Linear Algebra: Final Extra Credit

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The inner product is essential for many of the concepts used in Chapter 6, but so far we have only defined the inner product for real number spaces, $\mathbb{R}^n$. In order to expand the notion of an inner product to other vector spaces we define a set of axioms that a space's inner product must satisfy, so that we can define an inner product for other vector spaces. A inner product is an operation that takes two vectors in the considered vectors space as operands and associated with them a real number. This inner product can be used to define what a vector's length is, distances in the space as well as orthogonality. For example, be defining a inner product for $\mathbb{P}^n$, we define what it means for two polynomials to be orthogonal. After defining the inner product we can utilize concepts such as the Gram-Schmidt process, or solving Least-Squares problems in spaces that are not $\mathbb{R}^n$.

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Section 6.7 # 1, 3, 4, 7, 10, 13, 18, 21, 25

1) a) $\|\vec{x}\| = \sqrt{\langle \vec{x},\vec{x}\rangle} = \sqrt{4x_1^2+5x_2^2}$
$= \sqrt{9} = 3$
$\|\vec{x}\| = 3$

$\|\vec{y}\| = \sqrt{\langle \vec{y},\vec{y}\rangle} = \sqrt{4y_1^2+5y_2^2} = \sqrt{100+5} = \sqrt{105}$
$\|\vec{y}\| = \sqrt{105}$

$|\langle \vec{x},\vec{y}\rangle|^2 = |4x_1y_1+5x_2y_2|^2 = |20-5|^2 = |15|^2 = 225$
$|\langle \vec{x},\vec{y}\rangle|^2 = 225$

b) $\langle \vec{z},\vec{y}\rangle = 0 = 4z_1y_1 + 5z_2y_2 = 0$
$= 20z_1 - 5z_2$
$0 = 4z_1 - z_2$

$z_1 = \frac{1}{4} z_2$

All $\vec{z} \in \text{Span}\left\{\begin{bmatrix}1\\4\end{bmatrix}\right\}$ are orthogonal to $\vec{y}$

3) $P(t) = 4+t$, $q(t) = 5-4t^2$

$\langle P, q\rangle = P(t_0)q(t_0) + P(t_1)q(t_1) + \dots + P(t_n)q(t_n)$
$\{t_n\} = \{-1, 0, 1\}$
$\langle P, q\rangle = P(-1)q(-1) + P(0)q(0) + P(1)q(1)$
$= 3(1) + 4(5) + 5(1)$
$= 3+20+5 = 28$
$\langle P, q\rangle = 28$

4) $P(t) = 3t - t^2$, $q(t) = 3+2t^2$
$\langle P, q\rangle = P(-1)q(-1) + P(0)q(0) + P(1)q(1)$
$= -4(5) + 0(q(0)) + 2(5)$
$= -20 + 10 = -10$
$\langle P, q\rangle = -10$

7) $\text{Proj}_P\, q = \langle P,q\rangle \cdot \frac{P}{\|P\|^2} = 28 \cdot \frac{4+t}{\langle P,P\rangle} = 28 \cdot \frac{4+t}{9+16+25} = \frac{28(4+t)}{50} = \frac{14(4+t)}{25} = \frac{56+14t}{25}$

$\text{Proj}_P\, q(t) = \frac{56+14t}{25}$

10) $\{P_0, P_1, q\} = \{1, t, \frac{t^2-5}{4}\}$
$V = \text{Span}\{P_0, P_1, q\}$,
$P(t) = t^3$

$\text{Proj}_V P = \frac{\langle P, P_0\rangle}{\langle P_0, P_0\rangle} P_0 + \frac{\langle P, P_1\rangle}{\langle P_1, P_1\rangle} P_1 + \frac{\langle P, q\rangle}{\langle q, q\rangle} q$
$= \frac{0}{4} \cdot 1 + \frac{164}{20} \cdot t + \frac{0}{4} \cdot \frac{t^2-5}{4}$
$= \frac{41}{5} t$

The best approximation of $P(t) = t^3$ in $\text{Span}\{P_0, P_1, q\}$ is
$\text{Proj}_V P(t) = \frac{41}{5} t$

13) $\langle \vec{u}, \vec{v} \rangle = (A\vec{u}) \cdot (A\vec{v})$

1. $\langle \vec{u}, \vec{v} \rangle = (A\vec{u}) \cdot (A\vec{v}) = (A\vec{v}) \cdot (A\vec{u})$
   $\langle \vec{u}, \vec{v} \rangle = \langle \vec{v}, \vec{u} \rangle$

2. $\langle \vec{u}+\vec{v}, \vec{w} \rangle = (A(\vec{u}+\vec{v})) \cdot (A\vec{w})$
   $= (A\vec{u} + A\vec{v}) \cdot (A\vec{w})$
   $= (A\vec{u}) \cdot (A\vec{w}) + (A\vec{v}) \cdot (A\vec{w})$
   $= \langle \vec{u}, \vec{w} \rangle + \langle \vec{v}, \vec{w} \rangle$

3) $\langle c u, v \rangle = (A(c\vec{u})) \cdot (A\vec{v})$
   $= (cA\vec{u}) \cdot A\vec{v}$
   $= c\,((A\vec{u}) \cdot (A\vec{v}))$
   $= c \langle u, v \rangle$

4) let $(A\vec{u}) = [a_1 \; a_2 \; \dots \; a_n]$
   $\langle \vec{u}, \vec{u} \rangle = (A\vec{u})^T (A\vec{u}) = a_1^2 + a_2^2 + \dots + a_n^2$
   $\langle \vec{u}, \vec{u} \rangle \geq 0$ since $a_n^2 \geq 0 \; \forall_n$
   $\langle \vec{u}, \vec{u} \rangle = 0 \Rightarrow a_n = 0 \; \forall_n \Rightarrow \vec{u} = 0$ because A is invertible

$\boxed{\text{Therefore, } \langle u, v \rangle = (A\vec{u}) \cdot (A\vec{v}) \text{ defines a inner product space because it satisfies all of it's axioms.}}$

18) Show that $\|\vec{u}+\vec{v}\|^2 + \|\vec{u}-\vec{v}\|^2 = 2\|\vec{u}\|^2 + 2\|\vec{v}\|^2$

$\|\vec{u}+\vec{v}\|^2 + \|\vec{u}-\vec{v}\|^2 = \langle \vec{u}+\vec{v}, \vec{u}+\vec{v} \rangle + \langle \vec{u}-\vec{v}, \vec{u}-\vec{v} \rangle$

$= \langle \vec{u}, \vec{u}+\vec{v} \rangle + \langle \vec{v}, \vec{u}+\vec{v} \rangle + \langle \vec{u}, \vec{u}-\vec{v} \rangle - \langle \vec{v}, \vec{u}-\vec{v} \rangle$ (Axiom 2)

$= \langle \vec{u}+\vec{v}, \vec{u} \rangle + \langle \vec{u}+\vec{v}, \vec{v} \rangle + \langle \vec{u}-\vec{v}, \vec{u} \rangle - \langle \vec{u}-\vec{v}, \vec{v} \rangle$ (Axiom 1)

$= \underline{\langle \vec{u}, \vec{u} \rangle} + \langle \vec{v}, \vec{u} \rangle + \langle \vec{u}, \vec{v} \rangle + \underline{\langle \vec{v}, \vec{v} \rangle} + \underline{\langle \vec{u}, \vec{u} \rangle} - \langle \vec{v}, \vec{u} \rangle - \langle \vec{u}, \vec{v} \rangle + \underline{\langle \vec{v}, \vec{v} \rangle}$ (Axiom 2)

$= 2\langle \vec{u}, \vec{u} \rangle + 2\langle \vec{v}, \vec{v} \rangle$

$= 2\|\vec{u}\|^2 + 2\|\vec{v}\|^2$

21) In $C[0,1]$ $\langle f, g \rangle = \int_0^1 f(t)g(t)\,dt$, $f(t) = 1 - 3t^2$, $g(t) = t - t^3$

$\langle f, g \rangle = \int_0^1 (1-3t^2)(t-t^3)\,dt = \int_0^1 (3t^5 - 4t^3 + t)\,dt = \left[\frac{1}{2}t^6 - t^4 + \frac{1}{2}t^2\right]_0^1 = \frac{1}{2} - 1 + \frac{1}{2} - 0 + 0 - 0$
$= 0$

$\langle f, g \rangle = 0$

25) $\{x_1, x_2, x_3\} = \{1, t, t^2\}$, $\langle f, g \rangle = \int_{-1}^1 f(t)g(t)\,dt$

$v_1 = x_1 \quad v_2 = x_2 \quad v_3 = x_3 - \frac{\langle x_3, v_1 \rangle}{\langle v_1, v_1 \rangle} v_1 - \frac{\langle x_3, v_2 \rangle}{\langle v_2, v_2 \rangle} v_2$

$= t^2 - \frac{(2/3)}{2}(1) - \frac{(0)}{(2/3)}(t)$

$= t^2 - \frac{1}{3}$

A orthogonal basis for span$\{1, t, t^2\}$ in $C[-1,-1]$ is $\{1, t, t^2 - \frac{1}{3}\}$