About

Slides: https://bit.ly/team329

Me (twitter/github: story645)

- nth year grad student (on my 3rd EO)
- former adjunct at CCNY, former Digital Fellow
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Project

- Funded by Chan Zuckerberg Initiative EOSS 1 & 3
- paper rejected by vizweek last spring
- work has since gone all in on category theory

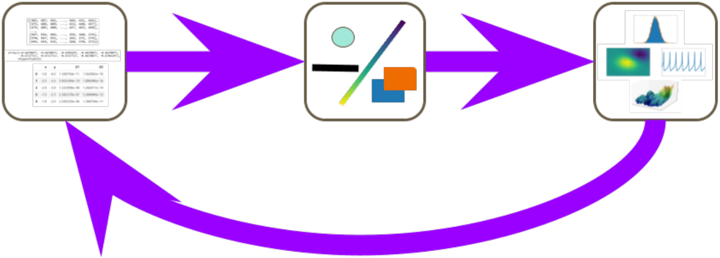
October 25, 2022

Hannah Aizenman, Tom Caswell, Michael Grossberg

What are we doing?

- develop a model for describing data to graphic transformations
- specify a visualization library architecture based on this model
- implement functional(ish) components based on this model using ideas from functional programming

What do visualization libraries do?



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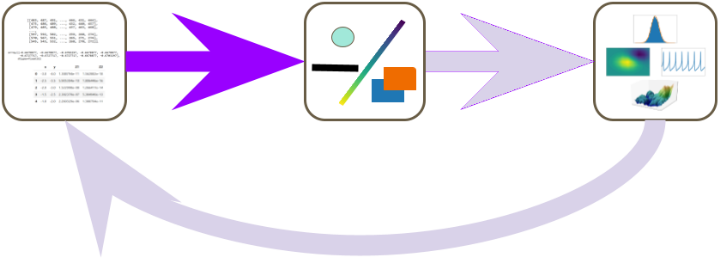

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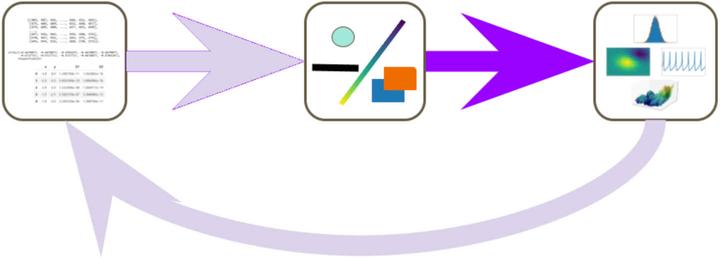

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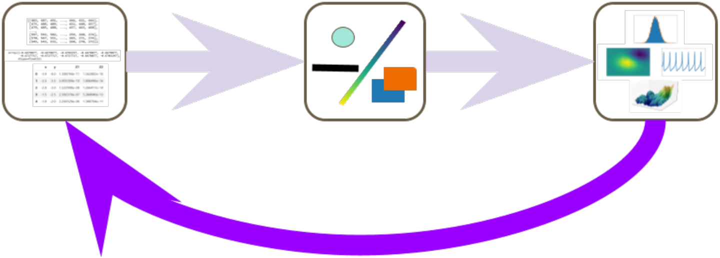
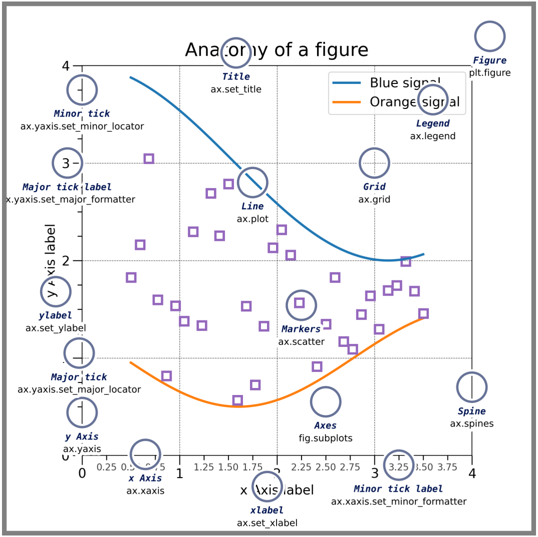
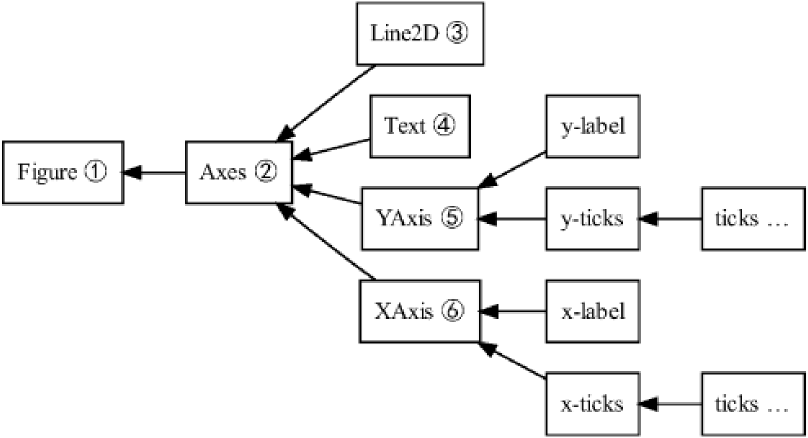


Matplotlib

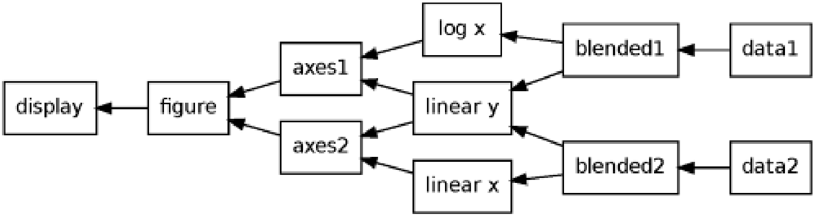
Everything is an Artist



Everything is an Artist



Transformations Change Artists into Different Coordinates



A Simple Artist

```python
class SomeArtist(Artist):
    'An example Artist that implements the draw method'

    def draw(self, renderer):
        # create some objects and use renderer to draw self here
        renderer.draw_path(graphics_context, path, transform)
```

Goals

How do we express structure?

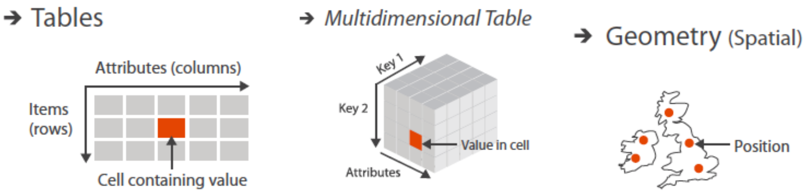
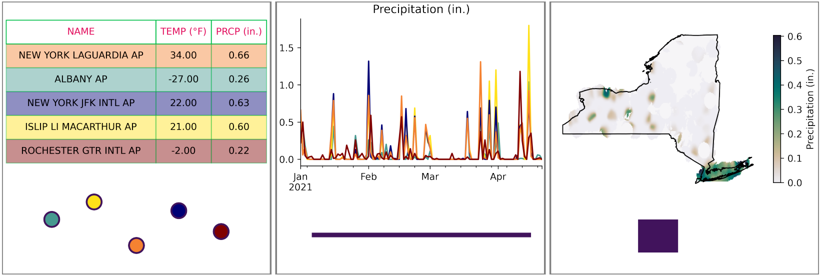


Figure: Figure 2.8 in Munzner's Visualization Analysis and Design[**munznerVisualizationAnalysisDesign2014**]

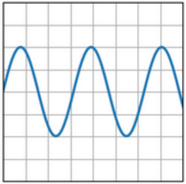
Continuity



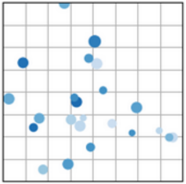
Topological Properties [**wilkinsonGrammarGraphics2005**]

how elements in a dataset are organized, e.g. discrete rows in a table, networked nodes, pixels in an image, points on a line

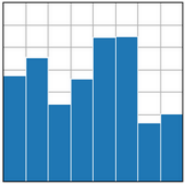
Visual Algorithms & Continuity



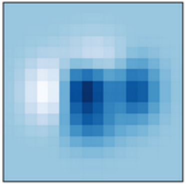
plot(x, y)



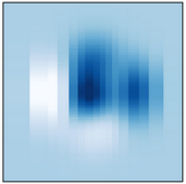
scatter(x, y)



bar(x, height) / barh(y, width)



imshow(Z)



pcolormesh(X, Y, Z)

contour(X, Y, Z)

Equivariance

What Retinal Variables & Marks: visual encodings should match properties of the data [**bertinSemiologyGraphicsDiagrams2011a**]

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Why Naturalness: easier to understand when properties match[**norman˙things˙smart**]

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Why Graphical Integrity: graphs show **only** the data[**tufteVisualDisplayQuantitative2001**]

Why Naturalness: easier to understand when properties match[**norman˙things˙smart**]

How Expressiveness: which structure preserving mappings can a tool implement[**mackinlayAutomatingDesignGraphical1986**]]

Domain Specific Library: library assumes structure [**HeerSoftware2006**]

DATE	LATITUDE	LONGITUDE	PRCP (in.)	NAME
2021-01-01	43.1167	-77.6767	0.2205	ROCHESTER GTR INTL AP
2021-01-01	41.5	-73.9	0.0000	STONYKILL NEW YORK
2021-01-01	42.7431	-73.8092	0.2598	ALBANY AP
2021-01-01	43.8	-73.7	0.0000	SCHROON LAKE NEW YORK
2021-01-01	43.0078	-73.6511	0.0000	SARA NEW YORK
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2021-01-01	40.6386	-73.7622	0.6299	NEW YORK JFK INTL AP
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ggplot[**wickhamGgplot2ElegantGraphics2016a**]
Vega[**satyanarayanDeclarativeInteractionDesign2014**]
Altair[**vanderplasAltairInteractiveStatistical2018**]
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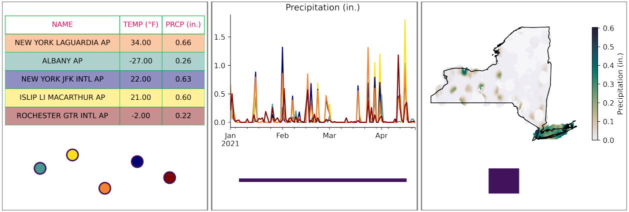
Building Block Library[**wongsuphasawatNavigatingWideWorld2021**]: visual algorithms assume structure [**toryRethinkingVisualizationHighlevel2004**]



1. Matplotlib[**hunterMatplotlib2DGraphics2007**] → Seaborn[**waskom2020seaborn**], xarray [**hoyer2017xarray**]
2. D3 [**bostockDataDrivenDocuments2011**]
3. VTK [**hanwellVisualizationToolkitVTK2015**, **geveciVTK2012**],MayaVi[**RamachandranMayaVI2011**]→ Titan[**brianwylieUnifiedToolkitInformation2009**], ParaView[**ahrens2005paraview**]

Design Composable Structure Preserving API

Fiber Bundles "unified, dimension-independent framework" that expresses data as the mapping between continuity and fields [**butlerVectorBundleClassesForm1992**, **butlerVisualizationModelBased1989**]

Category Theory Language express constraints in specifications [**wielsManagementEvolvingSpecifications1998**]

Sheaves on Bundles "algebraic data structure" for representing data over topological spaces [**ghristElementaryAppliedTopology2014**]

```
Artist(Data) -> Graphic
```

Category *Set*

Set_1 Set_2

Set_3

Category *Set*

$$\begin{tikzcd}
\mathbf{Set}_1 \arrow[loop left, "id_{\mathbf{Set}_1}"] & \mathbf{Set}_2 \arrow[loop above, "id_{\mathbf{Set}_2}"] \\
& \mathbf{Set}_3 \arrow[loop below, "id_{\mathbf{Set}_3}"]
\end{tikzcd}$$

Category $\textbf{\textit{Set}}$

$$\begin{tikzcd}
\textbf{\textit{Set}}_1 \arrow[loop left, "id_{\textbf{\textit{Set}}_1}"] \arrow[r, "r_{1,2}"] & \textbf{\textit{Set}}_2 \arrow[loop above, "id_{\textbf{\textit{Set}}_2}"] \arrow[d, "r_{2,3}"] \\
 & \textbf{\textit{Set}}_3 \arrow[loop below, "id_{\textbf{\textit{Set}}_3}"]
\end{tikzcd}$$

Category *Set*

$$
\begin{tikzcd}
\mathbf{Set}_1 \arrow[r, "r_{1,2}"] \arrow[loop left, "id_{\mathbf{Set}_1}"] \arrow[dr, "r_{1,2} \circ r_{2,3}"'] & \mathbf{Set}_2 \arrow[d, "r_{2,3}"] \arrow[loop above, "id_{\mathbf{Set}_2}"] \\
& \mathbf{Set}_3 \arrow[loop below, "id_{\mathbf{Set}_3}"]
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& \boldsymbol{Set}_3 \arrow[loop below, "id_{\boldsymbol{Set}_3}"]
\end{tikzcd}$$

associativity

if $r_{1,2} : \boldsymbol{Set}_1 \to \boldsymbol{Set}_2$,
$r_{2,3} : \boldsymbol{Set}_2 \to \boldsymbol{Set}_3$ and
$r_{3,4} : \boldsymbol{Set}_3 \to \boldsymbol{Set}_4$ then
$r_{3,4} \circ (r_{2,3} \circ r_{1,2}) = (r_{3,4} \circ r_{2,3}) \circ r_{1,2}$

Category *Set*

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$r_{3,4} \circ (r_{2,3} \circ r_{1,2}) = (r_{3,4} \circ r_{2,3}) \circ r_{1,2}$

identity

for every $r_{1,2} : \mathbf{Set}_1 \to \mathbf{Set}_2$ there
exists identity morphisms
$r_{1,2} \circ id_{\mathbf{Set}_1} = r_{1,2} = id_{\mathbf{Set}_2} \circ r_{1,2}$

Category $\textit{Set}^{op}$

$$\begin{tikzcd}
\textit{Set}_1 & \textit{Set}_2 \arrow[l] \\
& \textit{Set}_3 \arrow[ul] \arrow[u]
\end{tikzcd}$$

Presheaf Functor: $\mathcal{O} : \mathcal{C}^{op} \rightarrow \textbf{\textit{Set}}$

$$\begin{tikzcd} E \arrow[d, "\pi"'] \\ K \end{tikzcd}$$

Presheaf Functor: $\mathcal{O} : \mathcal{C}^{op} \to \textbf{\textit{Set}}$

$$\begin{tikzcd} E \arrow[d, "\pi"'] \\ K \arrow[u, bend right, "\tau \in \Gamma(K, E)"'] \end{tikzcd} \qquad \begin{array}{c} \Gamma(U_1, E\restriction_{U_1}) \\ \\ \\ U_1 \subset K \end{array}$$

Presheaf Functor: $\mathcal{O} : \mathcal{C}^{op} \rightarrow \boldsymbol{Set}$

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$$\Gamma(U_1, E\restriction_{U_1}) \in Ob(\boldsymbol{Set})$$

$$U_1 \in Ob(\mathcal{K}^{op})$$

Presheaf Functor: $\mathcal{O} : \mathcal{C}^{op} \to \mathbf{Set}$

$$
\begin{tikzcd}
E \arrow[d, "\pi"'] \\
K \arrow[u, bend right, "\tau \in \Gamma(K, E)"']
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
\Gamma(U_1, E\restriction_{U_1}) \in Ob(\mathbf{Set}) \\
U_1 \in Ob(\mathcal{K}^{op}) \arrow[u, mapsto, "\mathcal{O}_{K,E}"]
\end{tikzcd}
$$

Presheaf Functor: $\mathcal{O} : \mathcal{C}^{op} \to \textbf{\textit{Set}}$

$$\begin{tikzcd} E \arrow[d, "\pi"'] \\ K \arrow[u, bend right, "\tau \in \Gamma(K, E)"'] \end{tikzcd} \qquad \begin{tikzcd} \Gamma(U_1, E\restriction_{U_1}) \\ U_1 \arrow[u, mapsto, "\mathcal{O}_{K,E}"] \end{tikzcd} \qquad \begin{tikzcd} \Gamma(U_2, E\restriction_{U_2}) \\ U_2 \arrow[u, mapsto, "\mathcal{O}_{K,E}"'] \end{tikzcd}$$

Presheaf Functor: $\mathcal{O} : \mathcal{C}^{op} \to \textbf{Set}$

$$\begin{tikzcd} E \arrow[d, "\pi"'] \\ K \arrow[u, bend right, "\tau \in \Gamma(K, E)"'] \end{tikzcd} \qquad \begin{tikzcd} \Gamma(U_1, E\restriction_{U_1}) & & \Gamma(U_2, E\restriction_{U_2}) \arrow[ll, hook', "\iota^*"'] \\ U_1 \arrow[u, mapsto, "\mathcal{O}_{K,E}"] \arrow[rr, hook, "\iota"'] & & U_2 \arrow[u, mapsto, "\mathcal{O}_{K,E}"'] \end{tikzcd}$$

Presheaf Functor: $\mathcal{O} : \mathcal{C}^{op} \to \boldsymbol{Set}$

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stalk

$$\mathcal{O}_{K,E}\restriction_K := \lim_{U \ni K} \Gamma(U, E\restriction_U)$$

$$F_K \subset \mathcal{O}_{K,E}\restriction_K$$

Presheaf Functor: $\mathcal{O} : \mathcal{C}^{op} \to \mathbf{Set}$

$$\begin{tikzcd} F \arrow[r, hook] & E \arrow[d, "\pi"'] \\ & K \arrow[u, bend right, "\tau \in \Gamma(K,\, E)"'] \end{tikzcd}
\qquad
\begin{tikzcd} \Gamma(U_1,\, E\restriction_{U_1}) & & \Gamma(U_2,\, E\restriction_{U_2}) \arrow[ll, hook', "\iota^*"'] \\ U_1 \arrow[u, mapsto, "\mathcal{O}_{K,E}"] \arrow[rr, hook, "\iota"'] \arrow[r, phantom, "\mathcal{O}_{K,E}"] & {} & U_2 \arrow[u, mapsto, "\mathcal{O}_{K,E}"'] \end{tikzcd}$$

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$$\mathcal{O}_{K,E}\restriction_K := \lim_{U \ni K} \Gamma(U, E\restriction_U)$$

$$F_K \subset \mathcal{O}_{K,E}\restriction_K$$

germ

$$\tau(K) \in \mathcal{O}_{K,E}\restriction_K$$

Sheaves on Bundles

A sheaf is a presheaf that satisfies the following two axioms[**bakerMathsSheaf**]

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given $U = \bigcup_{i \in I} U_i$ and $\tau^a, \tau^b \in \mathcal{O}(U)$,

Sheaves on Bundles

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locality

given $U = \bigcup_{i \in I} U_i$ and $\tau^a, \tau^b \in \mathcal{O}(U)$,

if $\tau^a\restriction_{U_i} = \tau^b\restriction_{U_i}$ for each $U_i \in U$ then $\tau^a = \tau^b$

Sheaves on Bundles

A sheaf is a presheaf that satisfies the following two axioms[**bakerMathsSheaf**]

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given $U = \bigcup_{i \in I} U_i$ and $\tau^a, \tau^b \in \mathcal{O}(U)$,

gluing

given $\tau^i \in \mathcal{O}(U_i)$

Sheaves on Bundles

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given $\tau^i \in \mathcal{O}(U_i)$ s.t. $\tau^i\restriction_{U_i \cap U_j} = \tau^j\restriction_{U_i \cap U_j}$ for $U_i, U_j \in U$,

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gluing

given $\tau^i \in \mathcal{O}(U_i)$ s.t. $\tau^i\restriction_{U_i \cap U_j} = \tau^j\restriction_{U_i \cap U_j}$ for $U_i, U_j \in U$, there exists $\tau \in \mathcal{O}(U)$ such that $\tau\restriction_{U_i} = \tau^i$

Data

$$\begin{tikzcd} U \arrow[d, mapsto, "\mathcal{O}_{K,E}"] \\ \Gamma(U, E\restriction_U) \end{tikzcd}$$

- $F \hookrightarrow E \xrightarrow{\pi} K$
- $\mathcal{O}_{K,E} : U \mapsto \Gamma(U, E\restriction_U), U \subseteq K$
- $\tau : U \to F\restriction_U \in \Gamma(U, E\restriction_U)$
- $\tau(K) = \{f_0 : v_0, \cdots ,\}, K \in U$

Graphic

$$\begin{tikzcd} \Gamma(W, H\restriction_W) \\ W \arrow[u, mapsto, "\mathcal{O}_{S,H}"] \end{tikzcd}$$

- $D \hookrightarrow H \xrightarrow{\pi} S$
- $\mathcal{O}_{S,H} : W \mapsto \Gamma(W, E\restriction_W), W \subseteq S$
- $\rho : W \to D\restriction_W \in \Gamma(W, H\restriction_W)$
- $\rho(S) = \{d_0, \cdots\}, S \in W$

Function: $\xi : S \to K$

$$
\begin{tikzcd}
\Gamma(W, H\restriction_W) & & \\
\overbrace{U \times [0,1]^m}^{W} \arrow[u, mapsto, "\mathcal{O}_{S,H}"] \arrow[rr, "\xi"] & & U \arrow[d, mapsto, "\mathcal{O}_{K,E}"] \\
& & \Gamma(U, E\restriction_U)
\end{tikzcd}
$$

Pullback: data to region of the visualization

$$\begin{tikzcd}
W \arrow[r, "\xi"] \arrow[d, dashed, mapsto, "\mathcal{O}_{S,\xi^* E}"'] & U \arrow[d, mapsto, "\mathcal{O}_{K,E}"] \\
\Gamma(W, \xi^* E\restriction_W) & \Gamma(U, E\restriction_U) \arrow[l, "\xi^*"]
\end{tikzcd}$$

- $\xi^* F \hookrightarrow \xi^* E \xrightarrow{\pi} S$
- $\xi^* \mathcal{O}_{K,E} : W \mapsto \Gamma(W, \xi^* E\restriction_W), W \subseteq S$
- $\xi^* \tau : W \to \xi^* F\restriction_W \in \Gamma(W, \xi^* E\restriction_W)$
- $\xi^* \tau(S) = \tau(\xi(S)) = \tau(K)$

Pushforward: visualization to index of data

$$
\begin{tikzcd}
\Gamma(W, H\restriction_W) \arrow[r, "\xi_*"] & \Gamma(U, \xi_* H\restriction_U) \\
W \arrow[u, mapsto, "\mathcal{O}_{S,H}"] \arrow[r, "\xi"] & U \arrow[u, mapsto, dashed, "\mathcal{O}_{K,\xi_* H}"']
\end{tikzcd}
$$

- $\xi_* D \hookrightarrow \xi_* H \xrightarrow{\pi} K$
- $\xi_* \mathcal{O}_{S,H} : U \mapsto \Gamma(U, \xi_* H\restriction_U), U \subseteq K$
- $\xi_* \rho : U \to \xi_* D\restriction_U \in \Gamma(U, \xi_* H\restriction_U)$
- $\xi_* \rho(K) = \rho\restriction_{\xi^{-1}(K)} = \rho(S) \; \forall \; S \in \xi^{-1}(K)$

$$Hom_S(\mathcal{O}_{S,\xi^*E}, \mathcal{O}_{S,H}) = Hom_K(\mathcal{O}_{K,E}, \mathcal{O}_{K,\xi_* H})$$

$$\begin{tikzcd}
\mathcal{O}_{S,H} \arrow[r, "\xi_*"] & \mathcal{O}_{K,\xi_* H} \\
\mathcal{O}_{S,\xi^* E} \arrow[u, "A_S \in Hom_S"] & \mathcal{O}_{K,E} \arrow[l, "\xi^*"] \arrow[u, "A_K \in Hom_K"'] \arrow[ul, dashed, "A \in Hom_{K,S}"]
\end{tikzcd}$$

Data Space: $A_K : \mathcal{O}_{K,E} \Rightarrow \mathcal{O}_{K,\xi_* H}$

$$\Gamma(U, E\restriction U) \xleftarrow{\mathcal{O}_{K,E}} U \xmapsto{\mathcal{O}_{K,\xi_* H}} \Gamma(U, \xi_* H\restriction U)$$

Display Space: $A_S : \mathcal{O}_{S,\xi^* E} \Rightarrow \mathcal{O}_{S,H}$

$$\Gamma(U_1, E\restriction_{U_1}) \xrightarrow{A_{U_1}} \Gamma(U, \xi_* H\restriction_{U_1})$$

$$\Gamma(U_2, E\restriction_{U_2}) \xrightarrow{A_{U_2}} \Gamma(U, \xi_* H\restriction_{U_2})$$

$$\Gamma(U, E\restriction U) \xleftarrow{\mathcal{O}_{K,E}} U \xrightarrow{\mathcal{O}_{K,\xi_* H}} \Gamma(U, \xi_* H\restriction U)$$

$$
\begin{tikzcd}
U_1 & \Gamma(U_1, E\restriction_{U_1}) \arrow[r, "A_{U_1}"] & \Gamma(U, \xi_* H\restriction_{U_1}) \\
U_2 \arrow[u, hook, "\iota"] & \Gamma(U_2, E\restriction_{U_2}) \arrow[r, "A_{U_2}"] & \Gamma(U, \xi_* H\restriction_{U_2})
\end{tikzcd}
$$

$$
\Gamma(U, E\restriction U) \xleftarrow{\ \mathcal{O}_{K,E}\ }\!\!\dashv \; U \; \vdash\!\!\xrightarrow{\ \mathcal{O}_{K,\xi_* H}\ } \Gamma(U, \xi_* H\restriction U)
$$

$$
\begin{tikzcd}
U_1 \\
U_2 \arrow[u, hook, "\iota"]
\end{tikzcd}
\qquad
\begin{tikzcd}
\Gamma(U_1, E\restriction_{U_1}) \arrow[rr, "A_{U_1}"] \arrow[d, "\iota^*"'] \arrow[drr, phantom, "\overset{A_{\mathcal{O}}}{\cdots\cdots\rightarrow}"] & & \Gamma(U, \xi_* H\restriction_{U_1}) \arrow[d, "\iota^*"] \\
\Gamma(U_2, E\restriction_{U_2}) \arrow[rr, "A_{U_2}"] & & \Gamma(U, \xi_* H\restriction_{U_2})
\end{tikzcd}
$$

$$
\begin{tikzcd}
\Gamma(U, E\restriction U) & U \arrow[l, mapsto, "\mathcal{O}_{K,E}"'] \arrow[r, mapsto, "\mathcal{O}_{K,\xi_* H}"] & \Gamma(U, \xi_* H\restriction U)
\end{tikzcd}
$$

Artist: $A : \tau \mapsto \rho$

$$\mathcal{O}_{K,E} \xRightarrow{\quad A_K \quad} \mathcal{O}_{K,\xi_* H}$$

$$\begin{tikzcd}
U_1 & \Gamma(U_1, E\restriction_{U_1}) \arrow[r, "A_{U_1}"] \arrow[d, "\iota^*"'] & \Gamma(U, \xi_* H\restriction_{U_1}) \arrow[d, "\iota^*"] \\
& {} \arrow[r, dotted, "A_{\mathcal{O}}"] & {} \\
U_2 \arrow[uu, hook, "\iota"] & \Gamma(U_2, E\restriction_{U_2}) \arrow[r, "A_{U_2}"] & \Gamma(U, \xi_* H\restriction_{U_2})
\end{tikzcd}$$

$$\Gamma(U, E\restriction U) \xleftarrow{\ \mathcal{O}_{K,E}\ } U \xmapsto{\ \mathcal{O}_{K,\xi_* H}\ } \Gamma(U, \xi_* H\restriction U)$$

$$\mathcal{O}_{S,\xi^*E} \overset{A_S}{\Longrightarrow} \mathcal{O}_{S,H}$$

$$\begin{tikzcd}
W_1 & \Gamma(W_1, \xi^*E\restriction_{W_1}) \arrow[r, "A_{W_1}"] \arrow[d, "\iota^*"'] \arrow[dr, phantom, "\overset{A_{\mathcal{O}}}{\dashrightarrow}"] & \Gamma(W, H\restriction_{W_1}) \arrow[d, "\iota^*"] \\
W_2 \arrow[u, hook, "\iota"] & \Gamma(W_2, \xi^*E\restriction_{W_2}) \arrow[r, "A_{W_2}"] & \Gamma(W, H\restriction_{W_2})
\end{tikzcd}$$

$$\Gamma(W, \xi^*E\restriction W) \xleftarrow{\ \mathcal{O}_{S,\xi^*E}\ } W \xmapsto{\ \mathcal{O}_{S,H}\ } \Gamma(U, H\restriction W)$$

$$\begin{tikzcd}
\Gamma(W, H\restriction_W) \arrow[r, "\xi_*"] & \Gamma(U, \xi_* H\restriction_U) \\
\Gamma(W, \xi^* E\restriction_W) \arrow[u, "A_S"] & \Gamma(U, E\restriction_U) \arrow[l, "\xi^*"] \arrow[u, "A_K"'] \arrow[ul, "A"]
\end{tikzcd}$$

pull data section τ over graphic space $S \in S$

$$\xi^* \tau(S) = \tau(\xi(S)) = \tau(K)$$

push graphic ρ section over data space $K \in K$

$$\xi_* \rho(K) = \rho\restriction_{\xi^{-1}(K)} = \rho(S) \forall S \in \xi^{-1}(K)$$

Reachable ρ?

$$\begin{tikzcd} \Gamma(W, H\restriction_W) \\ W \arrow[u, mapsto, "\mho_{S,H}"] \end{tikzcd}$$

Reachable ρ?

$$\begin{tikzcd}
\Gamma(W, H\restriction_W) & \supset & Im_A(W, H\restriction_W) \\
W \arrow[u, mapsto, "\mathcal{O}_{S,H}"] \arrow[urr, dashed, mapsto, "A(\mathcal{O}_{K,E})"'] & &
\end{tikzcd}$$

Output Subtype

$$Im_A(W, H\restriction_W) = \{\rho | \, \exists \, \tau \in \Gamma(U, E\restriction_U) \; s.t. \; A(\tau) = \rho, \; \xi(W) = U\}$$

Expressing Equivariance via Morphisms ϕ

Fiber Category

The fiber F is a monoidal category (single object w/ bicartesian product operator on category) of an arbitrary type $\mathcal{C}$. The morphisms on the fiber are $\tilde{\phi} \in Hom(F, F)$ The category is equipped with the bifunctor $\otimes : F \times F \to F$

Fiber Bundle Category

object $F \hookrightarrow E \xrightarrow{\pi} K$

morphisms $\phi : (\hat{\phi}, \tilde{\phi})$

$\phi = (\hat{\phi}, \tilde{\phi})$

$$\begin{tikzcd} \Gamma(U, E\restriction_U) \arrow[r, "\hat{\phi}^*"] & \Gamma(U', \hat{\phi}^* E\restriction_{U'}) \\ U \arrow[u] & U' \arrow[l, "\hat{\phi}"'] \arrow[u] \end{tikzcd} \qquad \Gamma(U', \hat{\phi}^* E\restriction_{U'}) \xrightarrow{\tilde{\phi}} \Gamma(U', \hat{\phi}^* E\restriction_{U'})$$

Base Transformation: $\hat{\phi} : U' \to U$ where $U, U' \subseteq K$,
$\hat{\phi}^* \tau\restriction_U : \tau \mapsto \tau\restriction_U \circ \hat{\phi}$

Fiber Transformation:
$\tilde{\phi} : \hat{\phi}^* E_{K'} \to \hat{\phi}^* E_{K'} \in Hom(\hat{\phi}^* F\restriction_K, \hat{\phi}^* F\restriction_K), K' \in U'$
$\tilde{\phi} : \hat{\phi}^* \tau\restriction_U \mapsto \hat{\phi}^* \tau'\restriction_U,\ \tau, \tau' \in \Gamma(U', \hat{\phi}^* E\restriction_{U'})$

Section Transform: $\phi : \tau\restriction_U \mapsto \hat{\phi}^* \tau'\restriction_U$

Equivariant Artist

(A, A') are equivariant with respect to ϕ_E if a compatible transform ϕ_H can be defined such that

$$
\begin{tikzcd}
\Gamma(U, E\restriction_U) \arrow[rrr, "A"] \arrow[d, "\hat{\phi}^*{}_E"'] & & & Im_A(W, H\restriction_W) \arrow[d, dotted, "\hat{\phi}^*{}_H"] \\
\Gamma(U', \hat{\phi}^*{}_E E\restriction_{U'}) \arrow[dd, "\tilde{\phi}_E"'] & U & W \arrow[l, "\xi"'] & Im_A(W', \hat{\phi}^*{}_H H\restriction_{W'}) \arrow[dd, dotted, "\tilde{\phi}_H"] \\
 & U' \arrow[u, "\hat{\phi}_E"] & W' \arrow[l, "\xi"'] \arrow[u, dotted, "\hat{\phi}_H"'] & \\
\Gamma(U', E'\restriction_{U'}) \arrow[rrr, "A'"] & & & Im_A(W', H'\restriction_{W'})
\end{tikzcd}
$$

For all points $S' \in S'$:

$$A'(\tilde{\phi}_E(\tau(\hat{\phi}_E(\xi(S'))))) = \tilde{\phi}_H(A(\tau(\xi(\hat{\phi}_H(S')))))$$

Testing if A is equivariant

M is a (scaler, vector) measurable component (e.g. color, position, shape, texture, rotation,) of the rendered visual element.

$$
\begin{tikzcd}
\Gamma(U, E\restriction_U) \arrow[r, "A"] \arrow[d, "\eta"'] & Im_A(W, H\restriction_W) \arrow[r, "render"] \arrow[d, "\delta"] & visualization \arrow[dl, "measure"] \\
Hom(U, M) \arrow[r, "\xi^*"'] & Hom(W, M) &
\end{tikzcd}
$$

input $\eta : \tau \mapsto (U \xrightarrow{\eta_\tau} M)$

output $\delta : \rho \mapsto (W \xrightarrow{\delta_\rho} M)$

$\eta_\tau(K) = \delta_\rho(S)$ for all $\xi(S) = K, K \in K, S \in S$

Visual Measurement

$$
\begin{tikzcd}
M \arrow[r, "\tilde{\Phi}_M"] & M' \\
W \arrow[u, "\delta_{\rho}"] & W' \arrow[l, "\hat{\Phi}_H"] \arrow[u, "\delta_{\tilde{\Phi}_H \rho}"'] \arrow[ul, "\delta_{\hat{\Phi}^* \rho}"]
\end{tikzcd}
$$

$\delta_\rho \in Hom(W, M)$ is a mapping from graphic to visual measurement. δ_ρ is a map from an openset $W \subseteq S$ to a measurement M_W that corresponds to the graphic representation at that region $\rho\restriction_W$ and the corresponding data $\tau\restriction_{\xi\restriction_W}$.

Visual Measurement

$$
\begin{tikzcd}
M \arrow[r, "\tilde{\phi}_M"] & M' \\
W \arrow[u, "\delta_\rho"] & W' \arrow[l, "\hat{\phi}_H"] \arrow[ul, "\delta_{\hat{\phi}^*\rho}"] \arrow[u, "\delta_{\tilde{\phi}_H\rho}"']
\end{tikzcd}
\qquad
\begin{tikzcd}
M \arrow[r, "\tilde{\phi}_M"] & M' \\
U \arrow[u, "\eta_\tau"] & U' \arrow[l, "\hat{\phi}_E"] \arrow[ul, "\eta_{\hat{\phi}^*\tau}"] \arrow[u, "\eta_{\tilde{\phi}_E\tau}"']
\end{tikzcd}
$$

$\eta_\tau \in Hom(U, M)$ is a mapping from data to visual measurement.
$\eta_\tau : \tau \mapsto$ is a map from an openset $U \subseteq K$ to a measurement M_U that corresponds to the data record at that region $\tau\restriction_U$ and the corresponding graphic $\rho\restriction_{\xi^{-1}\restriction_U}$

Using output ρ to check if A is equivariant

$$
\begin{tikzcd}
\Gamma(U, E\restriction_U) \arrow[r, "A"] \arrow[d, "\phi_E"'] & Im_A(W, H\restriction_W) \arrow[r, "\delta"] \arrow[d, dotted, "\phi_H"'] & Hom(W, M) \arrow[d, "\phi_M"] \\
\Gamma(U', E'\restriction_{U'}) \arrow[r, "A"] & Im_A(W', H'\restriction_{W'}) \arrow[r, "\delta"] & Hom(W', M')
\end{tikzcd}
$$

$$
\begin{aligned}
A'(\tilde{\phi}_E(\tau(\hat{\phi}_E(\xi(S'))))) &= \tilde{\phi}_H(A(\tau(\xi(\hat{\phi}_H(S'))))) \\
\delta(\tilde{\phi}_H(\rho(\hat{\phi}_H)))(S') &= \phi_M(\delta(\rho))(S') = \delta_{\tilde{\phi}_H\rho}(S')
\end{aligned}
$$

Using input τ to check if A is equivariant

$$
\begin{tikzcd}
\Gamma(U, E\restriction_U) \arrow[r, "A"] \arrow[d, "\phi_E"'] \arrow[rr, bend left, "\eta"] & Im_A(W, H\restriction_W) \arrow[r, "\xi^{-1}\circ\delta"] \arrow[d, dotted, "\phi_H"'] & Hom(U, M) \arrow[d, "\phi_M"] \\
\Gamma(U', E'\restriction_{U'}) \arrow[r, "A"] \arrow[rr, bend right, "\eta"'] & Im_A(W', H'\restriction_{W'}) \arrow[r, "\xi^{-1}\circ\delta"] & Hom(U', M')
\end{tikzcd}
$$

equivariance $\eta(\tilde{\phi}_E(\tau(\hat{\phi}_E)))(K') = \phi_M(\eta(\tau))(K') = \eta_{\tilde{\phi}_E\tau}(K')$ for all $K' \in K'$

continuity $\lim_{x\to K} \eta_\tau(x) = \eta_\tau(K)$ for all $K \in K$

Complex data

combining continuities given $K_a \in K_c \subset K_a$ and $K_b \in K_c \subset K_b$, if $K_a = K_b$ then $\eta(\tau(K_a)) = \eta(\tau(K_b))$ for all $K_a, K_b \in K_a \bigsqcup_{K_c} K_b$

shared fibers if $p = p'$ then $\eta \circ q = \eta' \circ q'$

$$\begin{tikzcd}
& F^a \times F^b \arrow[dl, "p"'] \arrow[dr, "q\circ\eta"] & \\
F^a & & M^a \\
& F^a \times F^c \arrow[ul, "p'"] \arrow[ur, "q'\circ\eta'"'] &
\end{tikzcd}$$

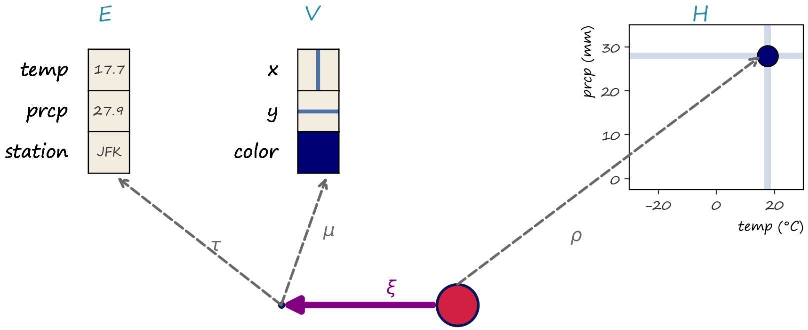
both given $\tau^d \in \mathcal{O}_{K_d, E_d}, \tau^e \in \mathcal{O}_{K_e, E_e}$ and $K \in K_d \bigsqcup_{K_f} K_e$ then $q(\eta(\tau^d(K))) = q'(\eta(\tau^e(K)))$ when $p(F^d \restriction_K) = p'(F^e \restriction_K)$

Composable $\phi = (\hat{\phi}, \prod_{i=0}^{n} \tilde{\phi}_i)$

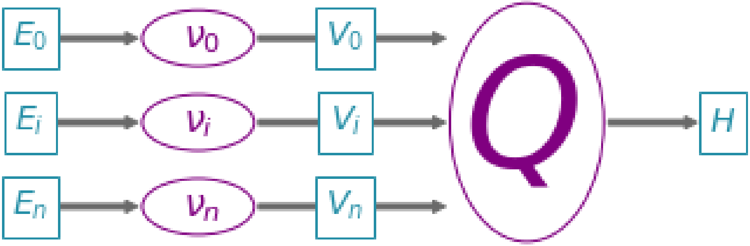
$$\begin{tikzcd}
E_a \arrow[d, "\phi a"'] & E_a \times E_b \arrow[l, "\pi_a"'] \arrow[r, "\pi_b"] \arrow[d, "\phi_{a,b}"] & E_b \arrow[d, "\phi_b"] \\
E_a & E_a \times E_b \arrow[l, "\pi_a"'] \arrow[r, "\pi_b"] & E_b
\end{tikzcd}$$

if there exists functions $\phi_{a,b} : E_a \times E_b \to E_a \times E_b$, $\phi_a : E_a \to E_a$ and $\phi_b : E_b \to E_b$ s.t. $\pi_a \circ \phi_a = \phi_{a,b} \circ \pi_a$ and $\pi_b \circ \phi_b = \phi_{a,b} \circ \pi_b$ then $\phi_{a,b} = (\phi_a, \phi_b)$

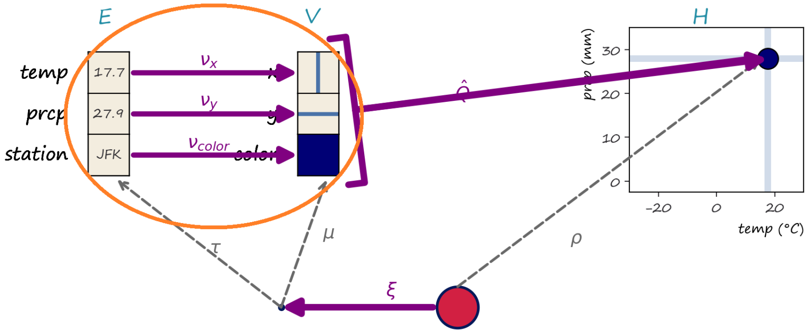
How Do We Get From Data to Graphic?



Building an equivariant A?



Data to Measurable Components



Typed Measurable Visual Components: V

$$\begin{tikzcd} P \arrow[r, hook] & V \arrow[d, "\pi"'] \\ & K \arrow[u, bend right, "\mu \in \Gamma(U, V\restriction_U)"'] \end{tikzcd}$$

Data V	continuity + visual fields
Continuity K	data continuity
components P	visual components **[bertinIIPropertiesGraphic2011]** of a graphic, e.g. x and y position, area, color, line thickness

Data to Visual Transformation $\nu : F_K \mapsto P_K$

$\pi(E) = \pi(\nu(E))$ and ν is composable s.t

$$\begin{tikzcd}
F_K \arrow[r, "\nu"] \arrow[d, "\tilde{\phi}_E"'] \arrow[dr, phantom, "\dashrightarrow"] & P_K \arrow[d, "\tilde{\phi}_V"] \\
F'_K \arrow[r, "\nu'"'] & P'_K
\end{tikzcd}
\qquad
\begin{tikzcd}
F_K \arrow[r, "\nu_E"] \arrow[dr, dashed, "\nu_{E,V}"'] & P_K \coloneqq {F_K}^V \arrow[d, "\nu_V"] \\
 & {P_K}^V
\end{tikzcd}$$

$\nu : \phi_E \to \phi_V$: Stevens' Scales [stevensTheoryScalesMeasurement1946]

scale	**group**	**constraint**
nominal	permutation	if $r_1 \neq r_2$ then $\nu(r_1) \neq \nu(r_2)$
ordinal	monotonic	if $r_1 \leqslant r_2$ then $\nu(r_1) \leqslant \nu(r_2)$
interval	translation	$\nu(r + c) = \nu(r) + \nu(c)$
ratio	scaling	$\nu(r * c) = \nu(r) * \nu(c)$

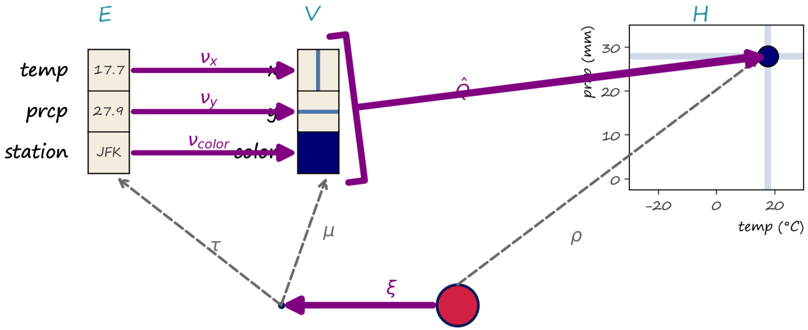
Shared Components: $\nu = \prod_{i=0}^{n} \nu_i$

$$\begin{tikzcd}
F_a \times F_b \arrow[rr, "\nu"] \arrow[dr, "p_F"] & & P_a \times P_b \arrow[dr, "p_P"] & \\
& F_a \arrow[rr, dashed, "\nu_a"] & & P_a \\
F_a \times F_c \arrow[rr, "\nu'"] \arrow[ur, "p_{F'}"'] & & {P_a}' \times {P_c}' \arrow[ur, "p_{P'}"'] &
\end{tikzcd} \tag{1}$$

Consistent Transformations [**hullmanKeeping2018**]

if $p_F = p_{F'}$ then $p_P(\nu(\tau)) = p_{P'}(\nu'(\tau'))$ s.t. there exists a transformation $\nu_a : F_a \to P_a$

Components to Graphic



Assembly Q

$$
\begin{tikzcd}
\Gamma(U, V\restriction_U) \arrow[rrr, "Q"] \arrow[ddd, "\tilde{\phi}_V \circ \hat{\phi}^*{}_V"'] & & & Im_A(W, H\restriction_W) \arrow[ddd, dotted, "\tilde{\phi}_H \circ \hat{\phi}^*{}_H"] \\
& U & W \arrow[l, "\xi"'] & \\
& U' \arrow[u, "\hat{\phi}_E"] & W' \arrow[l, "\xi"'] \arrow[u, dotted, "\hat{\phi}_H"'] & \\
\Gamma(U', V'\restriction_{U'}) \arrow[rrr, "Q'"] & & & Im_A(W', H'\restriction_{W'})
\end{tikzcd}
$$

equivariance

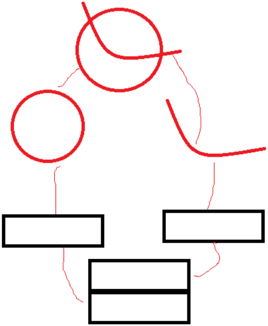
$$Q'(\tilde{\phi}_V(\mu(\hat{\phi}_E(\xi(S'))))) = \tilde{\phi}_H(Q(\mu(\xi(\hat{\phi}_H(S')))))$$

Combining Qs

The codomain of all Q targeting the same output space is the bundle H and $Hom(S_1, H) + Hom(S_2, H) = Hom(S_1 + S_2, H)$; therefore

$$\bigsqcup_i Q_i(\Gamma(U_i, E_i\restriction_{U_i})) = \Gamma(\bigsqcup_i W_i, H\restriction_{\sqcup_i W_i})$$

when $\xi(W_i) = U_i$



Compatible Qs

$$
\begin{tikzcd}
 & \Gamma(U, V^a\upharpoonright_U) \arrow[dd, "\nu^{a\to b}"] \arrow[dr, "{Q_U}^a"] \arrow[rr, bend left, "Q^a"] & & \\
\Gamma(U, E\upharpoonright_U) \arrow[ur, "\nu^a"] \arrow[dr, "\nu^b"'] & & Im_A(U, \xi_* H\upharpoonright_U) & Im_A \arrow[l, "\xi_*"'] \\
 & \Gamma(U, V^b\upharpoonright_U) \arrow[ur, "{Q_U}^b"'] \arrow[rr, bend right, "Q^b"'] & &
\end{tikzcd}
$$

Implementation Choices: $A_K = A_S$

$$
\begin{tikzcd}
\Gamma(W, \xi^* E\restriction_W) \arrow[r, "\nu_S"] \arrow[rr, bend left, "A_S"] & \Gamma(W, \xi^* V\restriction_W) \arrow[r, "Q_S"] & Im_A(W, H\restriction_W) \arrow[d, "\xi_*"] \\
\Gamma(U, E\restriction_U) \arrow[u, "\xi^*"] \arrow[r, "\nu_K"] \arrow[rr, bend right, "A_K"'] & \Gamma(U, V\restriction_U) \arrow[u, "\xi^*"] \arrow[r, "Q_K"] & Im_A(U, \xi_* H\restriction_U)
\end{tikzcd}
$$