} \begin{bmatrix} 1 & -5 & -7 & -2 \\ -3 & 7 & 5 & -2 \end{bmatrix} &\equiv \begin{bmatrix} 1 & -5 & -7 & -2 \\ 0 & -8 & -16 & -8 \end{bmatrix} && R_2 = R_2 + 3R_1 \\ &\equiv \begin{bmatrix} 1 & -5 & -7 & -2 \\ 0 & 1 & 2 & 1 \end{bmatrix} && R_2 = -\frac{R_2}{8} \\ &\equiv \begin{bmatrix} 1 & 0 & 3 & 3 \\ 0 & 1 & 2 & 1 \end{bmatrix} && R_1 = R_1 + 5R_2 \\ &\equiv \begin{cases} x_1 = 3 - 3x_3 \\ x_2 = 1 - 2x_3 \end{cases} \end{aligned}$$

The vectors whose image under  $T$  is  $\mathbf{b}$  are

$$\mathbf{x} = x_3 \begin{bmatrix} -3 \\ -2 \\ 1 \end{bmatrix} + \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$$

One such vector is  $\begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$ , but it is not a unique solution.

**Problem 21 (1.8#7)**

Let  $A$  be  $6 \times 5$  matrix. What must  $a$  and  $b$  be in order to define  $T : \mathbb{R}^a \rightarrow \mathbb{R}^b$  by  $T(x) = A\mathbf{x}$ ?

**Solution:**

$$a = 5, b = 6$$

**Problem 22 (1.8#9)**

Find all  $x$  in  $\mathbb{R}^4$  that are mapped into the zero vector by the transformation  $\mathbf{x} \mapsto A\mathbf{x}$  for the given matrix  $A$ .

$$\begin{bmatrix} 1 & -4 & 7 & -5 \\ 0 & 1 & -4 & 3 \\ 2 & -6 & 6 & -4 \end{bmatrix}$$

**Solution:**

$$\begin{aligned} & \begin{bmatrix} 1 & -4 & 7 & -5 & 0 \\ 0 & 1 & -4 & 3 & 0 \\ 2 & -6 & 6 & -4 & 0 \end{bmatrix} \equiv \begin{bmatrix} 1 & -4 & 7 & -5 & 0 \\ 0 & 1 & -4 & 3 & 0 \\ 0 & 2 & -8 & 6 & 0 \end{bmatrix} \\ & \qquad R_3 = R_3 - 2R_1 \\ & \equiv \begin{bmatrix} 1 & -4 & 7 & -5 & 0 \\ 0 & 1 & -4 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \equiv \begin{bmatrix} 1 & 0 & -9 & 7 & 0 \\ 0 & 1 & -4 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \\ & \qquad R_3 = R_3 - 2R_2 \qquad R_1 = R_1 + 4R_2 \\ & \equiv \begin{cases} x_1 = 9x_3 - 7x_4 \\ x_2 = 4x_3 - 3x_4 \\ x_3 = x_3 \\ x_4 = x_4 \end{cases} \end{aligned}$$

The  $\mathbf{x}$  in  $\mathbb{R}^4$  that are mapped into the zero vector by  $A$  are...

$$\mathbf{x} = x_3 \begin{bmatrix} 9 \\ 4 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -7 \\ -3 \\ 0 \\ 1 \end{bmatrix}$$



**Problem 23 (1.8#11)**

Let  $\mathbf{b} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ , and let  $A$  be the matrix be the matrix in Exercise 9. Is  $\mathbf{b}$  in the range of the linear transformation  $\mathbf{x} \mapsto A\mathbf{x}$ ? Why or why not?

**Solution:**

$$\begin{aligned}
 & \begin{bmatrix} 1 & -4 & 7 & -5 & -1 \\ 0 & 1 & -4 & 3 & 1 \\ 2 & -6 & 6 & -4 & 0 \end{bmatrix} \equiv \begin{bmatrix} 1 & -4 & 7 & -5 & -1 \\ 0 & 1 & -4 & 3 & 1 \\ 0 & 2 & -8 & 6 & 2 \end{bmatrix} \\
 & \qquad \qquad \qquad R_3 = R_3 - 2R_1 \\
 & \equiv \begin{bmatrix} 1 & -4 & 7 & -5 & -1 \\ 0 & 1 & -4 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \equiv \begin{bmatrix} 1 & 0 & -9 & 7 & 3 \\ 0 & 1 & -4 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \\
 & \qquad \qquad \qquad R_3 = R_3 - 2R_2 \qquad R_1 = R_1 + 4R_2 \\
 & \equiv \left\{ \begin{array}{l} x_1 = 9x_3 - 7x_4 + 3 \\ x_2 = 4x_3 - 3x_4 + 1 \\ x_3 = x_3 \\ x_4 = x_4 \end{array} \right\}
 \end{aligned}$$

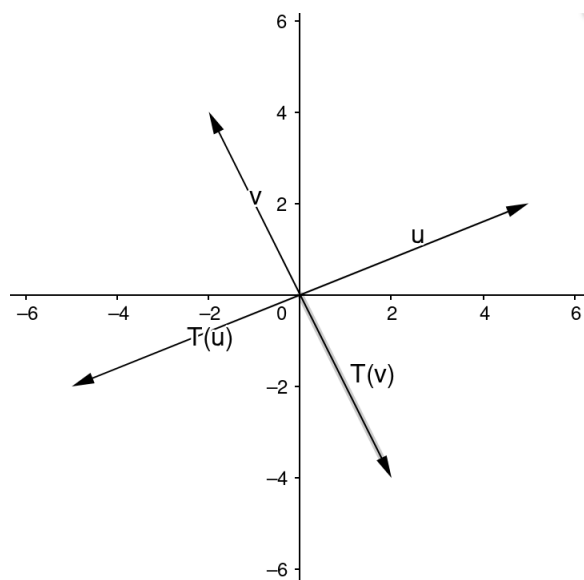
Yes,  $\mathbf{b}$  is in the range of the linear transformation  $\mathbf{x} \mapsto A\mathbf{x}$  because the system  $A\mathbf{x} = \mathbf{b}$  is consistent.

**Problem 24 (1.8#13)**

Use a rectangular coordinate system to plot  $\mathbf{u} = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$ ,  $\mathbf{v} = \begin{bmatrix} -2 \\ 4 \end{bmatrix}$ , and their images under the given transformation  $T$ . Describe geometrically what  $T$  does to each vector  $\mathbf{x}$  in  $\mathbb{R}^2$ .

$$T(\mathbf{x}) = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

**Solution:**



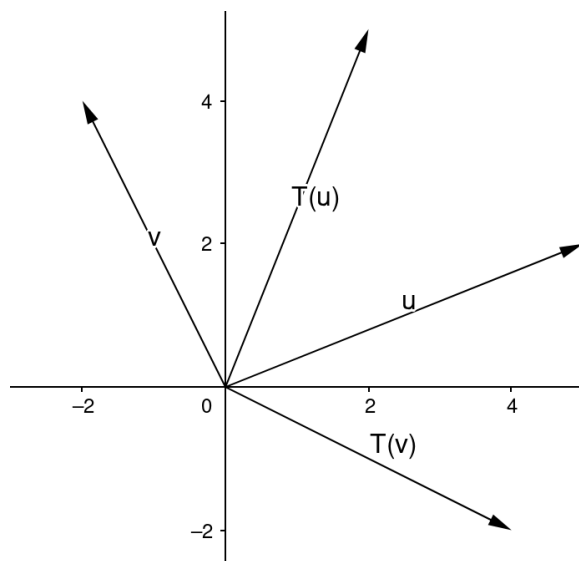
$T$  reflects each vector across the origin

### Problem 25 (1.8#16)

Use a rectangular coordinate system to plot  $\mathbf{u} = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$ ,  $\mathbf{v} = \begin{bmatrix} -2 \\ 4 \end{bmatrix}$ , and their images under the given transformation  $T$ . Describe geometrically what  $T$  does to each vector  $\mathbf{x}$  in  $\mathbb{R}^2$ .

$$T(\mathbf{x}) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

**Solution:**



$T$  swaps the  $x_1$  and  $x_2$  components of each vector.

### Problem 26 (1.8#30)

An *affine transformation*  $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$  has the form  $T(\mathbf{x}) = A\mathbf{x} + \mathbf{b}$ , with  $A$  an  $m \times n$  matrix and  $\mathbf{b}$  in  $\mathbb{R}^m$ . Show that  $T$  is *not* a linear transformation when  $\mathbf{b} \neq \mathbf{0}$

**Solution:**

*Proof.* Show that  $T(c\mathbf{x}) \neq cT(\mathbf{x})$

$$T(c\mathbf{x}) = A(c\mathbf{x}) + \mathbf{b} \tag{1}$$

$$cT(\mathbf{x}) = c(A\mathbf{x} + \mathbf{b}) \tag{2}$$

$$= A(c\mathbf{x}) + c\mathbf{b} \tag{3}$$

$$A(c\mathbf{x}) + \mathbf{b} \neq A(c\mathbf{x}) + c\mathbf{b} \tag{4}$$

Because it does not hold true that  $T(c\mathbf{x}) = cT(\mathbf{x})$  for the transformation  $T(\mathbf{x}) = A\mathbf{x} + \mathbf{b}$  it is not a linear transformation.  $\square$

**Problem 27 (1.8#33)**

Show that the transformation  $T$  defined by  $T(x_1, x_2) = (2x_1 - 3x_2, x_1 + 4, 5x_2)$  is not linear.

**Solution:**

*Proof.* Show that  $T(c\mathbf{x}) \neq cT(\mathbf{x})$

$$T(cx_1, cx_2) = (2cx_1 - 3cx_2, cx_1 + 4, 5cx_2) \quad (1)$$

$$cT(\mathbf{x}) = c(2x_1 - 3x_2, x_1 + 4, 5x_2) \quad (2)$$

$$= (2cx_1 - 3cx_2, cx_1 + c4, 5cx_2) \quad (3)$$

$$(2cx_1 - 3cx_2, cx_1 + 4, 5cx_2) \neq (2cx_1 - 3cx_2, cx_1 + c4, 5cx_2) \quad (4)$$

Because it does not hold true that  $T(c\mathbf{x}) = cT(\mathbf{x})$  for the transformation  $T(\mathbf{x}) = (2x_1 - 3x_2, x_1 + 4, 5x_2)$  it is not a linear transformation.  $\square$