# Topologically Equivariant Artist Model

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# About

## Me

- PhD candidate in Computer Science
- Matplotlib Community Manager & Core Developer
- socials/github: story645

## Project

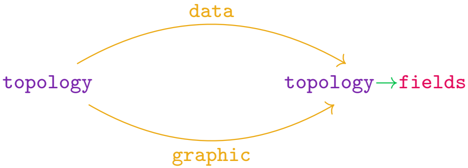
- Funded by Chan Zuckerberg Initiative EOSS 1 & 3

# Roadmap

$$\begin{tikzcd} \texttt{topology} \arrow[r, bend left, "\texttt{data}"] & \texttt{topology} \arrow[r] & \texttt{fields} \end{tikzcd}$$

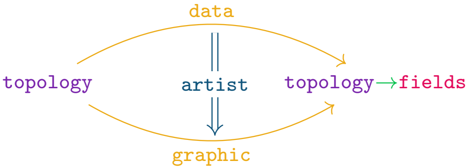
- uniform API for arbitrary datasets in arbitrary data containers

# Roadmap



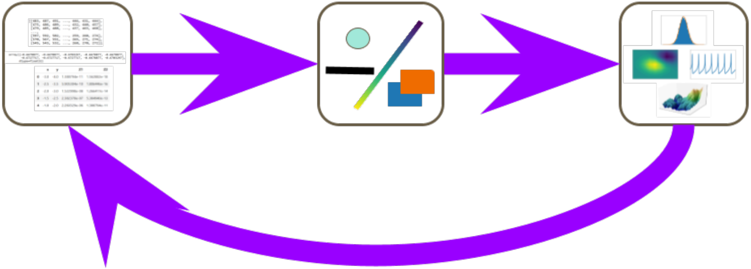

- uniform API for arbitrary datasets in arbitrary data containers
- generalizable methodology for expressing structure preservation

# Roadmap




- uniform API for arbitrary datasets in arbitrary data containers
- generalizable methodology for expressing structure preservation
- framework that translates into coding specifications

# What do visualization libraries do?



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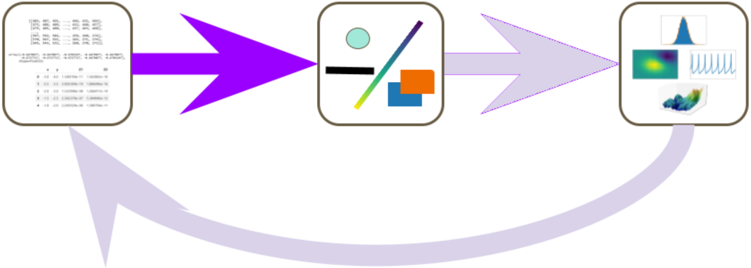

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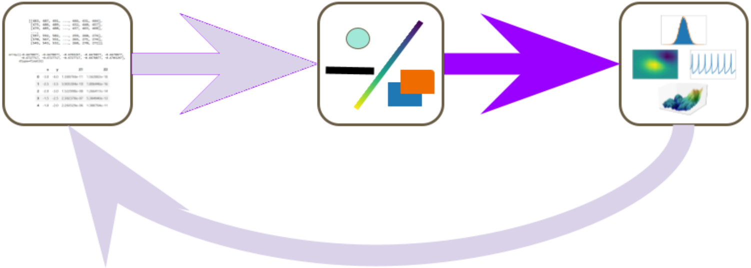
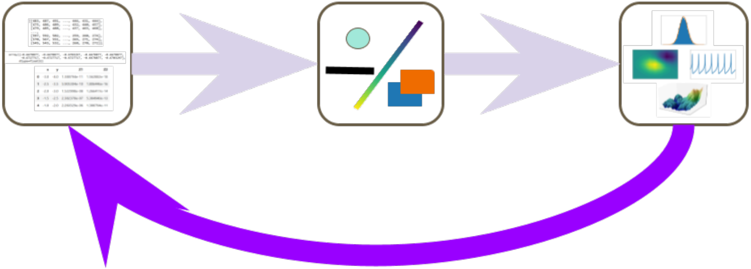


# What do visualization libraries do?



# Matplotlib

# How do we express structure?

**field** is a set of values of the same type, e.g. one column of a table or the pixels of an image

**topology** is the connectivity and relative positioning of elements in a dataset [1].

# Structure Preservation: Groups

## Definition

[2] An **action** of $G = (G, \circ, e)$ on $X$ is a function $\phi : G \times X \to X$. An action has the properties of identity $act(e, x) = x$ for all $x \in X$ and associativity $\phi(g, \phi(f, x)) = \phi(f \circ g, x)$ for $f, g \in G$.

# Structure Preservation: Groups

$\tau$ = data

| | | |
|---|---|---|
| 0 | 1 | 2 |
| 3 | 4 | 5 |

$\hat{\phi}_E^* \tau$ = data.T

| | |
|---|---|
| 0 | 3 |
| 1 | 4 |
| 2 | 5 |

$\widetilde{\phi}_E \tau$ = data*2

| | | |
|---|---|---|
| 0 | 2 | 4 |
| 6 | 8 | 10 |

$\phi_E \tau$ = data.T*2

| | |
|---|---|
| 0 | 6 |
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## Definition

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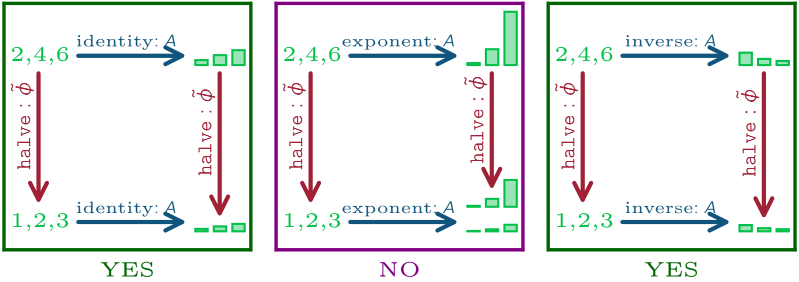
# Fields: Equivariance

**Definition**

Given a group $G$ that acts on both the input $X$ and the output $Y$ of a function $f: X \to Y$, a function $f$ is **equivariant** when $f(act(g, x)) = act(g, f(x))$ for all $g$ in $G$ and for all $x$ in $X$ [3]

# Fields: Equivariance

Is $\mathit{data} \mapsto \square$ equivariant w.r.t. scaling?




## Definition

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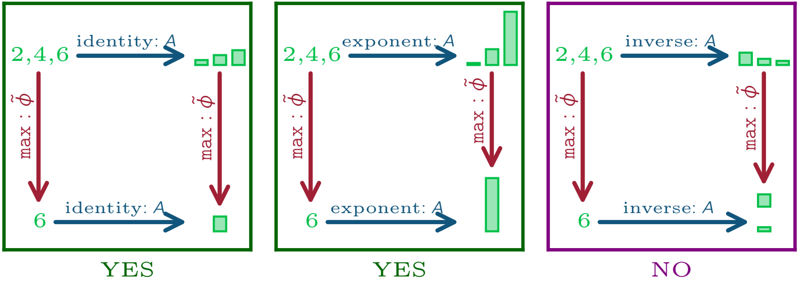
# Fields: Homomorphism

## Definition

Given the function $f : X \rightarrow Y$, with operators $(X, \circ)$ and $(Y, *)$, a function $f$ is **homomorphic** when $f(x_1 \circ x_2) = f(x_1) * f(x_2)$ and preserves identities $f(I_x) = I_y$ all $x, y \in X$ [2]

Is $\mathit{data} \mapsto \square$ homomorphic w.r.t. partial order?




## Definition

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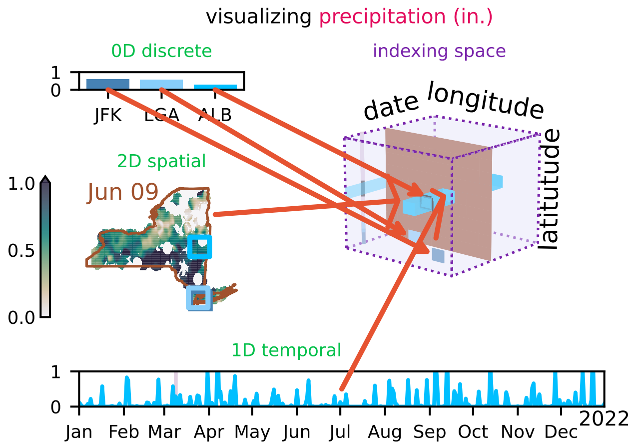
# Topology: Homeomorphism

**Definition**

A function $f$ is a *homeomorphism* if it is bijective, continuous, and has a continuous inverse function $f^{-1}$.

# Topology: Homeomorphism




**Definition**

A function $f$ is a *homeomorphism* if it is bijective, continuous, and has a continuous inverse function $f^{-1}$.

# Structure Preservation

- Bertin: equivalent data and visual field properties [4]
- Mackinlay: homomorphic fields and equivalent topology [5]
- Wilkinson: homomorphic scales and equivalent topology [1]
- Kindleman & Schieidegger: equivariant scales, invariant data representation, equivalent topology [6]

**contribution**

explicit homeomorphic topology, equivariant actions on fields and topology

# Domain Specific Library: library assumes structure [19]

| DATE | LATITUDE | LONGITUDE | PRCP (in.) | NAME |
|---|---|---|---|---|
| 2021-01-01 | 43.1167 | -77.6767 | 0.2205 | ROCHESTER GTR INTL AP |
| 2021-01-01 | 41.5 | -73.9 | 0.0000 | STONYKILL NEW YORK |
| 2021-01-01 | 42.7431 | -73.8092 | 0.2598 | ALBANY AP |
| 2021-01-01 | 43.8 | -73.7 | 0.0000 | SCHROON LAKE NEW YORK |
| 2021-01-01 | 43.0078 | -73.6511 | 0.0000 | SARA NEW YORK |
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[17, 18]

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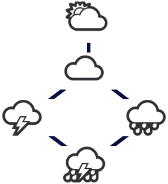


Gephi[9]
Graphviz[12]
Networkx[15]

# Building Block Library[29]: visual algorithms assume structure [30]

1. Matplotlib[20] $\rightarrow$ Seaborn[21], xarray [22]
2. D3 [23]
3. VTK [24, 25],MayaVi[26]$\rightarrow$ Titan[27], ParaView[28]

# Design Composable Structure Preserving API

Fiber Bundles Butler: "unified, dimension-independent framework" that expresses data as the mapping between continuity and fields [31, 32]

Simplicial Databases Spivak: Add rich typing for fields to Bulter [33]

Category Theory Language express constraints in specifications [34]

Sheaves Ghrist "algebraic data structure" for representing data over topological spaces [35]

# Expressive Types

$$\texttt{dataset} : \texttt{topology} \rightarrow \texttt{fields}$$

# Fiber Bundle

## Definition

A **fiber bundle** $(E, K, \pi, F)$ is a structure with topological spaces $E, F, K$ and bundle projection map $\pi : E \to K$ [36].

$$F \hookrightarrow E \xrightarrow{\pi} K \tag{1}$$

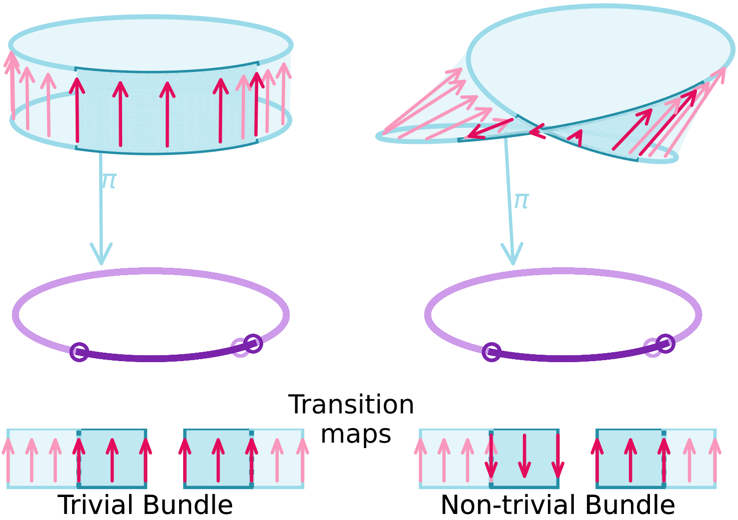
A continuous surjective map $\boldsymbol{\pi}$ is a **bundle projection** map when

1. all fibers in the bundle are isomorphic. Since all fibers are isomorphic $F \cong F_k$ for all points $k \in K$, there is a uniquely determined fiber space $F$ given by the preimage of the projection $\pi$ at any point $k$ in the base space $K$: $F = \pi^{-1}(k)$.
2. each point $k$ in the base space $K$ has an open neighborhood $U_k$ such that the total space $E$ over the neighborhood is locally trivial.

**Local triviality** means $E|_U = U \times F$.

# Fiber Bundle

$(E_c, K_c, \pi, F_c)$ $(E_m, K_m, \pi, F_m)$



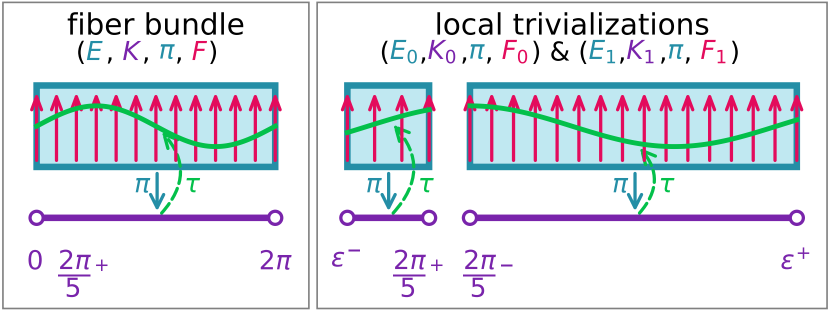
# Fiber Bundle: section

**Definition**

A **section** $\tau : K \to E$ over a fiber bundle is a smooth right inverse of $\pi(\tau(k)) = k$ for all $k \in K$

$$\begin{tikzcd} F \arrow[r, hook] & E \arrow[d, two heads, "\pi"'] \\ & K \arrow[u, bend right, dashed, "\tau"'] \end{tikzcd} \tag{2}$$

# Fiber Bundle: section



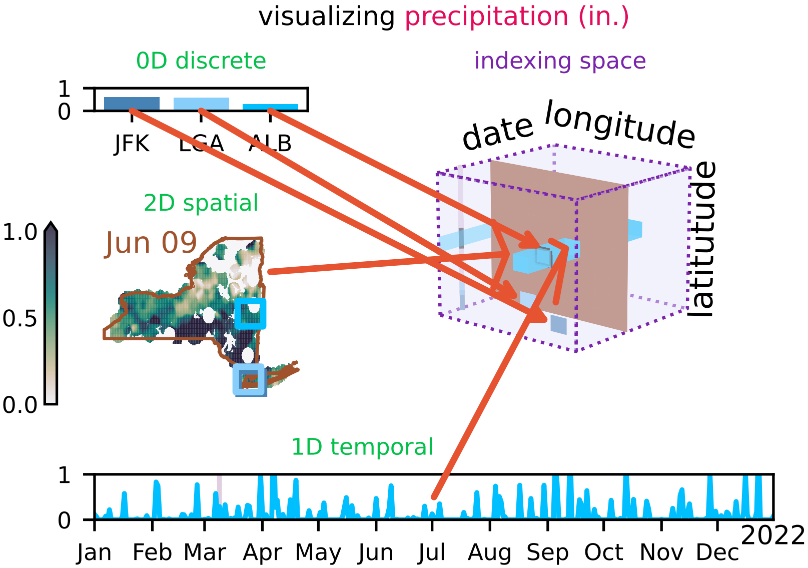
# Base space: topology

The base space of a fiber bundle is a quotient topology[37]. Given a set $X$ and a function $\mathcal{N} : X \to 2^{2^X}$ that assigns to any $x \in X$ a non-empty collection of subsets $\mathcal{N}(x)$, where each element of $\mathcal{N}(x)$ is a *neighborhood of* $x$, then $X$ with $\mathcal{N}$ is a topological space and $\mathcal{N}$ is a neighborhood *topology* if for each $x$ in $X$: [38]

## Definition

1. if $N$ is a neighborhood $N \in \mathcal{N}(x)$ of $x$ then $x \in N$
2. every superset of a neighborhood of $x$ is a neighborhood of $x$; therefore a union of a neighborhood and adjacent points in $X$ is also a neighborhood of $x$
3. the intersection of any two neighborhoods of $x$ is a neighborhood of $x$
4. any neighborhood $N$ of $x$ contains a neighborhood $M \subset N$ of $x$ such that $N$ is a neighborhood of each of the points in $M$

# Why neighborhood topology?



# Base TYPE $K$

An **category** $\mathcal{C}$ consists of the following *data*:

1. a collection of *objects* $X \in \mathbf{ob}(\mathcal{C})$
2. for every pair of objects $X, Y \in \mathbf{ob}(\mathcal{C})$, a set of *morphisms* $X \xrightarrow{f} Y \in Hom_{\mathcal{C}}(X, Y)$
3. for every object $X$, a distinct *identity morphism* $X \xrightarrow{id_x} X$ in $Hom_{\mathcal{C}}(X, X)$
4. a *composition function* $f \in Hom_{\mathcal{C}}(X, Y) \times g \in Hom_{\mathcal{C}}(Y, Z) \rightarrow g \circ f \in Hom_{\mathcal{C}}(X, Z)$

such that

1. *unitality:* for every morphism $X \xrightarrow{f} Y$, $f \circ id_x = f = id_y \circ f$
2. *associativity:* if any three morphisms $f, g, h$ are composable,

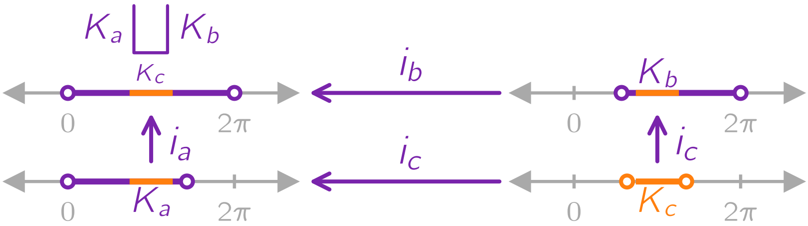
$$\begin{tikzcd} X \arrow[r, "f"] \arrow[rrr, dashed, bend right, "h \circ (g \circ f) = (h \circ g) \circ f"'] & Y \arrow[r, "g"] & Z \arrow[r, "h"] & W \end{tikzcd}$$

then they are associative such that $h \circ (g \circ f) = (h \circ g) \circ f$ [39, 40, 41, 42].

# Base TYPE $K$

The base space category consists of the openset object $K$ and morphisms between openset objects $\hat{\phi} \in Hom(K, K)$.
The base space category $\mathcal{K}$ is also equipped with functor $\oplus : \mathcal{K} \sqcup \mathcal{K} \to \mathcal{K}$ for combining base spaces.

# Add records → Add to base space



# Fiber space: Fields

| types and schema | record | table |
|---|---|---|
| $\begin{tikzcd} \mathcal{U}_\sigma \arrow[r, dotted] \arrow[d, two heads, "\pi_\sigma"'] \arrow[dr, dotted] & \mathcal{U} \arrow[d, two heads, "\pi"'] \\ C \arrow[r, "\sigma"'] & \texttt{DT} \end{tikzcd}$ | $\begin{tikzcd} \mathcal{U}_\sigma \arrow[d, two heads, "\pi_\sigma"'] \\ C \arrow[u, dashed, bend right, "r"'] \end{tikzcd}$ | $\begin{tikzcd} F \coloneqq \Gamma^\pi(\sigma) \arrow[d, two heads, "\pi"'] \\ K \arrow[u, dashed, bend right, "\tau"'] \end{tikzcd}$ |

$\mathcal{U}$ Space of all possible values, e.g. $\mathbb{R} \times \mathbb{R}^+$

$C$ Field names, e.g. *TEMP*, *PRCP*

$\texttt{DT}$ Data Type, e.g. `float`, `PosFloat`

| types and schema | record | table |
|---|---|---|
| $\begin{tikzcd} \mathcal{U}_\sigma \arrow[r, dotted] \arrow[d, two heads, "\pi_\sigma"'] \arrow[dr, dotted] & \mathcal{U} \arrow[d, two heads, "\pi"] \\ C \arrow[r, "\sigma"'] & \mathtt{DT} \end{tikzcd}$ | $\begin{tikzcd} \mathcal{U}_\sigma \arrow[d, two heads, "\pi_\sigma"'] \\ C \arrow[u, bend right, dashed, "r"'] \end{tikzcd}$ | $\begin{tikzcd} F \coloneqq \Gamma^\pi(\sigma) \arrow[d, two heads, "\pi"'] \\ K \arrow[u, bend right, dashed, "\tau"'] \end{tikzcd}$ |

A *schema* consists of a pair $(C, \sigma)$ where $C$ is the set of field names and $\sigma : C \to \mathbf{DT}$ is a function from field name to field data type[33] The function $\sigma$ is composed with $\pi$ such that $\pi^{-1}(\sigma(C)) \subseteq \mathscr{U}$; this composition induces a domain bundle $\pi_\sigma : \mathscr{U}_\sigma \to C$ that associates a field name $c \in C$ with its corresponding domain $\pi_\sigma^{-1}(C) \subseteq \mathscr{U}_\sigma$.

# Fiber space: Fields

| types and schema | record | table |
| --- | --- | --- |
| $\begin{tikzcd} \mathcal{U}_\sigma \arrow[r, dotted] \arrow[d, two heads, "\pi_\sigma"'] \arrow[dr, dotted] & \mathcal{U} \arrow[d, two heads, "\pi"'] \\ C \arrow[r, "\sigma"'] & \mathtt{DT} \end{tikzcd}$ | $\begin{tikzcd} \mathcal{U}_\sigma \arrow[d, two heads, "\pi_\sigma"'] \\ C \arrow[u, dashed, bend right, "r"'] \end{tikzcd}$ | $\begin{tikzcd} F \coloneqq \Gamma^\pi(\sigma) \arrow[d, two heads, "\pi"'] \\ K \arrow[u, dashed, bend right, "\tau"'] \end{tikzcd}$ |

A **record** is a function $r : C \to \mathscr{U}_\sigma$ and the set of records on $\pi_\sigma$ is denoted $\Gamma^\pi(\sigma)$. Records must return an object of type $\sigma(C) \in \mathbf{DT}$ for each field $c \in C$.

# Fiber space: Fields

| types and schema | record | table |
|---|---|---|
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Tables are sections $\tau : K \to \Gamma^\pi(\sigma)$ from an indexing space $K$ to the set of all possible records $\Gamma^\pi(\sigma)$ on the schema bundle

# Fiber space: Fields

| types and schema | record | table |
|---|---|---|
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We define the fiber space $F$ to be the space of all possible data records

$$F \coloneqq \{r : C \to \mathscr{U}_\sigma \mid \pi_\sigma(r(C)) = C \textit{ for all } C \in C\} \tag{3}$$

such that the preimage of a point is the corresponding data type domain $\pi^{-1}(k) = F_k = \mathscr{U}_{\sigma_k}$.

# Fiber category $F$

The fiber category has a single object $F$ of an arbitrary type and morphisms on the fiber object $\tilde{\phi} \in Hom(F, F)$.
The fiber category $\mathcal{F}$ is also equipped with a bifunctor $\otimes : \mathcal{F} \times \mathcal{F} \rightarrow \mathcal{F}$ for combining fiber types.

# Fiber morphism $\tilde{\phi}$

| scale | operators | sample constraint |
|---|---|---|
| nominal | $=, \neq$ | $\tau(k_1) \neq \tau(k_2) \implies \tilde{\phi}(\tau(k_1)) \neq \tilde{\phi}(\tau(k_2))$ |
| ordinal | $<, \leqslant, \geqslant, >$ | $\tau(k_1) \leqslant \tau(k_2) \implies \tilde{\phi}(\tau(k_1)) \leqslant \tilde{\phi}(\tau(k_2))$ |
| interval | $+, -$ | $\tilde{\phi}(\tau(k) + C) = \tilde{\phi}(\tau(k)) + C$ |
| ratio | $*, /$ | $\tilde{\phi}(\tau(k) * C) = \tilde{\phi}(\tau(k)) * C$ |

# Constraints as definition: Dates

- year: $\{y \in \mathbb{I} | 1992 \leqslant y \leqslant 2025\}$
- month: $\{m \in \mathbb{I} | 1 \leqslant m \leqslant 12\}$
- day: $\{d \in \mathbb{I} | 1 \leqslant d \leqslant 31\}$
- date: $\otimes : F_{year} \times F_{month} \times F_{day} \rightarrow F_{date}$

where the composition $\otimes$ includes the constrain to only return dates that have the right number of days for each month.

# Expand records → multiply fiber space

$$
\begin{tikzcd}
\overset{F_a \times_{F_c} Fb}{\text{59°F,Oct 02,0.07in.}} \arrow[r, "\pi_b"] \arrow[d, "\pi_a"] & \overset{F_b}{\text{Oct 02,0.07in.}} \arrow[d, "\pi_c"] \\
\underset{F_a}{\text{59°F,Oct 02}} \arrow[r, "\pi_c"] & \underset{F_c}{\text{Oct 02}}
\end{tikzcd}
$$

# local sections

We define section functions locally:

$$\Gamma(U, E\restriction_U) \coloneqq \left\{\tau : U \to E\restriction_U \;\middle|\; \pi(\tau(k)) = k \textit{ for all } k \in U\right\} \tag{4}$$

such that each section function $\tau : k \mapsto r$ maps from each point $k \in U$ to a corresponding record in the fiber space $r \in F_k$ over that point.
This encoding can be translated into a set of signatures $\{\texttt{data-subset} : \texttt{topology} \to \texttt{fields} \text{ s.t. } \texttt{data-subset} \subset \texttt{dataset}\}$.

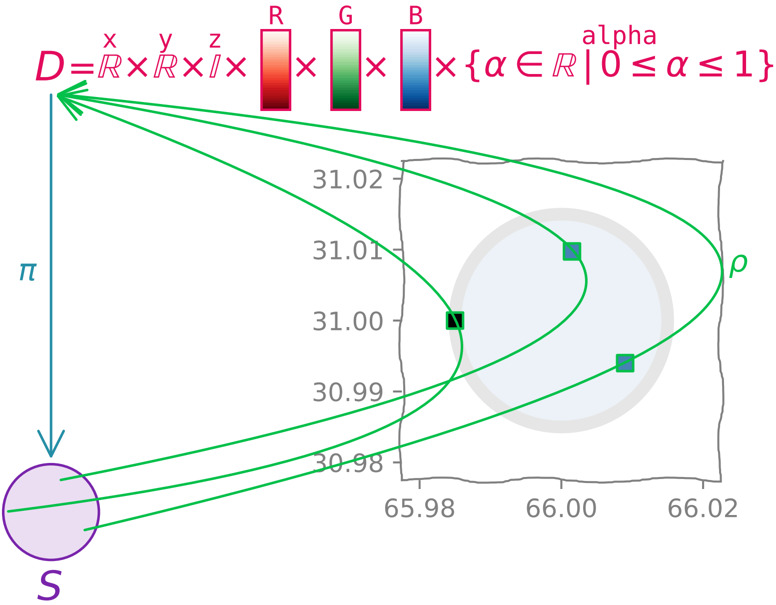
# global sections

When a bundle is trivial $E = K \times F$, we can defined a global sections $\tau : K \to F \in \Gamma(K, F)$ which we translate into a data signature of the form $\texttt{dataset} : \texttt{topology} \to \texttt{field}$ where $\tau = \texttt{dataset}$, $K = \texttt{topology}$ and $F = \texttt{fields}$

# Graphic bundle and section

$$D \hookrightarrow H \xrightarrow{\pi} S \tag{5}$$

$$\begin{aligned}\Gamma(W, H{\restriction}_W) &\coloneqq \\ &\left\{\rho : W \to H{\restriction}_W \;\middle|\; \pi(\rho(s)) = s \text{ for all } s \in W\right\}\end{aligned} \tag{6}$$



R
G
B
x
y
z
alpha
$D = \mathbb{R} \times \mathbb{R} \times \mathbb{I} \times \square \times \square \times \square \times \{\alpha \in \mathbb{R} \mid 0 \leq \alpha \leq 1\}$
π
ρ
31.02
31.01
31.00
30.99
30.98
65.98
66.00
66.02
S

## Definition

[43, 44] A **functor** is a map $F : \mathcal{C} \to \mathcal{D}$, which means it is a function between objects $F : \mathbf{ob}(\mathcal{C}) \mapsto \mathbf{ob}(\mathcal{D})$ and that for every morphism $f \in Hom(C_1, C_2)$ there is a corresponding function $F : Hom(C1, C2) \mapsto Hom(F(C_1), F(C_2))$. A **functor** must satisfy the properties

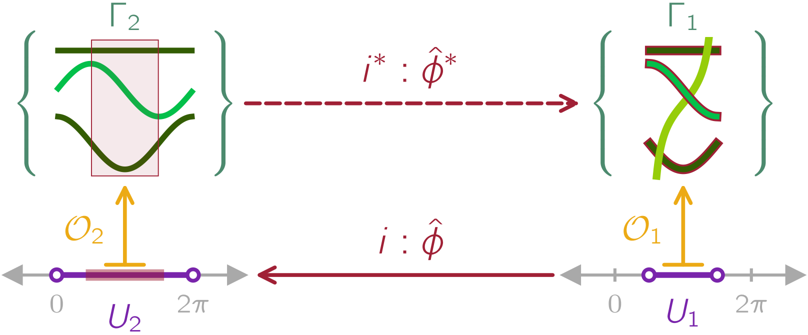
- *identity*: $F(id_C(C)) = id_D(F(C))$
- *composition*: $F(g) \circ F(f) = F(g \circ f)$ for any composable morphisms $C_1 \xrightarrow{f} C_2$, $C_2 \xrightarrow{g} C_3$

$F(C) \in \mathbf{ob}(\mathcal{D})$ denotes the object to which an object $C$ is mapped, and $F(f) \in Hom(F_( C_1), F_( C_2))$ denotes the morphism that $f$ is mapped to.

# topological equivariance: presheaf

**Definition**

A **presheaf** $F : \mathcal{C}^{op} \to \boldsymbol{Set}$ is a contravariant functor from an object in an arbitrary category to an object in the category $\boldsymbol{Set}$[36].




**Definition**

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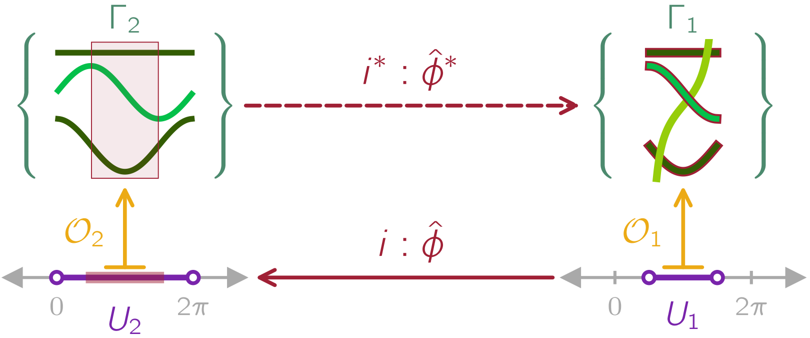
# topological equivariance: presheaf




The presheaf is contravariant when for every arbitrary morphism between input base spaces $\hat{\phi} : U_1 \to U_2$ there exists a corresponding pullback function between the sets of sections $\hat{\phi}^* : \Gamma(U_2, E\restriction_{U_2}) \to \Gamma(U_1, E\restriction_{U_c1})$.

# Container invariance: sheaf

## Definition

[45, 36] A **sheaf** is a presheaf that satisfies the following two axioms

- *locality* two sections in a sheaf are equal $\tau^a = \tau^b$ when they evaluate to the same values $\tau^a|_{U_i} = \tau^b|_{U_i}$ over the open cover $\bigcup_{i \in I} U_i \subset U$ (indexed by $I$).
- *gluing* the union of sections defined over subspaces $\tau^i \in \Gamma(U_i, E|_{U_i})$ is equivalent to a section defined over the whole space $\tau|_{U_i} = \tau^i$ for all $i \in I$ if all pairs of sections agree on overlaps $\tau^i|_{U_i \cap U_j} = \tau^j|_{U_i \cap U_j}$

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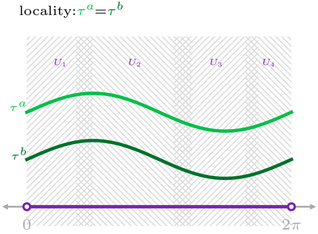

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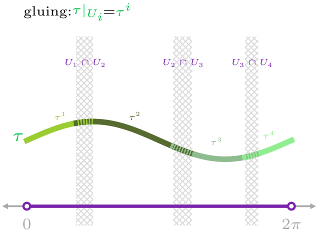

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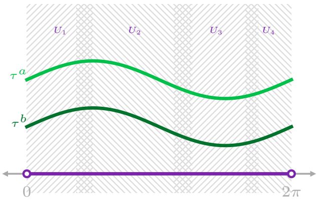
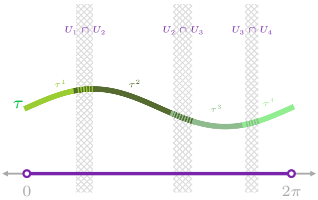
locality: $\tau^a = \tau^b$




gluing: $\tau|_{U_i} = \tau^i$



# What about needing more than one record?

The sheaf over a very small region surrounding a point $k$ is called a *stalk*[46]

$$\mathcal{O}_{K,E}\restriction_k \coloneqq \lim_{U \ni k} \Gamma(U, E\restriction_U) \tag{7}$$

where the fiber is contained inside the stalk $F_k \subset \mathcal{O}_{K,E}\restriction_k$. The *germ* is the section evaluated at a point in the stalk $\tau(k) \in \mathcal{O}_{K,E}\restriction_k$ and is the data.

# Homeomorphism

We define the mapping $\xi$ to be a surjective continuous map:

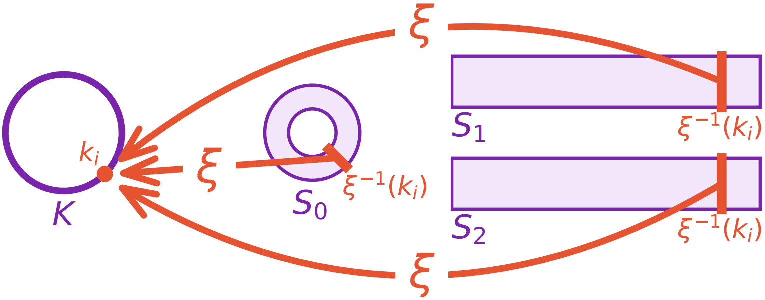
$$\xi : W \rightarrow U \tag{8}$$

between a graphic subspace $W \subseteq S$ and data subspace $U \subseteq K$. The set of points in graphic space that correspond to each point in data space is

$$\xi^{-1}(k) = \{s | \xi(s) = k \forall k \in K, s \in S\} \tag{9}$$

such that every point in a graphic space has a corresponding point in data space.

# Homeomorphism



# Pushforward: Graphic for Data

**Definition**

Given a sheaf $\mathcal{O}_{S,H}$ on $S$, the **pushforward** sheaf $\xi_*\mathcal{O}_{S,H}$ on $K$ is defined as

$$\xi_*(\mathcal{O}_{S,H})(U) = \mathcal{O}_{S,H}(\xi^{-1}(U)) \tag{10}$$

for all opensets $U \subset K$[46].

This provides a way to look up which graphic corresponds with a data index

$$\xi_*\rho(k) = \rho\restriction_{\xi^{-1}(k)} \tag{11}$$

such that $\xi_*\rho(k))(s) = \rho(s)$ for all $s \in \xi^{-1}(k)$.

# Pullback: Data at graphic

**Definition**

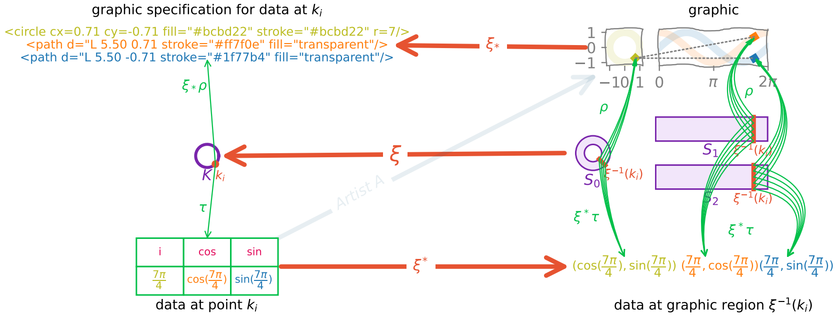
[46] Given a sheaf $\mathcal{O}_{K,E}$ on $K$, the **pullback** sheaf $\xi^*\mathcal{O}_{K,E}$ on $S$ is defined as the sheaf associated to the presheaf $\xi^*(\mathcal{O}_{K,E})(W) = \mathcal{O}_{K,E}(\xi(W))$ for $\xi(W) \in K$.

This provides a way to then look up what data values correspond with a graphic index

$$\xi^*\tau(s) = \tau(\xi(s)) = \tau(k) \tag{12}$$

As $\xi$ is surjective, there are many points $s \in W \subseteq S$ in the graphic space that correspond to a single point $\xi(s) = k$.

# Homeomorphism




| i | cos | sin |
|---|---|---|
| $\frac{7\pi}{4}$ | $\cos(\frac{7\pi}{4})$ | $\sin(\frac{7\pi}{4})$ |

# Artist

## Definition (natural transform)

Given two functors $F, G : \mathcal{C} \to \mathcal{D}$, a **natural transformation** $\alpha : F \to G$ is a function which assigns to each object $c$ of $\mathcal{C}$ a morphism $\alpha_c : F(input) \to G(c), G(c) \in \mathcal{D}$, in such a way that for every morphism $f : c \to c', c' \in \mathcal{C}$, the morphisms in $\mathcal{D}$ commute such that $\alpha'_c(F(f)(F(c))) = G(f)(\alpha_c(F(c)).$ When this holds, $\alpha_c$ is *natural* in $c$.[41].

| $\mathcal{C}$ | $\mathcal{D}$ |
| --- | --- |
| $\begin{tikzcd} c \arrow[d, "f"] \\ c' \end{tikzcd}$ | $\begin{tikzcd} F(c) \arrow[r, "\alpha_c"] \arrow[d, "F(f)"] & G(c) \arrow[d, "G(f)"] \\ F(c') \arrow[r, "\alpha_{c'}"] & G(c') \end{tikzcd}$ |

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# Artist

$$
\begin{tikzcd}
\texttt{topology} \arrow[rr, bend left, "\texttt{data}"] \arrow[rr, bend right, "\texttt{graphic}"'] \arrow[rr, phantom, "\Downarrow\texttt{artist}"] & & \texttt{topology} \arrow[r] & \texttt{fields}
\end{tikzcd}
$$

# How?

$$
\begin{tikzcd}
\Gamma(U, \xi_* H\restriction_U) & \Gamma(W, , )H\restriction_W \arrow[l, "\xi_*"'] \\
\Gamma(U, E\restriction_U) \arrow[u, "Hom_{\mathcal{O}_K}"] \arrow[ur, "Hom_{\mathcal{O}_K, \mathcal{O}_S}"] \arrow[r, "\xi^*"] & \Gamma(W, \xi^* E\restriction_W) \arrow[u, "Hom_{\mathcal{O}_S}"']
\end{tikzcd}
\tag{13}
$$

# Artist

$$A : \Gamma(K, E) \rightarrow Im_A(S, H), Im_A(S, H) \subset \Gamma(S, H) \tag{14}$$

where:

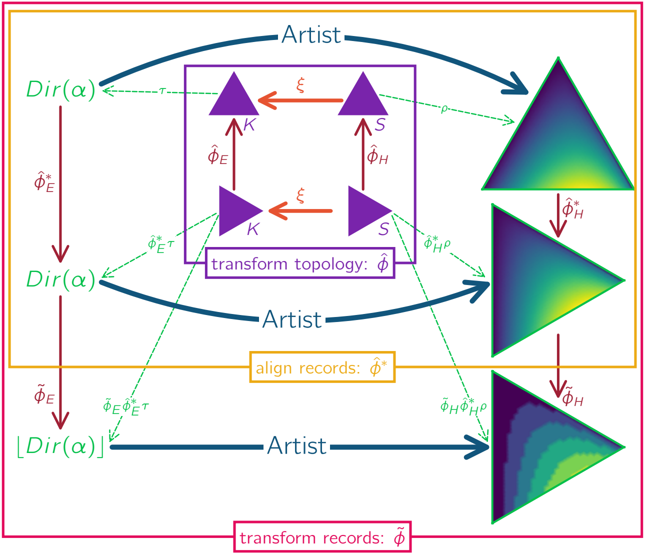
$$Im_A(S, H) \coloneqq \{\rho \mid \exists \, \tau \in \Gamma(K, E) \; s.t. \; A(\tau) = \rho, \; \xi(S) = K\} \tag{15}$$

# Equivariant Artist

$$
\begin{tikzcd}
\Gamma(K, E) \arrow[rrr, "A"] \arrow[d, "\hat{\phi}^*{}_E"'] & & & Im_A(S, H) \arrow[d, dotted, "\hat{\phi}^*{}_H"] \\
\Gamma(K', \hat{\phi}^*{}_E E) \arrow[dd, "\tilde{\phi}_E"'] & K & S \arrow[l, "\xi"'] & Im_A(S', \hat{\phi}^*{}_H H) \arrow[dd, dotted, "\tilde{\phi}_H"] \\
& K' \arrow[u, "\hat{\phi}_E"] & S' \arrow[l, "\xi"'] \arrow[u, dotted, "\hat{\phi}_H"'] & \\
\Gamma(K', E') \arrow[rrr, "A"] & & & Im_A(S', H')
\end{tikzcd}
\tag{16}
$$

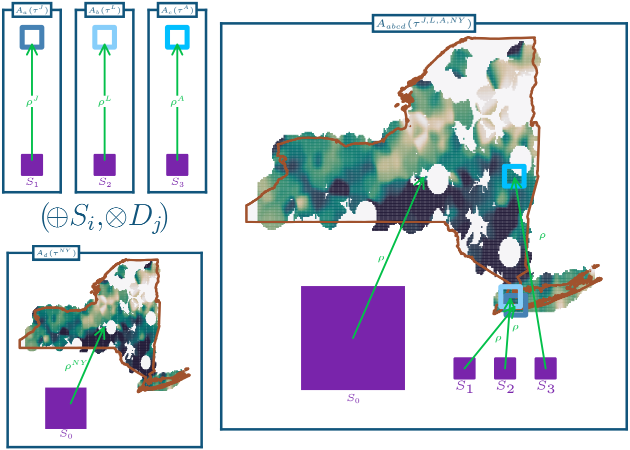
# Equivariant Artist

## Visualizing the Dirichlet Distribution



# artist composition

$$\bigsqcup_i \prod_j A(Hom(K_i, F_j)) = A(Hom(\bigsqcup_i K_i, \prod_j F_j)) \implies$$




$$\bigsqcup_i \prod_j Hom(S_i, D_j)) = Hom(\bigsqcup_i S_i, \prod_j D_j)$$

# Animation and Interactivity

pan, zoom, scroll sheaf: locality + gluing 9

selection and hover pushforward 11, pullback 12

brushing, linking, annotation composition of artists **??**

# Testing if $A$ is equivariant

$M$ is a (scaler, vector) measurable component (e.g. color, position, shape, texture, rotation, ) of the rendered visual element.

$$\begin{tikzcd}
\Gamma(U, E\restriction_U) \arrow[r, "A"] \arrow[d, "\eta"'] & Im_A(W, H\restriction_W) \arrow[r, "render"] \arrow[d, "\delta"] & visualization \arrow[dl, "measure"] \\
Hom(U, M) \arrow[r, "\xi^*"'] & Hom(W, M) &
\end{tikzcd}$$

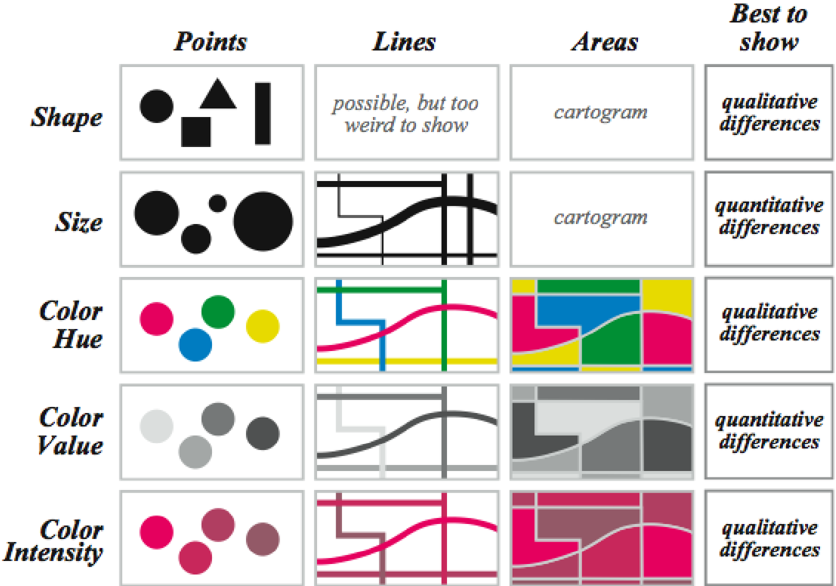
$$\text{input } \eta : \tau \mapsto (U \xrightarrow{\eta_\tau} M)$$

$$\text{output } \delta : \rho \mapsto (W \xrightarrow{\delta_\rho} M)$$

$$\eta_\tau(k) = \delta_\rho(s) \text{ for all } \xi(s) = k, k \in K, s \in S$$

# Visual Space: Specialized data space

$P \hookrightarrow V \xrightarrow{\pi} K$



# Encoding

$$\nu : F_k \to P_k \tag{17}$$

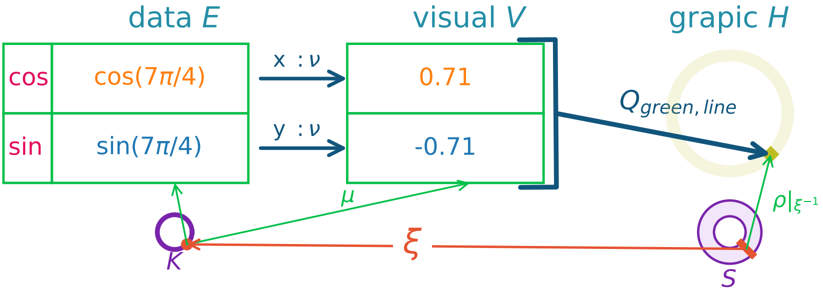
$$\begin{tikzcd}
F_k \arrow[r, "\nu"] \arrow[rr, bend right, dashed, "\nu''"] & P_k \coloneqq F'_k \arrow[r, "\nu'"] & P'_k
\end{tikzcd} \tag{18}$$

# Composition

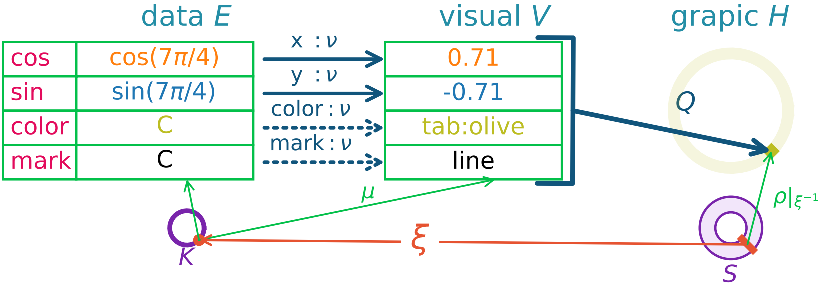
$$Q : \Gamma(K, V) \to \Gamma(S, H) \tag{19}$$

$$\begin{tikzcd}
\Gamma(K, V) \arrow[r, "\nu"] \arrow[rr, bend right, dashed, "Q'"'] & \Gamma(K, V') \arrow[r, "Q"] & \Gamma(S, H)
\end{tikzcd} \tag{20}$$

# Construction Stages



# Construction Stages: composition



# Verify encoder

$$
\begin{tikzcd}
F^a_k \times F^b_k \arrow[r, "\nu_{ab}"] \arrow[d, "\pi_a"'] \arrow[rr, bend left, "\eta_{ab}"] & P^a_k \times P^b_k \arrow[d, "\pi_a"] & M^{ab}_k \arrow[d, "M\restriction_a"] \\
F^a_k \arrow[r, dashed, "\nu_a"] & P^a_k \arrow[r, dotted, "\simeq"] & M^a_k \\
F^a_k \times F^c_k \arrow[r, "\nu_{ac}"] \arrow[u, "\pi_a"] \arrow[rr, bend right, "\eta_{ac}"'] & P^a_k \times P^c_k \arrow[u, "\pi_a"'] & M^{ac}_k \arrow[u, "M\restriction_a"']
\end{tikzcd}
\tag{21}
$$

# Verify encoder

encoding $\nu$

$$
\begin{tikzcd}[column sep=large, row sep=large]
\begin{matrix} name \times element : F_b \\ \text{JFK INTL AP, PRCP} \end{matrix} \arrow[r, "\nu_{color,\,height}"'] \arrow[d, "\pi_{F_a}"'] \arrow[rr, bend left, "\eta_{color,\,scatter}"] & \begin{matrix} color \times markerstyle : P_b \\ \text{steelblue, 's'} \end{matrix} \arrow[d, "\pi_{P_a}"'] & \text{color:\#4682b4, mark: 's'} \arrow[d, "M|_{color}"'] \\
\begin{matrix} name : F_a \\ \text{JFK INTL AP} \end{matrix} \arrow[r, dashed, "\nu_{color}"'] & \begin{matrix} color : P_a \\ \text{steelblue} \end{matrix} \arrow[r, dotted, "\simeq"] & \text{color: \#4682b4} \\
\begin{matrix} name \times element : F_b \\ \text{JFK INTL AP, TEMP} \end{matrix} \arrow[r, "\nu_{color,\,x,\,y}"'] \arrow[u, "\pi_{F_a}"] \arrow[rr, bend right, "\eta_{color,\,scatter}"'] & \begin{matrix} color \times markerstyle : P_b \\ \text{steelblue, 'o'} \end{matrix} \arrow[u, "\pi_{P_a}"] & \text{color:\#4682b4, mark:'o'} \arrow[u, "M|_{color}"]
\end{tikzcd}
$$

data $E$ visual $V$

# Verify compositor

$$
\begin{tikzcd}
\Gamma(K, V^a \times V^b) \arrow[r, "Q_{ab}"] \arrow[d, "\pi_a"'] & Im_{\underset{ab}{A}}(S, H) \arrow[d, "M\restriction_a \circ \delta_{ab}"] \\
\Gamma(K, V^a) \arrow[r, dotted, "\simeq"] & Hom(K, M^a) \\
\Gamma(K, V^a \times V^c) \arrow[r, "Q_{ac}"] \arrow[u, "\pi_a"] & Im_{\underset{ac}{A}}(S, H) \arrow[u, "M\restriction_a \circ \delta_{ac}"']
\end{tikzcd}
\tag{22}
$$

# Verify compositor

composition $Q$

$$
\begin{tikzcd}
\begin{array}{c} \mathit{color} \times \mathit{markerstyle} : P_b \\ \text{steelblue, 's'} \end{array} \arrow[r, "Q_{square}"] \arrow[d, "\pi_{P_a}"'] & \square \arrow[d, "M|_{P_a} \circ \delta_{P_a,\, square}"] \\
\begin{array}{c} \mathit{color} : P_a \\ \text{steelblue} \end{array} \arrow[r, dotted, "\simeq"] & \text{color: \#4682b4} \\
\begin{array}{c} \mathit{color} \times \mathit{markerstyle} : P_b \\ \text{steelblue, 'o'} \end{array} \arrow[r, "Q_{circle}"] \arrow[u, "\pi_{P_a}"] & \bigcirc \arrow[u, "M|_{P_a} \circ \delta_{P_a,\, circle}"']
\end{tikzcd}
$$

visual $V$ graphic $H$

# Implementation Choices: $A_K = A_S$

$$
\begin{tikzcd}
\Gamma(W, \xi^* E\restriction_W) \arrow[r, "\nu_S"] \arrow[rr, bend left, "A_S"] & \Gamma(W, \xi^* V\restriction_W) \arrow[r, "Q_S"] & Im_A(W, H\restriction_W) \arrow[d, "\xi_*"] \\
\Gamma(U, E\restriction_U) \arrow[r, "\nu_K"] \arrow[u, "\xi^*"] \arrow[rr, bend right, "A_K"'] & \Gamma(U, V\restriction_U) \arrow[r, "Q_K"] \arrow[u, "\xi^*"] & Im_A(U, \xi_* H\restriction_U)
\end{tikzcd}
$$

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