Carson Storm
Section 6.7 # 1, 3, 4, 7, 10, 13, 18, 21, 25

1) a) $\|\vec{x}\| = \sqrt{\langle \vec{x}, \vec{x} \rangle} = \sqrt{4x_1^2 + 5x_2^2}$
$= \sqrt{9} = 3$
$\|\vec{x}\| = 3$

$\|\vec{y}\| = \sqrt{\langle \vec{y}, \vec{y} \rangle} = \sqrt{4y_1^2 + 5y_2^2} = \sqrt{100 + 5} = \sqrt{105}$
$\|\vec{y}\| = \sqrt{105}$

$|\langle \vec{x}, \vec{y} \rangle|^2 = |4x_1y_1 + 5x_2y_2|^2 = |20 - 5|^2 = |15|^2 = 225$
$|\langle \vec{x}, \vec{y} \rangle|^2 = 225$

b) $\langle \vec{z}, \vec{y} \rangle = 0 = 4z_1y_1 + 5z_2y_2 = 0$
$= 20z_1 - 5z_2$
$0 = 4z_1 - z_2$

$z_1 = \frac{1}{4} z_2$

All $\vec{z} \in \text{Span}\left\{\begin{bmatrix} 1 \\ 4 \end{bmatrix}\right\}$ are orthogonal to $\vec{y}$

3) $P(t) = 4 + t$, $q(t) = 5 - 4t^2$
$\langle P, q \rangle = P(t_0)q(t_0) + P(t_1)q(t_1) + \dots + P(t_n)q(t_n)$
$\{t_n\} = \{-1, 0, 1\}$
$\langle P, q \rangle = P(-1)q(-1) + P(0)q(0) + P(1)q(1)$
$= 3(1) + 4(5) + 5(1)$
$= 3 + 20 + 5 = 28$
$\langle P, q \rangle = 28$

4) $P(t) = 3t - t^2$, $q(t) = 3 + 2t^2$
$\langle P, q \rangle = P(-1)q(-1) + P(0)q(0) + P(1)q(1)$
$= -4(5) + 0(q(0)) + 2(5)$
$= -20 + 10 = -10$
$\langle P, q \rangle = -10$

7) $\text{Proj}_P\, q = \langle P, q \rangle \cdot \frac{P}{\|P\|^2} = 28 \cdot \frac{4+t}{\langle P, P \rangle} = 28 \cdot \frac{4+t}{9+16+25} = \frac{28(4+t)}{50} = \frac{14(4+t)}{25} = \frac{56+14t}{25}$

$\text{Proj}_P\, q(t) = \frac{56+14t}{25}$

10) $\{P_0, P_1, q\} = \{1, t, \frac{t^2-5}{4}\}$
$V = \text{Span}\{P_0, P_1, q\}$
$P(t) = t^3$

$\text{Proj}_V\, P = \frac{\langle P, P_0 \rangle}{\langle P_0, P_0 \rangle} P_0 + \frac{\langle P, P_1 \rangle}{\langle P_1, P_1 \rangle} P_1 + \frac{\langle P, q \rangle}{\langle q, q \rangle} q$
$= \frac{0}{4} \cdot 1 + \frac{164}{20} \cdot t + \frac{0}{4} \cdot \frac{t^2-5}{4}$
$= \frac{41}{5} t$

The best approximation of $P(t) = t^3$ in $\text{Span}\{P_0, P_1, q\}$ is
$\text{Proj}_V\, P(t) = \frac{41}{5} t$

13) $\langle \vec{u},\vec{v}\rangle = (A\vec{u})\cdot(A\vec{v})$

1. $\langle \vec{u},\vec{v}\rangle = (A\vec{u})\cdot(A\vec{v}) = (A\vec{v})\cdot(A\vec{u})$
   $\langle \vec{u},\vec{v}\rangle = \langle \vec{v},\vec{u}\rangle$

2. $\langle \vec{u}+\vec{v},\vec{w}\rangle = (A(\vec{u}+\vec{v}))\cdot(A\vec{w})$
   $= (A\vec{u}+A\vec{v})\cdot(A\vec{w})$
   $= (A\vec{u})\cdot(A\vec{w}) + (A\vec{v})\cdot(A\vec{w})$
   $= \langle \vec{u},\vec{w}\rangle + \langle \vec{v},\vec{w}\rangle$

3) $\langle cu,v\rangle = (A(c\vec{u}))\cdot(A\vec{v})$
   $= (cA\vec{u})\cdot A\vec{v}$
   $= c\,((A\vec{u})\cdot(A\vec{v}))$
   $= c\langle u,v\rangle$

4) let $(A\vec{u}) = [a_1\ a_2\ \dots\ a_n]$
   $\langle \vec{u},\vec{u}\rangle = (A\vec{u})^T(A\vec{u}) = a_1^2+a_2^2+\dots+a_n^2$
   $\langle \vec{u},\vec{u}\rangle \geq 0$ Since $a_n^2 \geq 0\ \forall_n$
   $\langle \vec{u},\vec{u}\rangle = 0 \Rightarrow a_n = 0\ \forall_n \Rightarrow \vec{u} = 0$ because A is invertible

$\boxed{\text{Therefore, } \langle u,v\rangle = (A\vec{u})\cdot(A\vec{v}) \text{ defines a inner product space because it satisfies all of it's axioms.}}$

18) Show that $\|\vec{u}+\vec{v}\|^2 + \|\vec{u}-\vec{v}\|^2 = 2\|\vec{u}\|^2 + 2\|\vec{v}\|^2$

$\|\vec{u}+\vec{v}\|^2 + \|\vec{u}-\vec{v}\|^2 = \langle \vec{u}+\vec{v},\vec{u}+\vec{v}\rangle + \langle \vec{u}-\vec{v},\vec{u}-\vec{v}\rangle$

$= \langle \vec{u},\vec{u}+\vec{v}\rangle + \langle \vec{v},\vec{u}+\vec{v}\rangle + \langle \vec{u},\vec{u}-\vec{v}\rangle - \langle \vec{v},\vec{u}-\vec{v}\rangle$ (Axiom 2)

$= \langle \vec{u}+\vec{v},\vec{u}\rangle + \langle \vec{u}+\vec{v},\vec{v}\rangle + \langle \vec{u}-\vec{v},\vec{u}\rangle - \langle \vec{u}-\vec{v},\vec{v}\rangle$ (Axiom 1)

$= \langle \vec{u},\vec{u}\rangle + \langle \vec{v},\vec{u}\rangle + \langle \vec{u},\vec{v}\rangle + \langle \vec{v},\vec{v}\rangle + \langle \vec{u},\vec{u}\rangle - \langle \vec{v},\vec{u}\rangle - \langle \vec{u},\vec{v}\rangle + \langle \vec{v},\vec{v}\rangle$ (Axiom 2)

$= 2\langle \vec{u},\vec{u}\rangle + 2\langle \vec{v},\vec{v}\rangle$

$= 2\|\vec{u}\|^2 + 2\|\vec{v}\|^2$

21) In $C[0,1]$ $\langle f,g\rangle = \int_0^1 f(t)g(t)\,dt$, $f(t) = 1-3t^2$, $g(t) = t-t^3$

$\langle f,g\rangle = \int_0^1 (1-3t^2)(t-t^3)\,dt = \int_0^1 (3t^5-4t^3+t)\,dt = \left[\frac{1}{2}t^6 - t^4 + \frac{1}{2}t^2\right]_0^1 = \frac{1}{2} - 1 + \frac{1}{2} - 0 + 0 - 0$
$= 0$

$\langle f,g\rangle = 0$

25) $\{x_1,x_2,x_3\} = \{1,t,t^2\}$, $\langle f,g\rangle = \int_{-1}^{1} f(t)g(t)\,dt$

$V_1 = x_1$ $\quad V_2 = x_2$ $\quad V_3 = x_3 - \frac{\langle x_3,V_1\rangle}{\langle V_1,V_1\rangle}V_1 - \frac{\langle x_3,V_2\rangle}{\langle V_2,V_2\rangle}V_2$

$= t^2 - \frac{(2/3)}{2}(1) - \frac{(0)}{(2/3)}(t)$

$= t^2 - \frac{1}{3}$

A orthogonal basis for span$\{1,t,t^2\}$ in $C[-1,-1]$ is $\{1,t,t^2-\frac{1}{3}\}$