

Linear Algebra: Homework #1

Due on August 28, 2019

Professor MacArthur

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Problem 1 (1.1#1)

Solve the following system by using elementary row operations on the equations

$$\begin{aligned}x_1 + 5x_2 &= 7 \\ -2x_1 - 7x_2 &= -5\end{aligned}$$

Solution:

The system of equations can be represented by the following augmented matrices

$$\begin{aligned}\begin{bmatrix} 1 & 5 & 7 \\ -2 & -7 & -5 \end{bmatrix} &\equiv \begin{bmatrix} 1 & 5 & 7 \\ 0 & 3 & 9 \end{bmatrix} && R2 = 2 * R1 + R2 \\ &\equiv \begin{bmatrix} 1 & 5 & 7 \\ 0 & 1 & 3 \end{bmatrix} && R2 = \frac{1}{3} * R2 \\ &\equiv \begin{bmatrix} 1 & 0 & -8 \\ 0 & 1 & 3 \end{bmatrix} && R1 = -5 * R2 + R1 \\ &\equiv \begin{Bmatrix} x_1 = -8 \\ x_2 = 3 \end{Bmatrix}\end{aligned}$$

This is equivalent to the point $(-8, 3)$, which is the solution to the system of equations.

Problem 2 (1.1#3)

Find the point of intersection of the lines $x_1 - 5x_2 = 1$ and $3x_1 - 7x_2 = 5$ by using elementary row operations on the equations

Solution:

The system of equations can be represented by the following augmented matrices

$$\begin{aligned}\begin{bmatrix} 1 & 5 & 7 \\ 1 & -2 & -2 \end{bmatrix} &\equiv \begin{bmatrix} 1 & 5 & 7 \\ 0 & -7 & -9 \end{bmatrix} && R2 = -1 * R1 + R2 \\ &\equiv \begin{bmatrix} 1 & 5 & 7 \\ 0 & 1 & \frac{9}{7} \end{bmatrix} && R2 = \frac{1}{7} * R2 \\ &\equiv \begin{bmatrix} 1 & 0 & \frac{4}{7} \\ 0 & 1 & \frac{9}{7} \end{bmatrix} && R1 = -5 * R2 + R1 \\ &\equiv \begin{Bmatrix} x_1 = \frac{4}{7} \\ x_2 = \frac{9}{7} \end{Bmatrix}\end{aligned}$$

This is equivalent to the point $(\frac{4}{7}, \frac{9}{7})$, which is the point of intersection.

Problem 3 (1.1#7)

The augmented matrix of a linear system has been reduced by row operations to the form shown. In each case, continue the appropriate row operations and describe the solution set of the original system.

$$\begin{bmatrix} 1 & 7 & 3 & -4 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & -2 \end{bmatrix}$$

Solution:

There is no solution to the original system. Row 3 shows that this is an inconsistent system ($0 \neq 1$). This means that there is no solution to the system.

Problem 4 (1.1#8)

The augmented matrix of a linear system has been reduced by row operations to the form shown. In each case, continue the appropriate row operations and describe the solution set of the original system.

$$\begin{bmatrix} 1 & -4 & 9 & 0 \\ 0 & 1 & 7 & 0 \\ 0 & 0 & 2 & 0 \end{bmatrix}$$

Solution:

$$\begin{aligned} \begin{bmatrix} 1 & -4 & 9 & 0 \\ 0 & 1 & 7 & 0 \\ 0 & 0 & 2 & 0 \end{bmatrix} &\equiv \begin{bmatrix} 1 & -4 & 9 & 0 \\ 0 & 1 & 7 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} & R3 = \frac{1}{2} * R3 \\ &\equiv \begin{bmatrix} 1 & -4 & 0 & 0 \\ 0 & 1 & 7 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} & R1 = -9 * R3 + R1 \\ &\equiv \begin{bmatrix} 1 & -4 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} & R2 = -7 * R3 + R2 \\ &\equiv \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} & R1 = 4 * R2 + R1 \\ &\equiv \begin{cases} x_1 = 0 \\ x_2 = 0 \\ x_3 = 0 \end{cases} \end{aligned}$$

The solution to the original system is the point $(0, 0, 0)$.

Problem 5 (1.1#11)

Solve the following system

$$\begin{aligned}x_2 + 4x_3 &= -5 \\x_1 + 3x_2 + 5x_3 &= -2 \\3x_1 + 7x_2 + 7x_3 &= 6\end{aligned}$$

Solution:

The system can be represented by the following augmented matrices

$$\begin{aligned}\begin{bmatrix} 0 & 1 & 4 & 5 \\ 1 & 3 & 5 & -2 \\ 3 & 7 & 7 & 6 \end{bmatrix} &\equiv \begin{bmatrix} 1 & 3 & 5 & -2 \\ 0 & 1 & 4 & -5 \\ 3 & 7 & 7 & 6 \end{bmatrix} && \text{Swap } R2 \text{ and } R1 \\ &\equiv \begin{bmatrix} 1 & 3 & 5 & -2 \\ 0 & 1 & 4 & -5 \\ 0 & -2 & -8 & 12 \end{bmatrix} && R3 = -3 * R1 + R3 \\ &\equiv \begin{bmatrix} 1 & 3 & 5 & -2 \\ 0 & 1 & 4 & -5 \\ 0 & 0 & 0 & 2 \end{bmatrix} && R3 = 2 * R2 + R3\end{aligned}$$

There is no solution to this system, as evident by row 3, which shows that the system is inconsistent ($0 \neq 2$).

Problem 6 (1.1#18)

Do the three planes $x_1 + 2x_2 + x_3 = 4$, $x_2 - x_3 = 1$, and $x_1 + 3x_2 = 0$ have at least one common point of intersection? Explain.

Solution:

The equations of the three planes can be represented by the following augmented matrices

$$\begin{aligned}\begin{bmatrix} 1 & 2 & 1 & 4 \\ 0 & 1 & -1 & 1 \\ 1 & 3 & 0 & 0 \end{bmatrix} &\equiv \begin{bmatrix} 1 & 2 & 1 & 4 \\ 0 & 1 & -1 & 1 \\ 0 & 1 & -1 & -4 \end{bmatrix} && R3 = -1 * R1 + R3 \\ &\equiv \begin{bmatrix} 1 & 2 & 1 & 4 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & -5 \end{bmatrix} && R3 = -1 * R2 + R3\end{aligned}$$

There are no common points of intersection between the three planes because they produce an inconsistent system, as evident by row 3 of the augmented matrix ($0 \neq -5$).

Problem 7 (1.1#20)

Determine the value(s) of h such that the matrix is the augmented matrix of a consistent linear system.

$$\begin{bmatrix} 1 & h & -3 \\ -2 & 4 & 6 \end{bmatrix}$$

Solution:

$$\begin{bmatrix} 1 & h & -3 \\ -2 & 4 & 6 \end{bmatrix} \equiv \begin{bmatrix} 1 & h & -3 \\ 0 & 4 + 2h & 0 \end{bmatrix} \quad R2 = -1 * R1 + R2$$

The system is consistent for all possible values of h .

Problem 8 (1.1#23)

For each statement determine if it is True or False, and *justify* your answer. (If true, give the approximate location where a similar statement appears, or refer to a definition or theorem. If false, give the location of a statement that has been quoted or used incorrectly, or cite an example that shows the statement is not true in all cases.)

- (a) Every elementary row operation is reversible.
- (b) A 5×6 matrix has six rows.
- (c) The solution set of a linear system involving variables x_1, \dots, x_n is a list of numbers (s_1, \dots, s_n) that make each equation in the system a true statement when the values s_1, \dots, s_n are substituted for x_1, \dots, x_n , respectively.
- (d) Two fundamental questions about a linear system involve existence and uniqueness.

Solution:

- (a) This statement is true, as stated on pg. 6 "It is important to note that row operations are *reversible*".
- (b) This statement is false, a 5×6 matrix has 5 rows and 6 columns. According to pg. 4 "an $m \times n$ matrix is a rectangular array of numbers with m rows and n columns."
- (c) This statement is true, according to pg. 3, where it is stated "A **solution** of the system is a list (s_1, \dots, s_n) of numbers that makes each equation a true statement when the values s_1, \dots, s_n are substituted for x_1, \dots, x_n , respectively."
- (d) This statement is true, according to pg. 7, the two fundamental questions about a linear system are "is the system consistent" and "is the system unique".

Problem 9 (1.1#27)

Suppose the system below is consistent for all possible values of f and g . What can you say about the coefficients c and d ? Justify your answer.

$$\begin{aligned}x_1 + 3x_2 &= f \\ cx_1 + dx_2 &= g\end{aligned}$$

Solution:

The system can be represented by the following augmented matrices

$$\begin{bmatrix} 1 & 3 & f \\ c & d & g \end{bmatrix} \equiv \begin{bmatrix} 1 & 3 & f \\ 0 & d-3c & g-fc \end{bmatrix} \quad R2 = -c * R1 + R2$$

In order for the system to be consistent for all possible values of f and g , $d - 3c \neq 0$. In other words, so long as the values of c and d satisfy $d - 3c \neq 0$, the system will be consistent.

Problem 10 (1.2#1)

Determine which matrices are in reduced echelon form and which others are only in echelon form.

(a) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

(d) $\begin{bmatrix} 1 & 1 & 0 & 1 & 1 \\ 0 & 2 & 0 & 2 & 2 \\ 0 & 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 0 & 4 \end{bmatrix}$

Solution:

(a) RREF

(b) RREF

(c) Nothing

(d) REF

Problem 11 (1.2#3)

Row reduce the matrix to reduced row echelon form. Circle the pivot positions in the final matrix and in the original matrix, and list the pivot columns.

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 4 & 5 & 6 & 7 \\ 6 & 7 & 8 & 9 \end{bmatrix}$$

Solution:

$$\begin{aligned} \begin{bmatrix} \textcircled{1} & 2 & 3 & 4 \\ 4 & \textcircled{5} & 6 & 7 \\ 6 & 7 & 8 & 9 \end{bmatrix} &\equiv \begin{bmatrix} 1 & 2 & 3 & 4 \\ 4 & 5 & 6 & 7 \\ 0 & -5 & -10 & -15 \end{bmatrix} & R3 = -6 * R1 + R3 \\ &\equiv \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -3 & -6 & -9 \\ 0 & -5 & -10 & -15 \end{bmatrix} & R2 = -4 * R1 + R2 \\ &\equiv \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & -5 & -10 & -15 \end{bmatrix} & R2 = -\frac{1}{3}R2 \\ &\equiv \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} & R3 = 5 * R2 + R3 \\ &\equiv \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix} & R3 = \frac{1}{2}R3 \\ &\equiv \begin{bmatrix} \textcircled{1} & 0 & -1 & -2 \\ 0 & \textcircled{1} & 2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{aligned}$$

The pivot columns are C1, C2, and C4.

Problem 12 (1.2#7)

Find the general solution to the system whose augmented matrix is given below

$$\begin{bmatrix} 1 & 3 & 4 & 7 \\ 3 & 9 & 7 & 6 \end{bmatrix}$$

Solution:

$$\begin{aligned} \begin{bmatrix} 1 & 3 & 4 & 7 \\ 3 & 9 & 7 & 6 \end{bmatrix} &\equiv \begin{bmatrix} 1 & 3 & 4 & 7 \\ 0 & 0 & -5 & -15 \end{bmatrix} & R2 = -3 * R1 + R2 \\ &\equiv \begin{bmatrix} 1 & 3 & 4 & 7 \\ 0 & 0 & 1 & 3 \end{bmatrix} & R2 = -\frac{1}{3}R2 \\ &\equiv \begin{bmatrix} 1 & 3 & 0 & -5 \\ 0 & 0 & 1 & 3 \end{bmatrix} & R1 = -4 * R2 + R1 \\ &\equiv \begin{cases} x_1 = -5 - 3x_2 \\ x_3 = 3 \end{cases} \end{aligned}$$

The general solution to is all points that satisfy the system $\begin{cases} x_1 = -5 - 3x_2 \\ x_3 = 3 \end{cases}$, in other words all the points $(-5 - 3u, u, 3)$ for all possible values of u .

Problem 13 (1.2#11)

Find the general solution to the system whose augmented matrix is given below

$$\begin{bmatrix} 3 & -4 & 2 & 0 \\ -9 & 12 & -6 & 0 \\ -6 & 8 & -4 & 0 \end{bmatrix}$$

Solution:

$$\begin{aligned} \begin{bmatrix} 3 & -4 & 2 & 0 \\ -9 & 12 & -6 & 0 \\ -6 & 8 & -4 & 0 \end{bmatrix} &\equiv \begin{bmatrix} 3 & -4 & 2 & 0 \\ -9 & 12 & -6 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} & R3 = 2 * R1 + R3 \\ &\equiv \begin{bmatrix} 3 & -4 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} & R2 = 3 * R1 + R2 \\ &\equiv \begin{cases} 3x_1 - 4x_2 + 2x_3 = 0 \end{cases} \\ &\equiv \begin{cases} x_1 = \frac{4}{3}x_2 - \frac{2}{3}x_3 \end{cases} \end{aligned}$$

The general solution to is all points that satisfy the system $\left\{x_1 = \frac{4}{3}x_2 - \frac{2}{3}x_3\right\}$, in other words all the points $(\frac{4}{3}u - \frac{2}{3}v, u, v)$ for all possible values of u and v .

Problem 14 (1.2#19)

Choose h and k such that the system has (a) no solution, (b) a unique solution, and (c) many solutions. Give separate answers for each part.

$$\begin{aligned}x_1 + hx_2 &= 2 \\ 4x_1 + 8x_2 &= k\end{aligned}$$

Solution:

$$\begin{bmatrix} 1 & h & 2 \\ 4 & 8 & k \end{bmatrix} \equiv \begin{bmatrix} 1 & h & 2 \\ 0 & 8 - 4h & k - 8 \end{bmatrix} \quad R2 = -4 * R1 + R2$$

- (a) If $h = 2$ and $k = 1$, then the system becomes inconsistent and there is no solution
- (b) If $h = \frac{7}{4}$ and $k = 7$, then the system has the unique solution $(\frac{1}{4}, 1)$.
- (c) If $h = 2$ and $k = 8$, then the system has infinite solutions of the form $x_1 = 2 - 2x_2$.

Problem 15 (1.2#24)

Suppose a system of linear equations has a 3×5 *augmented* matrix whose fifth column is a pivot column. Is the system consistent? Why or why not?

Solution:

If the fifth column is a pivot column, then there is a pivot position in the third row and fifth column. This means that the last row is $\begin{bmatrix} 0 & 0 & 0 & 0 & 1 \end{bmatrix}$. This augmented matrix represents an inconsistent system.

Problem 16 (1.2#28)

What would you have to know about the pivot columns in an augmented matrix in order to know that the system is consistent and has a unique solution?

Solution:

If the last column of an augmented matrix is a pivot column, then the system is inconsistent. For example:

$$\begin{bmatrix} \textcircled{1} & 0 & 0 & 2 \\ 0 & \textcircled{1} & 0 & 3 \\ 0 & 0 & 0 & \textcircled{1} \end{bmatrix}$$

If every column except for the last is a pivot column then the system has a unique solution. For example:

$$\begin{bmatrix} \textcircled{1} & 0 & 0 & 1 \\ 0 & \textcircled{1} & 0 & 2 \\ 0 & 0 & \textcircled{1} & 3 \end{bmatrix}$$

Problem 17 (1.2#33)

Find the interpolating polynomial $p(t) = a_0 + a_1t + a_2t^2$ for the data $(1, 12)$, $(2, 15)$, $(3, 16)$. That is find a_0 , a_1 , and a_2 such that

$$a_0 + a_1(1) + a_2(1)^2 = 12$$

$$a_0 + a_1(2) + a_2(2)^2 = 15$$

$$a_0 + a_1(3) + a_2(3)^2 = 16$$

Solution:

The system, $\begin{cases} a_0 + a_1 + a_2 = 12 \\ a_0 + 2a_1 + 4a_2 = 15 \\ a_0 + 3a_1 + 9a_2 = 16 \end{cases}$ can be represented by the following augmented matrices

$$\begin{aligned}
\begin{bmatrix} 1 & 1 & 1 & 12 \\ 1 & 2 & 4 & 15 \\ 1 & 3 & 9 & 16 \end{bmatrix} &\equiv \begin{bmatrix} 1 & 1 & 1 & 12 \\ 1 & 2 & 4 & 15 \\ 0 & 2 & 8 & 4 \end{bmatrix} && R3 = -1 * R1 + R3 \\
&\equiv \begin{bmatrix} 1 & 1 & 1 & 12 \\ 0 & 1 & 3 & 3 \\ 0 & 2 & 8 & 4 \end{bmatrix} && R2 = -1 * R1 + R2 \\
&\equiv \begin{bmatrix} 1 & 1 & 1 & 12 \\ 0 & 1 & 3 & 3 \\ 0 & 0 & 2 & -2 \end{bmatrix} && R3 = -2 * R2 + R3 \\
&\equiv \begin{bmatrix} 1 & 1 & 1 & 12 \\ 0 & 1 & 3 & 3 \\ 0 & 0 & 1 & -1 \end{bmatrix} && R3 = \frac{1}{2} * R3 \\
&\equiv \begin{bmatrix} 1 & 1 & 1 & 12 \\ 0 & 1 & 0 & 6 \\ 0 & 0 & 1 & -1 \end{bmatrix} && R2 = -3 * R3 + R2 \\
&\equiv \begin{bmatrix} 1 & 1 & 0 & 13 \\ 0 & 1 & 0 & 6 \\ 0 & 0 & 1 & -1 \end{bmatrix} && R1 = -1 * R3 + R1 \\
&\equiv \begin{bmatrix} 1 & 0 & 0 & 7 \\ 0 & 1 & 0 & 6 \\ 0 & 0 & 1 & -1 \end{bmatrix} && R1 = -1 * R2 + R1 \\
&\equiv \begin{bmatrix} a_0 = 7 \\ a_1 = 6 \\ a_2 = -1 \end{bmatrix}
\end{aligned}$$

The interpolating polynomial $p(t) = a_0 + a_1t + a_2t^2$ for the data $(1, 12)$, $(2, 15)$, $(3, 16)$, is $p(t) = 7 + 6t - t^2$.

Problem 18 (1.3#1)

Compute $\mathbf{u} + \mathbf{v}$ and $\mathbf{u} - 2\mathbf{v}$

$$\mathbf{u} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} -3 \\ -1 \end{bmatrix}$$

Solution:

$$\mathbf{u} + \mathbf{v} = \begin{bmatrix} -1 \\ 2 \end{bmatrix} + \begin{bmatrix} -3 \\ -1 \end{bmatrix} = \begin{bmatrix} -4 \\ 1 \end{bmatrix}$$

$$\mathbf{u} - 2\mathbf{v} = \begin{bmatrix} -1 \\ 2 \end{bmatrix} - 2 \begin{bmatrix} -3 \\ -1 \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \end{bmatrix}$$

Problem 19 (1.3#5)

Write a system of equations that is equivalent to the given vector equation.

$$x_1 \begin{bmatrix} 6 \\ -1 \\ 5 \end{bmatrix} + x_2 \begin{bmatrix} -3 \\ 4 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -7 \\ -5 \end{bmatrix}$$

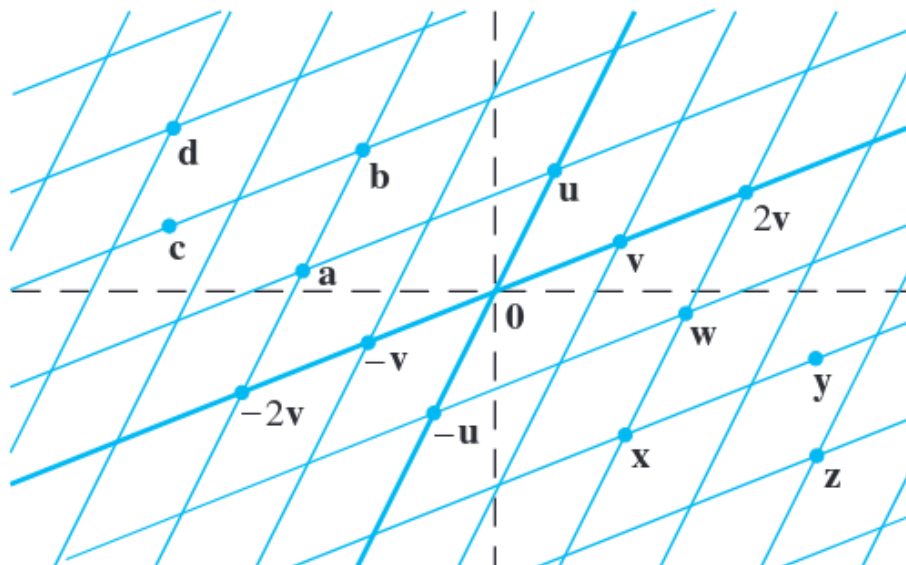
Solution:

The system of equations that is equivalent to the given vector equation is

$$\begin{cases} 6x_1 - 3x_2 = 1 \\ -x_1 + 4x_2 = -7 \\ 5x_1 = -5 \end{cases}$$

Problem 20 (1.3#7)

Use the accompanying figure to write vectors \mathbf{a} , \mathbf{b} , \mathbf{c} and \mathbf{d} as a linear combination of \mathbf{u} and \mathbf{v} .



Solution:

$$\mathbf{a} = \mathbf{u} - 2\mathbf{v}$$

$$\mathbf{b} = 2\mathbf{u} - 2\mathbf{v}$$

$$\mathbf{c} = 2\mathbf{u} - \frac{7}{2}\mathbf{v}$$

$$\mathbf{d} = 3\mathbf{u} - 4\mathbf{v}$$

Problem 21 (1.3#9)

Write a vector equation that is equivalent to the given system of equations.

$$x_2 + 5x_3 = 0$$

$$4x_1 + 6x_2 - x_3 = 0$$

$$-x_1 + 3x_2 - 8x_3 = 0$$

Solution:

The vector equation that is equivalent to the given system of equations is

$$x_1 \begin{bmatrix} 0 \\ 4 \\ -1 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 6 \\ 3 \end{bmatrix} + x_3 \begin{bmatrix} 5 \\ -1 \\ -8 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Problem 22 (1.3#13)

Determine if \mathbf{b} is a linear combination of the vectors formed from the columns of the matrix A .

$$A = \begin{bmatrix} 1 & -4 & 2 \\ 0 & 3 & 5 \\ -2 & 8 & -4 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 3 \\ -7 \\ -3 \end{bmatrix}$$

Solution:

$$\begin{bmatrix} 1 & -4 & 2 & 3 \\ 0 & 3 & 5 & 7 \\ -2 & 8 & -4 & -3 \end{bmatrix} \equiv \begin{bmatrix} 1 & -4 & 2 & 3 \\ 0 & 3 & 5 & 7 \\ 0 & 0 & 0 & 3 \end{bmatrix} \quad R3 = 2 * R1 + R3$$

\mathbf{b} is not a linear combination of the vectors formed from the columns of the matrix A .

Problem 23 (1.3#17)

Let $\mathbf{a}_1 = \begin{bmatrix} 1 \\ 4 \\ -2 \end{bmatrix}$, $\mathbf{a}_2 = \begin{bmatrix} -2 \\ -3 \\ 7 \end{bmatrix}$, and $\mathbf{b} = \begin{bmatrix} 4 \\ 1 \\ h \end{bmatrix}$. For what value(s) of h is \mathbf{b} in the plane spanned by \mathbf{a}_1 and \mathbf{a}_2 ?

Solution:

$$\begin{aligned}
\begin{bmatrix} 1 & -2 & 4 \\ 4 & -3 & 1 \\ -2 & 7 & h \end{bmatrix} &\equiv \begin{bmatrix} 1 & -2 & 4 \\ 4 & -3 & 1 \\ 0 & 3 & h+8 \end{bmatrix} && R3 = 2 * R1 + R3 \\
&\equiv \begin{bmatrix} 1 & -2 & 4 \\ 0 & 5 & -15 \\ 0 & 3 & h+8 \end{bmatrix} && R2 = -4 * R1 + R2 \\
&\equiv \begin{bmatrix} 1 & -2 & 4 \\ 0 & 1 & -3 \\ 0 & 3 & h+8 \end{bmatrix} && R2 = \frac{1}{5} R2 \\
&\equiv \begin{bmatrix} 1 & -2 & 4 \\ 0 & 1 & -3 \\ 0 & 0 & h+17 \end{bmatrix} && R3 = -3 * R2 + R3 \\
&\equiv \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & -3 \\ 0 & 0 & h+17 \end{bmatrix} && R1 = 2 * R2 + R1
\end{aligned}$$

In order for \mathbf{b} to be in the plane spanned by \mathbf{a}_1 and \mathbf{a}_2 , $h + 17 = 0$, so $h = -17$.

Problem 24 (1.3#19)

Give a geometric description of the Span $\{\mathbf{v}_1, \mathbf{v}_2\}$ for vectors $\mathbf{v}_1 = \begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} -2 \\ 0 \\ 3 \end{bmatrix}$

Solution:

The geometric description of the Span $\{\mathbf{v}_1, \mathbf{v}_2\}$ is the plane that contains the vectors $\begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} -2 \\ 0 \\ 3 \end{bmatrix}$ and passes through the origin.

Problem 25 (1.3#24)

Determine whether each statement is True or False. Justify your answer.

- (a) Any list of five real numbers is a vector in \mathbb{R}^5 .
- (b) The vector \mathbf{u} results when a vector $\mathbf{u} - \mathbf{v}$ is added to the vector \mathbf{v} .
- (c) The weights c_1, \dots, c_p in a linear combination $c_1\mathbf{v}_1, \dots, c_p\mathbf{v}_p$ cannot be all zero.
- (d) When \mathbf{u} and \mathbf{v} are nonzero vectors, Span $\{\mathbf{u}, \mathbf{v}\}$ contains the line through \mathbf{u} and the origin.
- (e) Asking whether the linear system corresponding to an augmented matrix $\begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 & \mathbf{b} \end{bmatrix}$ has a solution amounts to asking if \mathbf{b} is in Span $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$

Solution:

- (a) True, \mathbb{R}^n is the set of all lists of real numbers of length n .
- (b) True, $(\mathbf{u} - \mathbf{v}) + \mathbf{v} = \mathbf{u} + (\mathbf{v} - \mathbf{v}) = \mathbf{u}$.
- (c) False, the zero matrix is a linear combination of any set of vectors with zero weights.
- (d) True, the Span $\{\mathbf{u}, \mathbf{v}\}$ contains all points that are a linear combination of \mathbf{u} and \mathbf{v} and all the points on the line through \mathbf{u} and the origin are a linear combination of \mathbf{u} and \mathbf{v} .
- (e) True, if \mathbf{b} is in Span $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$, then there is some linear combination of $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$ that is equal to \mathbf{b} , and in order for a linear combination to exist there must be a solution to the linear system.

Problem 26 (1.3#25)

Let $A = \begin{bmatrix} 1 & 0 & -4 \\ 0 & 3 & -2 \\ -2 & 6 & 3 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 4 \\ 1 \\ -4 \end{bmatrix}$. Denote the columns of A by $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$, and let $W = \text{Span}\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$.

- (a) is \mathbf{b} in $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$? How many vectors are in $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$?
- (b) is \mathbf{b} in W ? How many vectors are in W ?
- (c) Show that \mathbf{a}_1 is in W . [Hint: Row operations are unnecessary.]

Solution:

- (a) No, there are three vectors in the set $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$
- (b) Yes, there are an infinite number of vectors in W .
- (c) The vector \mathbf{a}_1 can be written as a linear combination of $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$, so it is in W ($\mathbf{a}_1 = 1\mathbf{a}_1 + 0\mathbf{a}_2 + 0\mathbf{a}_3$).

Problem 27 (1.3#29)

Let v_1, \dots, v_k be points in \mathbb{R}^3 and supposed that for $j = 1, \dots, k$ an object with mass m_j is located at point v_j . Physicists call such objects *point masses*. The total mass of the system of point masses is

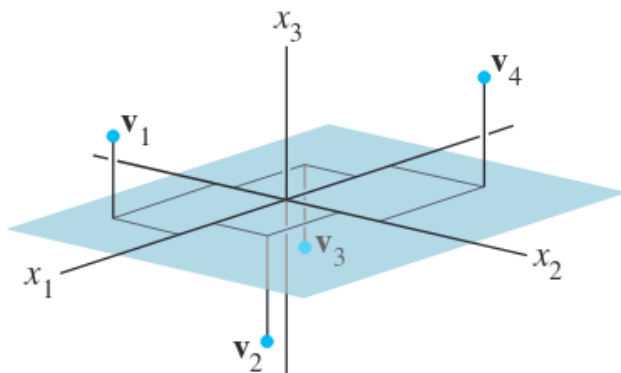
$$m = m_1 + \dots + m_k$$

The *center of mass* of the system is

$$\mathbf{v} = \frac{1}{m}(m_1 v_1 + m_2 v_2 + \dots + m_k v_k)$$

Compute the center of gravity of the system consisting of the following point masses

Point	Mass
$\mathbf{v}_1 = (5, -4, 3)$	2 g
$\mathbf{v}_2 = (4, 3, -2)$	5 g
$\mathbf{v}_3 = (-4, -3, -1)$	2 g
$\mathbf{v}_4 = (-9, 8, 6)$	1 g



Solution:

$$\mathbf{v} = \frac{1}{10} \left(2 \begin{bmatrix} 5 \\ -4 \\ 3 \end{bmatrix} + 5 \begin{bmatrix} 4 \\ 3 \\ -1 \end{bmatrix} + 2 \begin{bmatrix} -4 \\ -3 \\ -1 \end{bmatrix} + \begin{bmatrix} -9 \\ 8 \\ 6 \end{bmatrix} \right) = \begin{bmatrix} \frac{13}{10} \\ \frac{9}{10} \\ 0 \end{bmatrix}$$

The center of mass of the system is at $(\frac{13}{10}, \frac{9}{10}, 0)$.

Problem 28 (1.3#34)

Use the vector $\mathbf{u} = (u_1, \dots, u_n)$ to verify the following algebraic properties of \mathbb{R}^n .

- (a) $\mathbf{u} + (-\mathbf{u}) = (-\mathbf{u}) + \mathbf{u} = \mathbf{0}$
- (b) $c(d\mathbf{u}) = (cd)\mathbf{u}$ for all scalars c and d

Solution:

Proof. $\mathbf{u} + (-\mathbf{u}) = (-\mathbf{u}) + \mathbf{u}$

$$\begin{aligned}
 \mathbf{u} + (-\mathbf{u}) &= (u_1, \dots, u_n) - (u_1, \dots, u_n) && \text{(by substitution)} \\
 &= (u_1 - u_1, \dots, u_n - u_n) && \text{(by vector addition)} \\
 &= ((-u_1) + u_1, \dots, (-u_n) + u_n) && \text{(by associative property of addition)} \\
 &= (-\mathbf{u}) + \mathbf{u} && \text{(by vector addition)}
 \end{aligned}$$

□

Proof. $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$

$$\begin{aligned}
 \mathbf{u} + (-\mathbf{u}) &= (u_1, \dots, u_n) - (u_1, \dots, u_n) && \text{(by substitution)} \\
 &= (u_1 - u_1, \dots, u_n - u_n) && \text{(by vector addition)} \\
 &= (0, \dots, 0) && \text{(by addition)} \\
 &= \mathbf{0} && \text{(by def. of zero vector)}
 \end{aligned}$$

□

Proof. $c(d\mathbf{u}) = (cd)\mathbf{u}$ for all scalars c and d

$$\begin{aligned}
 c(d\mathbf{u}) &= c(du_1, \dots, cu_n) && \text{(by scalar multiplication)} \\
 &= (c(du_1), \dots, c(cu_n)) && \text{(by scalar multiplication of vector)} \\
 &= ((cd)u_1, \dots, (cd)u_n) && \text{(by commutative property of multiplication)} \\
 &= (cd)\mathbf{u} && \text{(by scalar multiplication of vector)}
 \end{aligned}$$

□