Topological Equivariant Artist Model

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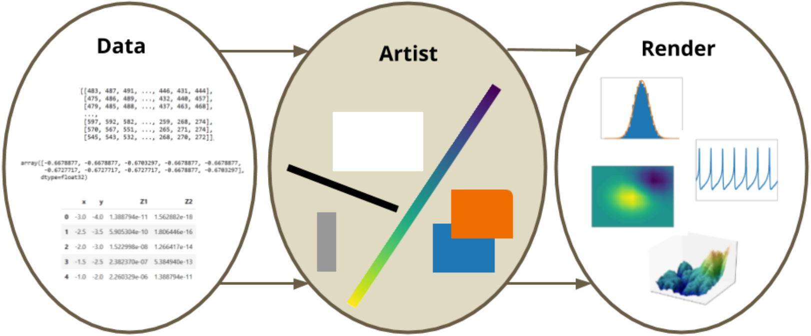
Visualizations are structure preserving maps



The aim of this work is to rearchitecture Matplotlib to take advantage of developments in software design, data structures, and visualization to improve consistency, reusability, and discoverability, so domain specific tool developers can build structure preserving visualization tools.

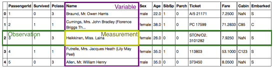
Visualization component constraints

equivariance properties of data and visual encoding match

continuity connectivity of data and visual encoding match

composibility structure preserved by individual components is preserved in combined components

Tools are tuned to the continuity of the date [15, 30]



	PassengerId	Survived	Pclass	Name	Sex	Age	SibSp	Parch	Ticket	Fare	Cabin	Embarked
0	1	0	3	Braund, Mr. Owen Harris	male	22.0	1	0	A/5 21171	7.2500	NaN	S
1	2	1	1	Cumings, Mrs. John Bradley (Florence Briggs Th...	female	38.0	1	0	PC 17599	71.2833	C85	C
2	3	1	3	Heikkinen, Miss. Laina	female	26.0	0	0	STON/O2. 3101282	7.9250	NaN	S
3	4	1	1	Futrelle, Mrs. Jacques Heath (Lily May Peel)	female	35.0	1	0	113803	53.1000	C123	S
4	5	0	3	Allen, Mr. William Henry	male	35.0	0	0	373450	8.0500	NaN	S

Figure: Based on fig 2.5 in Munzner's VAD[34]

1. ggplot[35]
2. protovis[3], D3 [4]
3. vega[23], altair[32]

1. ImageJ[24], ImagePlot[28]
2. Napari[25]

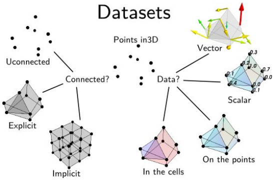


Figure: Data Representation, MayaVi 4.7.2 docs[11]

1. Matplotlib[16],
2. VTK [13, 14], MayaVi[22], ParaView[1], Titan[5]

Structure is encoded in variables and continuity

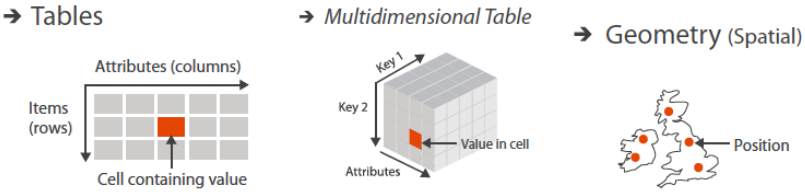


Figure: Image is figure 2.8 in Munzner's Visualization Analysis and Design[20]

binding metadata are structural *keys* with associated *values* (Munzner [20]))

continuity Fiber bundles can be a common data abstraction (Butler [6, 7])

variables Fibers can hold schema like encodings of variables (Spivak [26, 27])

Visualizations are (mostly) evaluated on equivariance

Expressiveness structure preserving mappings from data to graphic (Mackinlay [18])

Effectiveness design choices made in deference to perceptual saliency [**(**]Mackinlay [8–10, 20])

Naturalness easier to understand when properties match (Norman [21])

Graphical Integrity graphs show **only** the data (Tufte [31])

Models describe composition

language model APT: syntax and semantics of visualization (Mackinlay [18, 19])

functional dependencies constrained maps between data and visual representation(Sugibuchi [29])

category theory the semiotics of visualization are commutative (Vickers [33])

algebraic process data (α) and viz (ω) transforms are symmetric (Kindlmann and Scheidegger [17])

D data

R representations

V visualizations

$$\begin{tikzcd} D \arrow[r, "r_1"] \arrow[d, "\alpha"'] & R \arrow[r, "\nu"] & V \arrow[d, "\omega"] \\ D \arrow[r, "r_2"'] & R \arrow[r, "\nu"'] & V \end{tikzcd}$$

Contributions

Topological topology preserving relationship between data and graphic via continuous maps

Equivariant property preservation from data component to visual representation as equivariant maps that carry a homomorphism of monoid actions

Artist functional oriented visualization tool architecture built on the mathematical model to demonstrate the utility of the model

Model prototype of the architecture built on Matplotlib’s infrastructure to demonstrate the feasibility of the model

Topological Equivariant Artist Model

The Artist $\mathscr{A}$ is a map from data $\mathscr{E}$ to graphic $\mathscr{H}$

$$\mathscr{A} : \mathscr{E} \to \mathscr{H} \tag{1}$$

that carries a homomorphism of monoid actions

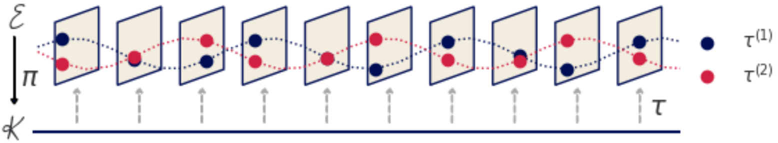
$$\varphi : M \to M' \tag{2}$$

such that artists are equivariant maps

$$\mathscr{A}(m \cdot r) = \varphi(m) \cdot \mathscr{A}(r) \tag{3}$$

with a deformation retraction from graphic to data space.

Data Bundle



A fiber bundle is a tuple (E, K, π, F) defined by the projection map π

$$F \hookrightarrow E \xrightarrow{\pi} K \tag{4}$$

where E is the total data space, F is the variable space, and K encodes the continuity.

Variables: Fiber

Given a space of all possible values $\mathbb{U}$

$$\begin{tikzcd} \mathbb{U}_\sigma \arrow[r] \arrow[d, "\pi_\sigma"'] & \mathbb{U} \arrow[d, "\pi"] \\ C \arrow[r, "\sigma"'] & \mathbf{DT} \end{tikzcd} \tag{5}$$

a fiber component is the restricted space $\mathbb{U}_{\sigma(c)}$.

$$F = \mathbb{U}_{\sigma(c)} = \mathbb{U}_T \tag{6}$$

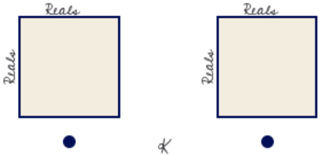
DT data types of the variables in the dataset

$\mathbb{U}$ disjoint union of all values of type $T \in \mathbf{DT}$

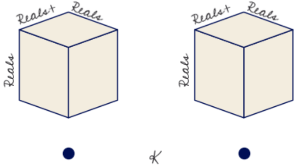
C variable names, $c \in C$

$\mathbb{U}_\sigma$ $\mathbb{U}$ restricted to the data type of a named variable

Variable types are dimensions of the fiber



(a)



(b)

Figure

4a $F = \mathbb{R} \times \mathbb{R}$, (time, temperature)

4b $\mathbb{R} \times \mathbb{R}^{+} \times \mathbb{R}$, (time, wind=(speed, direction))

Structure of Components: Monoid & Monoid Actions

A monoid M is a set with

associative binary operator $*: M \times M \to M$

identity element $e \in M$ such that $e * a = a * e = a$ for all $a \in M$.

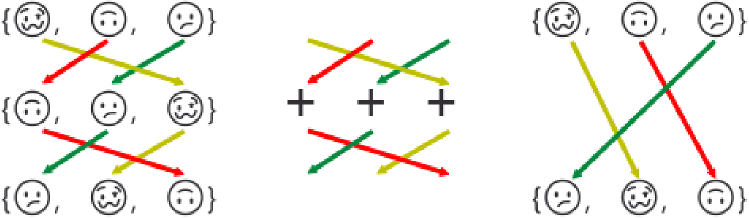
left monoid action

A set F with an action $\bullet : M \times F \to F$ with the properties:

associativity for all $f, g \in M$ and $x \in F$, $f \bullet (g \bullet x) = (f * g) \bullet x$

identity for all $x \in F, e \in M$, $e \bullet x = x$

Monoid Actions: Permutation



Why monoids? partial orders

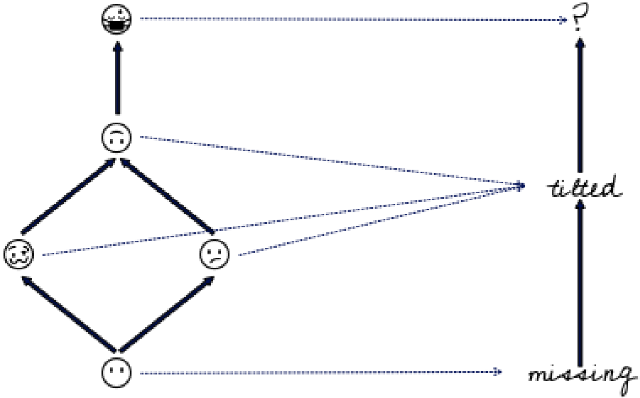
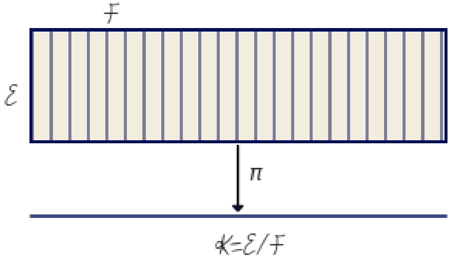


Figure: Inspired by definition 1.59 diagram in Spivak and Fong's An Invitation to Applied Category Theory [12]

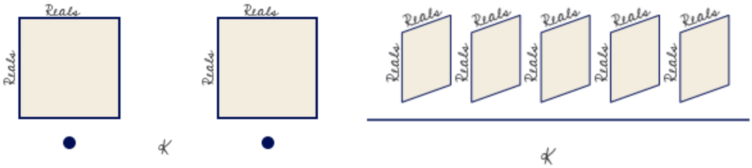
Data Continuity: Base space



where the total space can be decomposed into components

$$\pi : E_1 \oplus \ldots \oplus E_i \oplus \ldots \oplus E_n \rightarrow K \tag{7}$$

Data connectivity is encoded as the base space



(a)

(b)

Figure

6a data is discrete 0D points

6b data is lies on the 1D continuous interval K

Values: Section

For any fiber bundle, there exists a map

$$
\begin{tikzcd}
F \arrow[r, hook] & E \arrow[d, "\pi"'] \\
& K \arrow[u, bend right, "\tau"']
\end{tikzcd}
\tag{8}
$$

s.t. $\pi(\tau(k)) = k$. Set of all global sections is denoted $\Gamma(E)$.

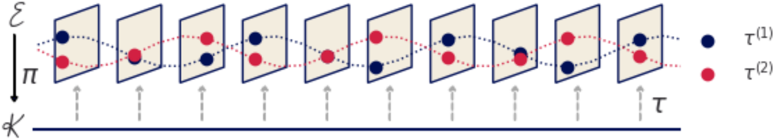
Record

Assuming a trivial fiber bundle $E = K \times F$, the section is

$$\tau(k) = (k, (g_{F_0}(k), \ldots, g_{F_n}(k))) \tag{9}$$

where $g : K \to F$ is the index function into the fiber.

Sample dataset



- F is $\mathbb{R} \times \mathbb{R}$
- K is interval $[0, 1]$
- $\tau^{(1)}$ is a *sin* function
- $\tau^{(2)}$ is a *cos* function
- $\tau^{(1)}, \tau^{(2)} \in \Gamma(E)$

Graphic Bundle

The graphics bundle is a tuple (H, S, π, D) defined by the projection map π

$$\begin{tikzcd} D \arrow[r, hook] & H \arrow[d, "\pi"'] \\ & S \arrow[u, bend right, "\rho"'] \end{tikzcd} \tag{10}$$

where ρ is the fully encoded graphic.

Example: 2D opaque image

The target display is $D = \mathbb{R}^5$ with elements

$$(x, y, r, g, b) \in D$$

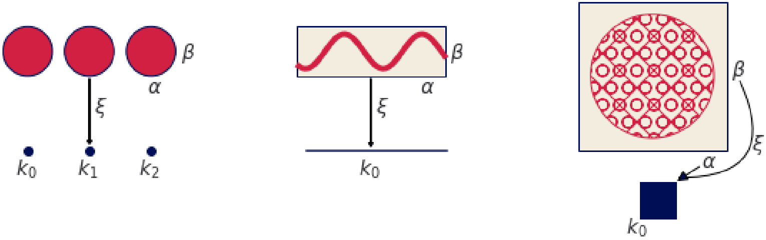
returned by ρ such that a graphic has color and 2D position.

Graphic Continuity

The surjective map $\xi : S \to K$

$$\begin{tikzcd} E \arrow[d, "\pi"'] & H \arrow[d, "\pi"'] \\ K & S \arrow[l, "\xi"'] \end{tikzcd} \tag{11}$$

goes from region $s \in S_k$ to its associated point k in data space.



Topological Equivariant Artist Model

The topological artist A is a monoid equivariant sheaf map

$$\begin{tikzcd}
E' \arrow[r, "\nu"] \arrow[dr, "\pi"'] & V \arrow[d, "\pi"] & \xi^*V \arrow[l, "\xi^*"'] \arrow[r, "Q"] \arrow[d, "\xi^*\pi"'] & H \arrow[dl, "\pi"] \\
& K & S \arrow[l, "\xi"'] &
\end{tikzcd} \tag{12}$$

where the artist $A : \mathcal{O}(E) \to \mathcal{O}(H)$ takes as input $E' = \mathcal{J}^2(E)$.

Visual Bundle

The visual bundle is a tuple (V, K, π, P) defined by the projection map π

$$\begin{tikzcd} P \arrow[r, hook] & V \arrow[d, "\pi"'] \\ & K \arrow[u, bend right, "\mu"'] \end{tikzcd} \tag{13}$$

where μ is the visual variable encoding[2] of the representation of the data section τ.

Example: position and color

Given an artist with parameters $\{xpos, ypos, color\}$, a sample visual section μ could be $\{.5, .5, (255, 20, 147)\}$

Visual Channel Encoders

We define the visual transformers ν on components of the data bundle τ_i

$$\{\nu_0, \ldots, \nu_n\} : \{\tau_0, \ldots, \tau_n\} \mapsto \{\mu_0, \ldots, \mu_n\} \tag{14}$$

as the set of equivariant maps with the constraint

$$\nu_i(m_r(E_i)) = \varphi(m_r)(\nu_i(E_i)) \tag{15}$$

where $\varphi : M \to M'$ carries a homomorphism of monoid actions.

Example: Nominal Equivariance

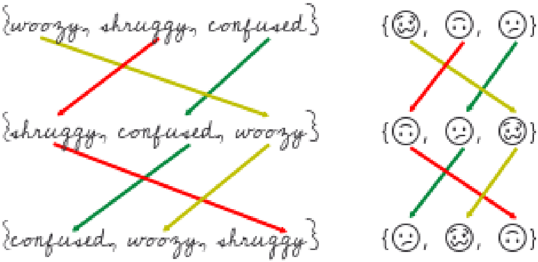
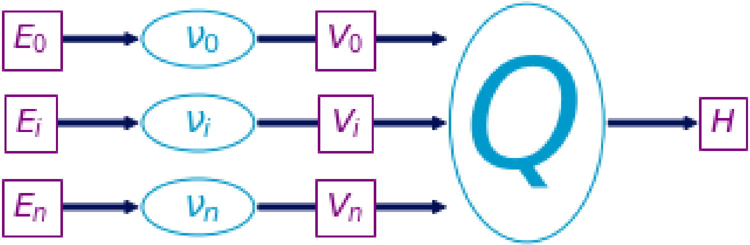


Figure: The actions on the text data are the same as the actions on the visual representation of that data as emojis.

Visualization Assembly Function



Glyph

The glyph is the graphic generated by $Q(S_j)$ where the path connected components $J \subset K$ are defined

$$J = \{j \in K \text{ s. t. } \exists\gamma \text{ s.t. } \gamma(0) = k \text{ and } \gamma(1) = j\} \tag{16}$$

such that the path γ from k to j is a continuous function from the interval [0,1] and S_j is the region

$$H \underset{\rho(S_j)}{\overset{}{\rightleftarrows}} S_j \underset{\xi^{-1}(J)}{\overset{\xi(s)}{\rightleftarrows}} J_k \tag{17}$$

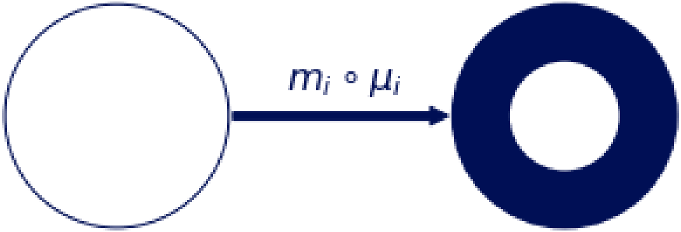
such that the glyph is differentiable, in keeping with Ziemkiewicz and Kosara's description of a glyph[36].

Visualization Equivariance

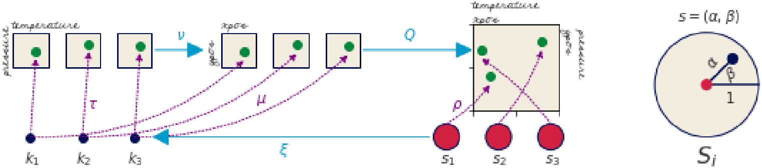
If Q is applied to μ, μ' that generate the same ρ

$$Q(\mu) = Q(\mu') \implies Q(m \circ \mu) = Q(m \circ \mu') \tag{18}$$

then the output of both sections acted on by the same monoid m must be the same.

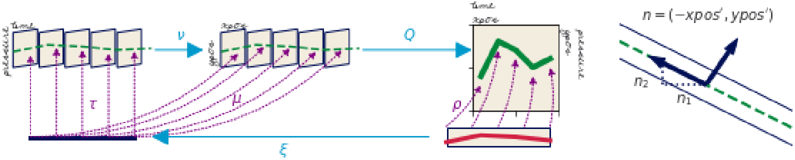


Scatter: $Q(xpos, ypos)(\alpha, \beta)$



$$x = size * \alpha \cos(\beta) + xpos$$
$$y = size * \alpha \sin(\beta) + ypos$$

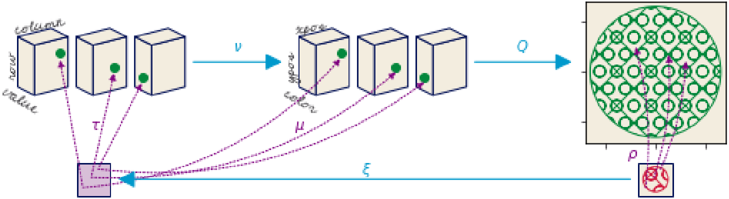
Line: $Q(xpos, \hat{n}_1, ypos, \hat{n}_2)(\alpha, \beta)$



$$|n| = \sqrt{{n_1}^2 + {n_2}^2},\ \hat{n}_1 = \frac{n_1}{|n|},\ \hat{n}_2 = \frac{n_2}{|n|}$$

$$x = xpos(\xi(\alpha)) + width * \beta \hat{n}_1(\xi(\alpha))$$
$$y = ypos(\xi(\alpha)) + width * \beta \hat{n}_2(\xi(\alpha))$$

Image $Q(xpos, yposcolor)$



$$R = R(\xi(\alpha, \beta))$$
$$G = G(\xi(\alpha, \beta))$$
$$B = B(\xi(\alpha, \beta))$$

References I

Rendering: Define a Pixel

Given a pixel

$$p = \left[y_{top}, y_{bottom}, x_{right}, x_{left}\right] \tag{19}$$

the inverse map of the bounding box

$$S_p = {\rho_{xy}}^{-1}(p) \tag{20}$$

is a region $S_p \subset S$ such that

$$r_p = \iint_{S_p} \rho_r(s)ds^2 \tag{21}$$

$$g_p = \iint_{S_p} \rho_g(s)ds^2 \tag{22}$$

$$b_p = \iint_{S_p} \rho_b(s)ds^2 \tag{23}$$

yields the color of the pixel.