1 Notation & Definitions

In this section we introduce a mathematical description of the visualization pipeline where artist A functions transform data of type $\Gamma(E)$ to an intermediate representation in prerendered display space of type $\Gamma(H)$:

$$A : O(E) \to O(H) \tag{1}$$

- A is the function that converts an instance of data $\Gamma(E)$ to an instance of a visual representation $\Gamma(H)$
- E is a locally trivial fiber bundle over K representing data space.
- K is a triangulizable space encoding the connectivity of the observations in the data.
- H is a fiber bundle over S representing visual space
- S is a simplacial complex of triangles encoding the connectivity of the visualization of the data in E
- $\tau : K \to E$ is the data being visualized
- $\rho : S \to H$ is the render map

When E is a trivial fiber bundle $E = F \times K$, it can be assumed that all fibers F_k over $k \in K$ are equal. Fiber bundles are product spaces of toplological spaces, which are a set of points with a set of neighborhoods for each point[5, 9].

1.1 Fiber Bundles

We provide a brief introduction to fiber bundles because we model data, visual transformations, and a prerendered visual graphic as fiber bundles. A fiber bundle is a structure (E, B, π, F) consisting of topological total space E, base space K, fiber space F and the map from total space to base space:

$$F \hookrightarrow E \xrightarrow{\pi} B \tag{2}$$

where there is a bijection from F to every fiber F_b over point $b \in B$ in E and the function $\pi : E \to B$ is the map into the B quotient space of E.

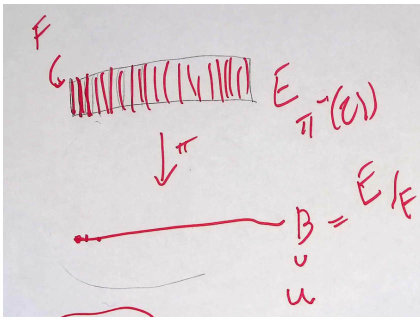
1.1.1 Base space

B is the quotient space of E, meaning it is the set of equivalence classes of elements p in E defined via the map $\pi : E \to B$ that sends each $p \in E$ to its equivalence class in $[p] \in B$ [8].

As shown in figure **??**, the fibers F divide E into smaller spaces consisisting of F and an open set neighborhood around F. This subdivision is projected down to the toplology $\mathcal{T}$

$$\mathcal{T}_b = \{U \subseteqq B : \{p \in E : [p] \in U\} \in \mathcal{T}_E\} \tag{3}$$

where $[p] \in U$ is the point $b \in B$ with an open set surrounding it that has an open preimage in E under the surjective map $\pi : p \to [p]$.



1.1.2 Fiber

As shown in equation!??, every point $b \in B$ has a local open set neighborhood U [5, 9]

$$\begin{tikzcd} \pi^{-1}(U) \arrow[r, "\varphi"] \arrow[d, "\pi"'] & U \times F \arrow[dl, "\mathrm{proj}_U"] \\ U & \end{tikzcd} \tag{4}$$

such that $\varphi : \pi^{-1}(U) \to U \times F$ is a homeomorphism where π and proj_U both map to U and the fiber over k $F_b = \pi^{-1}(b \in B)$ is homomorphic to the fiber F.

The section f is the mapping from base space to total space $f : B \to E$

$$\begin{tikzcd} F \arrow[r, hook] & E \arrow[d, "\pi"', shift right] \\ & B \arrow[u, "f"', bend right] \end{tikzcd} \tag{5}$$

such that it is the right inverse of π

$$\pi(f(b)) = b \text{ for all } b \in B \tag{6}$$

In a locally trivial fiber bundle, $E = B \times F$ [5, 9]:

$$f(b) = (b, g(b)) \tag{7}$$

where the domain of $g(b)$ is F_b and returns a point p in F_b. The space of all possible sections f of E is $\Gamma(E)$. All sections $f \in \Gamma(E)$ have the same fibers F and connectivity B.

1.1.3 Sheaf and Stalk

As described in equation 4, there is a local space $U \subset B$ around every b. The inclusion map $\iota : U \to B$ can be pulled back such that $\iota^* E$ is the space of E restricted over U.

$$\begin{tikzcd} \iota^* E \arrow[d, "\pi"', shift right] & E \arrow[l, "\iota^*"'] \arrow[d, "\pi"', shift right] \\ U \arrow[u, "\iota^* f"', bend right] & B \arrow[l, "\iota"'] \arrow[u, "f"', bend right] \end{tikzcd} \tag{8}$$

The localized section of fibers $\iota^* f : U \to \iota^* E$ is the sheaf $O(E)$ with germ of $\xi^* f$. The neighborhood of points the sheaf lies over is the stalk $\mathscr{F}_b$ [10, 12]

$$\iota^{-1}\mathscr{F}(\{b\}) = \varinjlim_{b\subseteq U} \mathscr{F}(U) = \varinjlim_{b\in U} = \mathscr{F}_b \tag{9}$$

which through ι gets the data in E at and near to b. Restricting the artist to the sheaf means the artist knows the data in F and also has access to derivatives of the data. This property is useful for some visual transformations.

1.2 Data Model

We use a fiber bundle model to represent the data, as proposed by Butler. [2, 3].

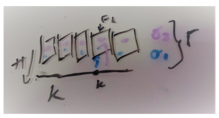
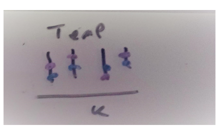
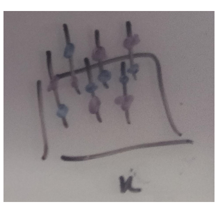


Figure 1: write up some words here

As illustrated by figure 1, the vertical lines F are the range of possible temperature values embedded in the total space E. The base space K of the fiber bundle is a line because the data points r in E are on a space that is continous in one dimension.

1.2.1 Base Space





in figure **??**, temperature is the only one data field in r but the K base spaces are different. subfig[1] is a timeseries, so the temperature in r at time t is dependent on the temperature in r_{t-1} and the temperature in r_{t+1} is dependent on r_t; this connectivity is expressed as a one dimensional K where K is the number line. In the case of the map, every temperature in r is dependent on its nearest neighbors on the plane, and one way to express this is by encoding K as a plane. K does not know the time or latitude or longitude of the point as those are metadata variables describing the k rather than the value of k. The mapping $\tau : K \to E$ provides the binding between the key $k \in K$ and the value r in E [6].

1.2.2 Fiber Space F

We use Spivak's formalization of data base schemas as the basis of our fiber space F [11]. He defines the type specification

$$\pi : U \to DT \tag{10}$$

where DT is the set of data types (as identified by their names) and U is the disjoint set of all possible objects x of all types in DT. This means that for each type $T \in DT$, the preimage $\pi^{-1}(T) \subset U$ is the domain of T, and $x \in \pi^{1}(T) \subset U$ is an object of type T. Spivak then defines a schema (C, σ) of type π, where π is the universe of all types, such that

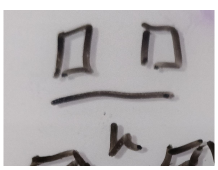
$$\sigma : C \to DT \tag{11}$$

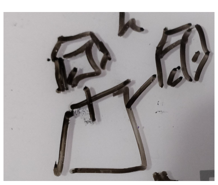
where C is the finite set of names of columuns, which we generalize to data fields in E. The set of all values restricted to the datatypes in DT is U_σ

$$\begin{array}{ccc} U_\sigma & \longrightarrow & U \\ {\scriptstyle \pi_\sigma}\downarrow & & \downarrow{\scriptstyle \pi} \\ C & \xrightarrow{\sigma} & DT \end{array} \tag{12}$$

The pullback $U_\sigma := \sigma^{-1}(U)$ restricts U to the datatypes of the fields in C such that U_σ is the fiber product $U \times_{DT} C$, and the pullback $\pi_\sigma : U_\sigma \to C$ specifies the domain bundle U_σ over C induced by σ. The fiber F is the cartesian product of all sets in the disjoint union U_σ.

For each field $c \in C$, the record function $r : C \to U_\sigma$ returns an object of type $\sigma(c) \in DT$. The set of all records $\Gamma(\sigma)$ is the set of all sections on U_σ. Spivak defines the τ mapping from an index of databases K to records $\Gamma(\sigma)$ as $\tau : K \to \Gamma(\sigma)$. This is equivalent to $\tau : k \to E$ since $F = \Gamma(\sigma)$ and F is the embedding in E on which the records r lie.





The fiber in figure **??** is the space of possible temperature values in degrees celsius, such that $F = [temp_{min}, temp_{max}]$ and is named Temp. In figure **??** time is encoded as a second dimension. This means that the set of possible values F with $C = \{\text{Temp}, \text{Time}\}$:

$$F = [temp_{min}, temp_{max}] \times [time_{min}, time_{max}] \tag{13}$$

and the function τ that retrieves records from F is

$$\tau(k) = (k, (r : \text{Temp} \to temp, r : \text{Time} \to time)) \tag{14}$$
$$temp \in [temp_{min}, temp_{max}], time \in [time_{min}, time_{max}]) \tag{15}$$

Since $\tau(k) = (k, r)$, *temp* is bound to a named data field and *sigma* binds *temp* to a temperature data type.

1.3 Prerender Space

Every point $k \in K$ maps to a space $S_k \in S$, which is the topology of the output of the artist A. The space H is a total space representing the predisplay space, with a fiber dependent on the render space and a base space of S:

$$D \hookrightarrow H \xrightarrow{\pi} S \tag{16}$$

Where the section $\rho : S \to H$

$$\begin{tikzcd} D \arrow[r, hook] & H \arrow[d, "\pi"'] \\ & S \arrow[u, bend right, "\rho"'] \end{tikzcd} \tag{17}$$

is mapping from a region s on a mathematical encoding of the image to a region xy on the screen that the renderer then maps to pixel space.

1.3.1 Section ρ

For a physical screen display, the predisplay space is a trivial fiber bundle $H = \mathbb{R}^7 \times S$ such that ρ is

$$\rho(s) = \{x, y, z, r, g, b, a\} \tag{18}$$

To draw an image, a region, H is inverse mapped into a region $s \in S$ where

$$s = \rho^{-1}_{XY}(xy) \tag{19}$$

such that the rest of the fields in $\mathbb{R}^7$ are then integrated over s to yield the remaining fields:

$$r = \iint_s \rho_R(s) ds^2 \tag{20}$$
$$g = \iint_s \rho_G(s) ds^2 \tag{21}$$
$$b = \iint_s \rho_B(s) ds^2 \tag{22}$$

Here we assume a single opaque 2D image such that the z and *alpha* fields can be omitted.

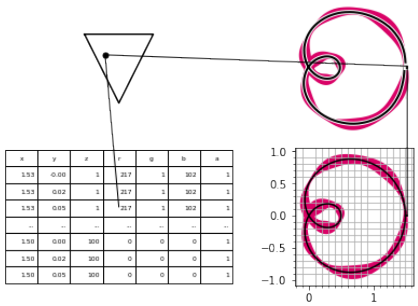


Figure 2

1.3.2 Example

As illustrated in figure 2, words.

1.4 Artist

The artist is a mapping from the sheaf $O(E)$ representing the data to a pre-render space $O(H)$.

$$A : O(E) \to O(H) \tag{23}$$

The artist is composed of two stages, data E to visual mappings V and V mappings composited to a visual in prerender space H. The visual fiber bundle V shares basespace K with E:

$$\begin{tikzcd} P \arrow[r, hook] & V \arrow[r, "\pi"] & K \end{tikzcd} \tag{24}$$

with fiber P that encodes visual paramters such as position or color. As in section 1.2.2, the fields in P are strongly typed. Inside the artist is also the equivalance class map $\xi : S \to K$ that goes from a region s to its associated point k

$$\begin{tikzcd} E \arrow[d, "\pi"'] & H \arrow[d, "\pi"'] \\ K & S \arrow[l, "\xi"'] \end{tikzcd} \tag{25}$$

such that when $\xi(s) = k$, $\xi^{*}\tau(s) = \tau(k)$.

1.4.1 Visual Mapping

The function $\nu : E \to V$ maps from data space to visual space [1, 4, 7] space

$$\begin{tikzcd} E \arrow[r, "\nu"] \arrow[d, "\pi"'] & V \arrow[dl, "\pi"'] \\ K & \end{tikzcd} \tag{26}$$

such that the visual section $\mu : K \to V$ is equivalent to the visual transformation of the data $\mu(k) = \nu \circ \tau(k)$

$$\begin{tikzcd} E \arrow[r, "\nu"] & V \\ K \arrow[u, "\tau"'] \arrow[ur, "\mu"'] & \end{tikzcd} \tag{27}$$

The constraint on $\nu : \tau \to \mu$ is that τ and μ must be equivariant.

1.4.2 Visual Composition

The visual fiber bundle V gets pulled back over S via ξ such that

$$\begin{tikzcd} \xi^*V \arrow[r, "Q"] \arrow[dr, "\xi^*\pi"'] & H \arrow[d, "\pi"] \\ & S \end{tikzcd} \tag{28}$$

the function $Q : \xi^*V \to H$ composites points in V into $\mathbb{R}^{\ntriangleright}$ tuples. The section μ is pulled over s

$$\begin{tikzcd} \xi^*V \arrow[r, "Q"] & H \\ & S \arrow[u, "\rho"'] \arrow[ul, "\xi^*\mu"] \end{tikzcd} \tag{29}$$

such that the composition Q of μ is equivalent to the render $\rho(s) = Q \circ \mu(s)$.

1.4.3 Pipeline

The full visualization path is

$$\begin{tikzcd} E \arrow[r, "\nu = transforms"] & V & \xi^*V \arrow[r, "Q = \text{set attrs}"] & H \\ & K \arrow[ul, "\tau = data"] \arrow[u, "\mu = \{\tau, \nu\}"'] & S \arrow[l, "\xi - init?draw"] \arrow[u, "\xi^*\mu = \nu \circ \tau"'] & \end{tikzcd} \tag{30}$$