Neal's Funnel

Jose Storopoli

Created on 09/03/2021. Updated on 09/03/2021

In this notebook we analyze Neal's Funnel (Neal, 2011). Neal (2011) defines a distribution that exemplifies the difficulties of sampling from some hierarchical models. Neal's example is fairly extreme, but can be trivially reparameterized in such a way as to make sampling straightforward. Neal's example has support for $y \in \mathbb{R}$ and $x \in \mathbb{R}^9$ with density

$$p(y, x) = \text{Normal}(y \mid 0, 3) \times \prod_{n=1}^{9} \text{normal}\left(x_n \mid 0, \exp\left(\frac{y}{2}\right)\right).$$

The probability contours are shaped like ten-dimensional funnels. The funnel's neck is particularly sharp because of the exponential function applied to y. I won't try to demonstrate it in 9 dimensions but I will show it in 3 dimensions. Below there is is the Neal's Funnel density in 3-D. This is partially taken from a StanCon 2018 YouTube video by Ben Goodrich[1]

```
using Distributions, Plots

x = -2:0.01:2;
kernel(x, y) = logpdf(Normal(0, exp(y / 2)), x)
surface(x, x, kernel)
```

So what if we reparameterize so that we can express y and x_n as standard normal distributions, by using a reparameterization trick[2]:

$$\begin{aligned} x^* &\sim \text{Normal}(0, 0) \\ x &= x^* \cdot \sigma_x + \mu_x \end{aligned}$$

So, we can provide the MCMC sampler a better-behaved posterior geometry to explore:

$$\begin{aligned} p(y^*, x^*) &= \text{Normal}(y^* \mid 0, 0) \times \prod_{n=1}^{9} \text{Normal}(x_n^* \mid 0, 0) \\ y &= 3y^* \\ x_n &= \exp\left(\frac{y}{2}\right)) x_n^*. \end{aligned}$$

Below there is is the Neal's Funnel reparameterized as standard normal density in 3-D.

[1] see from 45' onwards.
[2] this also works for multivariate distributions.

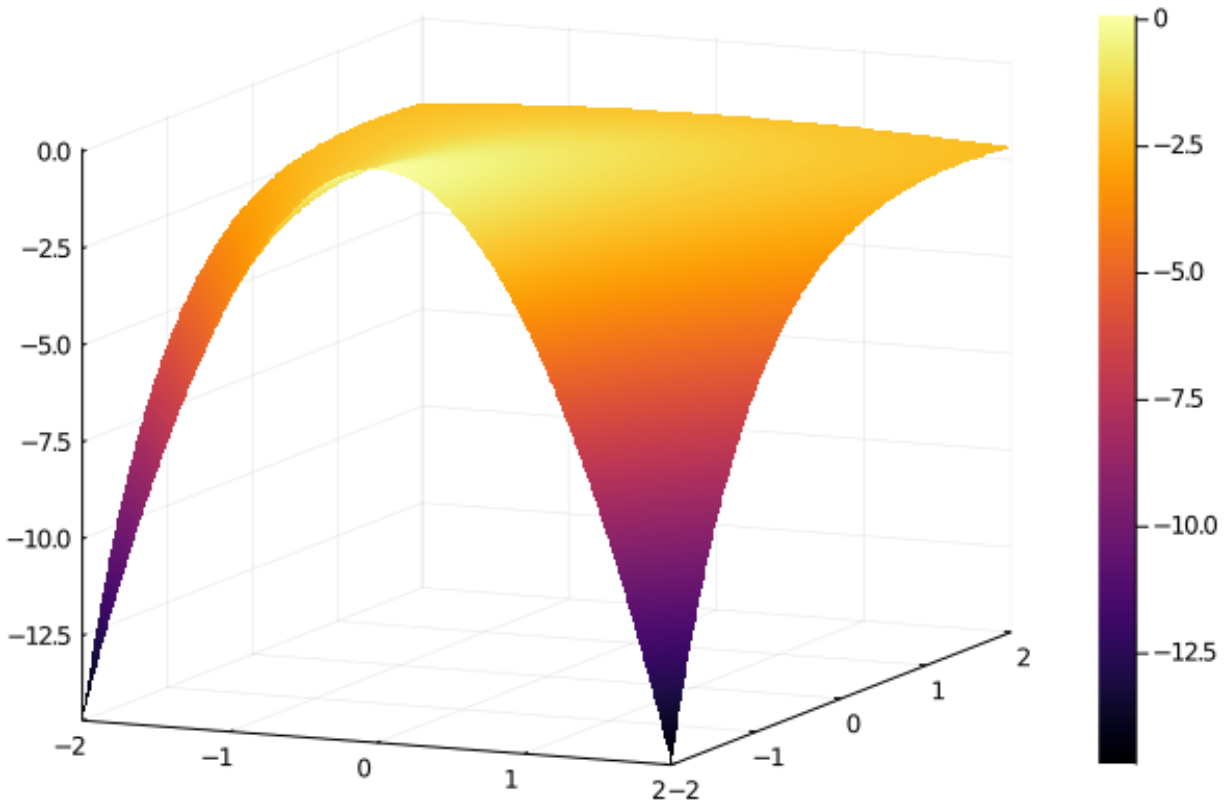

Figure 1: The Funnel

```
kernel_reparameterized(x, y) = logpdf(Normal(), x)
surface(x, x,  kernel_reparameterized)
```

Environment

```
using InteractiveUtils
versioninfo()
```

```
Julia Version 1.6.0-rc1
Commit a58bdd9010* (2021-02-06 15:49 UTC)
Platform Info:
  OS: macOS (x86_64-apple-darwin20.3.0)
  CPU: Intel(R) Core(TM) i5-8500B CPU @ 3.00GHz
  WORD_SIZE: 64
  LIBM: libopenlibm
  LLVM: libLLVM-11.0.1 (ORCJIT, skylake)
Environment:
  JULIA_NUM_THREADS = 6
```

References

Neal, R. M. (2011). MCMC using Hamiltonian dynamics. In S. Brooks, A. Gelman, G. L.

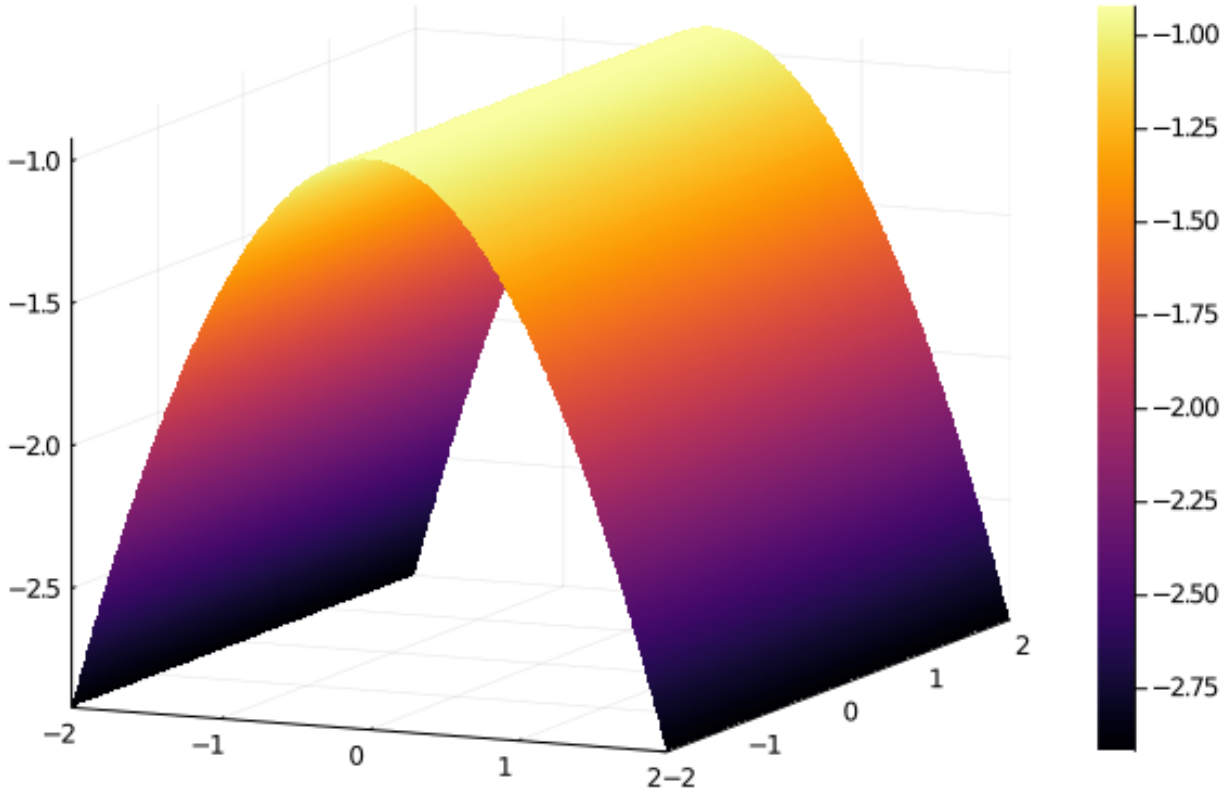

Figure 2: The Funnel Reparameterized

Jones, & X.-L. Meng (Eds.), *Handbook of markov chain monte carlo*.