Topologically Equivariant Artist Model

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About

Me

- PhD candidate in Computer Science
- Matplotlib Community Manager & Core Developer
- socials/github: story645

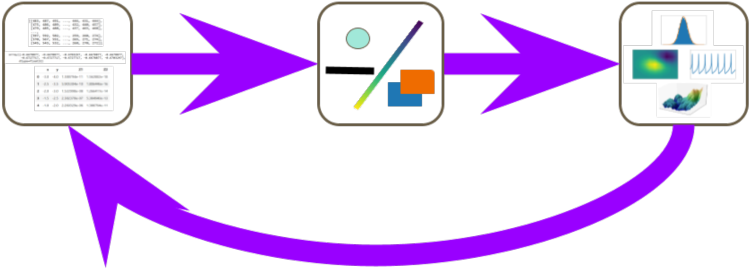
Project

- Funded by Chan Zuckerberg Initiative EOSS 1 & 3

Goals

- uniform API for arbitrary datasets in arbitrary data containers
- generalizable methodology for expressing structure preservation
- framework that translates into coding specifications

What do visualization libraries do?



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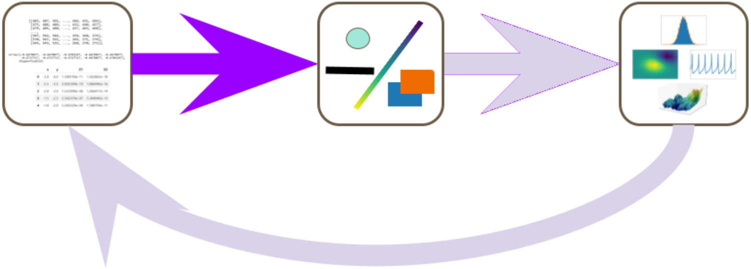

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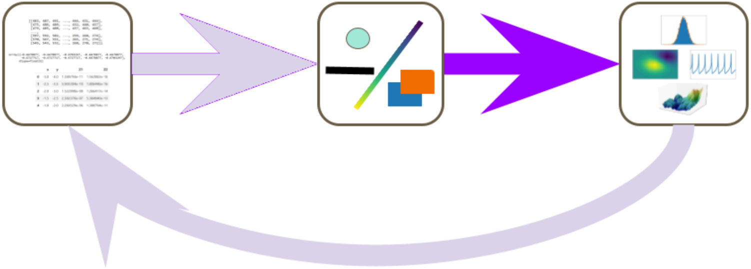

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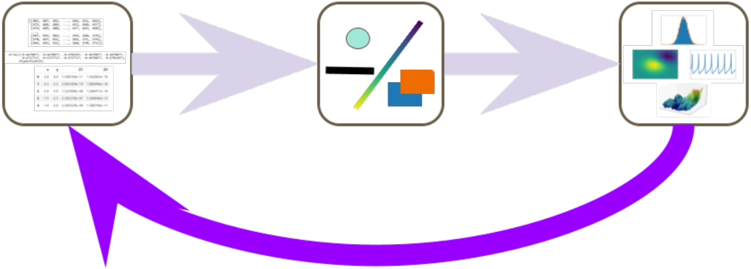


Matplotlib

How do we express structure?

field is a set of values of the same type, e.g. one column of a table or the pixels of an image

topology is the connectivity and relative positioning of elements in a dataset [1].

Structure Preservation: Equivalent scales and transforms

Definition

[2] An **action** of $G = (G, \circ, e)$ on X is a function $act : G \times X \to X$. An action has the properties of identity $act(e, x) = x$ for all $x \in X$ and associativity $act(g, act(f, x)) = act(f \circ g, x)$ for $f, g \in G$.

Fields: Equivariance

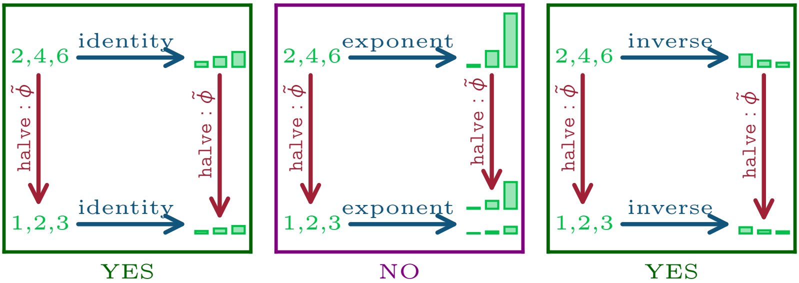
Given a group G that acts on both the input X and the output Y of a function $f : X \to Y$

Definition

A function f is **equivariant** when $f(act(g, x)) = act(g, f(x))$ for all g in G and for all x in X [3]

Fields: Equivariance

Is $\mathit{data} \mapsto \square$ equivariant w.r.t. scaling?



Fields: Homomorphism

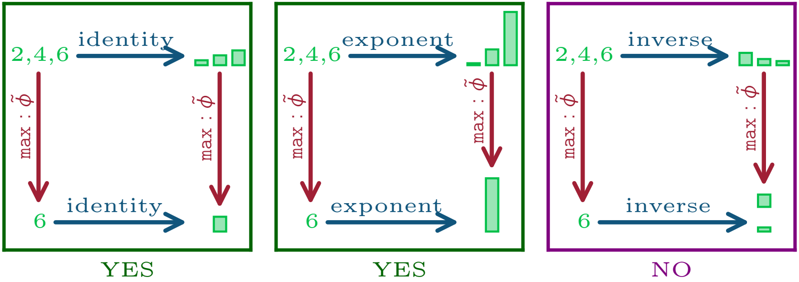
Given the function $f : X \to Y$, with operators $(X, \circ)$ and $(Y, *)$

Definition

A function f is **homomorphic** when $f(x_1 \circ x_2) = f(x_1) * f(x_2)$ and preserves identities $f(I_x) = I_y$ all $x, y \in X$ [2]

Fields: Homomorphism

Is $\mathit{data} \mapsto \square$ homomorphic w.r.t. partial order?

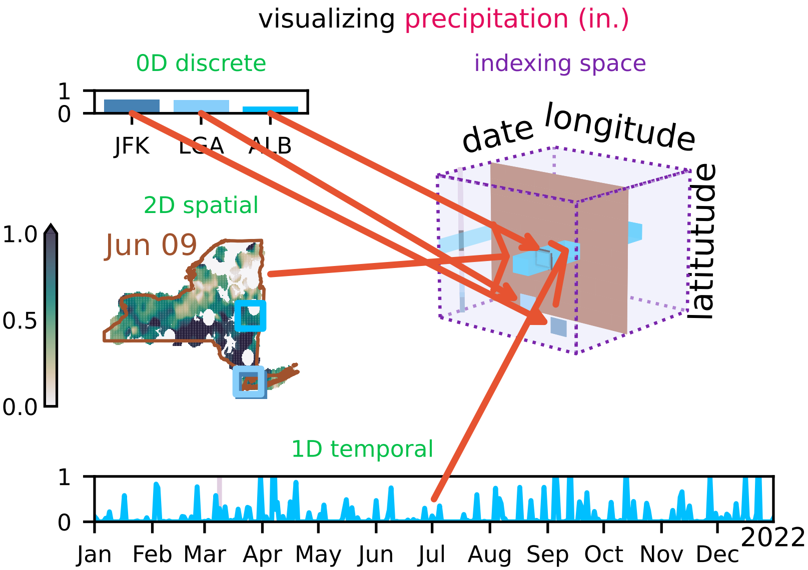


Topology: Homeomorphism

Definition

A function f is a *homeomorphism* if it is bijective, continuous, and has a continuous inverse function f^{-1}.

Homeomorphic topology



Structure Preservation

- Bertin: equivalent data and visual field properties [4]
- Mackinlay: homomorphic fields and equivalent topology [5]
- Wilkinson: homomorphic scales and equivalent topology [1]
- Kindleman & Schieidegger: equivariant scales, invariant data representation, equivalent topology [6]

contribution

explicit homeomorphic topology, equivariant actions on fields and topology

Domain Specific Library: library assumes structure [19]

DATE	LATITUDE	LONGITUDE	PRCP (in.)	NAME
2021-01-01	43.1167	-77.6767	0.2205	ROCHESTER GTR INTL AP
2021-01-01	41.5	-73.9	0.0000	STONYKILL NEW YORK
2021-01-01	42.7431	-73.8092	0.2598	ALBANY AP
2021-01-01	43.8	-73.7	0.0000	SCHROON LAKE NEW YORK
2021-01-01	43.0078	-73.6511	0.0000	SARA NEW YORK
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Vega[10]
Altair[13]
Tableau [16]
[17, 18]

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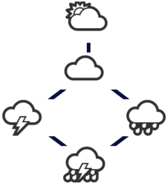


Gephi[9]
Graphviz[12]
Networkx[15]

Building Block Library[29]: visual algorithms assume structure [30]

1. Matplotlib[20] $\rightarrow$ Seaborn[21], xarray [22]
2. D3 [23]
3. VTK [24, 25],MayaVi[26]$\rightarrow$ Titan[27], ParaView[28]

Design Composable Structure Preserving API

Fiber Bundles: Butler: "unified, dimension-independent framework" that expresses data as the mapping between continuity and fields [31, 32]

Simplicial Databases: Spivak: Add rich typing for fields to Bulter [33]

Category Theory Language: express constraints in specifications [34]

Sheaves: Ghrist "algebraic data structure" for representing data over topological spaces [35]

Expressive Types

$$\texttt{dataset} : \texttt{topology} \rightarrow \texttt{fields}$$

Fiber Bundle

Definition

A **fiber bundle** (E, K, π, F) is a structure with topological spaces E, F, K and bundle projection map $\pi : E \to K$ [36].

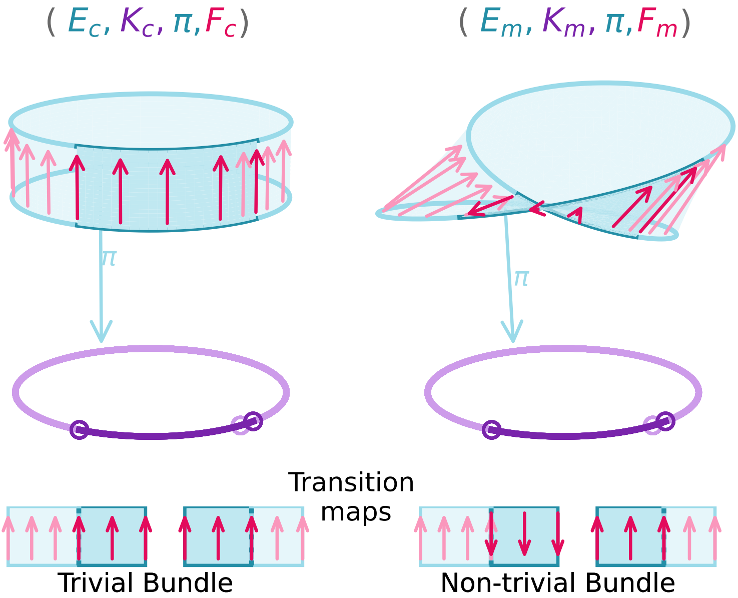
$$F \hookrightarrow E \xrightarrow{\pi} K \tag{1}$$

A continuous surjective map $\boldsymbol{\pi}$ is a **bundle projection** map when

1. all fibers in the bundle are isomorphic. Since all fibers are isomorphic $F \cong F_k$ for all points $k \in K$, there is a uniquely determined fiber space F given by the preimage of the projection π at any point k in the base space K: $F = \pi^{-1}(k)$.
2. each point k in the base space K has an open neighborhood U_k such that the total space E over the neighborhood is locally trivial.

Local triviality means $E|_U = U \times F$.

Fiber Bundle



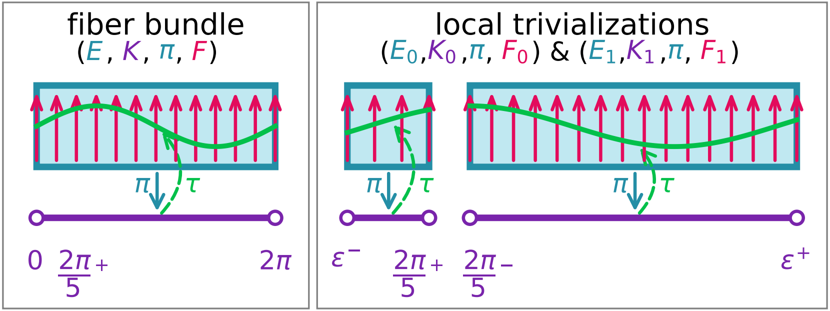
Fiber Bundle: section

Definition

A **section** $\tau : K \to E$ over a fiber bundle is a smooth right inverse of $\pi(\tau(k)) = k$ for all $k \in K$

$$\begin{tikzcd} F \arrow[r, hook] & E \arrow[d, two heads, "\pi"'] \\ & K \arrow[u, bend right, dashed, "\tau"'] \end{tikzcd} \tag{2}$$

Fiber Bundle: section



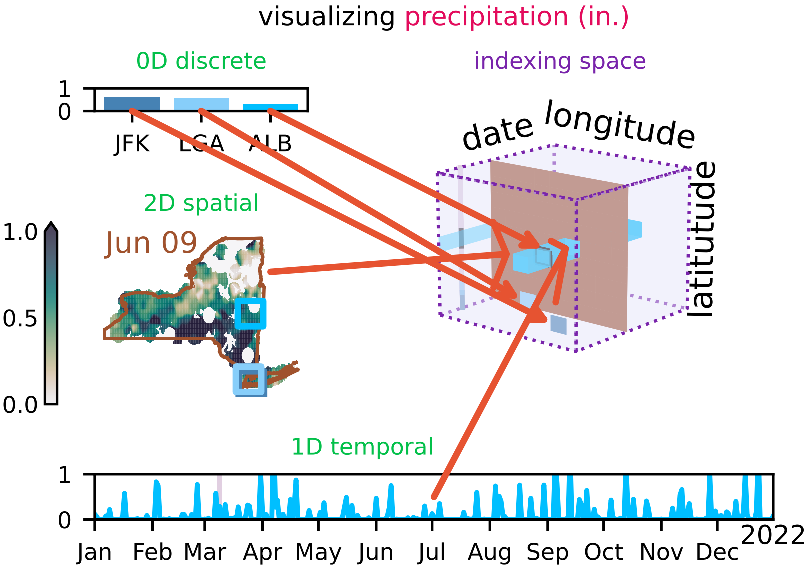
Base space: topology

The base space of a fiber bundle is a quotient topology[37]. Given a set X and a function $\mathcal{N} : X \to 2^{2^X}$ that assigns to any $x \in X$ a non-empty collection of subsets $\mathcal{N}(x)$, where each element of $\mathcal{N}(x)$ is a *neighborhood of* x, then X with $\mathcal{N}$ is a topological space and $\mathcal{N}$ is a neighborhood *topology* if for each x in X: [38]

Definition

1. if N is a neighborhood $N \in \mathcal{N}(x)$ of x then $x \in N$
2. every superset of a neighborhood of x is a neighborhood of x; therefore a union of a neighborhood and adjacent points in X is also a neighborhood of x
3. the intersection of any two neighborhoods of x is a neighborhood of x
4. any neighborhood N of x contains a neighborhood $M \subset N$ of x such that N is a neighborhood of each of the points in M

Why neighborhood topology?



Definition

An **category** $\mathcal{C}$ consists of the following *data*:

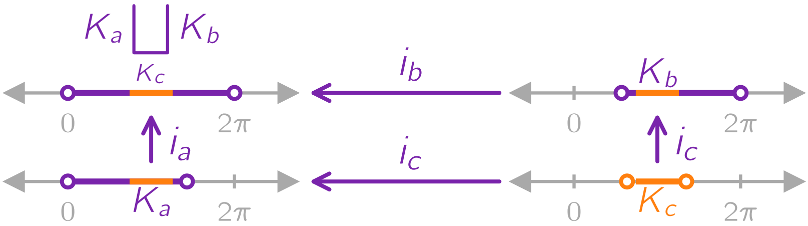
1. a collection of *objects* $X \in \mathbf{ob}(\mathcal{C})$
2. for every pair of objects $X, Y \in \mathbf{ob}(\mathcal{C})$, a set of *morphisms* $X \xrightarrow{f} Y \in Hom_{\mathcal{C}}(X, Y)$
3. for every object X, a distinct *identity morphism* $X \xrightarrow{id_x} X$ in $Hom_{\mathcal{C}}(X, X)$
4. a *composition function* $f \in Hom_{\mathcal{C}}(X, Y) \times g \in Hom_{\mathcal{C}}(Y, Z) \to g \circ f \in Hom_{\mathcal{C}}(X, Z)$

such that

1. *unitality:* for every morphism $X \xrightarrow{f} Y$, $f \circ id_x = f = id_y \circ f$
2. *associativity:* if any three morphisms f, g, h are composable,

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} W$$

Add records → Add to base space



Fiber space: Fields

types and schema	record	table
$\begin{tikzcd} \mathcal{U}_\sigma \arrow[r, dotted] \arrow[d, two heads, "\pi_\sigma"'] \arrow[dr, dotted] & \mathcal{U} \arrow[d, two heads, "\pi"] \\ C \arrow[r, "\sigma"'] & \mathtt{DT} \end{tikzcd}$	$\begin{tikzcd} \mathcal{U}_\sigma \arrow[d, two heads, "\pi_\sigma"'] \\ C \arrow[u, dashed, bend right, "r"'] \end{tikzcd}$	$\begin{tikzcd} F \coloneqq \Gamma^\pi(\sigma) \arrow[d, two heads, "\pi"'] \\ K \arrow[u, dashed, bend right, "\tau"'] \end{tikzcd}$

- $\mathcal{U}$ Space of all possible values, e.g. $\mathbb{R} \times \mathbb{R}^+$
- C Field names, e.g. *TEMP*, *PRCP*
- $\mathtt{DT}$ Data Type, e.g. `float`, `PosFloat`

Types and Schema

A *schema* consists of a pair (C, σ) where C is the set of field names and $\sigma : C \to \mathbf{DT}$ is a function from field name to field data type[33] The function σ is composed with π such that $\pi^{-1}(\sigma(C)) \subseteq \mathscr{U}$; this composition induces a domain bundle $\pi_\sigma : \mathscr{U}_\sigma \to C$ that associates a field name $c \in C$ with its corresponding domain $\pi_\sigma^{-1}(C) \subseteq \mathscr{U}_\sigma$.

Record

Definition

A **record** is a function $r : C \to \mathscr{U}_\sigma$ and the set of records on π_σ is denoted $\Gamma^\pi(\sigma)$. Records must return an object of type $\sigma(C) \in \mathbf{DT}$ for each field $c \in C$.

Tables are sections $\tau : K \to \Gamma^\pi(\sigma)$ from an indexing space K to the set of all possible records $\Gamma^\pi(\sigma)$ on the schema bundle

Fiber TYPE F

We define the fiber space F to be the space of all possible data records

$$F \coloneqq \{r : C \to \mathscr{U}_\sigma \mid \pi_\sigma(r(C)) = C \textit{ for all } C \in C\} \tag{3}$$

such that the preimage of a point is the corresponding data type domain $\pi^{-1}(k) = F_k = \mathscr{U}_{\sigma_k}$.

Fiber category F

The fiber category has a single object F of an arbitrary type and morphisms on the fiber object $\tilde{\phi} \in Hom(F, F)$.
The fiber category $\mathcal{F}$ is also equipped with a bifunctor $\otimes : \mathcal{F} \times \mathcal{F} \to \mathcal{F}$ for combining fiber types.

Fiber morphism $\tilde{\phi}$

scale	operators	sample constraint
nominal	$=, \neq$	$\tau(k_1) \neq \tau(k_2) \implies \tilde{\phi}(\tau(k_1)) \neq \tilde{\phi}(\tau(k_2))$
ordinal	$<, \leqslant, \geqslant, >$	$\tau(k_1) \leqslant \tau(k_2) \implies \tilde{\phi}(\tau(k_1)) \leqslant \tilde{\phi}(\tau(k_2))$
interval	$+, -$	$\tilde{\phi}(\tau(k) + C) = \tilde{\phi}(\tau(k)) + C$
ratio	$*, /$	$\tilde{\phi}(\tau(k) * C) = \tilde{\phi}(\tau(k)) * C$

Constraints as definition: Dates

- year: $\{y \in \mathbb{I} | 1992 \leqslant y \leqslant 2025\}$
- month: $\{m \in \mathbb{I} | 1 \leqslant m \leqslant 12\}$
- day: $\{d \in \mathbb{I} | 1 \leqslant d \leqslant 31\}$
- date: $\otimes : F_{year} \times F_{month} \times F_{day} \rightarrow F_{date}$

where the composition $\otimes$ includes the constrain to only return dates that have the right number of days for each month.

Expand records → multiply fiber space

$$
\begin{tikzcd}
\overset{F_a \times_{F_c} Fb}{\text{59°F,Oct 02,0.07in.}} \arrow[r, "\pi_b"] \arrow[d, "\pi_a"] & \overset{F_b}{\text{Oct 02,0.07in.}} \arrow[d, "\pi_c"] \\
\underset{F_a}{\text{59°F,Oct 02}} \arrow[r, "\pi_c"] & \underset{F_c}{\text{Oct 02}}
\end{tikzcd}
$$

local sections

We define section functions locally:

$$\Gamma(U, E\restriction_U) \coloneqq \left\{\tau : U \to E\restriction_U \;\middle|\; \pi(\tau(k)) = k \textit{ for all } k \in U\right\} \tag{4}$$

such that each section function $\tau : k \mapsto r$ maps from each point $k \in U$ to a corresponding record in the fiber space $r \in F_k$ over that point.

This encoding can be translated into a set of signatures $\{\texttt{data-subset} : \texttt{topology} \to \texttt{fields}$ s.t. $\texttt{data-subset} \subset \texttt{dataset}\}$.

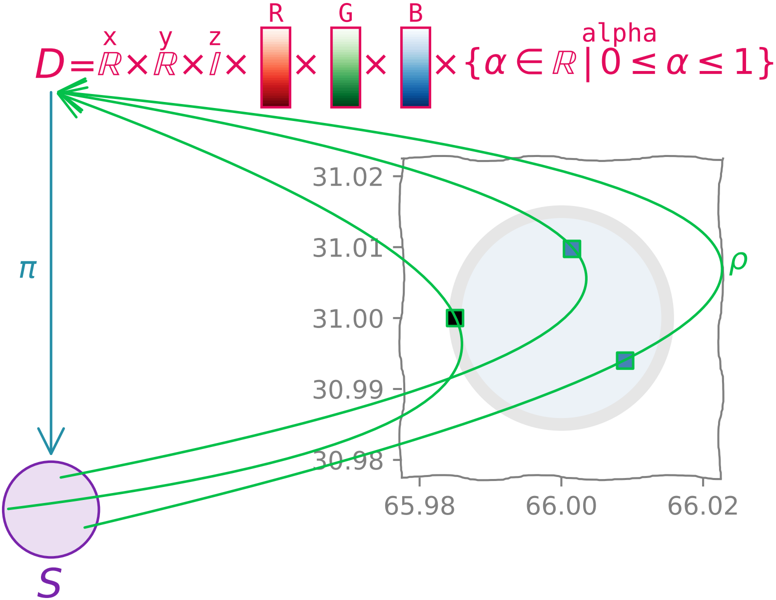
global sections

When a bundle is trivial $E = K \times F$, we can defined a global sections $\tau : K \to F \in \Gamma(K, F)$ which we translate into a data signature of the form $\texttt{dataset} : \texttt{topology} \to \texttt{field}$ where $\tau = \texttt{dataset}$, $K = \texttt{topology}$ and $F = \texttt{fields}$

Graphic bundle and section

$$D \hookrightarrow H \xrightarrow{\pi} \twoheadrightarrow S \tag{5}$$

$$\begin{aligned} \Gamma(W, H{\restriction}_W) \coloneqq & \\ & \left\{ \rho : W \to H{\restriction}_W \;\middle|\; \pi(\rho(s)) = s \textit{ for all } s \in W \right\} \end{aligned} \tag{6}$$


R
G
B
alpha
x
y
z
D = ℝ × ℝ × I × × × × {α ∈ ℝ | 0 ≤ α ≤ 1}
π
ρ
S
31.02
31.01
31.00
30.99
30.98
65.98
66.00
66.02

Definition

[43, 44] A **functor** is a map $F : \mathcal{C} \to \mathcal{D}$, which means it is a function between objects $F : \mathbf{ob}(\mathcal{C}) \mapsto \mathbf{ob}(\mathcal{D})$ and that for every morphism $f \in Hom(C_1, C_2)$ there is a corresponding function $F : Hom(C1, C2) \mapsto Hom(F(C_1), F(C_2))$. A **functor** must satisfy the properties

- *identity*: $F(id_C(C)) = id_D(F(C))$
- *composition*: $F(g) \circ F(f) = F(g \circ f)$ for any composable morphisms $C_1 \xrightarrow{f} C_2$, $C_2 \xrightarrow{g} C_3$

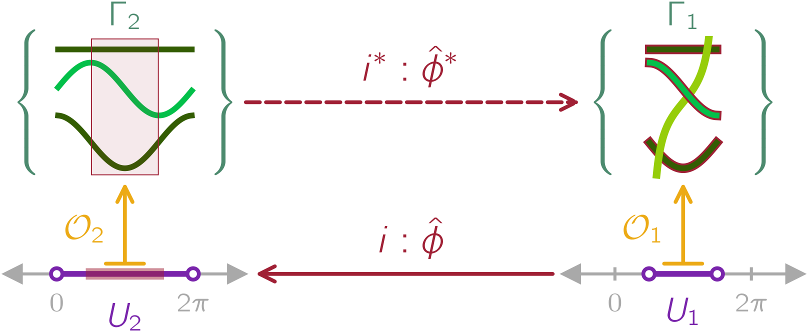
$F(C) \in \mathbf{ob}(\mathcal{D})$ denotes the object to which an object C is mapped, and $F(f) \in Hom(F_(C_1), F_(C_2))$ denotes the morphism that f is mapped to.

Definition

A **presheaf** $F : \mathcal{C}^{op} \rightarrow \boldsymbol{Set}$ is a contravariant functor from an object in an arbitrary category to an object in the category $\boldsymbol{Set}$[36].

The presheaf is contravariant when for every arbitrary morphism between input base spaces $\hat{\phi} : U_1 \rightarrow U_2$ there exists a corresponding pullback function between the sets of sections $\hat{\phi}^* : \Gamma(U_2, E\restriction_{U_2}) \rightarrow \Gamma(U_1, E\restriction_{U_c1})$.

topological equivariance: presheaf



Container invariance: sheaf

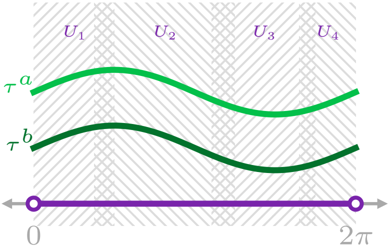
Definition

[45, 36] A **sheaf** is a presheaf that satisfies the following two axioms

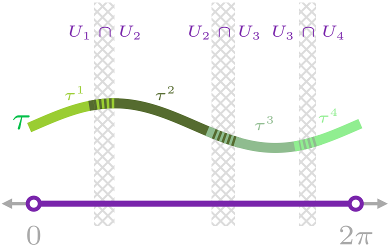
- *locality* two sections in a sheaf are equal $\tau^a = \tau^b$ when they evaluate to the same values $\tau^a|_{U_i} = \tau^b|_{U_i}$ over the open cover $\bigcup_{i \in I} U_i \subset U$ (indexed by I).
- *gluing* the union of sections defined over subspaces $\tau^i \in \Gamma(U_i, E|_{U_i})$ is equivalent to a section defined over the whole space $\tau|_{U_i} = \tau^i$ for all $i \in I$ if all pairs of sections agree on overlaps $\tau^i|_{U_i \cap U_j} = \tau^j|_{U_i \cap U_j}$

Container invariance: sheaf

locality: $\tau^a = \tau^b$



gluing: $\tau|_{U_i} = \tau^i$



What about needing more than one record?

The sheaf over a very small region surrounding a point k is called a *stalk*[46]

$$\mathcal{O}_{K,E}\restriction_k \coloneqq \lim_{U \ni k} \Gamma(U, E\restriction_U) \tag{7}$$

where the fiber is contained inside the stalk $F_k \subset \mathcal{O}_{K,E}\restriction_k$. The *germ* is the section evaluated at a point in the stalk $\tau(k) \in \mathcal{O}_{K,E}\restriction_k$ and is the data.

Homeomorphism

We define the mapping ξ to be a surjective continuous map:

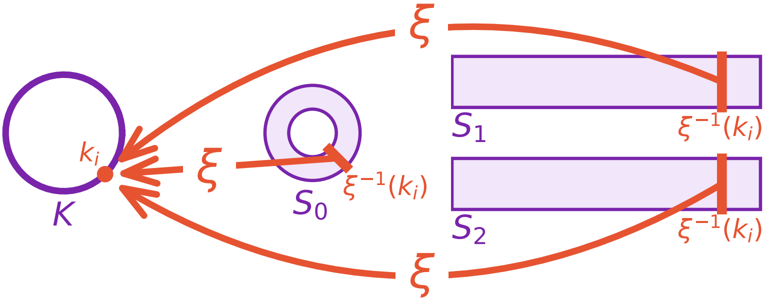
$$\xi : W \rightarrow U \tag{8}$$

between a graphic subspace $W \subseteq S$ and data subspace $U \subseteq K$.
The set of points in graphic space that correspond to each point in data space is

$$\xi^{-1}(k) = \{s | \xi(s) = k \forall k \in K, s \in S\} \tag{9}$$

such that every point in a graphic space has a corresponding point in data space.

Homeomorphism



Pushforward: Graphic for Data

Definition

Given a sheaf $\mathcal{O}_{S,H}$ on S, the **pushforward** sheaf $\xi_*\mathcal{O}_{S,H}$ on K is defined as

$$\xi_*(\mathcal{O}_{S,H})(U) = \mathcal{O}_{S,H}(\xi^{-1}(U)) \tag{10}$$

for all opensets $U \subset K$[46].

This provides a way to look up which graphic corresponds with a data index

$$\xi_*\rho(k) = \rho\restriction_{\xi^{-1}(k)} \tag{11}$$

such that $\xi_*\rho(k))(s) = \rho(s)$ for all $s \in \xi^{-1}(k)$.

Pullback: Data at graphic

Definition

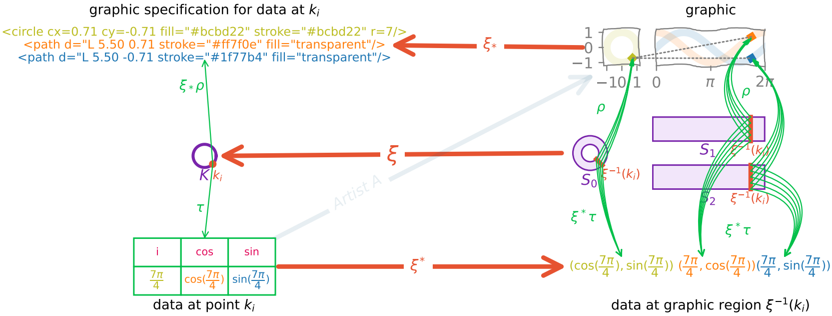
[46] Given a sheaf $\mathcal{O}_{K,E}$ on K, the **pullback** sheaf $\xi^*\mathcal{O}_{K,E}$ on S is defined as the sheaf associated to the presheaf $\xi^*(\mathcal{O}_{K,E})(W) = \mathcal{O}_{K,E}(\xi(W))$ for $\xi(W) \in K$.

This provides a way to then look up what data values correspond with a graphic index

$$\xi^*\tau(s) = \tau(\xi(s)) = \tau(k) \tag{12}$$

As ξ is surjective, there are many points $s \in W \subseteq S$ in the graphic space that correspond to a single point $\xi(s) = k$.

Homeomorphism



Definition

Given two functors $F, G : \mathcal{C} \rightarrow \mathcal{D}$, a **natural transformation** $\alpha : F \rightarrow G$ is a function which assigns to each object c of $\mathcal{C}$ a morphism $\alpha_c : F(input) \rightarrow G(c), G(c) \in \mathcal{D}$, in such a way that for every morphism $f : c \rightarrow c', c' \in \mathcal{C}$, the morphisms in $\mathcal{D}$ commute such that $\alpha'_c(F(f)(F(c))) = G(f)(\alpha_c(F(c)))$. When this holds, α_c is *natural* in c.[41].

Artist

$$
\begin{tikzcd}
\texttt{topology} \arrow[rr, bend left, "\texttt{data}"] \arrow[rr, bend right, "\texttt{graphic}"'] \arrow[rr, phantom, "\Downarrow\,\texttt{artist}"] & & \texttt{topology} \arrow[r] & \texttt{fields}
\end{tikzcd}
$$

How?

$$
\begin{tikzcd}
\Gamma(U, \xi_* H\restriction_U) & & \Gamma(W, ,)H\restriction_W \arrow[ll, "\xi_*"'] \\
\Gamma(U, E\restriction_U) \arrow[u, "Hom_{\mathcal{O}_K}"] \arrow[urr, "Hom_{\mathcal{O}_K, \mathcal{O}_S}"] \arrow[rr, "\xi^*"] & & \Gamma(W, \xi^* E\restriction_W) \arrow[u, "Hom_{\mathcal{O}_S}"']
\end{tikzcd}
\tag{13}
$$

Artist

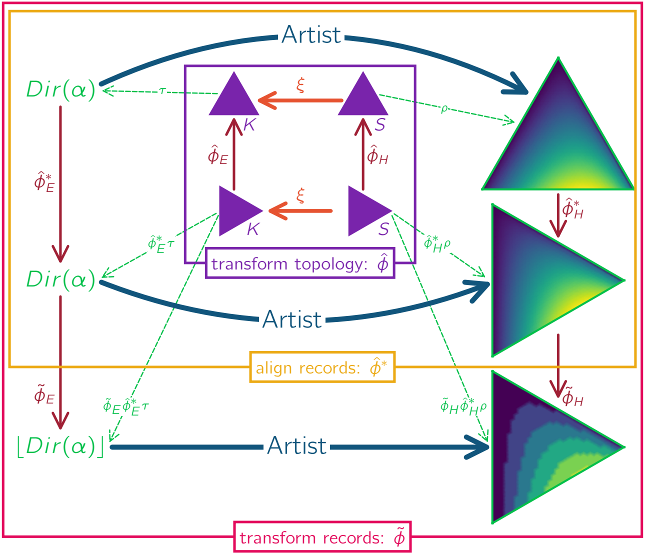
$$A : \Gamma(K, E) \rightarrow Im_A(S, H), Im_A(S, H) \subset \Gamma(S, H) \tag{14}$$

where:

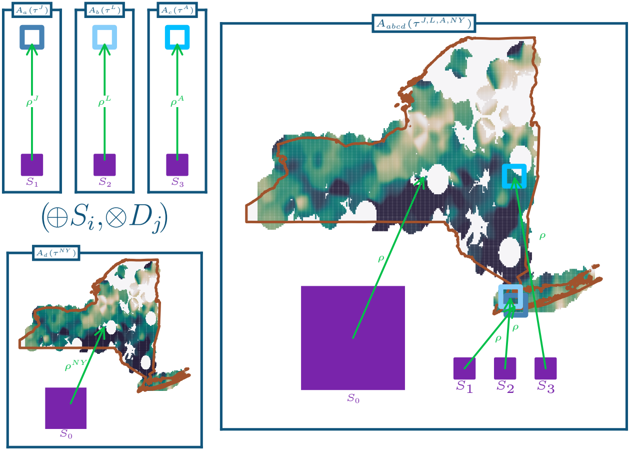
$$Im_A(S, H) \coloneqq \{\rho \mid \exists \tau \in \Gamma(K, E) \; s.t. \; A(\tau) = \rho, \; \xi(S) = K\} \tag{15}$$

$$
\begin{tikzcd}
\Gamma(K, E) \arrow[rrrr, "A"] \arrow[d, "\hat{\phi}^*{}_E"'] & & & & Im_A(S, H) \arrow[d, dotted, "\hat{\phi}^*{}_H"] \\
\Gamma(K', \hat{\phi}^*{}_E E) \arrow[dd, "\tilde{\phi}_E"'] & K & & S \arrow[ll, "\xi"'] & Im_A(S', \hat{\phi}^*{}_H H) \arrow[dd, dotted, "\tilde{\phi}_H"] \\
 & K' \arrow[u, "\hat{\phi}_E"] & & S' \arrow[ll, "\xi"'] \arrow[u, dotted, "\hat{\phi}_H"'] & \\
\Gamma(K', E') \arrow[rrrr, "A"] & & & & Im_A(S', H')
\end{tikzcd}
\tag{16}
$$

Visualizing the Dirichlet Distribution



$$\bigsqcup_i \prod_j A(Hom(K_i, F_j)) = A(Hom(\bigsqcup_i K_i, \prod_j F_j)) \implies$$



$$\bigsqcup_i \prod_j Hom(S_i, D_j)) = Hom(\bigsqcup_i S_i, \prod_j D_j)$$

Animation and Interactivity

pan, zoom, scroll sheaf: locality + gluing 12

selection and hover pushforward 11, pullback 12

brushing, linking, annotation composition of artists **??**

Testing if A is equivariant

M is a (scaler, vector) measurable component (e.g. color, position, shape, texture, rotation,) of the rendered visual element.

$$\begin{tikzcd}
\Gamma(U, E\restriction_U) \arrow[r, "A"] \arrow[d, "\eta"'] & Im_A(W, H\restriction_W) \arrow[r, "render"] \arrow[d, "\delta"] & visualization \arrow[dl, "measure"] \\
Hom(U, M) \arrow[r, "\xi^*"'] & Hom(W, M) &
\end{tikzcd}$$

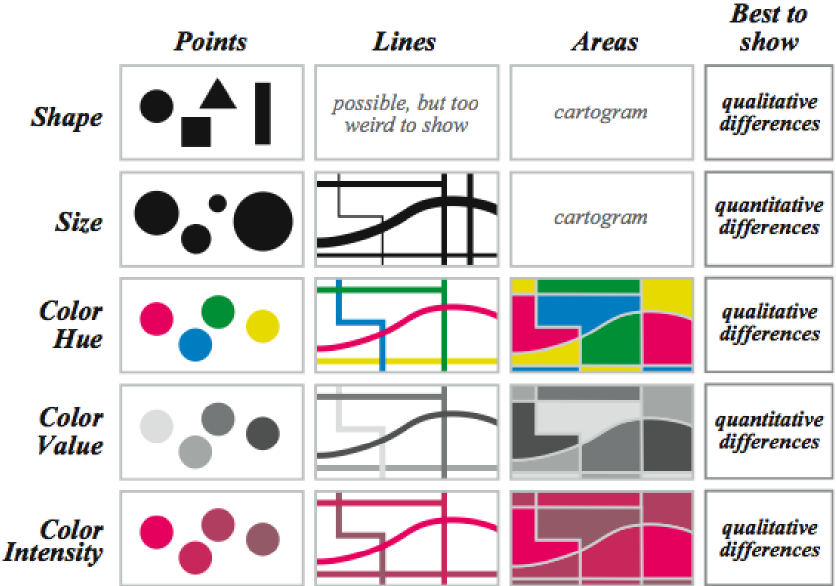
$$\text{input } \eta : \tau \mapsto (U \xrightarrow{\eta_\tau} M)$$

$$\text{output } \delta : \rho \mapsto (W \xrightarrow{\delta_\rho} M)$$

$$\eta_\tau(k) = \delta_\rho(s) \text{ for all } \xi(s) = k, k \in K, s \in S$$

Visual Space: Specialized data space

$P \hookrightarrow V \xrightarrow{\pi} K$



Encoding

$$\nu : F_k \to P_k \tag{17}$$

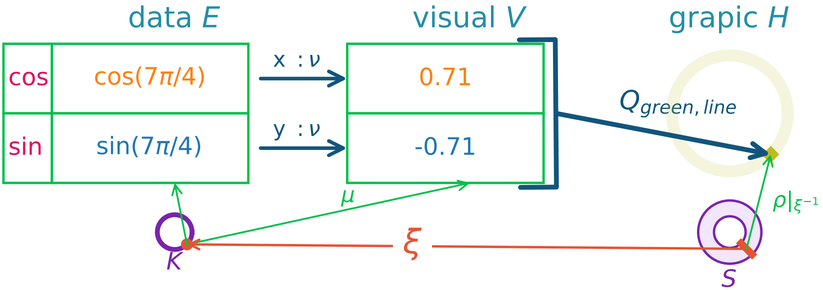
$$\begin{tikzcd}
F_k \arrow[r, "\nu"] \arrow[rr, bend right, dashed, "\nu''"'] & P_k \coloneqq F'_k \arrow[r, "\nu'"] & P'_k
\end{tikzcd} \tag{18}$$

Composition

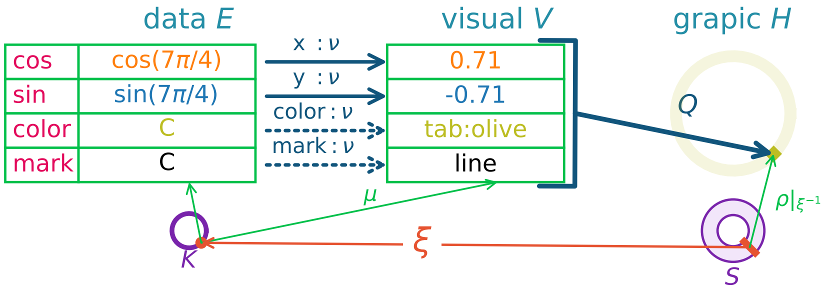
$$Q : \Gamma(K, V) \to \Gamma(S, H) \tag{19}$$

$$\begin{tikzcd}
\Gamma(K, V) \arrow[r, "\nu"] \arrow[rr, bend right, dashed, "Q'"] & \Gamma(K, V') \arrow[r, "Q"] & \Gamma(S, H)
\end{tikzcd} \tag{20}$$

Construction Stages



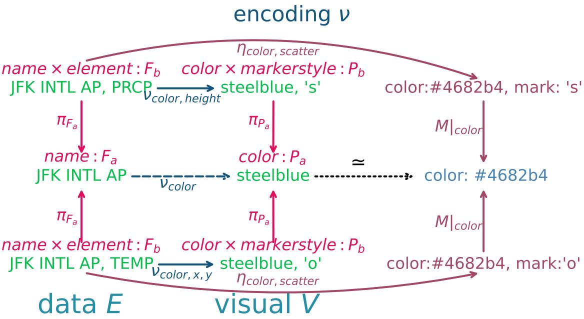
Construction Stages: composition



Verify encoder

$$
\begin{tikzcd}
F_k^a \times F_k^b \arrow[r, "\nu_{ab}"] \arrow[d, "\pi_a"'] \arrow[rr, bend left, "\eta_{ab}"] & P_k^a \times P_k^b \arrow[d, "\pi_a"] & M_k^{ab} \arrow[d, "M\restriction_a"] \\
F_k^a \arrow[r, dashed, "\nu_a"] & P_k^a \arrow[r, dotted, "\simeq"] & M_k^a \\
F_k^a \times F_k^c \arrow[r, "\nu_{ac}"] \arrow[u, "\pi_a"] \arrow[rr, bend right, "\eta_{ac}"'] & P_k^a \times P_k^c \arrow[u, "\pi_a"'] & M_k^{ac} \arrow[u, "M\restriction_a"']
\end{tikzcd}
\tag{21}
$$

Verify encoder



Verify compositor

$$
\begin{tikzcd}
\Gamma(K, V^a \times V^b) \arrow[r, "Q_{ab}"] \arrow[d, "\pi_a"'] & Im_{\underset{ab}{A}}(S, H) \arrow[d, "M\restriction_a \circ \delta_{ab}"] \\
\Gamma(K, V^a) \arrow[r, dotted, "\simeq"] & Hom(K, M^a) \\
\Gamma(K, V^a \times V^c) \arrow[r, "Q_{ac}"] \arrow[u, "\pi_a"] & Im_{\underset{ac}{A}}(S, H) \arrow[u, "M\restriction_a \circ \delta_{ac}"']
\end{tikzcd}
\tag{22}
$$

Verify compositor

composition Q

$$
\begin{tikzcd}[column sep=huge]
\begin{array}{c} \mathit{color} \times \mathit{markerstyle} : P_b \\ \text{steelblue, 's'} \end{array} \arrow[r, "Q_{square}"] \arrow[d, "\pi_{P_a}"'] & \square \arrow[d, "M|_{P_a} \circ \delta_{P_a,\, square}"] \\
\begin{array}{c} \mathit{color} : P_a \\ \text{steelblue} \end{array} \arrow[r, dashed, "\simeq"] & \text{color: \#4682b4} \\
\begin{array}{c} \mathit{color} \times \mathit{markerstyle} : P_b \\ \text{steelblue, 'o'} \end{array} \arrow[r, "Q_{circle}"] \arrow[u, "\pi_{P_a}"] & \bigcirc \arrow[u, "M|_{P_a} \circ \delta_{P_a,\, circle}"']
\end{tikzcd}
$$

visual V | graphic H

Implementation Choices: $A_K = A_S$

$$\begin{tikzcd}
\Gamma(W, \xi^* E\restriction_W) \arrow[r, "\nu_S"] \arrow[rr, bend left, "A_S"] & \Gamma(W, \xi^* V\restriction_W) \arrow[r, "Q_S"] & Im_A(W, H\restriction_W) \arrow[d, "\xi_*"] \\
\Gamma(U, E\restriction_U) \arrow[u, "\xi^*"] \arrow[r, "\nu_K"] \arrow[rr, bend right, "A_K"'] & \Gamma(U, V\restriction_U) \arrow[u, "\xi^*"] \arrow[r, "Q_K"] & Im_A(U, \xi_* H\restriction_U)
\end{tikzcd}$$

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