Topological Equivariant Artist Model

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Mathematical Data Abstraction

Fiber Bundles "unified, dimension-independent framework" that expresses data as the mapping between continuity and fields [**butlerVectorBundleClassesForm1992**, **butlerVisualizationModelBased1989**]

Category Theory Language express constraints in specifications [**wielsManagementEvolvingSpecifications1998**]

Sheaves on Bundles "algebraic data structure" for representing data over topological spaces [**ghristElementaryAppliedTopology2014**]

Fiber Bundle

$$\begin{tikzcd} F \arrow[r, hook] & E \arrow[d, "\pi"'] \\ & K \arrow[u, bend right, "\tau\in\Gamma(U,\, E\restriction_U)"'] \end{tikzcd}$$

Total Space $(E, \mathcal{T}_E)$

Base Space $(K, \mathcal{T}_K)$, $k \in U \subseteq K$

Fiber Space $F\restriction_k = \pi^{-1}(k), F = F\restriction_k \; \forall k \in K$

Sections

$\Gamma(U,\, E\restriction_U) \coloneqq \left\{\tau : U \to E\restriction_U \;\middle|\; \pi(\tau(k)) = k \textit{ for all } k \in U\right\}$

Locally Trivial for every point $k \in K$, there exists an open neighborhood $k \in U \subseteq K$ s.t. there is a homeomorphism $\pi^{-1}(U) \xrightarrow{\eta} U \times F$

(Globally) Trivial $E = K \times F$

Presheaf: $\mathcal{O} : \mathcal{C}^{op} \to \boldsymbol{Set}$

$$\begin{tikzcd} F \arrow[r, hook] & E \arrow[d, "\pi"'] \\ & K \arrow[u, bend right, "\tau \in \Gamma(K,\, E)"'] \end{tikzcd}$$

$$\begin{tikzcd} \boldsymbol{Set} \ni \Gamma(U_1,\, E\restriction_{U_1}) & & \Gamma(U_2,\, E\restriction_{U_2}) \arrow[ll, hook', "\iota^*"'] \\ Ob(\mathcal{K}^{op}) \ni U_1 \arrow[u, mapsto, "\mathcal{O}_{K,E}"] \arrow[rr, hook, "\iota"'] & \arrow[u, mapsto, "\mathcal{O}_{K,E}"] & U_2 \arrow[u, mapsto, "\mathcal{O}_{K,E}"'] \end{tikzcd}$$

stalk

$$\mathcal{O}_{K,E}\restriction_k \coloneqq \lim_{U \ni k} \Gamma(U,\, E\restriction_U)$$

$$F_k \subset \mathcal{O}_{K,E}\restriction_k$$

germ

$$\tau(k) \in \mathcal{O}_{K,E}\restriction_k$$

Sheaves on Bundles

A sheaf is a presheaf that satisfies the following two axioms[**bakerMathsSheaf**]

locality

given $U = \bigcup_{i \in I} U_i$ and $\tau^a, \tau^b \in \mathcal{O}(U)$,
if $\tau^a\restriction_{U_i} = \tau^b\restriction_{U_i}$ for each $U_i \in U$ then $\tau^a = \tau^b$

gluing

given $\tau^i \in \mathcal{O}(U_i)$ s.t. $\tau^i\restriction_{U_i \cap U_j} = \tau^j\restriction_{U_i \cap U_j}$ for $U_i, U_j \in U$,
there exists $\tau \in \mathcal{O}(U)$ such that $\tau\restriction_{U_i} = \tau^i$

Data

$$\begin{tikzcd} U \arrow[d, mapsto, "\mathcal{O}_{K,E}"] \\ \Gamma(U, E\restriction_U) \end{tikzcd}$$

- $F \hookrightarrow E \xrightarrow{\pi} K$
- $\mathcal{O}_{K,E} : U \mapsto \Gamma(U, E\restriction_U), U \subseteq K$
- $\tau : U \rightarrow F\restriction_U \in \Gamma(U, E\restriction_U)$
- $\tau(k) = \{f_0 : v_0, \cdots, \}, k \in U$

Graphic

$$\begin{array}{c} \Gamma(W, H\restriction_W) \\ \Big\uparrow{\scriptstyle \mathcal{O}_{S,H}} \\ W \end{array}$$

- $D \hookrightarrow H \xrightarrow{\pi} S$
- $\mathcal{O}_{S,H} : W \mapsto \Gamma(W, E\restriction_W), W \subseteq S$
- $\rho : W \to D\restriction_W \in \Gamma(W, H\restriction_W)$
- $\rho(s) = \{d_0, \cdots\}, s \in W$

Function: $\xi : S \to K$

$$
\begin{tikzcd}
\Gamma(W, H\restriction_{W}) & & \\
\overbrace{U \times [0,1]^m}^{W} \arrow[u, mapsto, "\mathcal{O}_{S,H}"] \arrow[rr, "\xi"] & & U \arrow[d, mapsto, "\mathcal{O}_{K,E}"] \\
& & \Gamma(U, E\restriction_{U})
\end{tikzcd}
$$

Pullback: data to region of the visualization

$$
\begin{tikzcd}
W \arrow[rr, "\xi"] \arrow[d, mapsto, dashed, "\mathcal{O}_{S,\xi^* E}"'] & & U \arrow[d, mapsto, "\mathcal{O}_{K,E}"] \\
\Gamma(W, \xi^* E\restriction_W) & & \Gamma(U, E\restriction_U) \arrow[ll, "\xi^*"]
\end{tikzcd}
$$

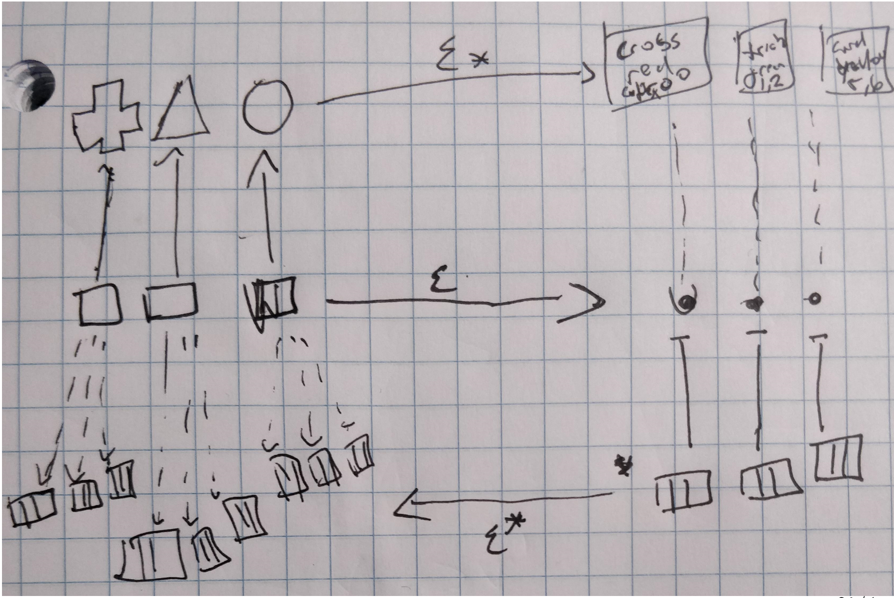
- $\xi^* F \hookrightarrow \xi^* E \xrightarrow{\pi} S$
- $\xi^* \mathcal{O}_{K,E} : W \mapsto \Gamma(W, \xi^* E\restriction_W), W \subseteq S$
- $\xi^* \tau : W \to \xi^* F\restriction_W \in \Gamma(W, \xi^* E\restriction_W)$
- $\xi^* \tau(s) = \tau(\xi(s)) = \tau(k)$

Pushforward: visualization to index of data

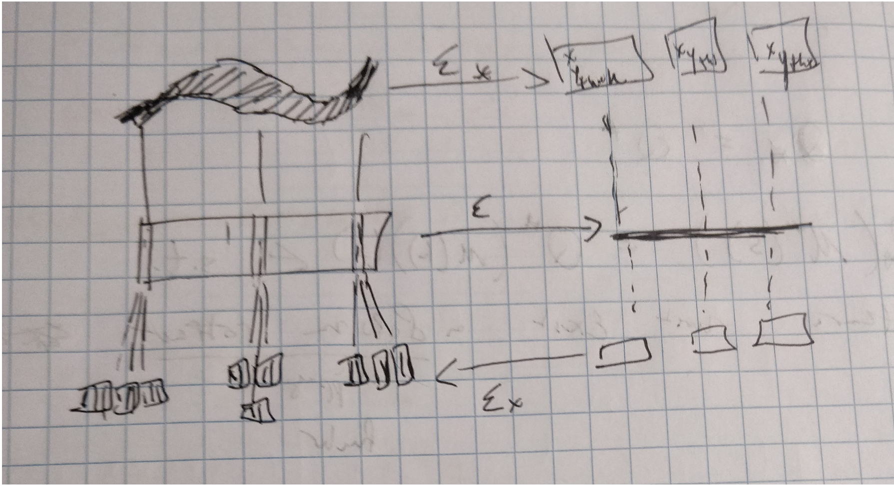
$$
\begin{tikzcd}
\Gamma(W, H\restriction_W) \arrow[r, "\xi_*"] & \Gamma(U, \xi_* H\restriction_U) \\
W \arrow[u, mapsto, "\mathcal{O}_{S,H}"] \arrow[r, "\xi"] & U \arrow[u, mapsto, dashed, "\mathcal{O}_{K,\xi_* H}"']
\end{tikzcd}
$$

- $\xi_* D \hookrightarrow \xi_* H \xrightarrow{\pi} K$
- $\xi_* \mathcal{O}_{S,H} : U \mapsto \Gamma(U, \xi_* H\restriction_U), U \subseteq K$
- $\xi_* \rho : U \to \xi_* D\restriction_U \in \Gamma(U, \xi_* H\restriction_U)$
- $\xi_* \rho(k) = \rho\restriction_{\xi^{-1}(k)} = \rho(s)\ \forall\ s \in \xi^{-1}(k)$

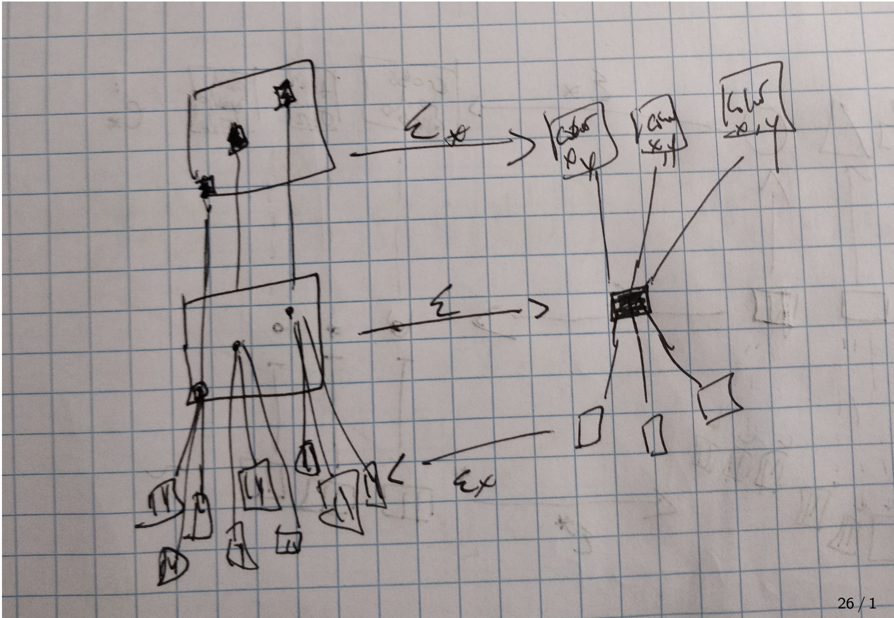
Example: Scatter Plot



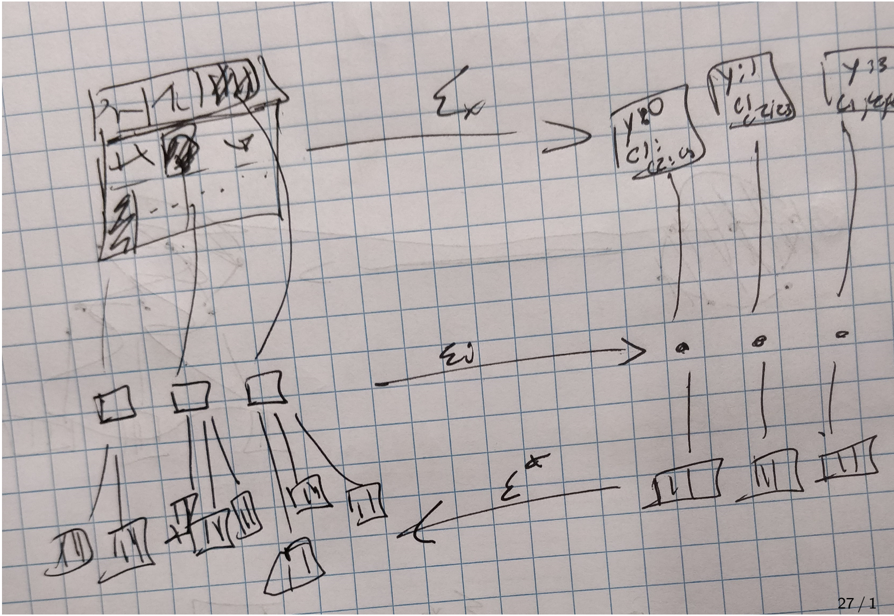
Example: Line Plot



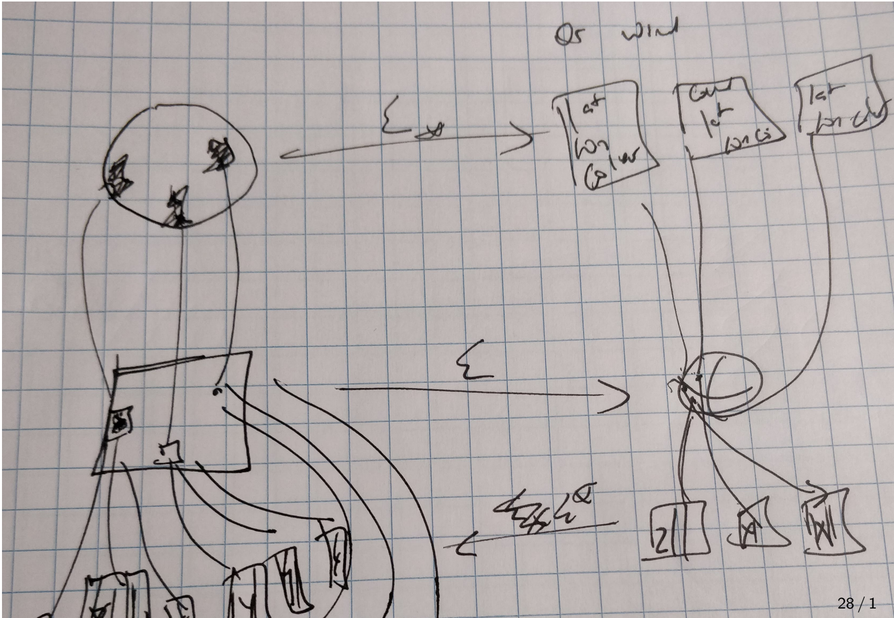
Example: Image



Example: Heatmap



Example: Sphere



Artist: $A : \mathcal{O}_{K,E} \to \mathcal{O}_{S,H}$

$$\begin{tikzcd}
\mathcal{O}_{S,H} \arrow[rr, "\xi_*"] & & \xi_*\mathcal{O}_{S,H} \\
& & \\
\xi^*\mathcal{O}_{K,E} \arrow[uu, Rightarrow, "A_W"] & & \mathcal{O}_{K,E} \arrow[ll, "\xi^*"'] \arrow[uu, Rightarrow, "A_U"'] \arrow[uull, Rightarrow, dashed, "A" description]
\end{tikzcd}$$

$$Nat_W(\xi^*\mathcal{O}_{K,E}, \mathcal{O}_{S,H}) = Nat_U(\mathcal{O}_{K,E}, \xi_*\mathcal{O}_{S,H})$$

Reachable ρ?

$$
\begin{tikzcd}
\Gamma(W, H\restriction_W) \arrow[r, phantom, "\supset"] & Im_A(W, H\restriction_W) \\
W \arrow[u, mapsto, "\mathcal{O}_{S,H}"] \arrow[ur, dashed, mapsto, "A(\mathcal{O}_{K,E})"']
\end{tikzcd}
$$

Output Subtype

$$Im_A(W, H\restriction_W) = \{\rho \mid \exists\ \tau \in \Gamma(U, E\restriction_U)\ s.t.\ A(\tau) = \rho,\ \xi(W) = U\}$$

Transform Data

Fiber Category

The fiber F is a monoidal category (single object w/ bicartesian product operator on category) of an arbitrary type $\mathcal{C}$. The morphisms on the fiber are $\tilde{\phi} \in Hom(F, F)$

Fiber Bundle Category

$$\begin{aligned} \text{object} &\quad F \hookrightarrow E \xrightarrow{\pi} K \\ \text{morphisms} &\quad \phi : (\hat{\phi}, \tilde{\phi}) \end{aligned}$$

$\phi = (\hat{\phi}, \tilde{\phi})$

$$
\begin{tikzcd}
\Gamma(U, E\restriction_U) \arrow[r, "\hat{\phi}^*"] & \Gamma(U', \hat{\phi}^* E\restriction_{U'}) \arrow[r, "\tilde{\phi}"] & \Gamma(U', \hat{\phi}^* E\restriction_{U'}) \\
U \arrow[u, mapsto, "\mathcal{O}_{K,E}"] & U' \arrow[l, "\hat{\phi}"] \arrow[u, mapsto, "\mathcal{O}_{K,\hat{\phi}^* E}"'] \arrow[ur, mapsto, "\mathcal{O}_{K,\hat{\phi}^* E}"'] &
\end{tikzcd}
$$

Base Transformation: $\hat{\phi} : U' \to U$ where $U, U' \subseteq K$,
$\hat{\phi}^* \tau\restriction_U : \tau \mapsto \tau\restriction_U \circ \hat{\phi}$

Fiber Transformation:
$\tilde{\phi} : \hat{\phi}^* E_{k'} \to \hat{\phi}^* E_{k'} \in Hom(\hat{\phi}^* F\restriction_k, F\restriction_k), k' \in U'$
$\tilde{\phi} : \hat{\phi}^* \tau\restriction_U \mapsto \hat{\phi}^* \tau'\restriction_U, \tau, \tau' \in \Gamma(U', \hat{\phi}^* E\restriction_{U'})$

Section Transform: $\phi : \tau\restriction_U \mapsto \hat{\phi}^* \tau'\restriction_U$

Equivariant Artist

(A, A') are equivariant with respect to ϕ_E if a compatible transform ϕ_H can be defined such that

$$
\begin{tikzcd}
\Gamma(U, E\restriction_U) \arrow[rrr, Rightarrow, "A"] \arrow[d, "\hat{\phi}^*{}_E"'] & & & Im_A(W, H\restriction_W) \arrow[d, dotted, "\hat{\phi}^*{}_H"] \\
\Gamma(U', \hat{\phi}^*{}_E E\restriction_{U'}) \arrow[dd, "\tilde{\phi}_E"'] & U \arrow[ul, dashed, mapsto] & W \arrow[l, "\xi"'] \arrow[ur, dashed, mapsto] & Im_A(W', \hat{\phi}^*{}_H H\restriction_{W'}) \arrow[dd, dotted, "\tilde{\phi}_H"] \\
 & U' \arrow[u, "\hat{\phi}_E"] \arrow[ul, dashed, mapsto] \arrow[dl, dashed, mapsto] & W' \arrow[l, "\xi"'] \arrow[u, dotted, "\hat{\phi}_H"'] \arrow[ur, dashed, mapsto] \arrow[dr, dashed, mapsto] & \\
\Gamma(U', E'\restriction_{U'}) \arrow[rrr, Rightarrow, "A'"] & & & Im_A(W', H'\restriction_{W'})
\end{tikzcd}
$$

Testing if A is equivariant

M is a (scaler, vector) measurable component (e.g. color, position, shape, texture, rotation,) of the rendered visual element.

$$
\begin{tikzcd}
\Gamma(U, E\restriction_U) \arrow[r, Rightarrow, "A"] \arrow[d, "\eta"'] & Im_A(W, H\restriction_W) \arrow[r, "render"] \arrow[d, "\delta"] & visualization \arrow[dl, "measure"] \\
Hom(U, M) \arrow[r, "\xi^*"'] & Hom(W, M) &
\end{tikzcd}
$$

input $\eta : \tau \mapsto (U \xrightarrow{\eta_\tau} M)$

output $\delta : \rho \mapsto (W \xrightarrow{\delta_\rho} M)$

$\eta_\tau(k) = \delta_\rho(s)$ for all $\xi(s) = k, k \in K, s \in S$

Using output ρ to check if A is equivariant

$$
\begin{tikzcd}
\Gamma(U, E\restriction_U) \arrow[r, Rightarrow, "A"] \arrow[d, "\phi_E"'] & Im_A(W, H\restriction_W) \arrow[r, "\delta"] \arrow[d, dotted, "\phi_H"'] & Hom(W, M) \arrow[d, "\phi_M"] \\
\Gamma(U', E'\restriction_{U'}) \arrow[r, Rightarrow, "A"] & Im_A(W', H'\restriction_{W'}) \arrow[r, "\delta"] & Hom(W', M')
\end{tikzcd}
$$

$$
\begin{aligned}
A'(\tilde{\phi}_E(\tau(\hat{\phi}_E(\xi(s'))))) &= \tilde{\phi}_H(A(\tau(\xi(\hat{\phi}_H(s'))))) \\
\delta(\tilde{\phi}_H(\rho(\hat{\phi}_H)))(s') &= \phi_M(\delta(\rho))(s') = \delta_{\tilde{\phi}_H \rho}(s')
\end{aligned}
$$

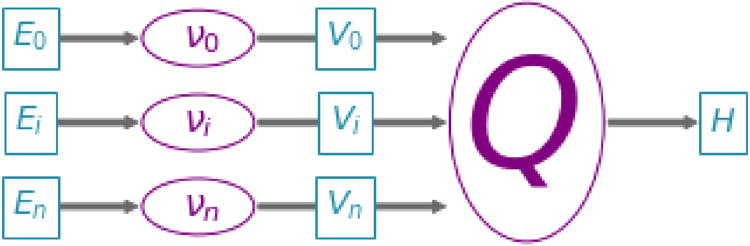
Using input τ to check if A is equivariant

$$
\begin{tikzcd}
\Gamma(U, E\restriction_U) \arrow[r, Rightarrow, "A"] \arrow[d, "\phi_E"'] \arrow[rr, bend left, "\eta"] & Im_A(W, H\restriction_W) \arrow[r, "\xi^{-1}\circ\delta"] \arrow[d, dotted, "\phi_H"'] & Hom(U, M) \arrow[d, "\phi_M"] \\
\Gamma(U', E'\restriction_{U'}) \arrow[r, Rightarrow, "A"] \arrow[rr, bend right, "\eta"'] & Im_A(W', H'\restriction_{W'}) \arrow[r, "\xi^{-1}\circ\delta"] & Hom(U', M')
\end{tikzcd}
$$

equivariance $\eta(\tilde{\phi}_E(\tau(\hat{\phi}_E)))(k') = \phi_M(\eta(\tau))(k') = \eta_{\tilde{\phi}_E\tau}(k')$ for all $k' \in K'$

continuity $\lim_{x \to k} \eta_\tau(x) = \eta_\tau(k)$ for all $k \in K$

Building an equivariant A?



Typed Measurable Visual Components: V

$$\begin{tikzcd} P \arrow[r, hook] & V \arrow[d, "\pi"'] \\ & K \arrow[u, bend right, "\mu\in\Gamma(U,\, V\restriction_U)"'] \end{tikzcd}$$

Data V continuity + visual fields

Continuity K data continuity

components P visual components **[bertinIIPropertiesGraphic2011]** of a graphic, e.g. x and y position, area, color, line thickness

visual components $\Gamma(U, V\restriction_U) \coloneqq \left\{\mu : U \to V\restriction_U \;\middle|\; \pi(\mu(k)) = k \text{ for all } k \in U\right\}$

Data to Visual Transformation: $\nu : E \to V$

$$
\begin{tikzcd}
E \arrow[rr, "\hat{\phi}^*"] \arrow[dd, "\nu_U"'] & & \hat{\phi}^* E \arrow[rr, "\tilde{\phi}_E"] & & E' \arrow[dd, "\nu'_{U'}"] \\
& U \arrow[ul, dotted, mapsto] \arrow[dl, dotted, mapsto] & U' \arrow[l, "\hat{\phi}"] \arrow[u, dotted, mapsto] \arrow[d, dotted, mapsto] \arrow[urr, dotted, mapsto] \arrow[drr, dotted, mapsto] & & \\
V \arrow[rr, "\hat{\phi}^*"'] & & \hat{\phi}^* V \arrow[rr, "\tilde{\phi}_V"'] & & V'
\end{tikzcd}
$$

Given $\tau(k) = \tau(\hat{\phi}(k'))$ and $\mu(k) = \mu(\hat{\phi}(k'))$

equivariance $\tilde{\phi}_V(\nu_U(\tau)) = \nu_U(\tilde{\phi}_E(\tau))$, $\nu_U = \nu'_{U'}$

continuity $\lim_{x \to k} (\nu(\tau(x))) = \nu(\tau(k))$ for all $k \in K$

Data to Visual Transformation $\nu : F_k \mapsto P_k$

$\pi(E) = \pi(\nu(E))$ and ν is composable s.t

$$
\begin{tikzcd}
F_k \arrow[r, mapsto, "\nu"] \arrow[d, "\tilde{\phi}_E"'] \arrow[dr, phantom, "\mapsto"] & P_k \arrow[d, "\tilde{\phi}_V"] \\
F'_k \arrow[r, mapsto, "\nu'"'] & P'_k
\end{tikzcd}
\qquad
\begin{tikzcd}
F_k \arrow[r, mapsto, "\nu_E"] \arrow[dr, mapsto, dashed, "\nu_{E,V}"'] & P_k \coloneqq F_k{}^V \arrow[d, mapsto, "\nu_V"] \\
& P_k{}^V
\end{tikzcd}
$$

$\nu : \phi_E \rightarrow \phi_V$: Stevens' Scales [stevensTheoryScalesMeasurement1946]

scale	**group**	**constraint**
nominal	permutation	if $r_1 \neq r_2$ then $\nu(r_1) \neq \nu(r_2)$
ordinal	monotonic	if $r_1 \leqslant r_2$ then $\nu(r_1) \leqslant \nu(r_2)$
interval	translation	$\nu(r + c) = \nu(r) + \nu(c)$
ratio	scaling	$\nu(r * c) = \nu(r) * \nu(c)$

Shared Components: $\nu = \prod_{i=0}^{n} \nu_i$

$$
\begin{tikzcd}
F_a \times F_b \arrow[rr, mapsto, "\nu"] \arrow[dr, "p_F"] & & P_a \times P_b \arrow[dr, "p_P"] & \\
& F_a \arrow[rr, mapsto, dashed, "\nu_a"] & & P_a \\
F_a \times F_c \arrow[rr, mapsto, "\nu'"] \arrow[ur, "p_{F'}"'] & & {P_a}' \times {P_c}' \arrow[ur, "p_{P'}"'] &
\end{tikzcd}
\tag{1}
$$

Consistent Transformations [**hullmanKeeping2018**]

if $p_F = p_{F'}$ then $p_P(\nu(\tau)) = p_{P'}(\nu'(\tau'))$ s.t. there exists a transformation $\nu_a : F_a \rightarrow P_a$

Assembly Q

$$\begin{tikzcd}
\Gamma(U, V\restriction_U) \arrow[rrr, Rightarrow, "Q"] \arrow[ddd, "\tilde{\phi}_V \circ \hat{\phi}^*{}_V"'] & & & \Gamma(W, H\restriction_W) \arrow[ddd, "\tilde{\phi}_H \circ \hat{\phi}_H"] \\
& U \arrow[ul, mapsto] & W \arrow[l, "\xi"'] \arrow[ur, mapsto] & \\
& U' \arrow[u, "\hat{\phi}_E"] \arrow[dl, mapsto] & W' \arrow[l, "\xi"'] \arrow[u, "\hat{\phi}_H"'] \arrow[dr, mapsto] & \\
\Gamma(U', V'\restriction_{U'}) \arrow[rrr, Rightarrow, "Q'"] & & & \Gamma(W', H'\restriction_{W'})
\end{tikzcd}$$

equivariance

$$Q'(\tilde{\phi}_V(\mu(\hat{\phi}_E(\xi(s'))))) = \tilde{\phi}_H(Q(\mu(\xi(\hat{\phi}_H(s')))))$$

Implementation Choices: $A_U = A_W$

$$
\begin{tikzcd}
\Gamma(W, \xi^* E\upharpoonright_W) \arrow[r, mapsto, "\nu_W"] \arrow[rrr, Rightarrow, bend left, "A_W"] & \Gamma(W, \xi^* V\upharpoonright_W) & \Gamma(W, \xi^* V\upharpoonright_W) \arrow[r, Rightarrow, "Q_W"] & \Gamma(W, H\upharpoonright_W) \arrow[d, "\xi_*"] \\
\Gamma(U, E\upharpoonright_U) \arrow[u, "\xi^*"] \arrow[r, mapsto, "\nu_U"] \arrow[rrr, Rightarrow, bend right, "A_U"'] & \Gamma(U, V\upharpoonright_U) \arrow[u, "\xi^*"'] & \Gamma(U, V\upharpoonright_U) \arrow[u, "\xi^*"] \arrow[r, Rightarrow, "Q_U"] \arrow[ur, Rightarrow, "Q"] & \Gamma(W, \xi_* H\upharpoonright_U)
\end{tikzcd}
$$

Components to Graphic