

Topological Equivariant Artist Model for Visualization Library Architecture

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Abstract—The abstract goes here.

Index Terms—

1 INTRODUCTION

This paper uses methods from algebraic topology and category theory to develop a model of the transformation from data to graphical representation. This model provides a language to specify how data is structured and how this structure is carried through in the visualization and serves as the basis for a functional approach to implementing visualization library components. Topology allows us to describe the structure of the data and graphics in a generalizable, scalable, and trackable way. Category theory provides a framework for separating the transformations implemented by visualization libraries from the various stages of visualization and therefore can be used to describe the constraints imposed on the library components [1], [2]. Well constrained modular components are inherently functional [3], and a functional framework yields a library implementation that is likely to be shorter, clearer, and more suited to distributed, concurrent, and on demand tasks [4]. Using this functional approach, this paper contributes a practical framework for decoupling data processing from visualization generation in a way that allows for modular visualization components that are applicable to a variety of data sets in different formats. **is it OK that this is something reviewer 4 wrote**

We restrict the properties of data that should be preserved to

continuity how elements in a dataset are organized, e.g. discrete rows in a table, networked nodes, pixels in an image, points on a line

equivariance functions on data that have an equivalent effect on the graphical representation, e.g. rotating a matrix has a matching rotation of the image, translating the points on a line has a matching visual shift in the line plot

2.1 Continuity

NAME	TEMP (°F)	PRCP (in.)
NEW YORK LAGUARDIA AP	61.00	0.4685
BINGHAMTON	-12.00	0.0315
NEW YORK JFK INTL AP	49.00	0.7402
ISLIP LI MACARTHUR AP	11.00	0.0709
SYRACUSE HANCOCK INTL AP	13.00	0.0118

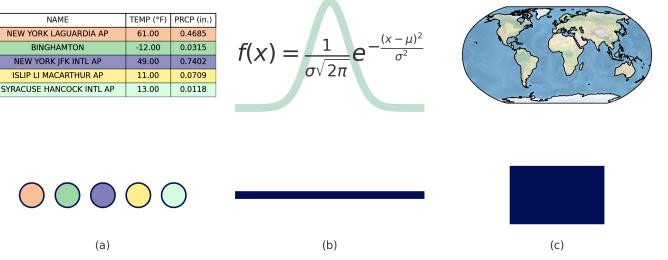


Fig. 1: Continuity is how elements in a data set are connected to each other, which is distinct from how the data is structured. The rows in (a) are discrete, therefore they have discrete continuity as illustrated by the discrete dots. The gaussian in (b) is a 1D continuous function, therefore the continuity of the elements of the gaussian can be represented as a line on an interval (0,1). In (c), every element of the globe is connected to its nearest neighbors, which yields a 2D continuous continuity as illustrated by the square.

Continuity describes elements in a data set are organized; this concept is termed topological properties by Wilkinson [6]. Wilkinson provides the examples of values that are isolated from each other, and therefore discrete, and values lying on a continuum or in a compact region. For example, in Figure 1, each station record in the table is independent of the others; therefore, the continuity of the table is discrete. The data provided by the gaussian are points sampled along the curve, therefore the continuity of the points on the line is 1D continuous. Every point on the globe is connected to its 6 nearest cardinal neighboring

2 RELATED WORK

This work aims to develop a model for describing visualization transformations that can serve as guidance for how to architecture a general purpose visualization library. We define a general purpose visualization library as one that provides non domain specific building block components [5] for building visualizations, for example functions for converting data to color or encoding data as dots. In this section, we describe how visualization libraries attempt this goal and discuss work that formally describes what properties of data should be preserved in a visualization.

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points (NW, N, NE, E, SE, S, SW, W). We propose that a robust model of continuity provides a way to develop library components that can work on the data in small pieces in a manner where the overall continuity of the data is preserved in the visual transformation.



Fig. 2: Continuity is implicit in choice of visualization rather than explicitly in choice of data container. The line plots in (b) are generated by a 2D table (a). Structurally this table can be identical to the 2D matrix (a) that generates the image in (c).

The preservation of continuity can be made explicit, as in the transformation of table to parallel coordinates in Ruchikachorn and Mueller [7], but is often expressed implicitly in the choice of visual algorithm (visualization type), as explored in taxonomies by Tory and Möller [8] and Chi [9].

For example, in Figure 2 the same table can be interpreted as a set of 1D continuous curves when visualized as a collection of line plots or as a 2D surface when visualized as an image. This means that often there is no way to express data continuity independent of visualization type, meaning most visualization libraries will allow, for example, visualizing discrete data as a line plot or an image. General purpose visualization libraries such as Matplotlib [10], Vtk [11], [12], and D3 [13]-carry distinct data models as part of the implementation of each visual algorithm. The lack of unified data model means that each plot in a linked [14], [15] visualization is treated as independent, as are the transforms converting each field in the data to a visual equivalent.

Domain specific libraries can often guarantee consistency because they have a single model of the data in their software design, as discussed in Heer and Agarwal [16]'s survey of visualization software design patterns. For example, the relational database is core to tools influenced by APT, such as Tableau [17], [18], [19] and the Grammar of Graphics [6] inspired ggplot [20], Vega [21] and Altair [22]. Images underpin scientific visualization tools such as Naspari [23] and ImageJ [24] and the digital humanities oriented ImagePlot [25] macro; the need to visualize and manipulate graphs has spawned tools like Gephi [26], Graphviz [27] and Networkx [28].

2.1.1 Fiber Bundles

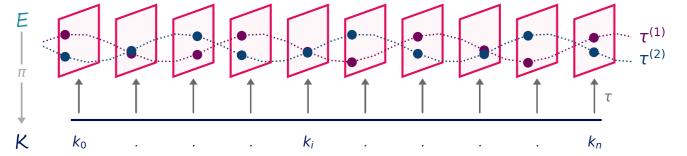


Fig. 3: A fiber bundle is mathematical construct that allows us to express the relationship between data and continuity. The **total** space E is the topological space in which the data is embedded. The **fiber** space F is embedded in E and is the set of all possible values that any **add big rectangle** E

The model described in this work provides a model for expressing data sets with different topological properties. We obtain this generality by using a mathematical structure called a fiber bundles as the basis of our abstraction, as proposed by Butler, Bryson, and Pendley [29], [30]. As described by them, a fiber bundle is a formal model of the mapping between data points and the underlying topological space they lie in. For example, nodes on a network and the graph of the network, an image and the underlying grid at which the image is sampled, or table columns and the table index.

Formally, a fiber bundle is a mathematical structure (E, K, π, F)

$$F \hookrightarrow E \xrightarrow{\pi} K \quad (1)$$

Definition 2.1. The **base space** K is a topological space, which means it is a set K with a collection of open sets [31] surrounding each element in the set. Open sets are a collection of subsets $\{U\}$ in a set K , including the empty set, the whole set K , and every union and intersection of its member subsets. For each point $k \in K$, there is a collection of subsets $\{U\} \subset K$ such that $k \in U \subseteq K$ [32], [33]

intuition about base space

Definition 2.2. The **fiber space** F is a topological space such that for every $k \in K$, the fiber over that point is isomorphic to the preimage of that point $F_k \cong \pi^{-1}(\{k\})$. All fibers F_k are homeomorphic to each other [32], [34]

The fibers can be thought of as encapsulating data types according to Spivak, who proposes that databases can be represented as fiber bundles [35], [36]. Spivak formulates the fiber as a (field name, data domain) simple schema where field names and field types can be formally mapped to the set of values associated with the field type, for example \mathbb{R} for a float column named *temperature*.

Definition 2.3. The **total space** E is the space reachable via the projection map π . The total space is always locally trivial, meaning $E = K \times F$ over any open neighborhood U_k .

The fiber bundle is trivial when $E = K \times F$ is true globally; a non-trivial bundle can be thought of as the twisted fiber product $E = K \times_T F$ [34], [37]. The twisting refers to the fiber F not aligning in the same direction over all $U \subseteq K$.

As proposed by Butler et al [29], [30], data can be represented as a map from the base space K to the fiber

space F . This map is called a section $\tau : K \rightarrow E$ of the fiber bundle.

$$\begin{array}{ccc} F & \xhookrightarrow{\quad} & E \\ & \pi \downarrow & \uparrow \tau \in \Gamma(K, E) \\ & & K \end{array} \quad (2)$$

A fiber bundle has many sections, and the set of all sections over an open set $openset \subseteq E$

$$\Gamma(U, E) := \{\tau^i : U \rightarrow E\}_{i \in I} \quad (3)$$

is the set of all data sets with a shared fiber F and continuity K . The set of all sections defined over the total base space is denoted $\Gamma(K, E)$.

2.1.2 Sheaves

Sheaves can be thought of as an "algebraic data structure" [38] for data that lives over topological spaces. A presheaf \mathcal{O} is a map from the open set to the set of sections $\mathcal{O} : U \rightarrow \Gamma(U, E)$ [32], [39], [40]. The presheaf preserves inclusion maps i between open sets and the inclusion map i^* between the sets of sections.

$$\begin{array}{ccccc} \Gamma(U_1, E) & \xleftarrow{i^*} & \Gamma(U_2, E) & & \\ \uparrow \mathcal{O}_E & \uparrow \mathcal{O}_E & \uparrow \mathcal{O}_E & & \\ U_1 & \xleftarrow{i} & U_2 & & \end{array} \quad (4)$$

This means that a smaller space U_1 is included in a larger space U_2 and a function that is continuous over a larger space U_2 is continuous over a subspace $U_1 \subset U_2$. Sheaves are presheaves where sets of sections are defined over unions of open sets over a topology $\bigcup_{j \in I} U_j \in E$ [32], [41]. Sheaves provide a mathematical abstraction of data, the space it lives over, and relationships between subsets.

2.2 Equivariance

When introducing the retinal variables, Bertin informally specifies that continuity is preserved in the mark and defines equivariance constraints in terms of data and visual variables being selective, associative, ordered, or quantitative [42]. In the *A Presentation Tool*(APT) model, Mackinlay embeds the continuity constraint in the choice of visualization type and generalizes the equivariance constraint to preserving a binary operator from one domain to another. The algebraic model of visualization [43], proposed by Kindermann and Scheidegger, restricts equivariance to invertible transformations.

2.2.1 Category Theory

In this work, we propose that equivariance constraints can be expressed using category theory. Vickers et. al [44] provide a brief introduction to category theory for visualization practitioners, but their work focuses on reasoning about visualization design, while this paper aims to provide guidance on designing visualization library components. Briefly, we introduce concepts in category theory that we use to describe our model in subsection 3.1 and section 4.

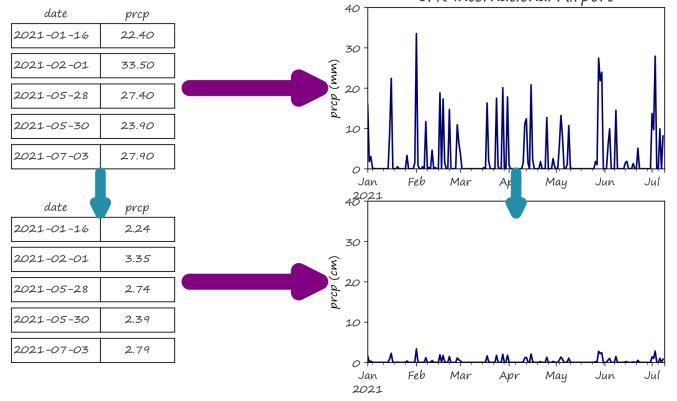


Fig. 4: Equivariance is that a transformation on the data has a corresponding transformation in the graphical representation. For example, in this figure the data is scaled by a factor 10. Equivalently the line plot is scaled by factor of 10, resulting in a shrunken line plot. Either a transformation on the data side can induce a transformation on the visual side, or a transformation on the visual side indicates that there is also a transformation on the data side.

Definition 2.4. A category \mathcal{C} consists of a collection of objects c with identity $id_c : C \rightarrow C$ and the set of morphisms between every two objects $f : C_1 \rightarrow C_2$, [45], [46].

The set of morphisms between objects is termed the hom set $hom_{\mathcal{C}}(C_1, C_2)$. The morphisms on the category compose

$$\begin{array}{ccccc} & & id_{C_2} & & \\ & & \swarrow & & \\ id_{C_1} & \curvearrowright & C_1 & \xrightarrow{f} & C_2 \\ & & \searrow & & \downarrow g \\ & & & & C_3 \\ & & & & \curvearrowright id_{C_3} \end{array}$$

and are constructed such that the following axioms hold [47]:

associativity if $f : C_1 \rightarrow C_2$, $g : C_2 \rightarrow C_3$ and $h : C_3 \rightarrow C_4$
then $h \circ (g \circ f) = (h \circ g) \circ f$

identity for every $f : C_1 \rightarrow C_2$ there exists identity morphisms $f \circ id_{C_1} = f = id_{C_2} \circ f$

Definition 2.5. An opposite category \mathcal{C}^{op} is a category with all the same objects of category \mathcal{C} and morphisms that are reversed. For example $f : C_1 \rightarrow C_2$ in \mathcal{C} is $f : C_2 \rightarrow C_1$ in \mathcal{C}^{op} .

Definition 2.6. A functor is a morphism between objects of any two categories $F : \mathcal{C} \rightarrow \mathcal{D}$. A functor has the properties of identity and composition [47]

A functor preserves the structure of the category, namely morphisms and the source and target objects of those morphisms, composition of morphisms, and identity morphisms [47]. For example, the functor $F : \mathcal{C} \rightarrow \mathcal{D}$ maps an

196 object in \mathcal{C} into an object in \mathcal{D}

$$\begin{array}{ccc} c \in \text{Ob}(\mathcal{C}) & \xrightarrow{F} & F(c) \in \text{Ob}(\mathcal{D}) \\ f \downarrow & & \downarrow F(f) \\ c' \in \text{Ob}(\mathcal{C}) & \xrightarrow{F} & F(c') \in \text{Ob}(\mathcal{D}) \end{array} \quad (5)$$

197 such that for every morphism $f : c \rightarrow c' \in \text{Hom}(\mathcal{C})$ there
198 is a compatible morphism $F(f) : F(c) \rightarrow F(c') \in \text{Hom}(\mathcal{D})$.
199 Functors $F : \mathcal{C}^{\text{op}} \rightarrow \mathcal{D}$ are called contravariant functors.

200 **Definition 2.7.** A presheaf \mathcal{O} as introduced in Equation 43
201 is a functor $\mathcal{O} : \mathcal{C}^{\text{op}} \rightarrow \text{Set}$. [40]

202 **Definition 2.8.** A natural transformation is a morphism 46
203 of functors $\alpha : F \rightarrow G$ where F, G are functors from \mathcal{C} to \mathcal{D} . [47]
204 [48]

205 The natural transformation composes with a functor such
206 that the output is a compatible functor between the same
207 categories. This means that for every morphism $f : c \rightarrow$
208 c' between objects in \mathcal{C} , there is a commuting naturality
209 square for the corresponding objects in \mathcal{D} [49].

$$\begin{array}{ccccc} & & G & & \\ & \swarrow & \downarrow & \searrow & \\ c & \xrightarrow{F} & F(c) & \xrightarrow{\alpha_c} & G(c) \\ f \downarrow & & \downarrow F(f) & \dashrightarrow & \downarrow G(f) \\ c' & \xrightarrow{F} & F(c') & \xrightarrow{\alpha_{c'}} & G(c') \\ & \searrow & \downarrow & \swarrow & \\ & & G & & \end{array}$$

210 unpack this a drop more

211 3 ARTIST

212 In this section we describe the properties a data to visuali55
213 zation transform must satisfy to be considered equivariant6
214 and continuity preserving. We propose that by explicitly6
215 defining these properties, we can better incorporate them13
216 into visualization library design.

217 We formulate the visualization transformation as a func60
218 tor, which we call the *Artist* A^1 . The artist A converts61
219 *data*, which we denote \mathcal{E} to *graphics* \mathcal{H} . We formulate this62
220 association as

$$A : \mathcal{E} \rightarrow \mathcal{H} \quad (8)$$

221 In this section we encapsulate the structure of the data
222 and graphic spaces as categories. We then use these defini267
223 tions to express the structure that an artist preserves. Finally268
224 we introduce rules for composing artists in a way where269

¹We call this transformation the artist because the *Artist* object Matplotlib [10]

225 the structure between the inputs to the individual artists is
226 preserved.

3.1 Natural Transformation of Data to Graphic

Fiber bundles, as introduced in subsubsection 2.1.1, serve as the model of data and graphic spaces in our model. In this section, we introduce categorical formulations of the topological spaces that make up a fiber bundle.

We introduce the category \mathcal{K} to encapsulate the subsets of the continuity and the inclusion maps that glue them together.

Definition 3.1. The category of open sets \mathcal{K} consists of

- objects open sets $U_i \in \{\text{openset}\}$ in the topological space (K, \mathcal{T}) , including the empty set \emptyset and the maximum set K .
- morphisms $\iota : U_i \hookrightarrow U_j$ for all $U_i, U_j \subset E$

We model data as the sheaf $\mathcal{O}(E) : \mathcal{K}^{\text{op}} \rightarrow \text{Set}$. The objects of Set in this sheaf are the sets of sections introduced in Equation 3 and the morphisms are the inclusion maps in Equation 4.

$$\begin{array}{c} \Gamma(U_i, E) \in \text{Ob}(\text{Set}) \\ \uparrow \quad \downarrow \\ \mathcal{O}(E) \xrightarrow{\varphi} \mathcal{O}(E) \\ \uparrow \quad \downarrow \\ U_i \in \text{Ob}(\mathcal{K}^{\text{op}}) \end{array} \quad (9)$$

244 The map $\varphi : \mathcal{O}(E) \rightarrow \mathcal{O}(E)$ is any arbitrary morphism
245 between sheaf functors $\mathcal{O}(E)$. Following from Equation 6,
246 φ is a natural transform, which means all inclusions ι, ι^* are
247 preserved as part of any φ transformation.

248 **Definition 3.2.** The fiber category \mathcal{F} is a monoidal category
249 [49], which means it is a category equipped with a bifunctor
250 $\otimes : \mathcal{F} \times \mathcal{F} \rightarrow \mathcal{F}$

- object a monoid object F . A monoid is a set with a binary operator that is associative, closed, and for which the set contains an identity element i [?].
- morphisms $F \otimes F \rightarrow F$ and unit $i \rightarrow F$

The bifunctor \otimes combines categories in a way that preserves identity and composition [45]. For example, given a pair of categories $\mathcal{F}_+, \mathcal{F}_-$, the bifunctor yields a set of pairs $\{(F_a, F_b) | F_a \in \mathcal{F}_+, F_b \in \mathcal{F}_-\}$. For the associated morphisms $g : \mathcal{F}_+ \rightarrow \mathcal{F}_+$ and $h : \mathcal{F}_- \rightarrow \mathcal{F}_-$, the function $(g \times h) : (\mathcal{F}_+ \times \mathcal{F}_-) \rightarrow (\mathcal{F}_+ \times \mathcal{F}_-)$ is pointwise $(g \times h)(F_a, F_b) := (g(F_a), h(F_b))$. Because of the bifunctor, objects of the category \mathcal{F} can be the product of any arbitrary number and types of categories, including Set , Graph , and topological spaces Top .

The set of all morphism of the fiber category $\text{Hom}_{\mathcal{F}}(F, F)$ are all possible morphisms in a given category. This includes any of the functions that serve as the basis of equivariance, such as the binary operations, group actions, and measurement scale discussed in subsection 2.2. The generalization to $\text{Hom}_{\mathcal{F}}$ also allows for the inclusion of structures such as monoid actions [50], which provide a way of applying partial order relations to data [45], such as to build multi-ranked indicators [51].

When a fiber bundle is trivial, we can dispense with localization and directly consider the fiber maps $\text{Hom}(F_k, F_k)$ over a point k as the basis of our equivariance. This is because a sheaf $\mathcal{O}(E)$ over a limit of open sets that contain a point $k \in K$ is approximately the same as a sheaf over a point k and is called the stalk $\mathcal{O}(E)|_k := \lim_{U \ni k} \Gamma(U, E)$ [52]. The fiber space over a point is embedded in the stalk over that point $F_k \hookrightarrow \mathcal{F}_k$. The sheaf maps φ restricted to the stalk are members of $\text{hom}_{\mathcal{F}}(F, F)$. The maps $\varphi \in \text{Hom}_{\mathcal{F}}(F, F)$ include the types of data transformations described in subsection 2.2. In section 4, we assume that the fiber bundles are trivial.

The continuity of the graphic is encapsulated as a category \mathcal{S} containing open set objects U' , the complete set of opensets S that is the full space of the graphic, and the empty set. As with \mathcal{K} , the morphisms between the open sets U' are inclusion morphisms ι . Over the graphic space $U' \subseteq S$ are a set of sections

$$\Gamma(U', H) := \{\rho^i : U' \rightarrow H\}_{i \in I} \quad (10)$$

which are the set of all graphics with a shared base space and fiber and are a subcategory of Set . This allows us to represent the full space of graphics as a sheaf $\mathcal{O}(H)$

$$\begin{array}{ccc} \text{Ob}(\text{Set}) & \ni & \Gamma(U'_i, H) \xleftarrow{\iota} \{\rho^j : U'_i \rightarrow H\}_{j \in J} \\ & \uparrow & \nearrow \text{A}(\mathcal{O}(E)) := \mathcal{O}_A(H) \\ \mathcal{O}(H) & & \\ & \uparrow & \\ U'_i & & \end{array} \quad (11)$$

The set of graphics reachable through an artist function is a subset $\{\rho^j\}_{j \in J}$ of the full set of graphics. These sets of reachable graphics over open sets U' are also objects of Set . This allows us to formulate the graphic output by the artist as a sheaf \mathcal{O}_A from the graphic base space to the set of reachable graphics.

To establish a mapping between graphic and data base spaces, we introduce the surjective map $\xi : S \rightarrow K$.

$$\begin{array}{ccc} E & \xleftarrow{\xi^*} & \xi^* E \\ \uparrow \tau \circ \xi & & \uparrow \xi^* \tau \\ K & \xleftarrow{\xi} & S \end{array} \quad (12)$$

As shown in Equation 12, the map between spaces ξ can be composed with section map τ such that there is a map from the graphic base space S to the fiber bundle in which the data lives E .

Since there is a mapping from the graphic space to the total bundle $\tau \circ \xi : S \rightarrow E$, the mapping ξ can be pulled through the composition such that there is a map ξ^* between a total space over graphics $\xi^* E$ and its equivalent space over data E . As illustrated in the example given in Figure ??, a bundle E over a point $k \in K$ is pulled back through ξ [53] such that it is copied over every point in the corresponding region in graphic space $\xi(s) = k, s \subset S$. The pullback ξ^* can then be composed with the elements of the set of sections

$$\Gamma(K, E) \xrightarrow{\xi^*} \Gamma(S, \xi^* E) \quad (13)$$

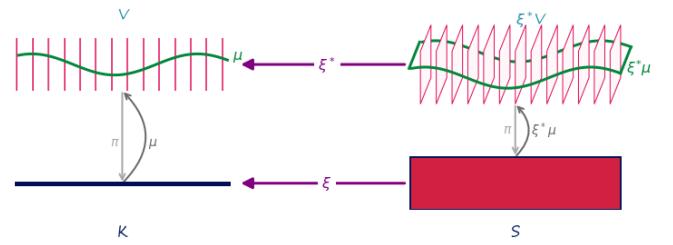


Fig. 5: The map ξ^* is a map from a repetition of the 1D fiber over every point $k \in K$ in data space, turning the fiber into a 2D plane of repeating values over the corresponding region $k = \xi(s)$ in the graphic space S .

to generate a set of equivalent sections over the graphic $\Gamma(H, \xi^* E)$. As a consequence of Equation 13, there exists a data sheaf over graphic space $\mathcal{O} : \mathcal{S}^\vee \rightarrow \Gamma(H, \xi^* E)$ which is equivalent to

$$\mathcal{O}_{\xi^*} : \mathcal{S}^\vee \rightarrow \text{Set} \quad (14)$$

Following from Equation 14 and the construction of the artist sheaf in Equation 11, we propose that the artist function A is a natural transformation

$$\begin{array}{ccc} \mathcal{S}^\text{op} & \xrightarrow{\mathcal{O}_{\xi^*}(E)} & \text{Set} \\ \downarrow & \text{A} \Downarrow & \downarrow \\ \mathcal{O}_A(H) & & \end{array} \quad (15)$$

of sheaf functors

$$\mathcal{A} : \mathcal{O}_{\xi^*}(E) \rightarrow \mathcal{O}_A(H) \quad (16)$$

where each sheaf is a map from an open set in the graphic base space $U' \subset S$ into a set of sections of a fiber bundle over that open set. As defined in Equation 6, the artist is a natural transformation. Following from Equation 7, the artist specifies that there are compatible inclusion maps in the data and graphic spaces

$$\begin{array}{ccccc} \Gamma(U_1, E) & \xleftarrow{\iota^*} & & & \Gamma(U_2, E) \\ \downarrow \mathcal{O}_{\xi^*} & & \nearrow \mathcal{O}_{\xi^*} & & \downarrow \mathcal{O}_{\xi^*} \\ U_1 & \xleftarrow{\iota} & U_2 & \xrightarrow{\iota} & \\ \downarrow \mathcal{O}_A & & \downarrow \mathcal{O}_A & & \downarrow \mathcal{O}_A \\ \Gamma(U_1, H) & \xleftarrow{\iota^*} & & & \Gamma(U_2, H) \end{array} \quad (17)$$

In Equation 9, φ was introduced as a morphism between sheaf functors \mathcal{O}_{ξ^*} . Since the artist is a functor $A : \mathcal{O}_{\xi^*} \rightarrow \mathcal{O}_A$, it follows from Equation 5 that

$$\begin{array}{ccc} \mathcal{O}_{\xi^*} & \xrightarrow{\mathcal{A}} & \mathcal{O}_A \\ \varphi \downarrow & & \downarrow \varphi' := A(\varphi) \\ \mathcal{O}_{\xi^*} & \xrightarrow{\mathcal{A}} & \mathcal{O}_A \end{array} \quad (18)$$

an arbitrary morphism φ on the data sheaf has a compatible morphism φ' on the graphic sheaf.

334 3.2 Composition of Artists

335 Visualizations generally consist of more than one visual element, for example line plots, a legend, axis ticks, spines, and
 336 labels. We propose that we can express expected consistency
 337 between these elements as a preservation of morphisms
 338 between objects of different data categories \mathcal{E} .
 339

340 3.2.1 Shared Index

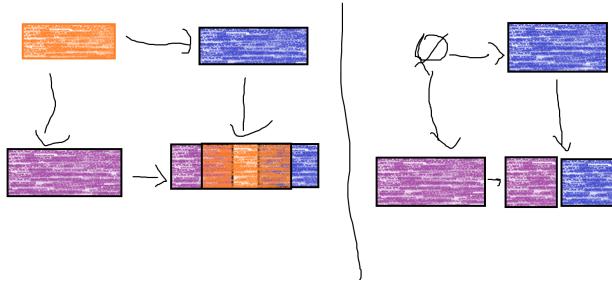


Fig. 6: In ??, the input object \mathcal{O}_E encodes a continuous function over spaces E_a, E_b and overlapping space E_c . In this figure, the visual transformation is to a point on the line on screen. An artist that preserves the continuity of the function must generate graphics such that functions evaluated $k \in K_a \cap K_b$

359
 360

361 An example of overlapping indexing is a parallelized
 362 version of a sliding window algorithm where window over-
 363 laps must resolve to the same value at the same position
 364 [54];

$$\begin{array}{ccc} K_c & \xleftarrow{i} & K_b \\ \downarrow i & & \downarrow i_{K_b} \\ K_a & \xrightarrow{i_{K_a}} & K_a \sqcup_{K_c} K_b \end{array} \quad (19)$$

365 where the projection functions i_{K_a}, i_{K_b} behave such that
 366 $i_{K_a}(k)|_{k \in K_b} = [k], i_{K_b}(k)|_{k \in K_b} = [k]$, meaning the projec-
 367 tion functions yield equivalent points in the disjoint union
 368 $K_a \sqcup_{K_c} K_b$. Data that is completely disjoint in the index, as
 369 shown in 12b, is a special case where $K_c = \emptyset$ and therefore
 370 there is no constraint on the artist with regards to how this
 371 data is rendered.

372
 373
 374
 375

376 3.2.2 Shared Fields

$$\begin{array}{ccc} F_a \times F_b & \xrightarrow{\text{proj}_a} & F_a \\ \downarrow \text{proj}_b & & \downarrow \text{proj}_c \\ F_b & \xrightarrow{\text{proj}_b} & F_c \end{array} \quad (20)$$

377 Equation 19 and Equation 20 can be composed to spec-
 378 ify how different aspects of the structure of a multivari-
 379 ate nested dimensional dataset, such as a spatio-temporal
 380 weather dataset, should be preserved in a visualization. this
 381 is probably the power of this model and should maybe get
 382 a figure?

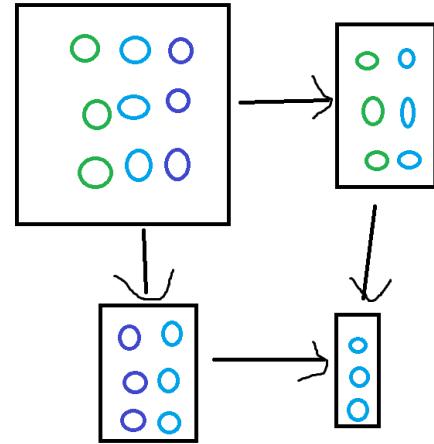


Fig. 7: The fiber spaces F_a, F_b have a shared fiber space F_c .

If they are input into the same artist A_1 , then an equivariant transformation is one where F_c is transformed to a visual element in a consistent manner. In this figure, F_c is mapped

to the x-axis for both F_a and F_b .

4 CONSTRUCTION AND FORMAL PROPERTIES OF ARTISTS

Following from ??, the artist A is a natural transformation

$$A : \mathcal{O}_{\mathcal{E}} \quad (21)$$

that we construct to preserve continuity and equivariance.

Definition 4.1. We define the natural transformation Artist A as the tuple $(\xi, \nu, Q, \mathcal{E}, \mathcal{V}, \mathcal{H})$ where

- 1) $\nu : \mathcal{E} \rightarrow \mathcal{V}$ is a bundle map from data values to the visual variables they are mapped to
- 2) $Q : \mathcal{O}_{\mathcal{V} \rightarrow \mathcal{O}_{\mathcal{A}}}$ is a sheaf map that builds a graphic generating function parameterized by the visual variables
- 3) $\xi : S \rightarrow K$ is a surjective map from the graphic topological base to the data topological base

and \mathcal{E}, \mathcal{V} , and \mathcal{H} are the categorical representations of fiber bundles that model the data, visual variable, and graphic space of the data to visual transformation. We construct the artist to have two stages

$$\begin{array}{ccc} \mathcal{E} & \xrightarrow{\xi} & \mathcal{V} \\ \pi \downarrow & \nearrow \pi & \uparrow \mathcal{V} \\ K & & \mathcal{K} \end{array} \quad \begin{array}{ccc} \mathcal{V} & \xrightarrow{Q} & \mathcal{H} \\ \uparrow \mathcal{O}_{\mathcal{V}} & \longrightarrow & \uparrow \mathcal{O}_{\mathcal{A}} \\ \mathcal{S} & \xrightarrow{\xi^*} & K \end{array} \quad (22)$$

which are the data to visual variable map ν at the bundle level and the visual variable to graphic map Q at the sheaf level. Usage of the bundle directly in an equation or diagram - V, H - denotes that the function can be evaluated pointwise over $k \in K$. Use of the categorical form - $\mathcal{E}, \mathcal{V}, \mathcal{H}$ - denotes that the function is evaluated over an openset

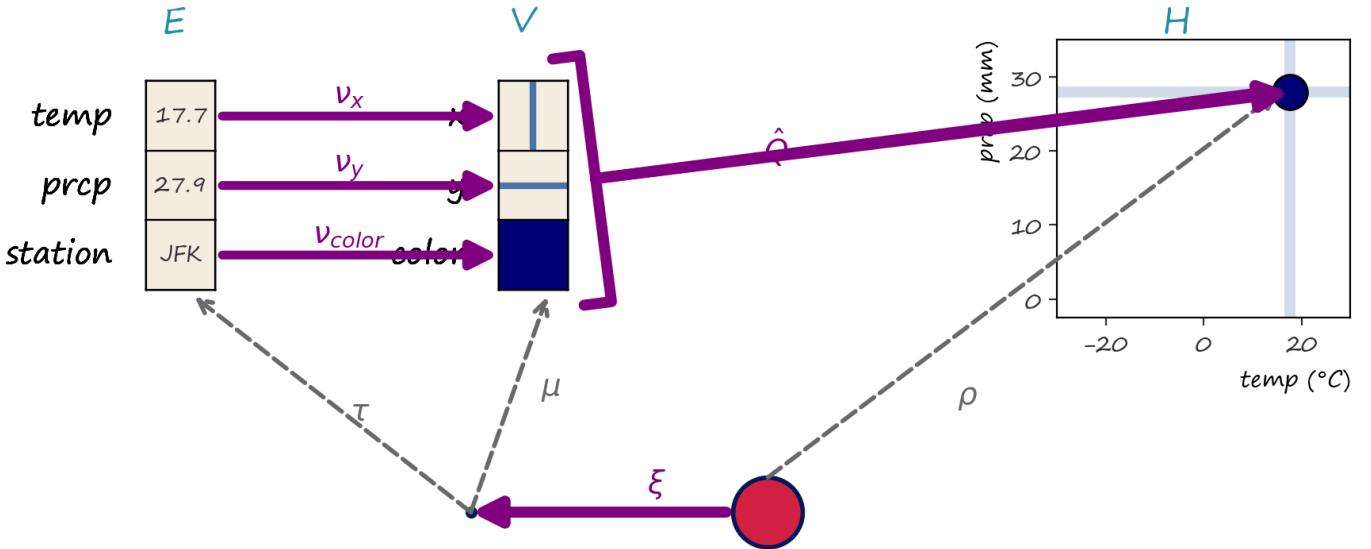


Fig. 8

object, meaning that the function needs knowledge of both a value over a point and some information about neighboring values.

4.1 Data Domain

We model data as sections of a fiber bundle (E, π, K, F) . We encode the continuity of the data as the **base space** K . We model the data as the section τ because, as described in subsubsection 2.1.1, the section is the map from the indexing space K to the space of possible data values F .

The section τ is the abstraction of the data being visualized. The section at a point $k \in K$ in the base space returns a value from each field in the fiber.

$$\tau(k) = ((\text{time}, t_k); (\text{precipitation}, p_k); (\text{station name}, n_k))$$

4.2 Graphic Codomain

The object of the graphic category \mathcal{H} is the fiber bundle H . The bundle H has the same structure as the data bundle E

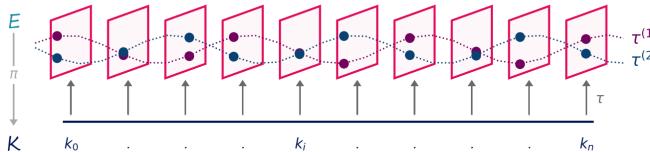


Fig. 9: replace with more concrete

One example of encoding data as a section of a fiber bundle is illustrated in Figure 9. In this example, the data is a **not totally decided yet** table of weather station data. Here we are interested in the time series of temperature values in the data, so we encode the continuity as the 1D interval K . In this multivariate data set, the fields we want to visualize are **time**, **temperature**, and **station**. The fiber space F is the cartesian cross product of the fibers of each field

$$F = F_{\text{time}} \times F_{\text{temperature}} \times F_{\text{station}}$$

where each field fiber is the set of values that are valid for the field:

$$\begin{aligned} F_{\text{time}} &= \mathbb{R} \\ F_{\text{temperature}} &= \mathbb{R} \\ F_{\text{station}} &= \{s_0, s_1, \dots, s_i, \dots, s_n\} \end{aligned}$$

with a fiber space D embedded in the total space H and a section map $\rho : S \rightarrow H$. The attributes of the graphic bundle (H, π, D, S) encode attributes of the graphic and display space

base space S continuity of display space (e.g. screen, 3D print)

fiber space D attributes of the display space (e.g. a pixel = (x, y, r, g, b, a))

section ρ graphic generating function

We represent the graphic output as a fiber bundle because it is an abstraction that is generalizable to various output mediums (screens, 3D prints) and types of graphics.

As illustrated in Figure 10, in this work, H assumes the display is an idealized 2D screen. **include some more about the figure** The graphic section ρ is an abstraction of rendering. For example, ρ can be a specification such as PDF [55], SVG [56] or an OpenGL scene graph [57], or a rendering engine such as Cairo [58] or AGG [59].

$$\begin{array}{ccc} D & \hookrightarrow & H \\ \pi \downarrow \rho & & \downarrow \\ S & & \end{array} \quad (23)$$

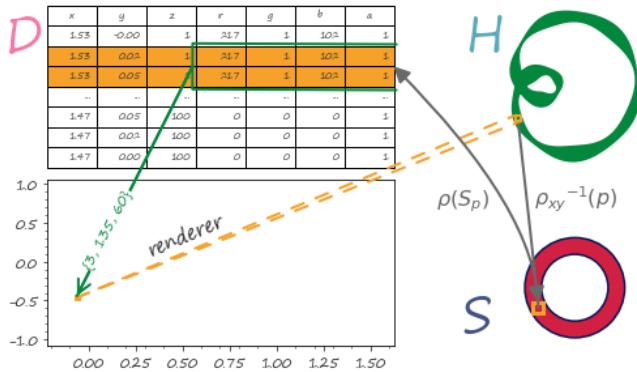


Fig. 10

4.3 Visualization Library Components

4.3.1 Visual Bundle V

$$P \longleftrightarrow V$$

$\pi \downarrow \quad \uparrow \mu$

$$K$$

4.3.2 Data to Visual Encodings: ✓

416 maybe figure out how to put these side by side nicely

$$\begin{array}{ccc} E & \xrightarrow{\nu} & V \\ \pi \downarrow & \nearrow \pi & \\ K & & \end{array} \qquad \begin{array}{ccc} F & \xrightarrow{\nu} & P \\ \tau \uparrow & \nearrow \mu & \\ K & & \end{array}$$

417 Since the functor ξ is bundle wise, it can be evaluated at
 418 the bundle at a point, which is the fiber space; therefore ξ
 419 is a functors from the product category \mathcal{F} to the product
 420 category \mathcal{P} . This constraint is expressed in our construction
 421 of γ

422 which means equivariance of

$$\begin{array}{ccc} \tau & \xrightarrow{\textcolor{violet}{\gamma}} & \mu \\ \varphi \downarrow & & \downarrow \textcolor{violet}{\gamma}_\varphi \\ \tau' & \xrightarrow{\textcolor{violet}{\gamma}} & \mu' \end{array} \quad (26)$$

423 Since the functor v acts on product categories, it can
 424 be decomposed into $n u_i$ component functors that act on
 425 corresponding sections τ_i of the fiber F_i .

$$\{\nu_0, \dots, \nu_n\} : \{\tau_0, \dots, \tau_n\} \mapsto \{\mu_0, \dots, \mu_n\} \quad (27)$$

426 One example of this is

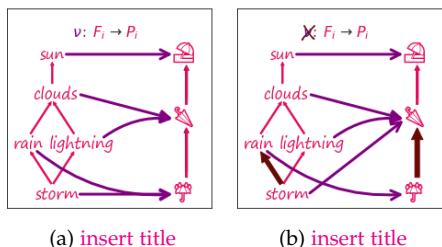


Fig. 11: is a weird nu, colors are a but much

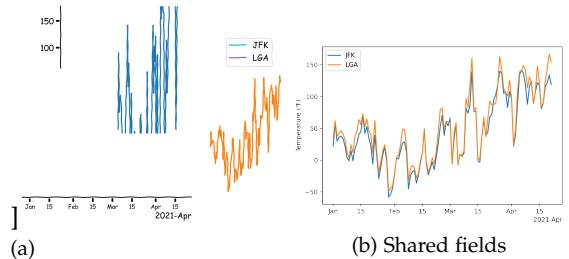


Fig. 12: In ??, the input object \mathcal{O}_E encodes a continuous function over spaces E_a , E_b and overlapping space E_c . In this figure, the visual transformation is to a point on the line on screen. An artist that preserves the continuity of the function must generate graphics such that functions evaluated $k \in K_a \cap K_b$ replace a) w/ fiber cross diagram?
- use first half introduced in Figure 7

(24)

4.3.2.1 Multiview Constraints: The concept that shared data fields should be encoded visually in a consistent manner is formally expressed by Qu and Hullman [60] as the notion that the same field should have the same scale.

We enforce the equivariance constraint specified in Equation 28 by constructing artists that apply ν encoders such that

$$\begin{array}{ccc} F_a & \xrightarrow{\quad v_c \quad} & P_c \\ \text{proj}_a \searrow & \nearrow \text{proj}_b & \\ & F_c & \\ \text{proj}_b \swarrow & \nearrow v_c' & \\ F_b & \xrightarrow{\quad v_c' \quad} & P_c' \end{array} \quad (28)$$

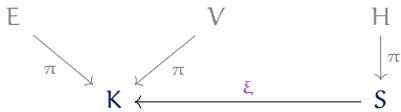
4.3.3 Visual to Graphic: Q

$$\begin{array}{ccc} \mathcal{V} & & \mathcal{H} \\ \uparrow & \xrightarrow{\textcolor{purple}{Q}} & \uparrow \\ \mathcal{O}_{\mathcal{V}} & & \mathcal{O}_{\mathcal{A}} \end{array} \quad (29)$$

$$\rho \xrightarrow{Q(m_j \circ \mu_i)} \rho'$$

Fig. 13: rework this as a commutative box w/ the r in E row associated w/ this $\hat{q}(k)$

$$\begin{array}{ccc}
 \mathcal{P}_{\dashv} & \xrightarrow{\quad Q_a \quad} & \mathcal{O}_A \\
 \searrow p_c & & \nearrow l_c \\
 & \mathcal{P}_{\lrcorner} & \\
 \swarrow p'_c & \nearrow l'_c & \\
 \mathcal{P}_{\vdash} & \xrightarrow{\quad Q_b \quad} & \mathcal{O}_{A'} \\
 \end{array} \tag{30}$$

435 4.3.4 Graphic to Data: ξ 

436 The functor ξ is a deformation retract, which means⁴⁵⁰
437 [32], [34]

438 By homotopy lifting?

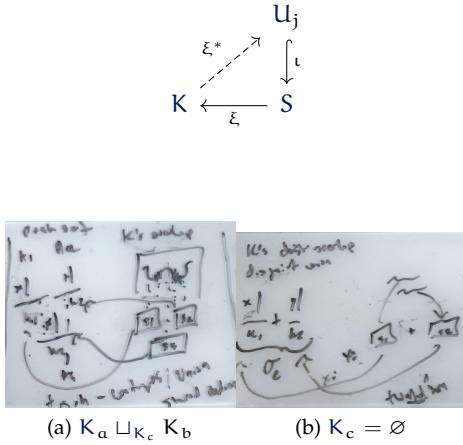
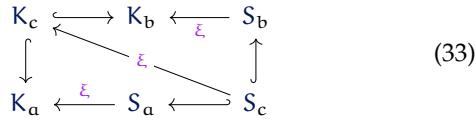


Fig. 14: probably doesn't need the k_0



439 talk about ξ even though is not explicitly implemented
440 The mapping between graphic and data space expresses⁴⁶²
441 how... This is what allows the artist to generate graphics
442 where the subset of data on view is dynamically updated,
443 such as pan and zoom navigation [61] and sliding windows
444 on streaming data [54], [62].

445 5 CASE STUDY

449 5.1 A

```
1   for local_tau in axesArtist.artist.data.query(screen_bounds, dpi):
2       mu = axesArtist.artist.graphic.mu(local_tau)
3       rho = axesArtist.artist.graphic.qhat(**mu)
4       H = rho(renderer)
```

450 where the artist is already parameterized with the ξ
451 functions and which fibers they are associated to:

1

(32)⁴⁵² 5.1.1 ξ
453 5.1.2 ν
454 5.1.3 \hat{Q}

455 6 DISCUSSION

456 6.1 Limitations

457 6.2 future work

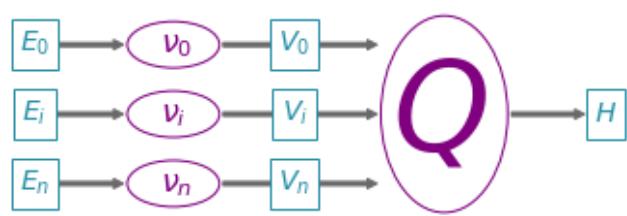
458 7 CONCLUSION

459 The conclusion goes here.

460 ACKNOWLEDGMENTS

461 The authors would like to thank...

Hannah Aizenman Biography text here.



464 Michael Grossberg Biography text here.

Fig. 15: v&

446 We implement the arrows in Figure 15. `axesArtist` is
447 a parent artist that acts as a screen. This allows for the
448 composition described in subsection 3.2