1 Notation & Definitions

In this section we introduce a mathematical description of the visualization pipeline where artist A functions transform data of type $\Gamma(E)$ to an intermediate representation in prerendered display space of type $\Gamma(H)$:

$$A : \Gamma(E) \to \Gamma(H) \tag{1}$$

- A is the function that converts an instance of data $\Gamma(E)$ to an instance of a visual representation $\Gamma(H)$
- E is a locally trivial fiber bundle over K representing data space.
- K is a triangulizable space encoding the connectivity of the points in E
- H is a fiber bundle over S representing visual space
- S is a simplacial complex encoding the visualization
- Γ is the global space of sections

When E is a trivial fiber bundle $E = F \times K$, it can be assumed that all fibers F_k over $k \in K$ are equal. Fiber bundles are product spaces of toplological spaces, which are a set of points with a set of neighborhoods for each point[7, 16].

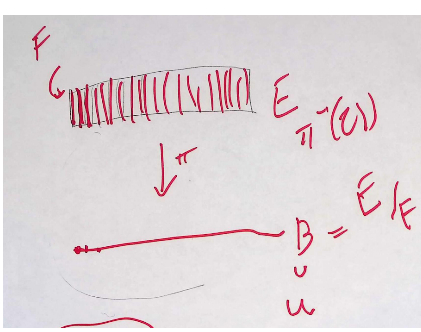
1.1 Fiber Bundles

We provide a brief description of fiber bundles because we model data, visual transformations, and a prerendered visual graphic as fiber bundles. A fiber bundle is a structure (E, B, π, F) consisting of topological total space E, base space K, fiber space F and the map from total space to base space:

$$F \hookrightarrow E \xrightarrow{\pi} B \tag{2}$$

where there is a bijection from F to every fiber F_b over point $b \in B$ in E and the function $\pi : E \to B$ is the map into the B quotient space of E. By defintion of a fiber bundle, π is always a mapping from total space to base space, independent of the points $p \in E$, and therefore we call this mapping π for all the fiber bundles in the model.

1.1.1 Base space



B is the quotient space of E, meaning it is the set of equivalence classes of elements p in E defined via the map $\pi : E \to B$ that sends each $p \in E$ to its equivalence class in $[p] \in B$ [14].

As shown in figure **??**, the fibers F divide E into smaller spaces consisisting of F and an open set neighborhood around F. This subdivision is projected down to the toplology $\mathcal{T}$

$$\mathcal{T}_b = \{U \subseteqq B : \{p \in E : [p] \in U\} \in \mathcal{T}_E\} \tag{3}$$

where $[p] \in U$ is the point $b \in B$ with an open set surrounding it that has an open preimage in E under the surjective map $\pi : p \to [p]$.

1.1.2 Fiber

As shown in equation!**??**, every point $b \in B$ has a local open set neighborhood U [7, 16]

$$\begin{tikzcd} \pi^{-1}(U) \arrow[r, "\varphi"] \arrow[d, "\pi"'] & U \times F \arrow[dl, "\mathrm{proj}_U"] \\ U & \end{tikzcd} \tag{4}$$

such that $\varphi : \pi^{-1}(U) \to U \times F$ is a homeomorphism where π and proj_U both map to U and the fiber over k $F_b = \pi^{-1}(b \in B)$ is homomorphic to the fiber F.

1.1.3 Section

The section f is the mapping from base space to total space $f : B \to E$

$$\begin{tikzcd} F \arrow[r, hook] & E \arrow[d, "\pi"'] \\ & B \arrow[u, bend right, "f"'] \end{tikzcd} \tag{5}$$

such that f is the right inverse of π

$$\pi(f(b)) = b \text{ for all } b \in B \tag{6}$$

In a locally trivial fiber bundle, $E = B \times F$ [7, 16]:

$$f(b) = (b, g(b)) \tag{7}$$

where the domain of $g(b)$ is F_b and returns a point p in F_b. The space of all possible sections f of E is $\Gamma(E)$. All sections $f \in \Gamma(E)$ have the same fibers F and connectivity B.

1.1.4 Sheaf and Stalk

As described in equation **??**, there is a local space $U \subset B$ around every b. The inclusion map $\iota : U \to B$ can be pulled back such that $\iota^* E$ is the space of E restricted over U.

$$\begin{tikzcd} \iota^* E \arrow[d, "\pi"'] & E \arrow[l, "\iota^*"'] \arrow[d, "\pi"'] \\ U \arrow[u, bend right, "\iota^* f"'] & B \arrow[l, "\iota"'] \arrow[u, bend right, "f"'] \end{tikzcd} \tag{8}$$

The localized section of fibers $\iota^* f : U \to \iota^* E$ is the sheaf $\mathcal{O}(E)$ with germ of $\xi^* f$. The neighborhood of points the sheaf lies over is the stalk $\mathcal{F}_b$ [18, 20]

$$\iota^{-1}\mathcal{F}(\{b\}) = \varinjlim_{b \subseteq U} \mathcal{F}(U) = \varinjlim_{b \in U} = \mathcal{F}_b \tag{9}$$

which through ι gets the data in E at and near to b. Restricting the artist to the sheaf means the artist knows the data in F and also has access to derivatives of the data. This property is useful for some visual transformations.

1.2 Data Model

As proposed by Butler [3, 4], we model data as a fiber bundle (E, K, π, F) with $\pi : E \to K$ where K which can be thought of as a set of keys k. A section $\tau : K \to E$ is an instance of the data that lies in E and is discussed in section 1.2.4.

1.2.1 Example

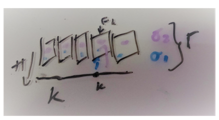


Figure 1: write up some words here

As illustrated by figure **??**, the vertical lines F are the range of possible temperature values embedded in the total space E. The base space K of the fiber bundle is a line because the data points r in E are on a space that is continous in one dimension.

1.2.2 Base Space

The base space K is a representation of the connectivity of the data, specifically whether the points in E are discrete or sampled from a continous space. The same dataset can be expressed with different K.

In our draft implementation of the data as fiber bundle model, we represent K as a simplacial complex.

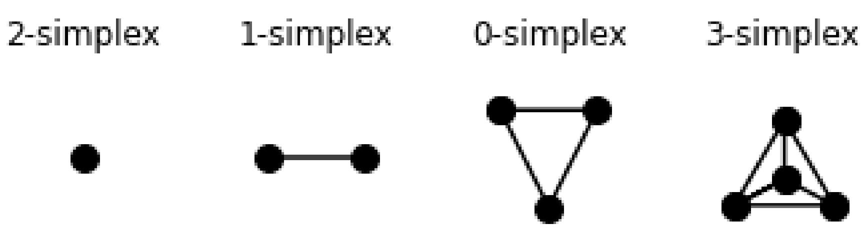


Figure 2: Simplices encode the connectivity of the data, from fully disconnected (0 simplex) observations to all observations are connected to at least 3 other observations. Higher order simplicies are outside the scope of this paper.

One way to represent the topological space K is as a set composed of simplices, such as those shown in figure **??**. Simplices are a way of encoding the connectivity of each observation ($\sigma(k)$) to another:

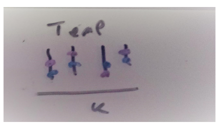
0-simplex discrete observations (inventory records)

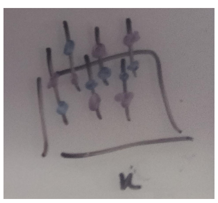
1-simplex 1D continuos data (timeseries)

2-simplex 2D continuos data (map)

3-simplex 3D continuos data (video)

1.2.3 Example





in figure **??**, temperature is the only one data field in r but the K base spaces are different. subfig[1] is a timeseries, so the temperature in r at time t is dependent on the temperature in r_{t-1} and the temperature in r_{t+1} is dependent on r_t; this connectivity is expressed as a one dimensional K where K is the number line. In the case of the map, every temperature in r is dependent on its nearest neighbors on the plane, and one way to express this is by encoding K as a plane. K does not know the time or latitude or longitude of the point as those are metadata variables describing the k rather than the value of k. The mapping $\tau : K \to E$ provides the binding between the key $k \in K$ and the value r in E [13].

1.2.4 Fiber and Sections

We use Spivak's formalization of data base schemas as the basis of our fiber space F [19]. He defines the type specification

$$\pi : U \to DT \tag{10}$$

where DT is the set of data types (as identified by their names) and U is the disjoint set of all possible objects x of all types in DT. This means that for each type $T \in DT$, the preimage $\pi^{-1}(T) \subset U$ is the domain of T, and $x \in \pi^{1}(T) \subset U$ is an object of type T. Spivak then defines a schema (C, σ) of type π, where π is the universe of all types, such that

$$\sigma : C \to DT \tag{11}$$

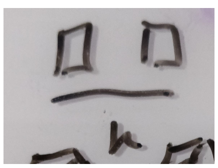
where C is the finite set of names of columuns, which we generalize to data fields in E. The set of all values restricted to the datatypes in DT is U_σ

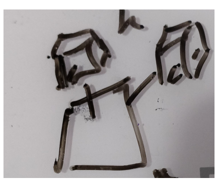
$$\begin{tikzcd} U_\sigma \arrow[r] \arrow[d, "\pi_\sigma"'] & U \arrow[d, "\pi"] \\ C \arrow[r, "\sigma"] & DT \end{tikzcd} \tag{12}$$

The pullback $U_\sigma \coloneqq \sigma^{-1}(U)$ restricts U to the datatypes of the fields in C such that U_σ is the fiber product $U \times_{DT} C$, and the pullback $\pi_\sigma : U_\sigma \to C$ specifies the domain bundle U_σ over C induced by σ. The fiber F is the cartesian product of all sets in the disjoint union U_σ.

For each field $c \in C$, the record function $r : C \to U_\sigma$ returns an object of type $\sigma(c) \in DT$. The set of all records $\Gamma(\sigma)$ is the set of all sections on U_σ. Spivak defines the τ mapping from an index of databases K to records $\Gamma(\sigma)$ as $\tau : K \to \Gamma(\sigma)$. This is equivalent to $\tau : k \to E$ since $F = \Gamma(\sigma)$ and F is the embedding in E on which the records r lie.

1.2.5 Example





The fiber in figure **??** is the space of possible temperature values in degrees celsius, such that $F = [temp_{min}, temp_{max}]$ and is named Temp. In figure **??** time is encoded as a second dimension. This means that the set of possible values F with $C = \{\text{Temp}, \text{Time}\}$:

$$F = [temp_{min}, temp_{max}] \times [time_{min}, time_{max}] \tag{13}$$

and the function τ that retrieves records from F is

$$\tau(k) = (k, (r : \text{Temp} \to temp, r : \text{Time} \to time)) \tag{14}$$

$$temp \in [temp_{min}, temp_{max}], time \in [time_{min}, time_{max}]) \tag{15}$$

Since $\tau(k) = (k, r)$, $temp$ is bound to a named data field and $sigma$ binds $temp$ to a temperature data type.

1.3 Prerender Space

We model the prerender space on which lives on ideal version of the visulaization as a fiber bundle (H, S, π, D). H is the predisplay space, with a fiber D dependent on the target physical display and a base space of S.

1.3.1 Base space

K can be considered a subspace of the screen base space S such that $\xi : S \to K$ is a deformation retraction [15]

$$\begin{tikzcd} E \arrow[d, "\pi"'] & H \arrow[d, "\pi"'] \\ K & S \arrow[l, "\xi"'] \end{tikzcd} \tag{16}$$

that goes from a region $s \in S_k$ to its associated point k, such that when $\xi(s) = k$, $\xi^* \tau(s) = \tau(k)$.

1.3.2 Fiber and Section

A section $\rho : S \to H$ is a mapping from a region s on a mathematical encoding of the image to a region xy on the screen that the renderer then maps to visual space as defined in D. For a physical screen display, the predisplay space is a trivial fiber bundle $H = \mathbb{R}^7 \times S$ such that ρ is

$$\rho(s) = \{x, y, z, r, g, b, a\} \tag{17}$$

To draw an image, a region, H is inverse mapped into a region $s \in S$ where

$$s = \rho^{-1}_{XY}(xy) \tag{18}$$

such that the rest of the fields in $\mathbb{R}^7$ are then integrated over s to yield the remaining fields:

$$r = \iint\limits_{s} \rho_R(s) ds^2 \tag{19}$$

$$g = \iint\limits_{s} \rho_G(s) ds^2 \tag{20}$$

$$b = \iint\limits_{s} \rho_B(s) ds^2 \tag{21}$$

Here we assume a single opaque 2D image such that the z and $alpha$ fields can be omitted.

1.3.3 Example

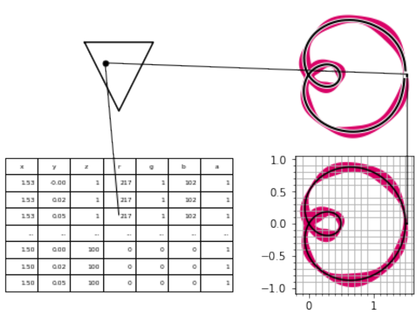


Figure 3

As illustrated in figure 3, words.

1.4 Artist

In this section we will define the artist as a mapping from a sheaf $\mathcal{O}(E)$ to $\mathcal{O}(H)$.

$$A : \mathcal{O}(E) \to \mathcal{O}(H) \tag{22}$$

The artist decomposes to mapping data to visual $\nu : E \to V$, then compositing V pulled back along ξ to ξ^*V to a visual mark in prerender space $Q : \xi^*V \to H$.

$$\begin{tikzcd}
E \arrow[r, "\nu"] \arrow[dr, "\pi"'] & V \arrow[d, "\pi"] & \xi^*V \arrow[r, "Q"] \arrow[d, "\xi^*\pi"'] & H \arrow[dl, "\pi"] \\
& K & S \arrow[l, "\xi"'] &
\end{tikzcd} \tag{23}$$

The visual fiber bundle $(V,\ K,\ \pi,\ P)$ has section $\mu : V \to K$ that resolves to a visual variable [2, 12] in fiber P. The visual transformer ν is a set of functions each targeting a different μ

$$\{\nu_0, \ldots, \nu_n\} : \{\tau_0, \ldots, \tau_n\} \mapsto \{\mu_0, \ldots, \mu_n\} \tag{24}$$

where μ_i are the visual parameters in the assembly function $Q(\mu_0, \ldots, \mu_n)(s) = \rho(s)$.

1.4.1 Example: Matplotlib Visual Fiber

For example, for Matplotlib [8], some of the possible types in P are:

$\boldsymbol{\nu_i}$	$\boldsymbol{\mu_i}$	$\boldsymbol{codomain(\nu_i)}$
position	x, y, z, theta, r	$\mathbb{R}$
size	linewidth, markersize	$\mathbb{R}^+$
shape	markerstyle	$\{f_0, \ldots, f_n\}$
color	color, facecolor, markerfacecolor, edgecolor	$\mathbb{R}^4$
texture	hatch	$\mathbb{N}^{10}$
	linestyle	$\{f_0, \ldots, f_n\} \times (\mathbb{R}, \mathbb{R}^{+n, n\%2=0})$

1.4.2 Visual Channels

$\nu : E \to V$ is an equivariant map such that there is a homomorphism from left monoid actions on E_i to left monoid actions on V_i where i identifies a field in the fiber. E_i and V_i

each contain a set of values as defined in F and P respectively. A validly constructed ν is one where the diagram

$$\begin{tikzcd} E_i \arrow[r, "\nu_i"] \arrow[d, "m_e"'] & V_i \arrow[d, "m_v"] \\ E_i \arrow[r, "\nu_i"] & V_i \end{tikzcd} \tag{25}$$

commutes such that $\nu_i(m_e(E_i)) = m_v(\nu_i(E_i))$.

Definition A monoid[11] M is a set that is closed under an associative binary operator $*$ and has an identity element $e \in M$ such that $e * a = a * e = a$ for all $a \in M$. A left monoid action [1, 17] of M is a set X with an action $\bullet$ with the properties:

closure $\bullet : M \times X \to X$,

associativity for all $m, t \in M$ and $x \in X, m \bullet (t \bullet x) = (m \bullet t) \bullet x$

identity for all $x \in X, e \in M, e \bullet x = x$

Example: Partial Order To preserve ordering of elements in E_i, ν must be a monotonic function such that given $e_1, e_2 \in E_i$

$$\text{if } e_1 \leq e_2 \text{ then } \nu(e_1) \leq \nu(e_2) \tag{26}$$

Example: Translation fairly certain I lost the thread here According to Stevens, interval data is a set with general linear group actions [10, 21]. Position is a visual variable that can support translation [2, 9, 12].

For example, here ν is a direct map $\nu(\tau_{i,k}) = \mu i, k$ and the transform function is a multiplicative shift $t(x) = x + x$ applied twice

$$\begin{align} \nu(t(t(\tau_{i,k}))) &= t(t(\nu(\tau_{i,k}))) \tag{27} \\ \nu(t(2\tau_{i,k})) &= t(t_v(\mu_{i,k})) \tag{28} \\ \nu(2\tau_{i,k} + 2\tau_{i,k}) &= t(2\mu_{i,k}) \tag{29} \\ \nu(4\tau_{i,k}) &= 2\mu_{i,k} + 2\mu_{i,k} \tag{30} \\ \nu(4\tau_{i,k}) &= 4\mu_{i,k} \tag{31} \\ 4\mu_{i,k} &= 4\mu_{i,k} \tag{32} \\ & \tag{33} \end{align}$$

ν is valid because scaling the data by a factor of 4 causes a scaling of the equivalent visualization by 4, which since the constant factor can be pulled out means they are equivalent.

Example: Invalid ν Given a transform $t(x) = 2 * x$, we construct a ν that always takes data to .5:

$$\begin{tikzcd} E_1 \arrow[r, "\lambda : e \mapsto .5"] \arrow[d, "2e"'] & V_i \arrow[d, "2v"] \\ E_1 \arrow[r, "\lambda"] & V_1 \end{tikzcd} \tag{34}$$

This ν is invalid because the graph does not commute for t:

$$\nu(t(e)) \stackrel{?}{=} t(\nu(e)) \tag{35}$$
$$.5 \stackrel{?}{=} t(.5) \tag{36}$$
$$.5 \neq 2 * .5 \tag{37}$$

To consutruct a valid ν, the diagram must commute for all monoid actions on the sets E_i, V_i.

1.4.3 Assembling Marks

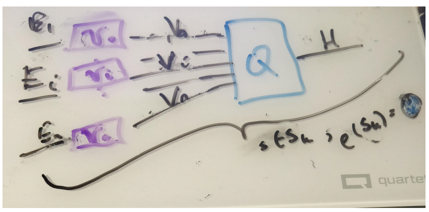


Figure 4: The ν functions convert data E to visual V. Q assembles the different types of visual parameters V_i into a graphic in H. $Q \circ \mu(\xi^{-1}J)$ forms a visual mark by applying Q to a region mapped to connected components $J \subset K$.

The assembly function $Q : \Gamma(V) \to \Gamma(H)$ composites the visual variables in V into an element in H. Given a monoid action on a set in V, there should be a monoid action on the corresponding $Q(\Gamma(V))$. While $\Gamma(V)$ holds for all cases, we can specialize to the bundle V or the sheaf (V) depending on the specific visualization.

Proposition If $\forall g \in M$ and $\forall x_1, x_2 \in \Gamma(V)$ then $Q(x_1) = Q(x_2)$ $impliesQ(g \circ x_1) = Q(g \circ x_2)$; therefore we can define a group action on $Q(\Gamma(V)) = Y$ as $g \circ y = y'$ where $y' = Q(g \circ x)$ with $x \in f^{-1}(y)$

For each region s in the display space H, the mark [2, 5] it belongs to can be found by mapping s back to K via the lookup on S then taking $\xi(s)$ back to a point on $k \in K$ which lies on the path connected component $J \subset K$.

$$H \underset{\rho(\xi^{-1}(J))}{\rightleftarrows} S \underset{\xi^{-1}(J)}{\overset{\xi(s)}{\rightleftarrows}} J_k = \{j \in K \text{ s. t. } \exists \gamma \text{ s.t. } \gamma(0) = k \text{ and } \gamma(1) = j\} \tag{38}$$

where the path[6] γ from k to j is a continous function from the interval [0,1]. To get back to the display space H from J, the inverse image of $J \in S, \xi^{-1}(J)$ is pushed back to S, and then $\rho(\xi^{-1}(J))$ maps it into R^7 such that $Q \circ \xi^* \mu(\xi^{-1}(J))$ generates a graphical mark.

1.4.4 Visual Idioms: Equivalance class of artists

n $O(E)$ of the same type, they output the same type of prerender $O(H)$:

Natural transformation + composition is partial ordering? Back and forth is equivalent