Mathematical Data Abstraction

Fiber Bundles "unified, dimension-independent framework" that expresses data as the mapping between continuity and fields [**butlerVectorBundleClassesForm1992**, **butlerVisualizationModelBased1989**]

Category Theory Language express constraints in specifications [**wielsManagementEvolvingSpecifications1998**]

Sheaves on Bundles "algebraic data structure" for representing data over topological spaces [**ghristElementaryAppliedTopology2014**]

Fiber Bundle

$$\begin{tikzcd} F \arrow[r, hook] & E \arrow[d, "\pi"'] \\ & K \arrow[u, bend right, "\tau\in\Gamma(U,\, E\restriction_U)"'] \end{tikzcd}$$

Total Space

$(E, \mathcal{T}_E)$

Base Space

$(K, \mathcal{T}_K),\ k \in U \subseteq K$

Fiber Space

$F\restriction_k = \pi^{-1}(k), F = F\restriction_k\ \forall k \in K$

Sections

$\Gamma(U,\, E\restriction_U) \coloneqq \left\{\tau : U \to E\restriction_U \;\middle|\; \pi(\tau(k)) = k \text{ for all } k \in U\right\}$

Locally Trivial

for every point $k \in K$, there exists an open neighborhood $k \in U \subseteq K$ s.t. there is a homeomorphism $\pi^{-1}(U) \xrightarrow{\varphi} U \times F$

(Globally) Trivial

$E = K \times F$

Data Bundle

$$
\begin{tikzcd}
F \arrow[r, hook] & E \arrow[d, "\pi"'] \\
& K \arrow[u, bend right, "\tau \in \Gamma(U, E\restriction_U)"']
\end{tikzcd}
$$

Data E

continuity + fields

Continuity K

how data elements are organized (topological properties) [**wilkinsonGrammarGraphics2005**], index (key) space [**munznerWhatDataAbstraction2014**])

Fields F

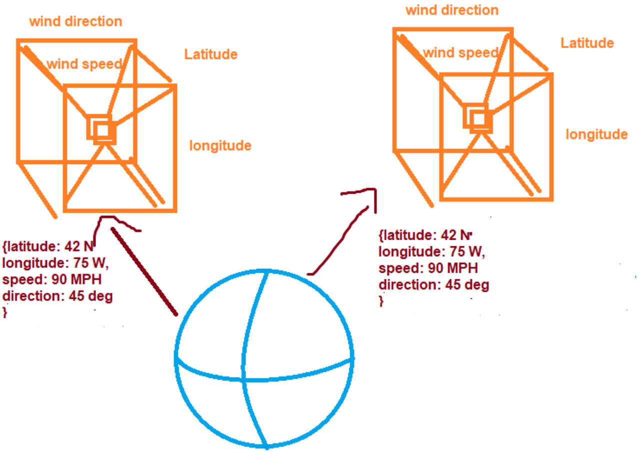
generalization of a schema - named and typed date fields [**spivakSIMPLICIALDATABASES**, **spivakDatabasesAreCategories2010**]

Data **[butlerVisualizationModelBased1989, butlerVectorBundleClassesForm1992]**

$\tau(k) = r,\ k \in U \subseteq K,\ r \in F\restriction_k$

$r = \{field : value, \ldots, field : value, \ldots, field : value\}$

Data Bundle


wind direction
wind speed
Latitude
longitude
{latitude: 42 N
longitude: 75 W,
speed: 90 MPH
direction: 45 deg
}
wind direction
wind speed
Latitude
longitude
{latitude: 42 N
longitude: 75 W,
speed: 90 MPH
direction: 45 deg
}

Graphic Bundle

$$\begin{tikzcd} D \arrow[r, hook] & H \arrow[d, "\pi"'] \\ & S \arrow[u, bend right, "\rho\in\Gamma(W,\, H\restriction_W)"'] \end{tikzcd}$$

Graphic H
: continuity + renderer fields

Continuity S
: parameterization of graphic area (e.g. "inked" bounding box[**CairographicsOrg**])

Display D
: renderer fields, e.g. {xy,rgba}, {xy, cymk}, {xyz, rgba}

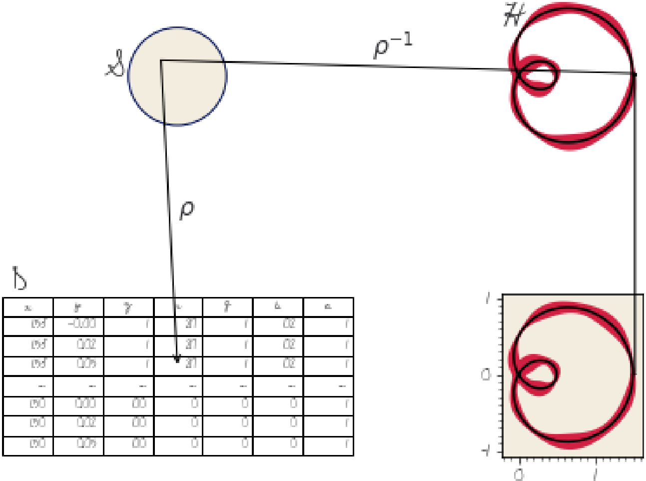
Graphic

$$\Gamma(W, H\restriction_W) \coloneqq \left\{\rho : W \to H\restriction_W \;\middle|\; \pi(\rho(s)) = s \textit{ for all } s \in W\right\}$$

$$\rho(s) = d,\ s \in W \subseteq S,\ d \in F\restriction_s$$

$$d = \{x, y, r, g, b\}$$

Graphic Bundle



```
Artist(data:Data) -> Graphic
```

Category $\mathcal{C}$

$$
\begin{tikzcd}
C_1 \arrow[loop left, "id_{C_1}"] \arrow[r, "f"] \arrow[dr, "g\circ f"'] & C_2 \arrow[loop above, "id_{C_2}"] \arrow[d, "g"] \\
& C_3 \arrow[loop below, "id_{C_3}"]
\end{tikzcd}
$$

associativity

if $f : C_1 \to C_2$, $g : C_2 \to C_3$ and $h : C_3 \to C_4$ then

$h \circ (g \circ f) = (h \circ g) \circ f$

identity

for every $f : C_1 \to C_2$ there exists identity morphisms

$f \circ id_{C_1} = f = id_{C_2} \circ f$

Opposite Category $\mathcal{C}^{op}$

$$
\begin{tikzcd}
C_1 \arrow[loop left, "id_{C_1}"] \arrow[r, "f"] \arrow[dr, "g \circ f"'] & C_2 \arrow[loop above, "id_{C_2}"] \arrow[d, "g"] \\
 & C_3 \arrow[loop below, "id_{C_3}"]
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
C_1 \arrow[loop left, "id_{C_1}"] & C_2 \arrow[loop above, "id_{C_2}"] \arrow[l, "f"'] \\
 & C_3 \arrow[loop below, "id_{C_3}"] \arrow[u, "g"'] \arrow[ul, "f \circ g"]
\end{tikzcd}
$$

Functor $F : \mathcal{C} \to \mathcal{D}$

composition

$$\begin{tikzcd} c \arrow[r, mapsto, "F"] \arrow[d, "f"'] & F(c) = d \arrow[d, "F(f)"] \\ c' \arrow[r, mapsto, "F"'] & F(c') = d' \end{tikzcd}$$

$$F(g) \circ F(f) = F(g \circ f)$$

identity

$$F(id_c) = id_{F(c)}$$

Natural Transformation $\alpha : F \Rightarrow G$

$$
\begin{tikzcd}
\mathcal{C} \arrow[r, bend left, mapsto, "F"] \arrow[r, bend right, mapsto, "G"'] \arrow[r, phantom, "\Downarrow\alpha"] & \mathcal{D}
\end{tikzcd}
$$

Natural Transformation $\alpha : F \Rightarrow G$

$$
\begin{tikzcd}
F(c) \arrow[d, "F(f)"'] \arrow[rr, Rightarrow, bend left, "\alpha_c"] & c \arrow[l, mapsto, "F"] \arrow[r, mapsto, "G"'] \arrow[d, "f"] & G(c) \arrow[d, mapsto, "G(f)"] \\
F(c') \arrow[rr, Rightarrow, bend right, "\alpha_{c'}"'] & c' \arrow[l, mapsto, "F"'] \arrow[r, mapsto, "G"] & G(c')
\end{tikzcd}
$$

Presheaf: $\mathcal{O} : \mathcal{C}^{op} \to \mathit{Set}$

$$
\begin{tikzcd}
F \arrow[r, hook] & E \arrow[d, "\pi"'] \\
& K \arrow[u, bend right, "\tau \in \Gamma(K, E)"']
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathit{Set} \ni \Gamma(U_1, E\restriction_{U_1}) & & \Gamma(U_2, E\restriction_{U_2}) \arrow[ll, hook', "\iota^*"'] \\
Ob(\mathcal{K}^{op}) \ni U_1 \arrow[u, mapsto, "\mathcal{O}_{K,E}"] \arrow[rr, hook, "\iota"'] & & U_2 \arrow[u, mapsto, "\mathcal{O}_{K,E}"']
\end{tikzcd}
$$

stalk

$$\mathcal{O}_{K,E}\restriction_k \coloneqq \lim_{U \ni k} \Gamma(U, E\restriction_U)$$

$$F_k \subset \mathcal{O}_{K,E}\restriction_k$$

germ

$$\tau(k) \in \mathcal{O}_{K,E}\restriction_k$$

Sheaves on Bundles

A sheaf is a presheaf that satisfies the following two axioms[**bakerMathsSheaf**]

locality

given $U = \bigcup_{i \in I} U_i$ and $\tau^a, \tau^b \in \mathcal{O}(U)$,
if $\tau^a\restriction_{U_i} = \tau^b\restriction_{U_i}$ for each $U_i \in U$ then $\tau^a = \tau^b$

gluing

given $\tau^i \in \mathcal{O}(U_i)$ s.t. $\tau^i\restriction_{U_i \cap U_j} = \tau^j\restriction_{U_i \cap U_j}$ for $U_i, U_j \in U$,
there exists $\tau \in \mathcal{O}(U)$ such that $\tau\restriction_{U_i} = \tau^i$

Data

$$\begin{tikzcd} U \arrow[d, mapsto, "\mathcal{O}_{K,E}"] \\ \Gamma(U, E\restriction_U) \end{tikzcd}$$

$F \hookrightarrow E \xrightarrow{\pi} K$

$\mathcal{O}_{K,E} : U \mapsto \Gamma(U, E\restriction_U), U \subset K$

$\Gamma(U, E\restriction_U) \ni \tau : U \to F\restriction_U$

$\tau(k) = \{f_0 : v_0, \cdots ,\}, k \in U$

Graphic

$$\begin{tikzcd} \Gamma(W, H\upharpoonright_W) \\ W \arrow[u, mapsto, "\mathcal{O}_{S,H}"] \end{tikzcd}$$

$$D \hookrightarrow H \xrightarrow{\pi} S$$

$$\mathcal{O}_{S,H} : W \mapsto \Gamma(W, E\upharpoonright_W), W \subset S$$

$$\Gamma(W, H\upharpoonright_W) \ni \rho : W \to D\upharpoonright_W$$

$$\rho(s) = \{d_0, \cdots\}, s \in W$$

Function: $\xi : S \to K$

$$\begin{tikzcd}
\Gamma(W, H\restriction_W) & & \\
W \arrow[u, mapsto, "\mathcal{O}_{S,H}"] \arrow[rr, "\xi"] & & U \arrow[d, mapsto, "\mathcal{O}_{K,E}"] \\
& & \Gamma(U, E\restriction_U)
\end{tikzcd}$$

Pullback: data to region of the visualization

$$\begin{tikzcd} W \arrow[r, "\xi"] \arrow[d, mapsto, dashed, "\mathcal{O}_{S,\xi^* E}"'] & U \arrow[d, mapsto, "\mathcal{O}_{K,E}"] \\ \Gamma(W, \xi^* E\restriction_W) & \Gamma(U, E\restriction_U) \arrow[l, "\xi^*"] \end{tikzcd}$$

$$\xi^* F \hookrightarrow \xi^* E \xrightarrow{\pi} S$$

$$\xi^* \mathcal{O}_{K,E} : W \mapsto \Gamma(W, \xi^* E\restriction_W), W \subset S$$

$$\Gamma(W, \xi^* E\restriction_W) \ni \xi^* \tau : W \to \xi^* F\restriction_W$$

$$\xi^* \tau(s) = \tau(\xi(s)) = \tau(k)$$

Pushforward: visualization to index of data

$$\begin{tikzcd}
\Gamma(W, H\restriction_W) \arrow[r, "\xi_*"] & \Gamma(U, \xi_* H\restriction_U) \\
W \arrow[u, mapsto, "\mathcal{O}_{S,H}"] \arrow[r, "\xi"] & U \arrow[u, mapsto, dashed, "\mathcal{O}_{K,\xi_* H}"']
\end{tikzcd}$$

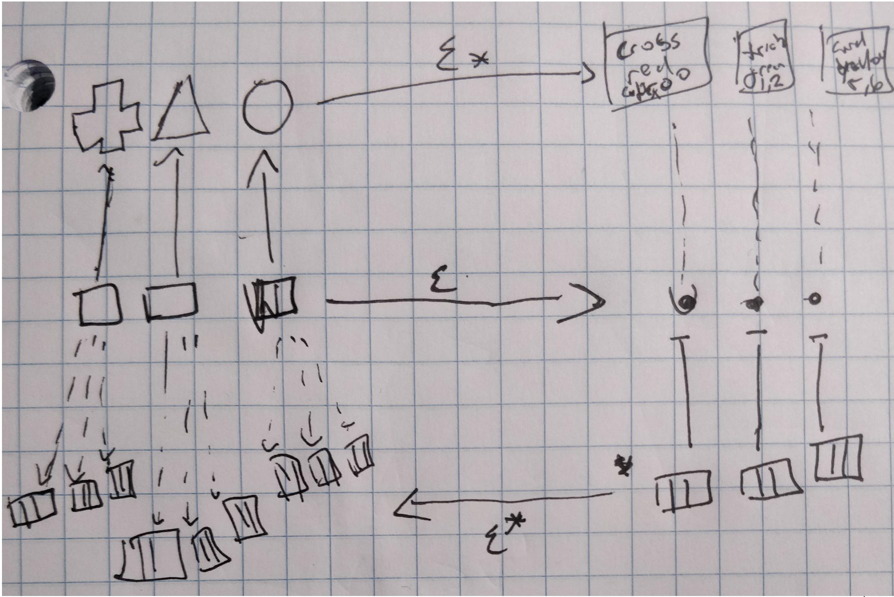
$$\xi_* D \hookrightarrow \xi_* H \xrightarrow{\pi} K$$

$$\xi_* \mathcal{O}_{S,H} : U \mapsto \Gamma(U, \xi_* H\restriction_U), U \subset K$$

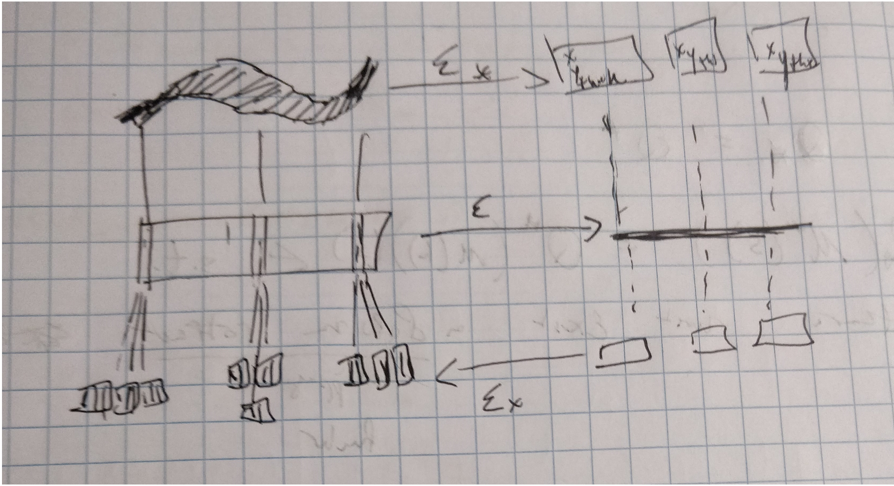
$$\Gamma(U, \xi_* H\restriction_U) \ni \xi_* \rho : U \to \xi_* D\restriction_U$$

$$\xi_* \rho(k) = \rho\restriction_{\xi^{-1}(k)} = \rho(s)\ \forall s \in \xi^{-1}(k)$$

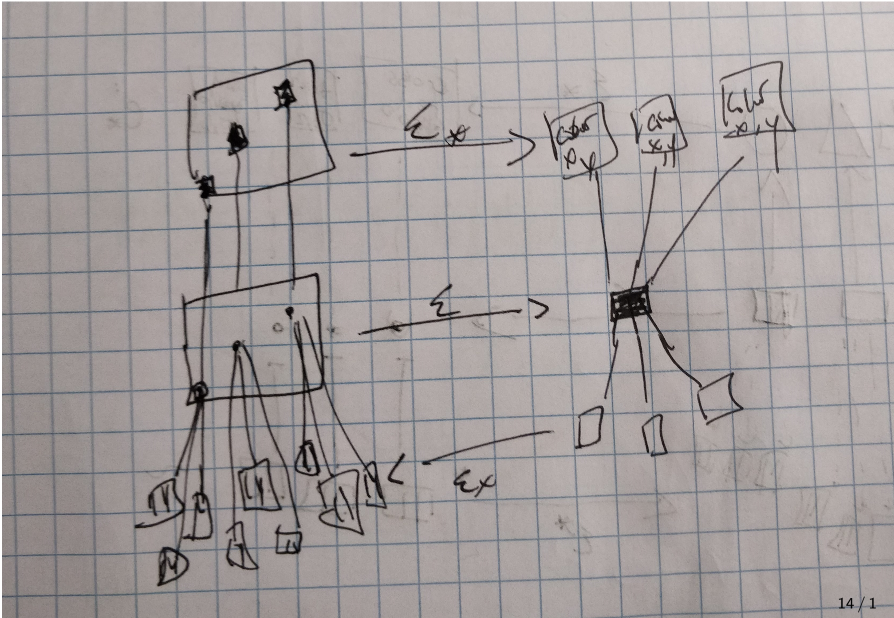
Example: Scatter Plot



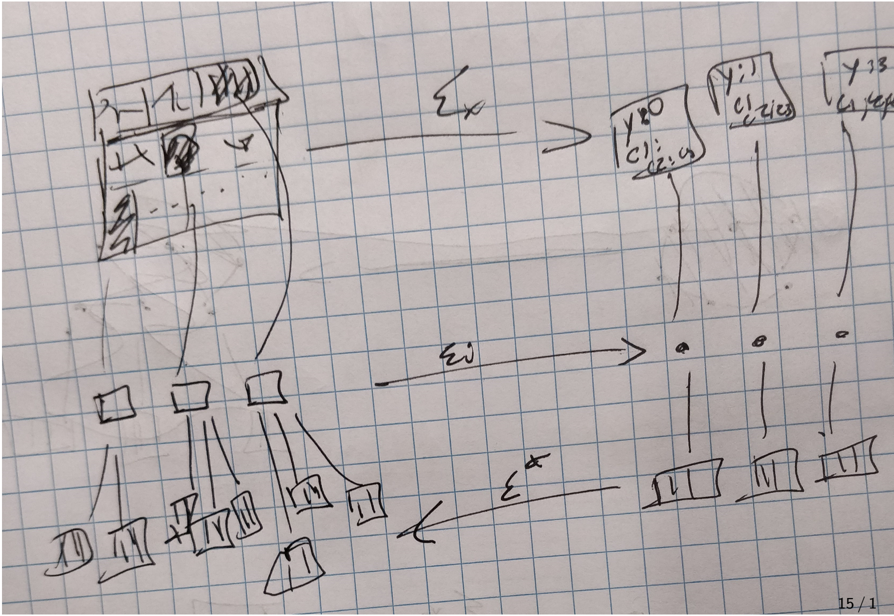
Example: Line Plot



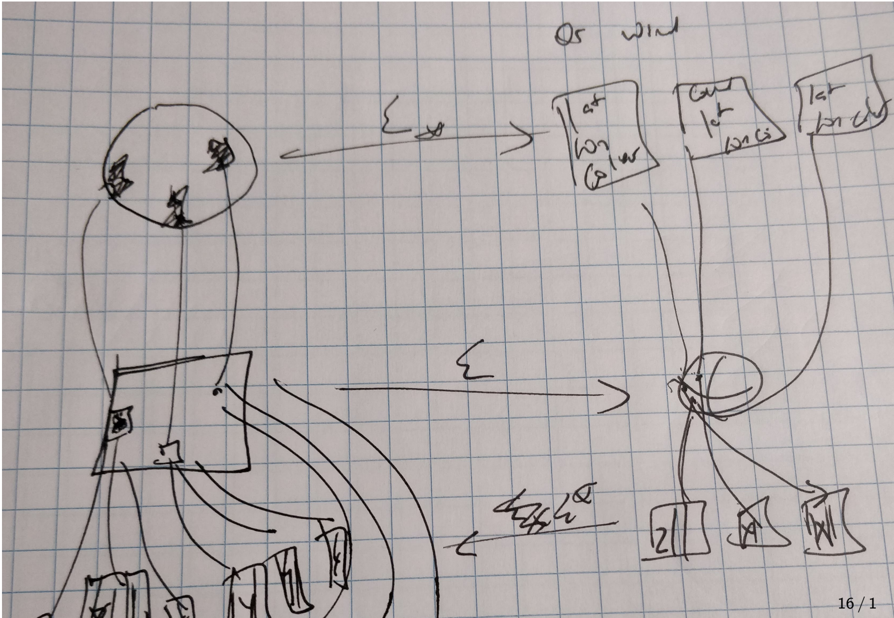
Example: Image



Example: Heatmap



Example: Sphere



Artist: $A : \mathcal{O}_{K,E} \to \mathcal{O}_{S,H}$

$$\begin{tikzcd}
\mathcal{O}_{S,H} \arrow[rr, "\xi_*"] & & \xi_*\mathcal{O}_{S,H} \\
\xi^*\mathcal{O}_{K,E} \arrow[u, Rightarrow, "A_W"] & & \mathcal{O}_{K,E} \arrow[ll, "\xi^*"] \arrow[u, Rightarrow, "A_U"'] \arrow[ull, Rightarrow, dashed, "A"]
\end{tikzcd}$$

$$Nat_W(\xi^*\mathcal{O}_{K,E}, \mathcal{O}_{S,H}) = Nat_U(\mathcal{O}_{K,E}, \xi_*\mathcal{O}_{S,H})$$

Data Space: $A_U : \mathcal{O}_{K,E} \Rightarrow \xi_* \mathcal{O}_{S,H}$

$$
\begin{tikzcd}[column sep=huge, row sep=huge]
\Gamma(U_1, E) \arrow[d, "\iota^*"'] \arrow[rr, Rightarrow, bend left, "A_{U_1}"] & U_1 \arrow[l, mapsto, "\mathcal{O}_{K,E}"] \arrow[r, mapsto, "\mathcal{O}_{K,\xi_* H}"'] & \Gamma(U_1, \xi_* H) \arrow[d, "\iota^*"] \\
\Gamma(U_2, E) \arrow[rr, Rightarrow, bend right, "A_{U_2}"'] & U_2 \arrow[u, hook, "\iota"] \arrow[l, mapsto, "\mathcal{O}_{K,E}"'] \arrow[r, mapsto, "\mathcal{O}_{K,\xi_* H}"] & \Gamma(U_2, \xi_* H)
\end{tikzcd}
$$

Display Space: $A_W : \xi^*\mathcal{O}_{K,E} \Rightarrow \mathcal{O}_{S,H}$

$$\begin{tikzcd}
\Gamma(W_1, \xi^*E\restriction_{W_1}) \arrow[dd, hook, "\iota^*"'] \arrow[rr, Rightarrow, bend left, "A_{W_1}"] & W_1 \arrow[l, mapsto, "\mathcal{O}_{S,\xi^*E}"] \arrow[r, mapsto, "\mathcal{O}_{S,H}"] & \Gamma(W_1, H\restriction_{W_1}) \arrow[dd, hook, "\iota^*"] \\
{} \arrow[rr, dotted, Rrightarrow] & & {} \\
\Gamma(W_2, \xi^*E\restriction_{W_2}) \arrow[rr, Rightarrow, bend right, "A_{W_2}"'] & W_2 \arrow[uu, hook, "\iota"] \arrow[l, mapsto, "\mathcal{O}_{S,\xi^*E}"'] \arrow[r, mapsto, "\mathcal{O}_{S,H}"'] & \Gamma(W_2, H\restriction_{W_2})
\end{tikzcd}$$

Artist: $A : \Gamma(U, E) \to \Gamma(W, H)$

$$\begin{tikzcd}
\Gamma(W, H\restriction_W) \arrow[rr, "\xi_*"] & & \Gamma(U, \xi_* H\restriction_U) \\
\Gamma(W, \xi^* E\restriction_W) \arrow[u, Rightarrow, "A_W"] & & \Gamma(U, E\restriction_U) \arrow[ll, "\xi^*"] \arrow[u, Rightarrow, "A_U"'] \arrow[ull, dashed, Rightarrow, "A"]
\end{tikzcd}$$

pull data section τ over graphic space $s \in S$

$$\xi^*\tau(s) = \tau(\xi(s)) = \tau(k)$$

push graphic ρ section over data space $k \in K$

$$\xi_*\rho(k) = \rho\restriction_{\xi^{-1}(k)} = \rho(s) \forall s \in \xi^{-1}(k)$$

Artist as Profunctor

$$\begin{tikzcd}
\Gamma(U, E\restriction_U) \arrow[rr, Rightarrow, "A"] & & \Gamma(W, E\restriction_W) \\
U \arrow[u, mapsto, "\mho_{K,E}"] & & W \arrow[ll, "\xi"'] \arrow[u, mapsto, "\mho_{S,H}"']
\end{tikzcd}$$

Transform Data

Fiber Bundle Category

object $F \hookrightarrow E \xrightarrow{\pi} K$

morphisms $\phi : E \to E$

Fiber Category

The fiber F is the sole object of an arbitrary category $\mathcal{C}$. The morphisms on the fiber are $Hom(F, F)$

Data Transformations

$\phi : E\restriction_k \to E\restriction_{k'} \in Hom(F\restriction_k, F\restriction_{k'})$

$\phi = (\hat{\phi}, \tilde{\phi})$

$$
\begin{tikzcd}
\Gamma(U, E\restriction_U) \arrow[r, "\hat{\phi}^*"] & \Gamma(U', \hat{\phi}^* E\restriction_{U'}) \arrow[r, "\tilde{\phi}"] & \Gamma(U', \hat{\phi}^* E\restriction_{U'}) \\
U \arrow[u, mapsto, "\mathcal{O}_{K,E}"] & U' \arrow[l, "\hat{\phi}"] \arrow[u, mapsto, "\hat{\phi}^*\mathcal{O}_{K,E}"'] \arrow[ur, mapsto, "\hat{\phi}^*\mathcal{O}_{K,E}"']
\end{tikzcd}
$$

Base K Type stays same: $\hat{\phi} : K \to K$

Fiber F Type stays the same: $\tilde{\phi} : \hat{\phi}^* E \to \hat{\phi}^* E$ s.t.
$\pi(\hat{\phi}^* E) = \pi(\tilde{\phi}(\hat{\phi}^* E))$

Section Transform: $\phi : \tau^a\restriction_U \mapsto \tau^b\restriction_{U'}$

Equivariant Artist

$$\begin{tikzcd}
\mathcal{O}_{K,E} \arrow[r, Rightarrow, "A"] \arrow[d, "\phi_E"'] & \mathcal{O}_{S,H} \arrow[d, dashed, "\phi_H"] \\
\mathcal{O}_{K',\hat{\phi}^*E} \arrow[r, Rightarrow, "A"] & \mathcal{O}_{S',\hat{\phi}^*H}
\end{tikzcd}$$

Artist Constraints

$$A(\tau^a) = A(\tau^b) \;\not\Longrightarrow\; \tau^a = \tau^b$$

$$A(\tau^a) = A(\tau^b) \;\Longrightarrow\; A(\phi_E(\tau^a)) = A(\phi_E(\tau^b))$$

Equivariant $\hat{\phi}$

$$
\begin{tikzcd}
W \arrow[r, "\xi"] \arrow[d, dashed, "\hat{\phi}_S"'] & U \arrow[d, "\hat{\phi}_K"] \\
W' \arrow[r, "\xi"'] & U'
\end{tikzcd}
$$

$$\underbrace{K \times [0,1]^m}_{S} \overset{\xi}{\longmapsto} K$$

Checking if A is equivariant

$$\begin{tikzcd} \Gamma(U, E\restriction_U) \arrow[r, Rightarrow, "A"] & \Gamma(W, H\restriction_W) \arrow[r, "render"] \arrow[dr, "\delta"'] & visualization \arrow[d, "measure"] \\ & & M \end{tikzcd}$$

Checking if A is equivariant

$$
\begin{tikzcd}
\Gamma(U, E\restriction_U) \arrow[r, Rightarrow, "A"'] \arrow[d, "\phi_E"'] \arrow[rr, dashed, bend left, "\varphi"] & \Gamma(W, H\restriction_W) \arrow[r, "\delta"'] & M \arrow[d, "\phi_M"] \\
\Gamma(U', \hat{\phi}^* E\restriction_{U'}) \arrow[r, Rightarrow, "A"] \arrow[rr, dashed, bend right, "\varphi"'] & \Gamma(W', \hat{\phi}^* H\restriction_{W'}) \arrow[r, "\delta"] & M'
\end{tikzcd}
$$

there exists a functor $\varphi : \phi_E \to \phi_M$ such that

equivariance $\varphi(\phi_E(\tau(k))) = \phi_M(\varphi(\tau(k)))$, for all $k \in K$

continuity $\lim_{x \to k} (\varphi(\tau(x))) = \varphi(\tau(k))$ for all $k \in K$

Multiple Bundles: $\phi = (\hat{\phi}, \prod_{i=0}^{n} \tilde{\phi}_i)$

$$
\begin{tikzcd}
\tilde{\phi}_0\hat{\phi}^* E_0 \cdots \otimes \cdots \tilde{\phi}_i\hat{\phi}^* E_i \cdots \otimes \cdots \tilde{\phi}_n\hat{\phi}^* E_n \arrow[d, "\pi"] \arrow[d, "\pi"'] \arrow[d, "\pi"] & = & \prod_{i=0}^{n} \tilde{\phi}_i\hat{\phi}^* E \arrow[d, "\pi"] \\
(\hat{\phi} = \hat{\phi}_{i\in[0,n]})K & = & \hat{\phi}K
\end{tikzcd}
$$

Complex data

combining continuities given $k_a \in K_c \subset K_a$ and $k_b \in K_c \subset K_b$,
if $k_a = k_b$ then $\varphi(\tau(k_a)) = \varphi(\tau(k_b))$ for all
$k_a, k_b \in K_a \bigsqcup_{K_c} K_b$

shared fibers if $p = p'$ then $\varphi \circ q = \varphi' \circ q'$

$$
\begin{tikzcd}
 & F^a \times F^b \arrow[dl, "p"'] \arrow[dr, "q\circ\varphi"] & \\
F^a & & M^a \\
 & F^a \times F^c \arrow[ul, "p'"] \arrow[ur, "q'\circ\varphi'"'] &
\end{tikzcd}
$$

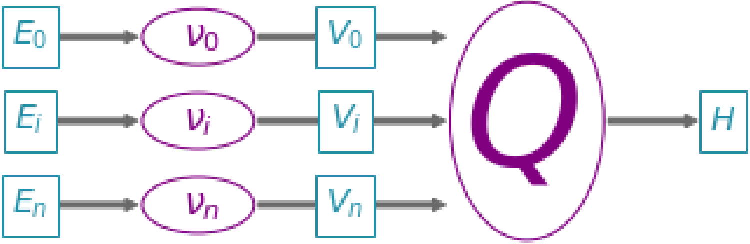
both given $\tau^d \in \mathcal{O}_{K_d,E_d}, \tau^e \in \mathcal{O}_{K_e,E_e}$ and $k \in K_d \bigsqcup_{K_f} K_e$ then

$$q(\varphi(\tau^d(k))) = q'(\varphi(\tau^e(k))) \text{ when } p(F^d\restriction_k) = p'(F^e\restriction_k)$$

"Valid" viz?

$$
\begin{tikzcd}
\Gamma(W, H\restriction_W) & \supset & \left\{\rho \;\middle|\; \delta(A(\phi_E(\tau))) = \phi_M(\delta(\rho))\right\} \\
W \arrow[u, mapsto, "\mathcal{O}_{S,H}"] \arrow[urr, dashed, mapsto, "A(\mathcal{O}_E) \coloneqq \mathcal{O}_A"']
\end{tikzcd}
$$

What's in the A black box?



Measurable Visual Components

$$\begin{tikzcd} P \arrow[r, hook] & V \arrow[d, "\pi"'] \\ & K \arrow[u, bend right, "\mu\in\Gamma(U,\, V\restriction_U)"'] \end{tikzcd}$$

Data V

continuity + visual fields

Continuity K

data continuity

components P

visual components [**bertinIIPropertiesGraphic2011**] of a graphic, e.g. x and y position, area, color, line thickness

visual components

$$\Gamma(U, V\restriction_U) \coloneqq \left\{\mu : U \to V\restriction_U \;\middle|\; \pi(\mu(k)) = k \textit{ for all } k \in U\right\}$$

Data to Visual Transformation: $\nu : \tau \mapsto \mu$

$$\begin{tikzcd}
\Gamma(U, E\restriction_U) \arrow[rr, mapsto, "\nu"] & & \Gamma(U, V\restriction_U) \\
& U \arrow[ul, mapsto, "\mathcal{O}_{K,E}"] \arrow[ur, mapsto, "\mathcal{O}_{K,V}"'] &
\end{tikzcd}$$

constraints

equivariance $\nu(\phi_E(\tau(k))) = \phi_V(\nu(\tau(k)))$, for all $k \in K$

continuity $\lim\limits_{x \to k}(\nu(\tau(x))) = \nu(\tau(k))$ for all $k \in K$

pointwise $\nu : \tau(k) \mapsto \mu(k) = \nu : \xi^*\tau(s) \mapsto \xi^*\mu(s)$ when $\xi(s) = k$

Functor $\nu : \tau \mapsto \mu$

$$\begin{tikzcd}
\Gamma(U, E\restriction_U) \arrow[r, mapsto, "\nu_{F,P}"] \arrow[dr, dotted, "\nu_{F,P'}"'] & \Gamma(U, V\restriction_U) \coloneqq \Gamma(U, E'\restriction_U) \arrow[d, mapsto, "\nu_{F',P'}"] \\
& \Gamma(W, V')
\end{tikzcd}$$

$\nu : \phi_E \mapsto \phi_V$: Stevens' Scales [stevensTheoryScalesMeasurement1946]

scale	**group**	**constraint**
nominal	permutation	if $r_1 \neq r_2$ then $\nu(r_1) \neq \nu(r_2)$
ordinal	monotonic	if $r_1 \leqslant r_2$ then $\nu(r_1) \leqslant \nu(r_2)$
interval	translation	$\nu(r + c) = \nu(r) + \nu(c)$
ratio	scaling	$\nu(r * c) = \nu(r) * \nu(c)$

Shared Components: $\nu = \prod_{i=0}^{n} \nu_i$

$$
\begin{tikzcd}
F_a \times F_b \arrow[rr, mapsto, "\nu"] \arrow[dr, "p_F"] & & P_a \times P_b \arrow[dr, "p_P"] & \\
& F_a \arrow[rr, mapsto, dashed, "\nu_a"] & & P_a \\
F_a \times F_c \arrow[rr, mapsto, "\nu'"] \arrow[ur, "p_{F'}"'] & & {P_a}' \times {P_c}' \arrow[ur, "p_{P'}"'] &
\end{tikzcd}
\tag{1}
$$

Consistent Transformations [**hullmanKeeping2018**]

if $p_F = p_{F'}$ then $p_P(\nu(\tau)) = p_{P'}(\nu'(\tau'))$ s.t. there exists a transformation $\nu_a : F_a \to P_a$

Building a Visual Representation

$$\begin{tikzcd}
\Gamma(U, E\restriction_U) \arrow[rr, mapsto, "\nu"] & & \Gamma(U, V\restriction_U) \\
& U \arrow[ul, mapsto, "\mathcal{O}_{K,E}"] \arrow[ur, mapsto, "\mathcal{O}_{K,V}"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
\Gamma(U, V\restriction_U) \arrow[r, "Q"] & \Gamma(W, H\restriction_W) \\
U \arrow[u, mapsto, "\mathcal{O}_{K,V}"] & W \arrow[l, "\xi"'] \arrow[u, mapsto, "\mathcal{O}_{S,H}"']
\end{tikzcd}$$

equivariance $\delta(Q(\nu\phi_E(\tau))) = \phi_M(\delta(\tau))$

continuity the assembly function Q is defined as

$$Q_*(\mu(k))(s) = Q(\mu(\xi(s))) \tag{2}$$

such that there must exist a deformation retract $\xi : S \to K$ for all points $s \in S$.

Implementation Choices: $A_U = A_W$

$$
\begin{tikzcd}
\Gamma(W, \xi^* E\restriction_W) \arrow[r, mapsto, "\nu"] \arrow[rrr, Rightarrow, bend left, "A_W"] & \Gamma(W, \xi^* V\restriction_W) & \Gamma(W, \xi^* V\restriction_W) \arrow[r, Rightarrow, "Q^*"] & \Gamma(W, H\restriction_W) \arrow[d, "\xi_*"] \\
\Gamma(U, E\restriction_U) \arrow[r, mapsto, "\nu"] \arrow[u, "\xi^*"] \arrow[urrr, Rightarrow, bend right, "A_U"'] & \Gamma(U, V\restriction_U) \arrow[u, "\xi^*"'] & \Gamma(U, V\restriction_U) \arrow[u, "\xi^*"] \arrow[ur, Rightarrow, "Q"] \arrow[r, Rightarrow, "Q_*"] & \Gamma(W, \xi_* H
\end{tikzcd}
$$