

1 Notation & Definitions

In this section we introduce a mathematical description of the visualization pipeline where artist A functions transform data of type $\Gamma(E)$ to an intermediate representation in prerendered display space of type $\Gamma(H)$:

$$A : \mathcal{O}(E) \rightarrow \mathcal{O}(H) \quad (1)$$

- A is the function that converts an instance of data $\Gamma(E)$ to an instance of a visual representation $\Gamma(H)$
- E is a locally trivial fiber bundle over K representing data space.
- K is a triangulizable space encoding the connectivity of the points in E
- H is a fiber bundle over S representing visual space
- S is a simplicial complex encoding the visualization

When E is a trivial fiber bundle $E = F \times K$, it can be assumed that all fibers F_k over $k \in K$ are equal. Fiber bundles are product spaces of topological spaces, which are a set of points with a set of neighborhoods for each point[4, 10].

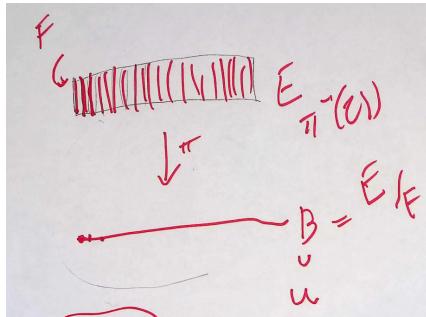
1.1 Fiber Bundles

We provide a brief description of fiber bundles because we model data, visual transformations, and a prerendered visual graphic as fiber bundles. A fiber bundle is a structure (E, B, π, F) consisting of topological total space E , base space B , fiber space F and the map from total space to base space:

$$F \hookrightarrow E \xrightarrow{\pi} B \quad (2)$$

where there is a bijection from F to every fiber F_b over point $b \in B$ in E and the function $\pi : E \rightarrow B$ is the map into the B quotient space of E . By defintion of a fiber bundle, π is always a mapping from total space to base space, independent of the points $p \in E$, and therefore we call this mapping π for all the fiber bundles in the model.

1.1.1 Base space



B is the quotient space of E , meaning it is the set of equivalence classes of elements p in E defined via the map $\pi : E \rightarrow B$ that sends each $p \in E$ to its equivalence class in $[p] \in B$ [8].

As shown in figure ??, the fibers F divide E into smaller spaces consisting of F and an open set neighborhood around F . This subdivision is projected down to the topology \mathcal{T}

$$\mathcal{T}_b = \{U \subseteq B : \{p \in E : [p] \in U\} \in \mathcal{T}_E\} \quad (3)$$

where $[p] \in U$ is the point $b \in B$ with an open set surrounding it that has an open preimage in E under the surjective map $\pi : p \rightarrow [p]$.

1.1.2 Fiber

As shown in equation!??, every point $b \in B$ has a local open set neighborhood U [4, 10]

$$\begin{array}{ccc} \pi^{-1}(U) & \xrightarrow{\varphi} & U \times F \\ \pi \downarrow & \nearrow \text{proj}_U & \\ U & & \end{array} \quad (4)$$

such that $\varphi : \pi^{-1}(U) \rightarrow U \times F$ is a homeomorphism where π and proj_U both map to U and the fiber over k $F_b = \pi^{-1}(b \in B)$ is homomorphic to the fiber F .

1.1.3 Section

The section f is the mapping from base space to total space $f : B \rightarrow E$

$$\begin{array}{ccc} F & \hookrightarrow & E \\ \pi \downarrow & \nearrow f & \\ B & & \end{array} \quad (5)$$

such that f is the right inverse of π

$$\pi(f(b)) = b \text{ for all } b \in B \quad (6)$$

In a locally trivial fiber bundle, $E = B \times F$ [4, 10]:

$$f(b) = (b, g(b)) \quad (7)$$

where the domain of $g(b)$ is F_b and returns a point p in F_b . The space of all possible sections f of E is $\Gamma(E)$. All sections $f \in \Gamma(E)$ have the same fibers F and connectivity B .

1.1.4 Sheaf and Stalk

As described in equation ??, there is a local space $U \subset B$ around every b . The inclusion map $\iota : U \rightarrow B$ can be pulled back such that $\iota^* E$ is the space of E restricted over U .

$$\begin{array}{ccc} \iota^* E & \xleftarrow{\iota^*} & E \\ \pi \downarrow & \nearrow \iota^* f & \pi \downarrow \\ U & \xleftarrow{\iota} & B \end{array} \quad (8)$$

The localized section of fibers $\iota^* f : U \rightarrow \iota^* E$ is the sheaf $\mathcal{O}(E)$ with germ of $\xi^* f$. The neighborhood of points the sheaf lies over is the stalk \mathcal{F}_b [11, 13]

$$\iota^{-1} \mathcal{F}(\{b\}) = \varinjlim_{b \subseteq U} \mathcal{F}(U) = \varinjlim_{b \in U} = \mathcal{F}_b \quad (9)$$

which through ι gets the data in E at and near to b . Restricting the artist to the sheaf means the artist knows the data in F and also has access to derivatives of the data. This property is useful for some visual transformations.

1.2 Data Model

As proposed by Butler [2, 3], we model data as a fiber bundle (E, K, π, F) with $\pi : E \rightarrow K$ where K which can be thought of as a set of keys k . A section $\tau : K \rightarrow E$ is an instance of the data that lies in E and is discussed in section 1.2.4.

1.2.1 Example



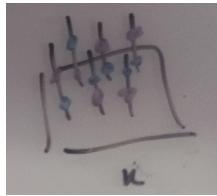
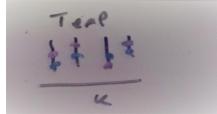
Figure 1: write up some words here

As illustrated by figure ??, the vertical lines F are the range of possible temperature values embedded in the total space E . The base space K of the fiber bundle is a line because the data points r in E are on a space that is continuous in one dimension.

1.2.2 Base Space

The base space K is a representation of the connectivity of the data, specifically whether the points in E are discrete or sampled from a continuous space. The same dataset can be expressed with different K .

1.2.3 Example



in figure ??, temperature is the only one data field in r but the K base spaces are different. subfig[1] is a timeseries, so the temperature in r at time t is dependent on the temperature in r_{t-1} and the temperature in r_{t+1} is dependent on r_t ; this connectivity is expressed as a one dimensional K where K is the number line. In the case of the map, every temperature in r is dependent on its nearest neighbors on the plane, and one way to express this is by encoding K as a plane. K does not know the time or latitude or longitude of the point as those are metadata variables describing the k rather than the value of k . The mapping $\tau : K \rightarrow E$ provides the binding between the key $k \in K$ and the value r in E [7].

1.2.4 Fiber and Sections

We use Spivak's formalization of data base schemas as the basis of our fiber space F [12]. He defines the type specification

$$\pi : U \rightarrow DT \quad (10)$$

where DT is the set of data types (as identified by their names) and U is the disjoint set of all possible objects x of all types in DT . This means that for each type $T \in DT$, the preimage $\pi^{-1}(T) \subset U$ is the domain of T , and $x \in \pi^{-1}(T) \subset U$ is an object of type T . Spivak then defines a schema (C, σ) of type π , where π is the universe of all types, such that

$$\sigma : C \rightarrow DT \quad (11)$$

where C is the finite set of names of columuns, which we generalize to data fields in E . The set of all values restricted to the datatypes in DT is U_σ

$$\begin{array}{ccc} U_\sigma & \longrightarrow & U \\ \pi_\sigma \downarrow & & \downarrow \pi \\ C & \xrightarrow{\sigma} & DT \end{array} \quad (12)$$

The pullback $U_\sigma := \sigma^{-1}(U)$ restricts U to the datatypes of the fields in C such that U_σ is the fiber product $U \times_{DT} C$, and the pullback $\pi_\sigma : U_\sigma \rightarrow C$ specifies the domain bundle U_σ over C induced by σ . The fiber F is the cartesian product of all sets in the disjoint union U_σ .

For each field $c \in C$, the record function $r : C \rightarrow U_\sigma$ returns an object of type $\sigma(c) \in DT$. The set of all records $\Gamma(\sigma)$ is the set of all sections on U_σ . Spivak defines the τ mapping from an index of databases K to records $\Gamma(\sigma)$ as $\tau : K \rightarrow \Gamma(\sigma)$. This is equivalent to $\tau : k \rightarrow E$ since $F = \Gamma(\sigma)$ and F is the embedding in E on which the records r lie.

1.2.5 Example



The fiber in figure ?? is the space of possible temperature values in degrees celsius, such that $F = [\text{temp}_{\min}, \text{temp}_{\max}]$ and is named Temp. In figure ?? time is encoded as a second dimension. This means that the set of possible values F with $C = \{\text{Temp}, \text{Time}\}$:



$$F = [temp_{min}, temp_{max}] \times [time_{min}, time_{max}] \quad (13)$$

and the function τ that retrieves records from F is

$$\tau(k) = (k, (r : Temp \rightarrow temp, r : Time \rightarrow time)) \quad (14)$$

$$temp \in [temp_{min}, temp_{max}], time \in [time_{min}, time_{max}] \quad (15)$$

Since $\tau(k) = (k, r)$, $temp$ is bound to a named data field and $sigma$ binds $temp$ to a temperature data type.

1.3 Prerender Space

We model the prerender space on which lives on ideal version of the visualization as a fiber bundle (H, S, π, D) . H is the predisplay space, with a fiber D dependent on the target physical display and a base space of S .

1.3.1 Base space

K can be considered a subspace of the screen base space S such that $\xi : S \rightarrow K$ is a deformation retraction [9]

$$\begin{array}{ccc} E & & H \\ \pi \downarrow & & \pi \downarrow \\ K & \xleftarrow{\xi} & S \end{array} \quad (16)$$

that goes from a region $s \in S_k$ to its associated point k , such that when $\xi(s) = k$, $\xi^*\tau(s) = \tau(k)$.

1.3.2 Fiber and Section

A section $\rho : S \rightarrow H$ is a mapping from a region s on a mathematical encoding of the image to a region xy on the screen that the renderer then maps to visual space as defined in D . For a physical screen display, the predisplay space is a trivial fiber bundle $H = \mathbb{R}^7 \times S$ such that ρ is

$$\rho(s) = \{x, y, z, r, g, b, a\} \quad (17)$$

To draw an image, a region, H is inverse mapped into a region $s \in S$ where

$$s = \rho_{XY}^{-1}(xy) \quad (18)$$

such that the rest of the fields in \mathbb{R}^7 are then integrated over s to yield the remaining fields:

$$r = \iint_s \rho_R(s) ds^2 \quad (19)$$

$$g = \iint_s \rho_G(s) ds^2 \quad (20)$$

$$b = \iint_s \rho_B(s) ds^2 \quad (21)$$

Here we assume a single opaque 2D image such that the z and *alpha* fields can be omitted.

1.3.3 Example

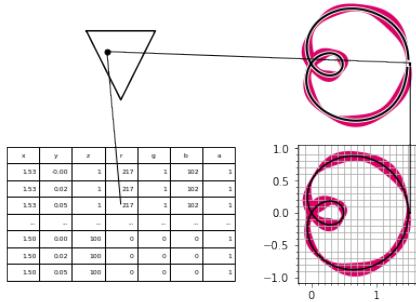


Figure 2

As illustrated in figure 2, words.

1.4 Artist

In this section we will define the artist as a mapping from a sheaf $\mathcal{O}(E)$ to $\mathcal{O}(H)$.

$$A : \mathcal{O}(E) \rightarrow \mathcal{O}(H) \quad (22)$$

The artist decomposes to mapping data to visual $\nu : E \rightarrow V$, then compositing V pulled back along ξ to ξ^*V to a visual mark in prerender space $Q : \xi^*V \rightarrow H$.

$$\begin{array}{ccc} E & \xrightarrow{\nu} & V \\ & \searrow \pi & \downarrow \pi \\ & & K \end{array} \quad \begin{array}{ccc} \xi^*V & \xrightarrow{Q} & H \\ \xi^*\pi \downarrow & & \swarrow \pi \\ S & \xleftarrow{\xi} & \end{array} \quad (23)$$

The visual fiber bundle (V, K, π, P) has section $\mu : V \rightarrow K$ that resolves to a visual variable [1, 6] in fiber P . The visual transformer ν is a set of functions each targeting a different μ

$$\nu : \{\nu_0, \dots, \nu_n\} \rightarrow \{\mu_0, \dots, \mu_n\} \quad (24)$$

where μ_i are the visual parameters in the assembly function $Q(\mu_0, \dots, \mu_n)(s) = \rho(s)$.

1.4.1 Example: Matplotlib Visual Fiber

For example, for Matplotlib [5], some of the possible types in P are:

ν_i	μ_i	$dom(\nu_i)$
position	x, y, z	\mathbb{R}
size	linewidth, markersize	\mathbb{R}
shape	markerstyle	$((\mathbb{R}, \mathbb{R}), x \in \{0, 1, 2, 3, 4, 79\})^n$ [†]
color	color, colors, linecolor, facecolor, markerfacecolor, edgecolor	\mathbb{R}^4
texture	hatch	$/* \backslash ^ - * + * x * o * O * . * * \in \Sigma^+$ $\Sigma = \{/, \backslash, , -, +, x, o, O, ., *\}$ [‡]
	linestyle	$(\mathbb{Z}, (\mathbb{Z}^n))^\dagger$

[†] $n > 0$

[‡] $\Sigma^+ = \bigcup_{n \in \mathbb{Z}^+} \Sigma^n$

1.4.2 Visual Channels

The function $\nu : E \rightarrow V$ is a mapping from the data bundle to the visual bundle

$$\begin{array}{ccc} E_1 & \xrightarrow{\nu} & V_1 \\ f \downarrow & & \downarrow f \\ E_2 & \xrightarrow{\nu} & V_2 \end{array} \quad (25)$$

such that a visual section $\mu : K \rightarrow V$ is equivalent to the visual transformation of the data $\mu(k) = \nu \circ \tau(k)$

+2 on data side means +2 on visual side

.2 = E1 - lambda e - ζ .2 V1 = .2 + .2 + .2 .4 = E2 - lambda e - ζ .2 V2 = .2

.2 = E1 - lambda e - ζ .2 V1 = .2

V2 = .2

1 = E1 - lambda e - ζ e/5 = .25

+ 1 + .25

2 = E2 - lambda e - ζ e/5 = .5

preserve unique mapping? does ν need to be invertable?

shifts in either space need to be mirrored in the other space, this is not valid here's why:

$\lambda e : .2$

$$\begin{array}{ccc} E_1 & \xrightarrow{\lambda : e \mapsto .5} & V_1 \\ e + 2 \downarrow & & \downarrow h + 2 \\ E_2 & \xrightarrow{\lambda} & V_2 \end{array} \quad (26)$$

$x,y = 1,2$ $y,x = 2,1$ $y,x = 1,2$

The constraint on $\nu : \tau \rightarrow \mu$

Allowable Maps For example, stevens: preserve structure Some examples of allowable maps are described in Bertin/Munzner: nus are constrained to preserve..., expect functorial nus - transform on input side to transform on output side (E1-E1, V1-V1, nu) \dashv condition that's how we're constraining nu. functorial in ν component

1.4.3 Constructing Marks

The visual fiber bundle V gets pulled back over S via ξ such that

$$\begin{array}{ccc} \xi^*V & \xrightarrow{Q} & H \\ & \searrow \xi^*\pi & \downarrow \pi \\ & S & \end{array} \quad (27)$$

the function $Q : \xi^*V \rightarrow H$ composites points in V into \mathbb{R}^\neq tuples. The section μ is pulled over s

$$\begin{array}{ccc} \xi^*V & \xrightarrow{Q} & H \\ \nearrow \xi^*\mu & & \uparrow \rho \\ S & & \end{array} \quad (28)$$

such that the composition Q of μ is equivalent to the render $\rho(s) = Q \circ \mu(s)$. (In practice $\mu \approx \xi^*\mu$) Q is the constructor for the mark, one Q per artist, $A = \text{sum of artists } Q$ as there's default line, point, etc that's stretched and tweaked and whatever else by the μ visual variable + structure preserving, Q must have map that commutes such that a change in μ carries over somehow to ρ , functorial in its arguments - commutative diagram - into rho