Linear Algebra: Homework #2

Due on September 4, 2019

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Problem 1 (1.4#1)

Compute the product using (a) the definition, as in Example 1, and (b) the row–vector rule for computing $A\boldsymbol{x}$. If a product is undefined, explain why.

$$\begin{bmatrix} -4 & 2 \\ 1 & 6 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ -2 \\ 7 \end{bmatrix}$$

Solution:

The product is undefined because the number of columns in A is not equal to the number of rows in $\boldsymbol{x}$.

Problem 2 (1.4#3)

Compute the product using (a) the definition, as in Example 1, and (b) the row–vector rule for computing $A\boldsymbol{x}$. If aproduct is undefined, explain why.

$$\begin{bmatrix} 6 & 5 \\ -4 & 4 \\ 7 & 6 \end{bmatrix} \begin{bmatrix} 2 \\ -3 \end{bmatrix}$$

Part A:

$$\begin{bmatrix} 6 & 5 \\ -4 & 4 \\ 7 & 6 \end{bmatrix} \begin{bmatrix} -2 \\ 3 \end{bmatrix} = 2 \begin{bmatrix} 6 \\ -4 \\ 7 \end{bmatrix} - 3 \begin{bmatrix} 5 \\ -3 \\ 6 \end{bmatrix} = \begin{bmatrix} -3 \\ 1 \\ -4 \end{bmatrix}$$

Part B:

$$\begin{bmatrix} 2(6) - 3(5) \\ 2(-4) - 3(-3) \\ 2(7) - 3(6) \end{bmatrix} = \begin{bmatrix} -3 \\ 1 \\ -4 \end{bmatrix}$$

Problem 3 (1.4#5)

Use the definition of $A\boldsymbol{x}$ to write the matrix equation as a vector equation.

$$\begin{bmatrix} 5 & 1 & -8 & 4 \\ -2 & -7 & 3 & -5 \end{bmatrix} \begin{bmatrix} 5 \\ -1 \\ 3 \\ -2 \end{bmatrix} = \begin{bmatrix} -8 \\ 16 \end{bmatrix}$$

Solution:

$$5 \begin{bmatrix} 5 \\ -2 \end{bmatrix} - \begin{bmatrix} 1 \\ -7 \end{bmatrix} + 3 \begin{bmatrix} -8 \\ 3 \end{bmatrix} - 2 \begin{bmatrix} 4 \\ -5 \end{bmatrix} = \begin{bmatrix} -8 \\ 16 \end{bmatrix}$$

Problem 4 (1.4#7)

Use the definition of $A\boldsymbol{x}$ to write the vector equation as a matrix equation.

$$\begin{bmatrix} 4 \\ -1 \\ 7 \\ -4 \end{bmatrix} x_1 + \begin{bmatrix} -5 \\ 3 \\ -5 \\ 1 \end{bmatrix} x_2 + \begin{bmatrix} 7 \\ -8 \\ 0 \\ 2 \end{bmatrix} x_3 = \begin{bmatrix} 6 \\ -8 \\ 0 \\ -7 \end{bmatrix}$$

Solution:

$$\begin{bmatrix} 4 & -5 & 7 \\ -1 & 3 & -8 \\ 7 & -5 & 0 \\ -4 & 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 6 \\ -8 \\ 0 \\ -7 \end{bmatrix}$$

Problem 5 (1.4#11)

Given A and $\boldsymbol{b}$, write the augmented matrix for the linear system that corresponds to the matrix equation $A\boldsymbol{x} = \boldsymbol{b}$. Then solve the system and write the solution as a vector.

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 5 \\ -2 & -4 & -3 \end{bmatrix}, \boldsymbol{b} = \begin{bmatrix} -2 \\ 2 \\ 9 \end{bmatrix}$$

Solution:

$$\begin{aligned}
\begin{bmatrix} 1 & 2 & 4 & -2 \\ 0 & 1 & 5 & 2 \\ -2 & -4 & -3 & 9 \end{bmatrix} &\equiv \begin{bmatrix} 1 & 2 & 4 & -2 \\ 0 & 1 & 5 & 2 \\ 0 & 0 & 5 & 5 \end{bmatrix} && R3 = 2 * R1 + R3 \\
&\equiv \begin{bmatrix} 1 & 2 & 4 & -2 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 5 & 5 \end{bmatrix} && R2 = -R3 + R2 \\
&\equiv \begin{bmatrix} 1 & 2 & 4 & -2 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 1 \end{bmatrix} && R3 = \frac{1}{5} R3 \\
&\equiv \begin{bmatrix} 1 & 0 & 4 & 4 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 1 \end{bmatrix} && R1 = -2 * R2 + R1 \\
&\equiv \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 1 \end{bmatrix} && R1 = -4 * R3 + R1
\end{aligned}$$

$$\boldsymbol{x} = \begin{bmatrix} 0 \\ -3 \\ 1 \end{bmatrix}$$

Problem 6 (1.4#15)

Let $A = \begin{bmatrix} 2 & -1 \\ -6 & 3 \end{bmatrix}$ and $\boldsymbol{b} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$. Show that the equation $A\boldsymbol{x} = \boldsymbol{b}$ does not have a solution for all possible $\boldsymbol{b}$, and describe the solution set of all $\boldsymbol{b}$ for which $A\boldsymbol{x} = \boldsymbol{b}$ *does* has a solution.

Solution:

$$\begin{bmatrix} 2 & -1 & b_1 \\ -6 & 3 & b_2 \end{bmatrix} \equiv \begin{bmatrix} 2 & -1 & b_1 \\ 0 & 0 & 3b_1 + b_2 \end{bmatrix} \qquad R2 = 3 * R1 + R2$$

$A\boldsymbol{x} = \boldsymbol{b}$ has a solution for all b_1 and b_2 such that $3b_1 + b_2 = 0$.

Problem 7 (1.4#19)

Can each vector in $\mathbb{R}^4$ be written as a linear combination of the columns of the matrix A? Do the columns of A span $\mathbb{R}^4$?

$$A = \begin{bmatrix} 1 & 3 & 0 & 3 \\ -1 & -1 & -1 & 1 \\ 0 & -4 & 2 & -8 \\ 2 & 0 & 3 & -1 \end{bmatrix}$$

Solution:

$$\begin{bmatrix} 1 & 3 & 0 & 3 \\ -1 & -1 & -1 & 1 \\ 0 & -4 & 2 & -8 \\ 2 & 0 & 3 & -1 \end{bmatrix} \equiv \begin{bmatrix} 1 & 3 & 0 & 3 \\ 0 & 2 & -1 & 4 \\ 0 & -4 & 2 & -8 \\ 2 & 0 & 3 & -1 \end{bmatrix} \qquad R2 = R1 + R2$$

$$\equiv \begin{bmatrix} 1 & 3 & 0 & 3 \\ 0 & 2 & -1 & 4 \\ 0 & 0 & 0 & 0 \\ 2 & 0 & 3 & -1 \end{bmatrix} \qquad R3 = 2 * R2 + R3$$

No, the columns of A do not span $\mathbb{R}^4$ because they are not linearly independent.

Problem 8 (1.4#21)

Let $\boldsymbol{v_1} = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}$, $\boldsymbol{v_2} = \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}$, $\boldsymbol{v_3} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}$. Does $\{\boldsymbol{v_1}, \boldsymbol{v_2}, \boldsymbol{v_3}\}$ span $\mathbb{R}^3$? Why or why not?

Solution:

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 1 & -1 \end{bmatrix} \equiv \begin{bmatrix} 1 & 0 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix} \qquad R3 = R1 + R3$$

$$\equiv \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & -1 \\ 0 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix} \qquad R2 = R4 + R2$$

$$\equiv \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 1 & -1 \end{bmatrix} \qquad R3 = R2 + R3$$

$$\equiv \begin{bmatrix} 1 & 0 & 1 & & \\ 0 & 1 & -1 & & \\ 0 & 0 & 00 & 0 & -1 \end{bmatrix} \qquad \text{Swap R2 \& R4}$$

$$\equiv \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix} \qquad \text{Swap R3 \& R4}$$

$$\equiv \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \qquad R3 = -R3$$

$$\equiv \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \qquad R2 = R3 + R2$$

$$\equiv \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \qquad R1 = R1 - R2$$

No, $\{\boldsymbol{v_1}, \boldsymbol{v_2}, \boldsymbol{v_3}\}$ does not spans all of $\mathbb{R}^3$ because there is not a pivot position in each row.

Problem 9 (1.4#25)

Note that $\begin{bmatrix} 4 & -3 & 1 \\ 5 & -2 & 5 \\ -6 & 2 & -3 \end{bmatrix} \begin{bmatrix} -3 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} -7 \\ -3 \\ 10 \end{bmatrix}$. Use this fact (and no row operations) to find scalars c_1, c_2, c_3 such that $\begin{bmatrix} -7 \\ -3 \\ 10 \end{bmatrix} = c_1 \begin{bmatrix} 4 \\ 5 \\ -6 \end{bmatrix} + c_2 \begin{bmatrix} -3 \\ -2 \\ 2 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ 5 \\ -3 \end{bmatrix}$.

Solution:

The scalars are $c_1 = -3, c_2 = -1, c_3 = 2$.

Problem 10 (1.5#1)

Determine if the system has a nontrival solution. Try to use as few row operations as possible.

$$\begin{aligned} 2x_1 - 5x_2 + 8x_3 &= 0 \\ -2x_2 - 7x_2 + x_3 &= 0 \\ 4x_1 + 2x_2 + 7x_3 &= 0 \end{aligned}$$

Solution:

$$\begin{aligned}
\left\{ \begin{aligned} 2x_1 - 5x_2 + 8x_3 &= 0 \\ -2x_2 - 7x_2 + x_3 &= 0 \\ 4x_1 + 2x_2 + 7x_3 &= 0 \end{aligned} \right\} &\equiv \begin{bmatrix} 2 & -5 & 8 & 0 \\ -2 & -7 & 1 & 0 \\ 4 & 2 & 7 & 0 \end{bmatrix} \\
&\equiv \begin{bmatrix} 2 & -5 & 8 & 0 \\ 0 & -12 & 9 & 0 \\ 4 & 2 & 7 & 0 \end{bmatrix} && R2 = R1 + R2 \\
&\equiv \begin{bmatrix} 2 & -5 & 8 & 0 \\ 0 & -12 & 9 & 0 \\ 0 & 12 & -9 & 0 \end{bmatrix} && R3 = -2 * R1 + R3 \\
&\equiv \begin{bmatrix} 2 & -5 & 8 & 0 \\ 0 & -12 & 9 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} && R3 = R2 + R3
\end{aligned}$$

The system has a nontrival solution because there is a free variable.

Problem 11 (1.5#3)

Determine if the system has a nontrival solution. Try to use as few row operations as possible.

$$\begin{aligned} -3x_1 + 5x_2 - 7x_3 &= 0 \\ -6x_1 + 7x_2 + x_3 &= 0 \end{aligned}$$

Solution:

The system has a nontrival solution because it is undetermined, so it has a free variable.

Problem 12 (1.5#7)

Describe all solutions of $A\boldsymbol{x} = \mathbf{0}$ in parametric form, where A is row equivalent to the given matrix.

$$\begin{bmatrix} 1 & 3 & -3 & 7 \\ 0 & 1 & -4 & 5 \end{bmatrix}$$

Solution:

$$A = \begin{bmatrix} 1 & 3 & -3 & 7 \\ 0 & 1 & -4 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 9 & -8 \\ 0 & 1 & -4 & 5 \end{bmatrix} \qquad R1 = -3 * R2 + R1$$

$$A\boldsymbol{x} = 0 \equiv \begin{Bmatrix} x_1 + 9x_3 - 8x_4 = 0 \\ x_2 - 4x_3 + 5x_4 = 0 \end{Bmatrix}$$

$$\equiv \begin{Bmatrix} x_1 = 8x_4 - 9x_3 \\ x_2 = 4x_3 - 5x_4 \end{Bmatrix}$$

$$\implies \boldsymbol{x} = x_3 \begin{bmatrix} -9 \\ 4 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 8 \\ -5 \\ 0 \\ 1 \end{bmatrix}$$

Problem 13 (1.5#11)

Describe all solutions of $A\boldsymbol{x} = \mathbf{0}$ in parametric form, where A is row equivalent to the given matrix.

$$\begin{bmatrix} 1 & -4 & -2 & 0 & 3 & -5 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & -4 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Solution:

$$A = \begin{bmatrix} 1 & -4 & -2 & 0 & 3 & -5 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & -4 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & -4 & 0 & 0 & 3 & -7 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & -4 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad R1 = 2 * R2 + R1$$

$$= \begin{bmatrix} 1 & -4 & 0 & 0 & 0 & 5 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & -4 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad R1 = -3 * R23 + R1$$

$$A\boldsymbol{x} = 0 \equiv \begin{Bmatrix} x_1 - 4x_2 + 5x_6 = 0 \\ x_3 - x_6 = 0 \\ x_5 - 4x_6 = 0 \end{Bmatrix}$$

$$\equiv \begin{Bmatrix} x_1 = 4x_2 - 5x_6 \\ x_3 = x_6 \\ x_5 = 4x_6 \end{Bmatrix}$$

$$\implies \boldsymbol{x} = x_2 \begin{bmatrix} 4 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_6 \begin{bmatrix} -5 \\ 0 \\ 1 \\ 0 \\ 4 \\ 1 \end{bmatrix}$$

Problem 14 (1.5#13)

Suppose the solution set of a certain linear system of equations can be described as $x_1 = 5 + 4x_3$, $x_2 = -2 - 7x_3$, with x_3 free. Use vectors to describe this set as a line in $\mathbb{R}^3$.

Solution:

$$\boldsymbol{x}(t) = t \begin{bmatrix} 4 \\ -7 \\ 1 \end{bmatrix} + \begin{bmatrix} 5 \\ -2 \\ 0 \end{bmatrix}$$

Problem 15 (1.5#15)

Follow the method of Example 3 to describe the solution of the following system in parametric form. Also, give a geometric description of the solution set and compare it to that in Exercise 5.

$$\begin{aligned} x_1 + 3x_2 + x_3 &= 1 \\ -4x_1 - 9x_2 + 2x_3 &= -1 \\ -3x_2 - 6x_3 &= -3 \end{aligned}$$

Solution:

$$\begin{aligned}
\left\{\begin{matrix} x_1 + 3x_2 + x_3 = 1 \\ -4x_1 - 9x_2 + 2x_3 = -1 \\ -3x_2 - 6x_3 = -3 \end{matrix}\right\} &\equiv \begin{bmatrix} 1 & 3 & 1 & 1 \\ -4 & -9 & 2 & -1 \\ 0 & -3 & -6 & -3 \end{bmatrix} \\
&\equiv \begin{bmatrix} 1 & 3 & 1 & 1 \\ 0 & 3 & 6 & 3 \\ 0 & -3 & -6 & -3 \end{bmatrix} R2 = 4 * R1 + R2 \\
&\equiv \begin{bmatrix} 1 & 3 & 1 & 1 \\ 0 & 3 & 6 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} R3 = R2 + R3 \\
&\equiv \left\{\begin{matrix} x_1 - 5x_3 = 2 \\ x_2 + 2x_3 = 1 \end{matrix}\right\} \\
&\equiv \left\{\begin{matrix} x_1 = 5x_3 - 2 \\ x_2 = 1 - 2x_3 \end{matrix}\right\} \\
&\equiv \boldsymbol{x}(t) = t \begin{bmatrix} 5 \\ -2 \\ 1 \end{bmatrix} + \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}
\end{aligned}$$

The solution set to this system is a line in $\mathbb{R}^3$ that goes through $\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$ and parallel to $\begin{bmatrix} 5 \\ -2 \\ 1 \end{bmatrix}$.

Problem 16 (1.5#19)

Find the parametric equation of the line through $\boldsymbol{a}$ and parallel to $\boldsymbol{b}$.

$$\boldsymbol{a} = \begin{bmatrix} -2 \\ 0 \end{bmatrix}, \boldsymbol{b} = \begin{bmatrix} -5 \\ 3 \end{bmatrix}$$

Solution:

$$\boldsymbol{x}(t) = t\boldsymbol{b} + \boldsymbol{a} = t \begin{bmatrix} -5 \\ 3 \end{bmatrix} + \begin{bmatrix} -2 \\ 0 \end{bmatrix}$$

Problem 17 (1.5#25)

Prove the second part of Theorem 6: Let $\boldsymbol{w}$ be any solution of $A\boldsymbol{x} = \boldsymbol{b}$, and define $\boldsymbol{v_h} = \boldsymbol{w} - \boldsymbol{p}$. Show that $\boldsymbol{v_h}$ is a solution of $A\boldsymbol{x} = \mathbf{0}$. This shows that every solution of $A\boldsymbol{x} = \boldsymbol{b}$ has the form $\boldsymbol{w} = \boldsymbol{p} + \boldsymbol{v_h}$, with $\boldsymbol{p}$ a particular solution of $A\boldsymbol{x} = \boldsymbol{b}$ and $\boldsymbol{v_h}$ a solution of $A\boldsymbol{x} = 0$.

Solution:

Proof. Given $\boldsymbol{p}$ is a solution to $A\boldsymbol{x} = \boldsymbol{b}$, and $\boldsymbol{w}$ is a solution to $A\boldsymbol{x} = \boldsymbol{b}$, $\boldsymbol{v_h} = \boldsymbol{w} - \boldsymbol{p}$ is a solution to $A\boldsymbol{x} = 0$

$$
\begin{aligned}
0 &= A\boldsymbol{v_h} && && (1)\\
&= A(\boldsymbol{w} - \boldsymbol{p}) && \text{(by substitution)} && (2)\\
&= A\boldsymbol{w} - A\boldsymbol{p} && \text{(by distributive property)} && (3)\\
0 &= \boldsymbol{b} - A\boldsymbol{p} && \text{(by substitution)} && (4)\\
A\boldsymbol{p} &= \boldsymbol{b} && \text{(by addition)} && (5)\\
\boldsymbol{b} &= \boldsymbol{b} && \text{(by substitution)} && (6)
\end{aligned}
$$

□

Therefore if $\boldsymbol{p}$ is a solution to $A\boldsymbol{x} = \boldsymbol{b}$, and $\boldsymbol{w}$ is a solution to $A\boldsymbol{x} = \boldsymbol{b}$, then $\boldsymbol{v_h} = \boldsymbol{w} - \boldsymbol{p}$ is a solution to $A\boldsymbol{x} = 0$

Problem 18 (1.5#27)

Suppose A is a 3×3 *zero* matrix (with all zero entries). Describe the solution set of the equation $A\boldsymbol{x} = \mathbf{0}$.

Solution:

The solution set of the equation $A\boldsymbol{x} = 0$ when $A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ is all vectors in $\mathbb{R}^3$.

Problem 19 (1.5#39s)

Let A be an $m \times n$ matrix, and let $\boldsymbol{u}$ be a vector in $\mathbb{R}^n$ that satisfies the equation $A\boldsymbol{x} = \mathbf{0}$. Show that for any scalar c the vector $c\boldsymbol{u}$ also satisfies $A\boldsymbol{x} = \mathbf{0}$. [That is, show that $A(c\boldsymbol{u}) = \mathbf{0}$].

Solution:

Given $A\boldsymbol{u} = 0$

$$
\begin{aligned}
A(c\boldsymbol{u}) &= (A\boldsymbol{u})c && \text{(by Associative property)}\\
&= (0)c && \text{(by substitution)}\\
&= 0 && \text{(by multiplication of zero)}
\end{aligned}
$$