

Topologically Equivariant Artist Model

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About

Me

- PhD candidate in Computer Science
- Matplotlib Community Manager & Core Developer
- socials/github: story645

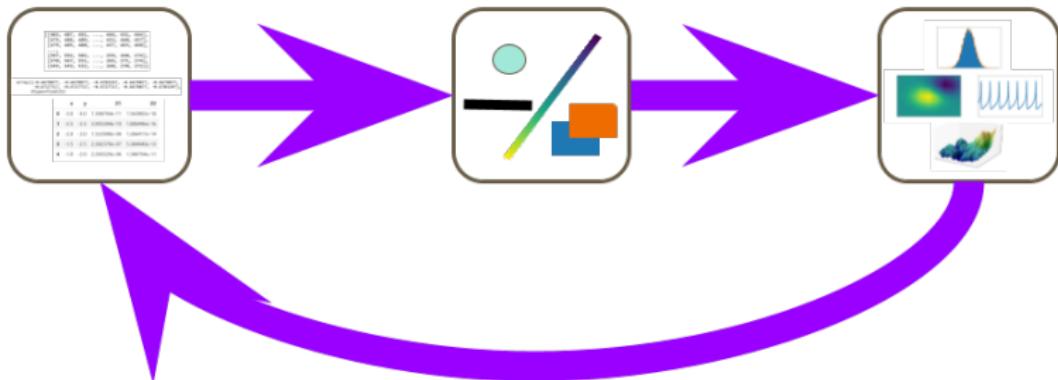
Project

- Funded by Chan Zuckerberg Initiative EOSS 1 & 3

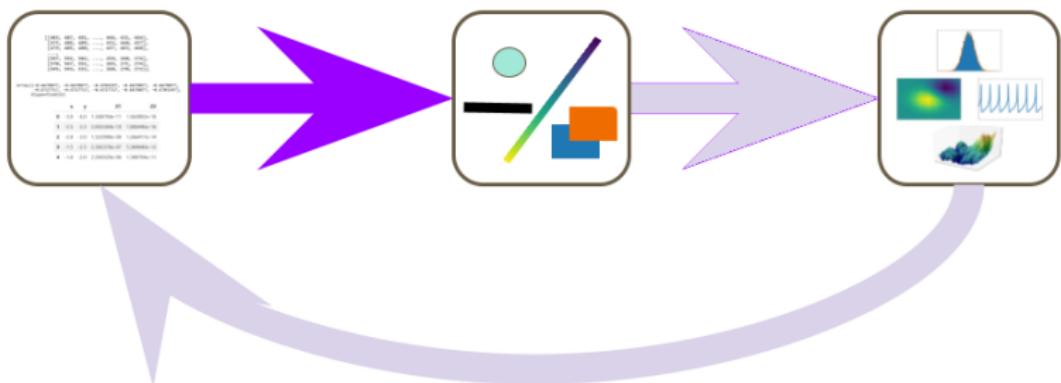
Goals

- uniform API for arbitrary datasets in arbitrary data containers
- generalizable methodology for expressing structure preservation
- framework that translates into coding specifications

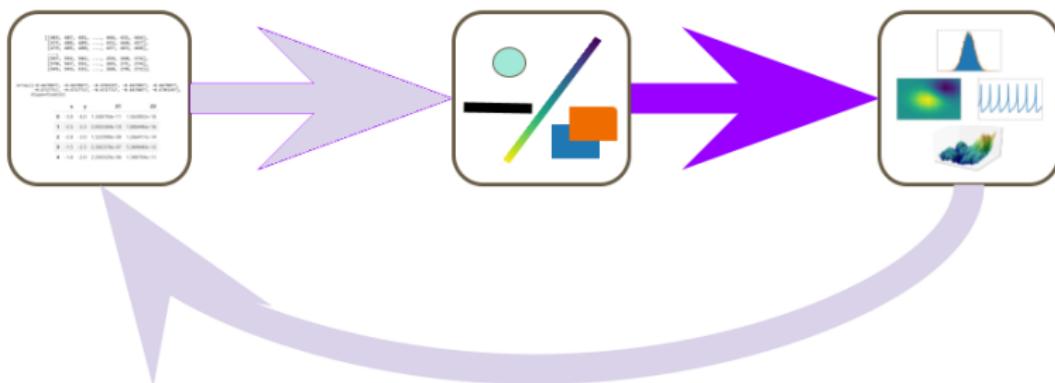
What do visualization libraries do?



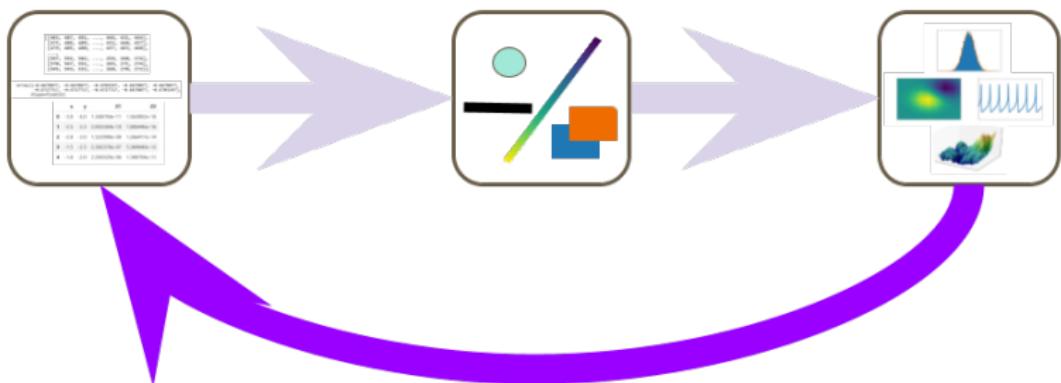
What do visualization libraries do?



What do visualization libraries do?



What do visualization libraries do?





How do we express structure?

field is a set of values of the same type, e.g. one column of a table or the pixels of an image

topology is the connectivity and relative positioning of elements in a dataset [1].

Structure Preservation: Equivalent scales and transforms

Definition

[2] An **action** of $G = (G, \circ, e)$ on X is a function $act : G \times X \rightarrow X$.
An action has the properties of identity $act(e, x) = x$ for all $x \in X$
and associativity $act(g, act(f, x)) = act(f \circ g, x)$ for $f, g \in G$.

Fields: Equivariance

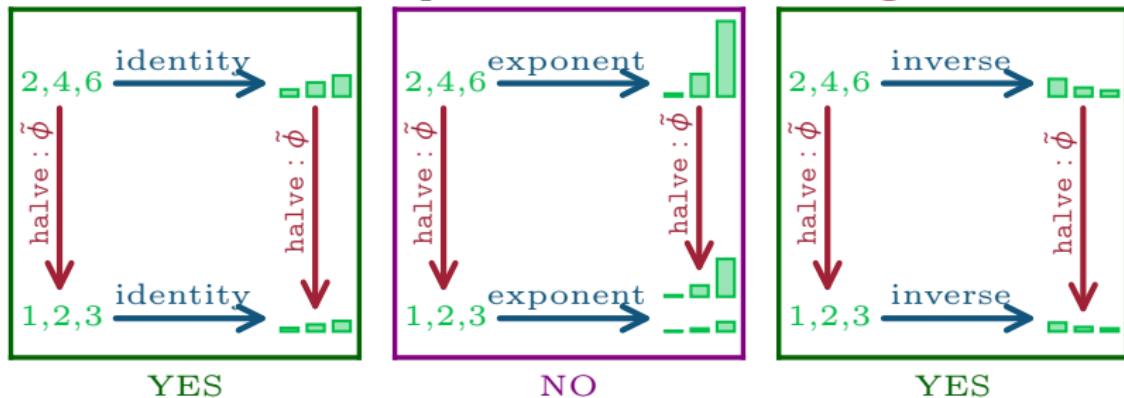
Given a group G that acts on both the input X and the output Y of a function $f : X \rightarrow Y$

Definition

A function f is **equivariant** when $f(\text{act}(g, x)) = \text{act}(g, f(x))$ for all g in G and for all x in X [3]

Fields: Equivariance

Is $\text{data} \mapsto \square$ equivariant w.r.t. scaling?



Fields: Homomorphism

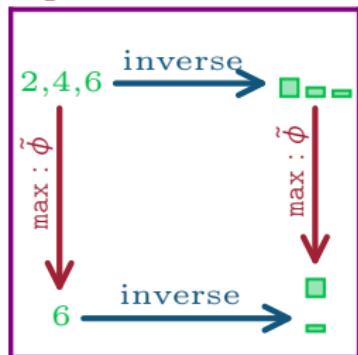
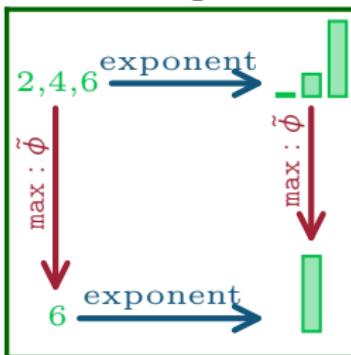
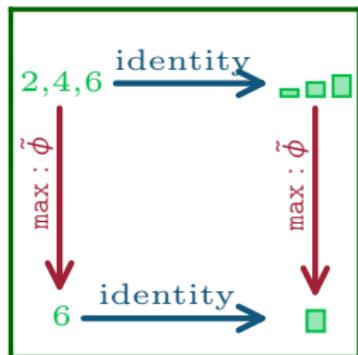
Given the function $f : X \rightarrow Y$, with operators (X, \circ) and $(Y, *)$

Definition

A function f is **homomorphic** when $f(x_1 \circ x_2) = f(x_1) * f(x_2)$ and preserves identities $f(I_x) = I_y$ all $x, y \in X$ [2]

Fields: Homomorphism

Is $\text{data} \mapsto \square$ homomorphic w.r.t. partial order?



YES

YES

NO

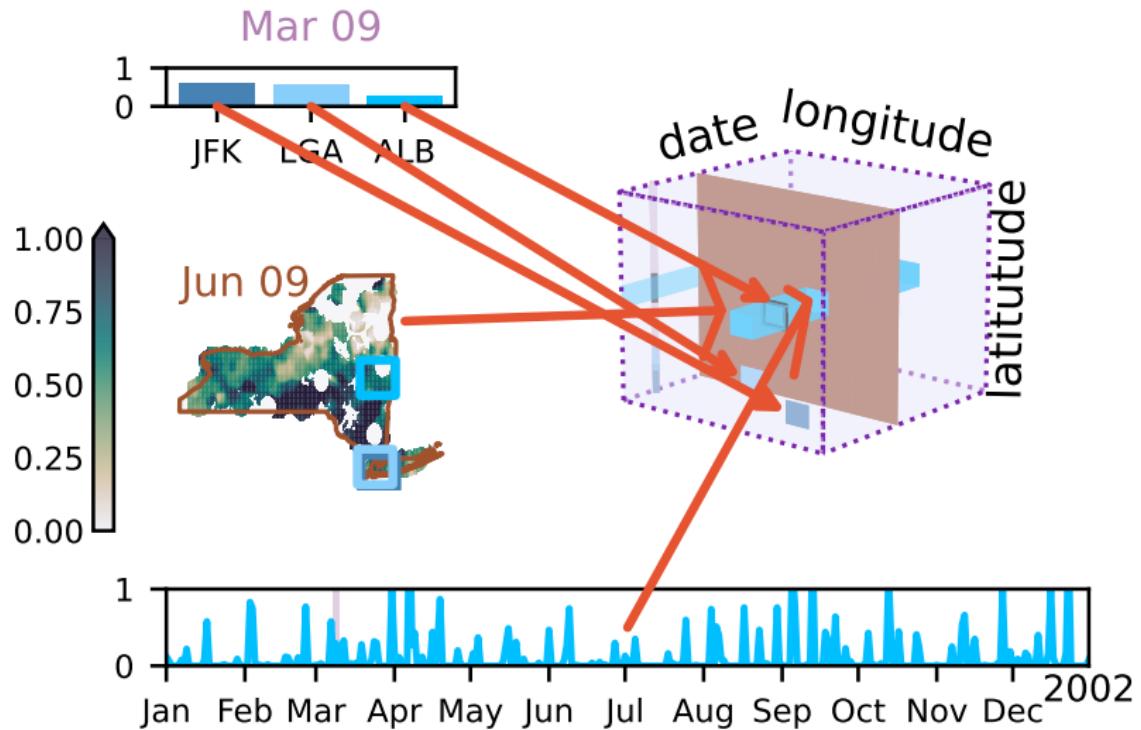
Topology: Homeomorphism

Definition

A function f is a *homeomorphism* if it is bijective, continuous, and has a continuous inverse function f^{-1} .

Homeomorphic topology

graphs show precipitation (in.)



Structure Preservation

- Bertin: equivalent data and visual field properties [4]
- Mackinlay: homomorphic fields and equivalent topology [5]
- Wilkinson: homomorphic scales and equivalent topology [1]
- Kindleman & Schieidegger: equivariant scales, invariant data representation, equivalent topology [6]

contribution

explicit homeomorphic topology, equivariant actions on fields and topology

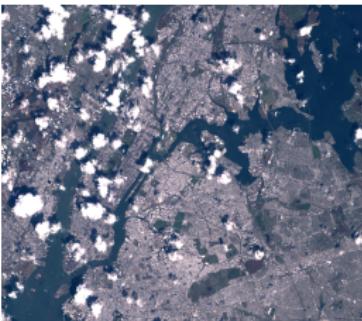
Domain Specific Library: library assumes structure [19]

| DATE | LATITUDE | LONGITUDE | PREC (in) | NAME |
|------------|----------|-----------|-----------|---------------------------|
| 2023-01-01 | 43.15457 | -77.8765 | 0.2209 | ROCHESTER GTR INTL AP |
| 2023-01-01 | 45.5 | -35.9 | 0.0000 | STONYBROOK NEW YORK |
| 2023-01-01 | 40.7431 | -73.8892 | 0.2998 | AUSTRALIAN AP |
| 2023-01-01 | 43.8 | -75.7 | 0.0000 | SCHROON LAKE NEW YORK |
| 2023-01-01 | 43.9978 | -73.6511 | 0.0000 | SARATOGA NEW YORK |
| 2023-01-01 | 40.7794 | -73.8863 | 0.4614 | NEW YORK LAGUARDIA AP |
| 2023-01-01 | 40.7794 | -73.8863 | 0.4614 | NEW YORK LAGUARDIA AP |
| 2023-01-01 | 43.1111 | -76.1109 | 0.0000 | SYRACUSE HAMILTON INTL AP |
| 2023-01-01 | 40.7939 | -73.1632 | 0.2984 | ISLE OF MACARTHUR AP |
| 2023-01-01 | 43.35 | -73.6187 | 0.1181 | GLENNS FALLS AP |

- ggplot[7]
- Vega[10]
- Altair[13]
- Tableau [16]
- [17, 18]

Domain Specific Library: library assumes structure [19]

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| 2023-01-01 | 40.7431 ¹ | -73.8892 ¹ | 0.2998 | AUSTRALIAN AP |
| 2023-01-01 | 43.8 ¹ | -75.7 ¹ | 0.0000 | SCHROON LAKE NEW YORK |
| 2023-01-01 | 43.9378 ¹ | -73.6511 ¹ | 0.0000 | SARATOGA SPRINGS NEW YORK |
| 2023-01-01 | 40.7794 ¹ | -73.8883 ¹ | 0.4614 | NEW YORK LAGUARDIA AP |
| 2023-01-01 | 40.7794 ¹ | -73.8883 ¹ | 0.4614 | NEW YORK J F肯尼迪 AP |
| 2023-01-01 | 43.1111 ¹ | -76.1109 ¹ | 0.0000 | SYRACUSE HAMILTON INTL AP |
| 2023-01-01 | 40.7939 ¹ | -73.1632 ¹ | 0.2984 | ISLE OF MACARTHUR AP |
| 2023-01-01 | 43.35 ¹ | -73.6181 ¹ | 0.1181 | GLENS FALLS AP |

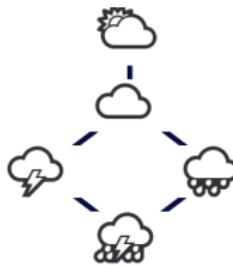
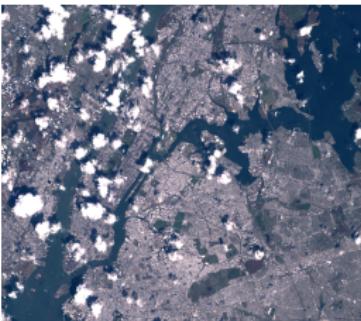


ggplot[7]
Vega[10]
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[17, 18]

ImageJ[8]
ImagePlot[11]
Napari[14]

Domain Specific Library: library assumes structure [19]

| DATE | LATITUDE | LONGITUDE | PREC (in) | NAME |
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| 2023-01-01 | 45.5 | -35.9 | 0.0000 | STONYBROOK NEW YORK |
| 2023-01-01 | 40.7431 | -73.8892 | 0.2998 | AUSTRALY AP |
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| 2023-01-01 | 43.1113 | -76.1109 | 0.0000 | SYRACUSE HANCOCK INTL AP |
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ggplot[7]
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[17, 18]

ImageJ[8]
ImagePlot[11]
Napari[14]

Gephi[9]
Graphviz[12]
Networkx[15]

Building Block Library[29]: visual algorithms assume structure [30]

1. Matplotlib[20] → Seaborn[21], xarray [22]
2. D3 [23]
3. VTK [24, 25], MayaVi[26] → Titan[27], ParaView[28]

Design Composable Structure Preserving API

Fiber Bundles Butler: "unified, dimension-independent framework" that expresses data as the mapping between continuity and fields [31, 32]

Simplicial Databases Spivak: Add rich typing for fields to Butler [33]

Category Theory Language express constraints in specifications [34]

Sheaves Ghrist "algebraic data structure" for representing data over topological spaces [35]

Expressive Types

```
dataset : topology → fields
```

Fiber Bundle

Definition

A **fiber bundle** (E, K, π, F) is a structure with topological spaces E, F, K and bundle projection map $\pi: E \rightarrow K$ [36].

$$F \hookrightarrow E \xrightarrow{\pi} K \tag{1}$$

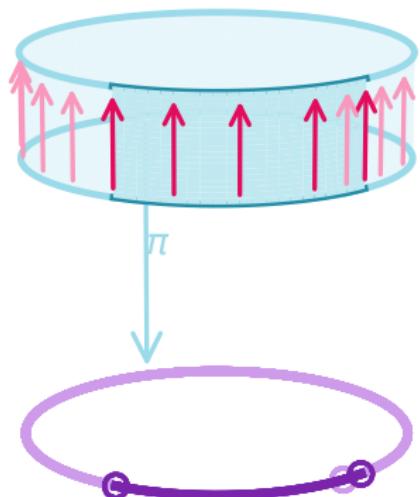
A continuous surjective map π is a **bundle projection** map when

1. all fibers in the bundle are isomorphic. Since all fibers are isomorphic $F \cong F_k$ for all points $k \in K$, there is a uniquely determined **fiber space** F given by the preimage of the projection π at any point k in the **base space** K : $F = \pi^{-1}(k)$.
2. each point k in the **base space** K has an open neighborhood U_k such that the **total space** E over the neighborhood is locally trivial.

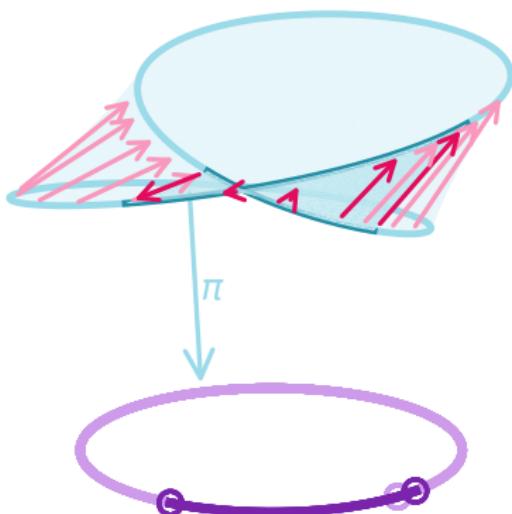
Local triviality means $E|_U = U \times F$.

Fiber Bundle

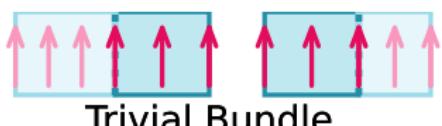
(E_c, K_c, π, F_c)



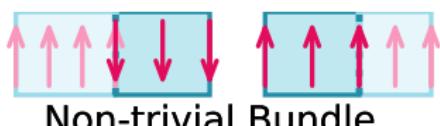
(E_m, K_m, π, F_m)



Transition
maps



Trivial Bundle

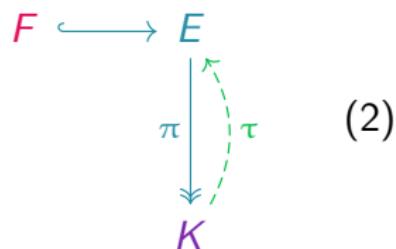


Non-trivial Bundle

Fiber Bundle: section

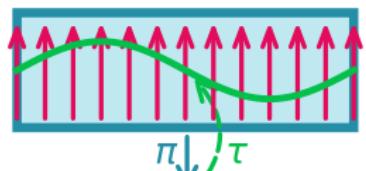
Definition

A **section** $\tau: K \rightarrow E$ over a fiber bundle is a smooth right inverse of $\pi(\tau(k)) = k$ for all $k \in K$



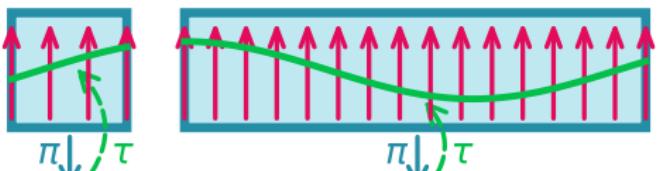
Fiber Bundle: section

fiber bundle
 (E, K, π, F)



$0 \frac{2\pi}{5}+ \frac{2\pi}{5}- 2\pi$

local trivializations
 (E_0, K_0, π, F_0) & (E_1, K_1, π, F_1)



$\varepsilon^- \frac{2\pi}{5}+ \frac{2\pi}{5}- \varepsilon^+$

Base space: topology

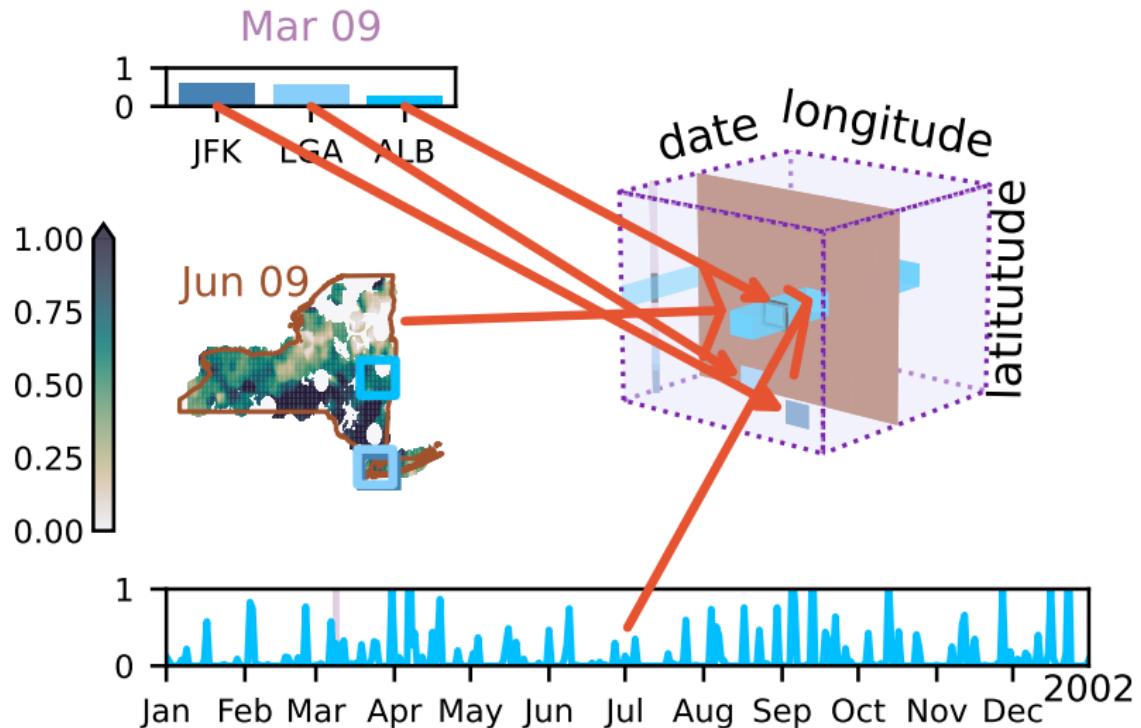
The base space of a fiber bundle is a quotient topology[37]. Given a set X and a function $\mathcal{N}: X \rightarrow 2^{2^X}$ that assigns to any $x \in X$ a non-empty collection of subsets $\mathcal{N}(x)$, where each element of $\mathcal{N}(x)$ is a *neighborhood of x* , then X with \mathcal{N} is a **topological space** and \mathcal{N} is a neighborhood *topology* if for each x in X : [38]

Definition

1. if N is a neighborhood $N \in \mathcal{N}(x)$ of x then $x \in N$
2. every superset of a neighborhood of x is a neighborhood of x ; therefore a union of a neighborhood and adjacent points in X is also a neighborhood of x
3. the intersection of any two neighborhoods of x is a neighborhood of x
4. any neighborhood N of x contains a neighborhood $M \subset N$ of x such that N is a neighborhood of each of the points in M

Why neighborhood topology?

graphs show precipitation (in.)



Base TYPE K

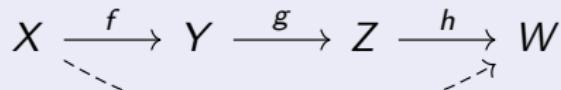
Definition

An **category** \mathcal{C} consists of the following *data*:

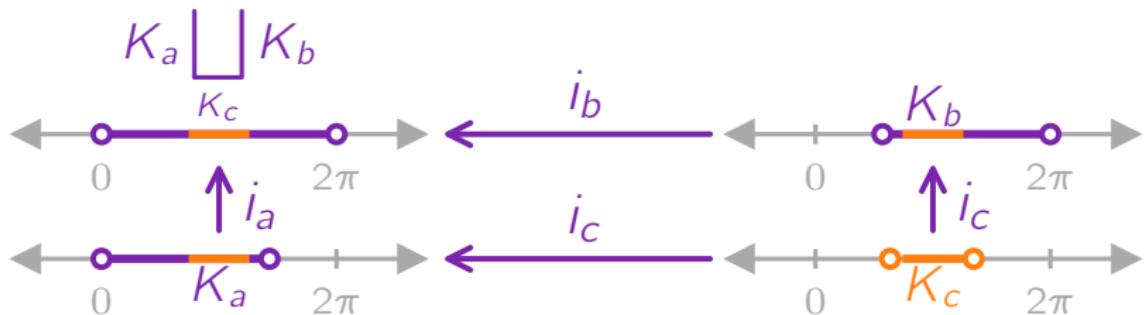
1. a collection of *objects* $X \in \mathbf{ob}(\mathcal{C})$
2. for every pair of objects $X, Y \in \mathbf{ob}(\mathcal{C})$, a set of *morphisms*
 $X \xrightarrow{f} Y \in \text{Hom}_{\mathcal{C}}(X, Y)$
3. for every object X , a distinct *identity morphism* $X \xrightarrow{id_X} X$ in
 $\text{Hom}_{\mathcal{C}}(X, X)$
4. a *composition function*
 $f \in \text{Hom}_{\mathcal{C}}(X, Y) \times g \in \text{Hom}_{\mathcal{C}}(Y, Z) \rightarrow g \circ f \in \text{Hom}_{\mathcal{C}}(X, Z)$

such that

1. *unitality*: for every morphism $X \xrightarrow{f} Y$, $f \circ id_X = f = id_Y \circ f$
2. *associativity*: if any three morphisms f, g, h are composable,

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} W$$


Add records \rightarrow Add to base space



Field types

A *schema* consists of a pair (C, σ) where C is the set of field names and $\sigma : C \rightarrow \mathbf{DT}$ is a function from field name to field data type[33]. The function σ is composed with π such that $\pi^{-1}(\sigma(C)) \subseteq \mathcal{U}$; this composition induces a domain bundle $\pi_\sigma : \mathcal{U}_\sigma \rightarrow C$ that associates a field name $c \in C$ with its corresponding domain $\pi_\sigma^{-1}(c) \subseteq \mathcal{U}_\sigma$.

Record

Definition

A **record** is a function $r : C \rightarrow \mathcal{U}_\sigma$ and the set of records on π_σ is denoted $\Gamma^\pi(\sigma)$. Records must return an object of type $\sigma(C) \in \mathbf{DT}$ for each field $c \in C$.

Tables are sections $\tau : K \rightarrow \Gamma^\pi(\sigma)$ from an indexing space K to the set of all possible records $\Gamma^\pi(\sigma)$ on the schema bundle

Fiber TYPE F

We define the fiber space F to be the space of all possible data records

$$F := \{r : C \rightarrow \mathcal{U}_\sigma \mid \pi_\sigma(r(C)) = C \text{ for all } C \in C\} \quad (3)$$

such that the preimage of a point is the corresponding data type domain $\pi^{-1}(k) = F_k = \mathcal{U}_{\sigma_k}$.

Fiber category \mathcal{F}

The fiber category has a single object \mathcal{F} of an arbitrary type and morphisms on the fiber object $\tilde{\phi} \in \text{Hom}(\mathcal{F}, \mathcal{F})$.

The fiber category \mathcal{F} is also equipped with a bifunctor $\otimes : \mathcal{F} \times \mathcal{F} \rightarrow \mathcal{F}$ for combining fiber types.

Fiber morphism $\tilde{\phi}$

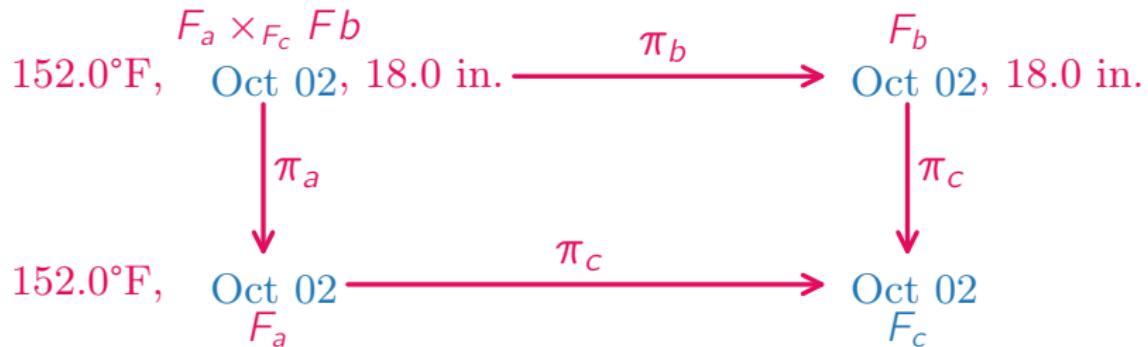
| scale | operators | sample constraint |
|----------|------------------------------|--|
| nominal | $=, \neq$ | $\tau(\textcolor{violet}{k}_1) \neq \tau(\textcolor{violet}{k}_2) \implies \tilde{\phi}(\tau(\textcolor{violet}{k}_1)) \neq \tilde{\phi}(\tau(\textcolor{violet}{k}_2))$ |
| ordinal | $<, \leqslant, \geqslant, >$ | $\tau(\textcolor{violet}{k}_1) \leqslant \tau(\textcolor{violet}{k}_2) \implies \tilde{\phi}(\tau(\textcolor{violet}{k}_1)) \leqslant \tilde{\phi}(\tau(\textcolor{violet}{k}_2))$ |
| interval | $+, -$ | $\tilde{\phi}(\tau(\textcolor{violet}{k}) + C) = \tilde{\phi}(\tau(\textcolor{violet}{k})) + C$ |
| ratio | $*, /$ | $\tilde{\phi}(\tau(\textcolor{violet}{k}) * C) = \tilde{\phi}(\tau(\textcolor{violet}{k})) * C$ |

Constraints as definition: Dates

- year: $\{y \in \mathbb{I} | 1992 \leq y \leq 2025\}$
- month: $\{m \in \mathbb{I} | 1 \leq m \leq 12\}$
- day: $\{d \in \mathbb{I} | 1 \leq d \leq 31\}$
- date: $\otimes : F_{year} \times F_{month} \times F_{day} \rightarrow F_{date}$

where the composition \otimes includes the constraint to only return dates that have the right number of days for each month.

Expand records → multiply fiber space



local sections

We define section functions locally:

$$\Gamma(U, E|_U) \coloneqq \{\tau : U \rightarrow E|_U \mid \pi(\tau(k)) = k \text{ for all } k \in U\} \quad (4)$$

such that each section function $\tau : k \mapsto r$ maps from each point $k \in U$ to a corresponding record in the fiber space $r \in F_k$ over that point.

This encoding can be translated into a set of signatures

{`data-subset` : `topology` → `fields` s.t. `data-subset` ⊂ `dataset`}.

global sections

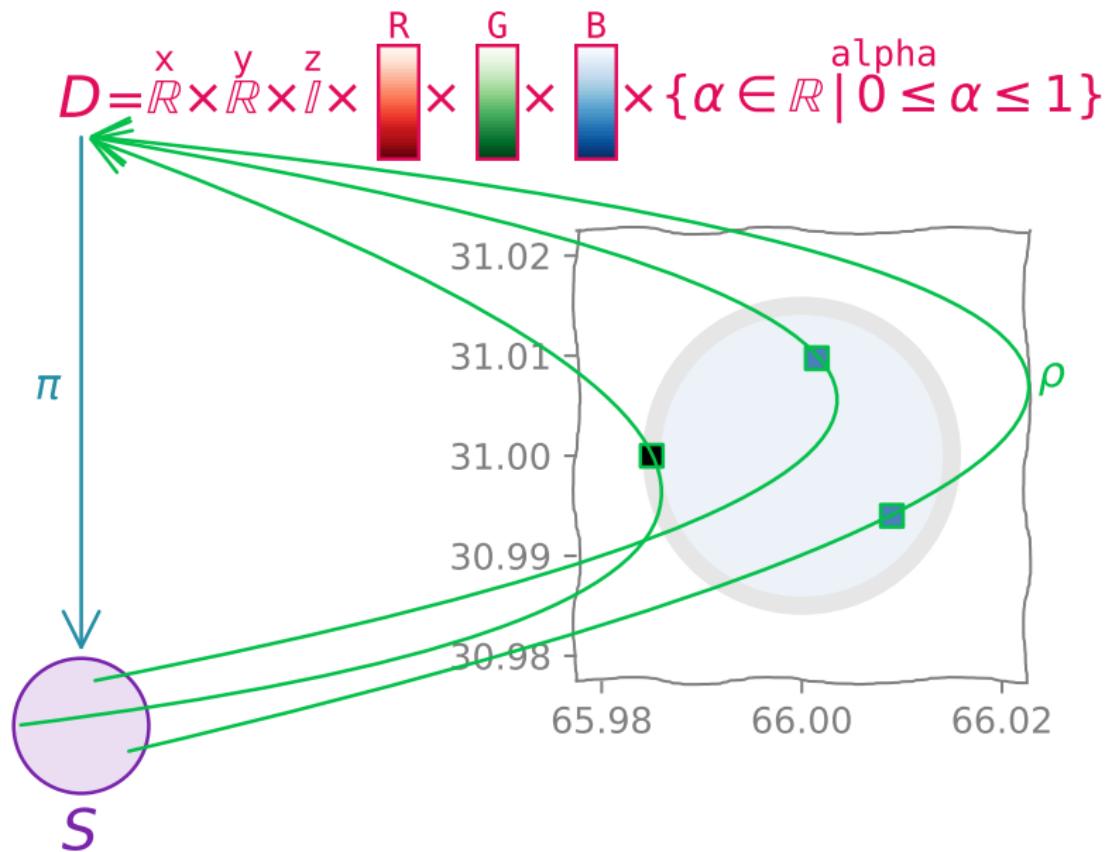
When a bundle is trivial $E = K \times F$, we can define a global sections $\tau : K \rightarrow F \in \Gamma(K, F)$ which we translate into a data signature of the form `dataset : topology → field` where $\tau = \text{dataset}$, $K = \text{topology}$ and $F = \text{fields}$

Graphic bundle and section

$$D \hookrightarrow H \xrightarrow{\pi} S \quad (5)$$

$$\begin{aligned} \Gamma(W, H|_W) &\coloneqq \\ &\{ \rho : W \rightarrow H|_W \mid \pi(\rho(s)) = s \text{ for all } s \in W \} \end{aligned} \quad (6)$$

Graphic section



Definition

[43, 44] A **functor** is a map $F : \mathcal{C} \rightarrow \mathcal{D}$, which means it is a function between objects $F : \mathbf{ob}(\mathcal{C}) \mapsto \mathbf{ob}(\mathcal{D})$ and that for every morphism $f \in Hom(C_1, C_2)$ there is a corresponding function $F : Hom(C_1, C_2) \mapsto Hom(F(C_1), F(C_2))$. A **functor** must satisfy the properties

- *identity*: $F(id_{\mathcal{C}}(C)) = id_{\mathcal{D}}(F(C))$
- *composition*: $F(g) \circ F(f) = F(g \circ f)$ for any composable morphisms $C_1 \xrightarrow{f} C_2, C_2 \xrightarrow{g} C_3$

$F(C) \in \mathbf{ob}(\mathcal{D})$ denotes the object to which an object C is mapped, and $F(f) \in Hom(F(C_1), F(C_2))$ denotes the morphism that f is mapped to.

topological equivariance: presheaf

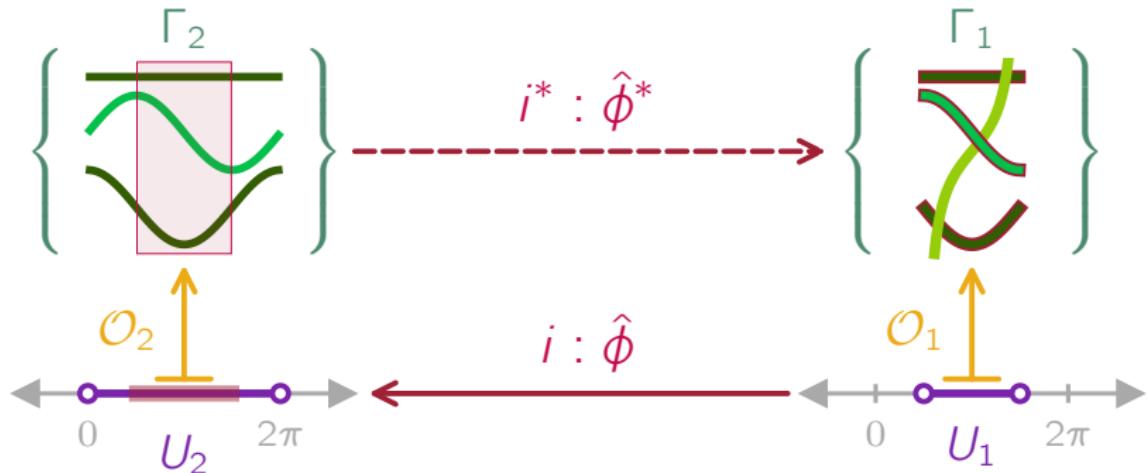
Definition

A **presheaf** $F : \mathcal{C}^{op} \rightarrow \mathbf{Set}$ is a contravariant functor from an object in an arbitrary category to an object in the category \mathbf{Set} [36].

The presheaf is contravariant when for every arbitrary morphism between input base spaces $\hat{\phi} : U_1 \rightarrow U_2$ there exists a corresponding pullback function between the sets of sections

$$\hat{\phi}^* : \Gamma(U_2, E|_{U_2}) \rightarrow \Gamma(U_1, E|_{U_1}).$$

topological equivariance: presheaf



Container invariance: sheaf

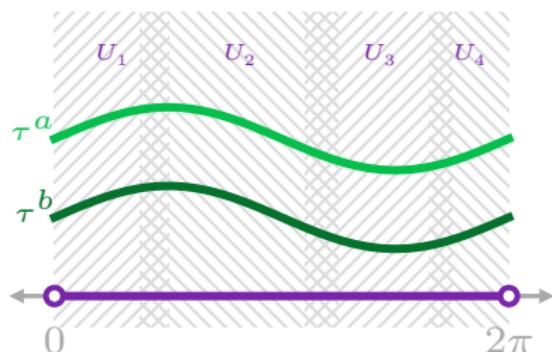
Definition

[45, 36] A **sheaf** is a presheaf that satisfies the following two axioms

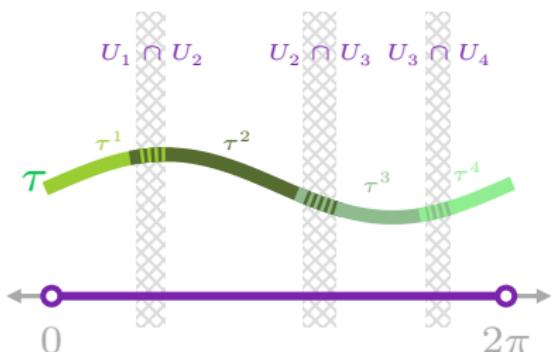
- *locality* two sections in a sheaf are equal $\tau^a = \tau^b$ when they evaluate to the same values $\tau^a|_{U_i} = \tau^b|_{U_i}$ over the open cover $\bigcup_{i \in I} U_i \subset U$ (indexed by I).
- *gluing* the union of sections defined over subspaces $\tau^i \in \Gamma(U_i, E|_{U_i})$ is equivalent to a section defined over the whole space $\tau|_U = \tau^i$ for all $i \in I$ if all pairs of sections agree on overlaps $\tau^i|_{U_i \cap U_j} = \tau^j|_{U_i \cap U_j}$

Container invariance: sheaf

locality: $\tau^a = \tau^b$



gluing: $\tau|_{U_i} = \tau^i$



What about needing more than one record?

The sheaf over a very small region surrounding a point k is called a *stalk*[46]

$$\mathcal{O}_{K,E}|_k := \lim_{U \ni k} \Gamma(U, E|_U) \quad (7)$$

where the fiber is contained inside the stalk $F_k \subset \mathcal{O}_{K,E}|_k$. The *germ* is the section evaluated at a point in the stalk $\tau(k) \in \mathcal{O}_{K,E}|_k$ and is the data.

Homeomorphism

We define the mapping ξ to be a surjective continuous map:

$$\xi : W \rightarrow U \quad (8)$$

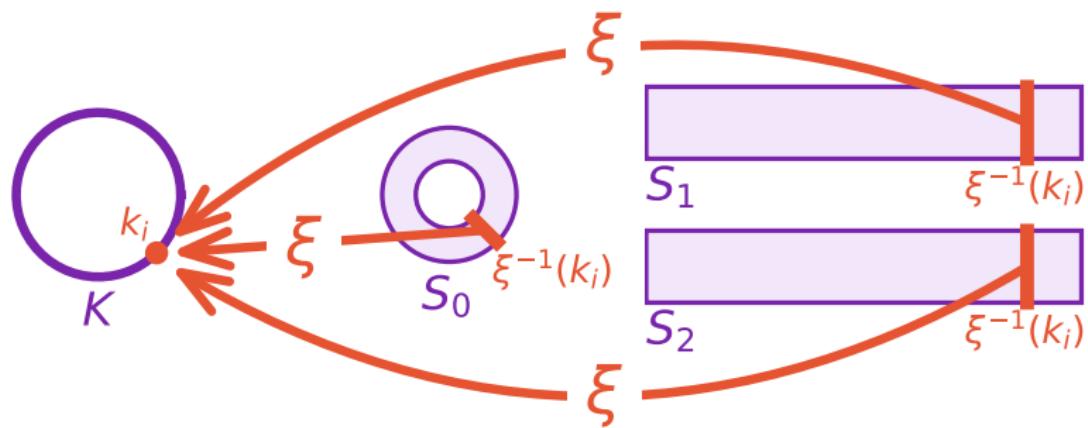
between a graphic subspace $W \subseteq S$ and data subspace $U \subseteq K$.

The set of points in graphic space that correspond to each point in data space is

$$\xi^{-1}(k) = \{s | \xi(s) = k \forall k \in K, s \in S\} \quad (9)$$

such that every point in a graphic space has a corresponding point in data space.

Homeomorphism



Pushforward: Graphic for Data

Definition

Given a sheaf $\mathcal{O}_{S,H}$ on S , the **pushforward** sheaf $\xi_* \mathcal{O}_{S,H}$ on K is defined as

$$\xi_*(\mathcal{O}_{S,H})(U) = \mathcal{O}_{S,H}(\xi^{-1}(U)) \quad (10)$$

for all opensets $U \subset K$ [46].

This provides a way to look up which graphic corresponds with a data index

$$\xi_* \rho(k) = \rho|_{\xi^{-1}(k)} \quad (11)$$

such that $\xi_* \rho(k)(s) = \rho(s)$ for all $s \in \xi^{-1}(k)$.

Pullback: Data at graphic

Definition

[46] Given a sheaf $\mathcal{O}_{K,E}$ on K , the **pullback** sheaf $\xi^*\mathcal{O}_{K,E}$ on S is defined as the sheaf associated to the presheaf

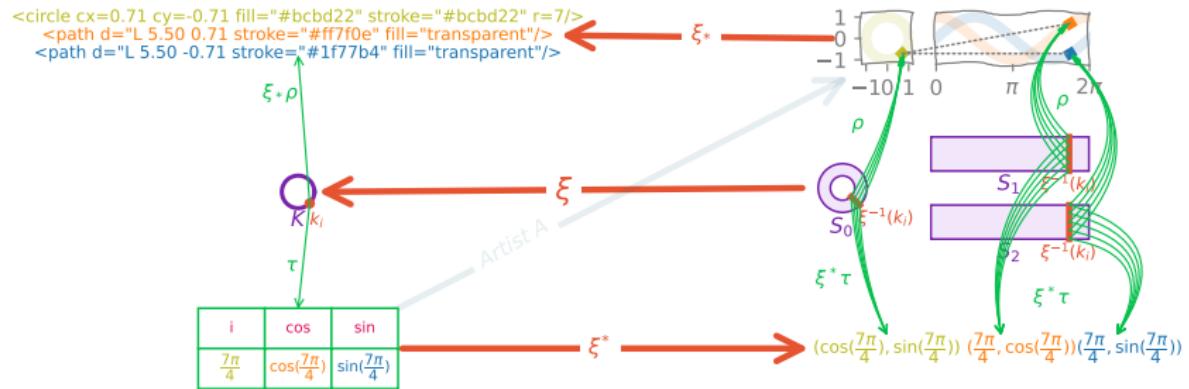
$$\xi^*(\mathcal{O}_{K,E})(W) = \mathcal{O}_{K,E}(\xi(W)) \text{ for } \xi(W) \in K.$$

This provides a way to then look up what data values correspond with a graphic index

$$\xi^*\tau(s) = \tau(\xi(s)) = \tau(k) \quad (12)$$

As ξ is surjective, there are many points $s \in W \subseteq S$ in the graphic space that correspond to a single point $\xi(s) = k$.

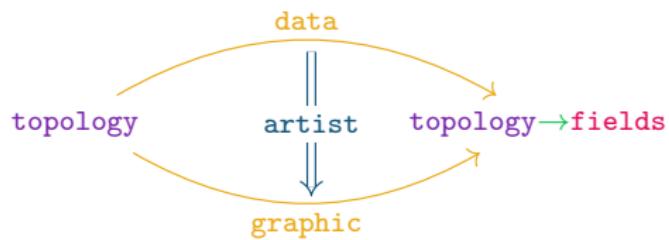
Homeomorphism



Definition

Given two functors $F, G : \mathcal{C} \rightarrow \mathcal{D}$, a **natural transformation** $\alpha : F \rightarrow G$ is a function which assigns to each object c of \mathcal{C} a morphism $\alpha_c : F(c) \rightarrow G(c)$, $G(c) \in \mathcal{D}$, in such a way that for every morphism $f : c \rightarrow c'$, $c' \in \mathcal{C}$, the morphisms in \mathcal{D} commute such that $\alpha'_{c'}(F(f)(F(c))) = G(f)(\alpha_c(F(c)))$. When this holds, α_c is *natural* in c .[41].

Artist



How?

$$\begin{array}{ccc} \Gamma(U, \xi_* H|_U) & \xleftarrow{\xi_*} & \Gamma(W, ,)H|_W \\ \uparrow Hom_{\mathcal{O}_K} & \nearrow Hom_{\mathcal{O}_K, \mathcal{O}_S} & \uparrow Hom_{\mathcal{O}_S} \\ \Gamma(U, E|_U) & \xrightarrow{\xi^*} & \Gamma(W, \xi^* E|_W) \end{array} \quad (13)$$

Artist

$$A : \Gamma(K, E) \rightarrow Im_A(S, H), Im_A(S, H) \subset \Gamma(S, H) \quad (14)$$

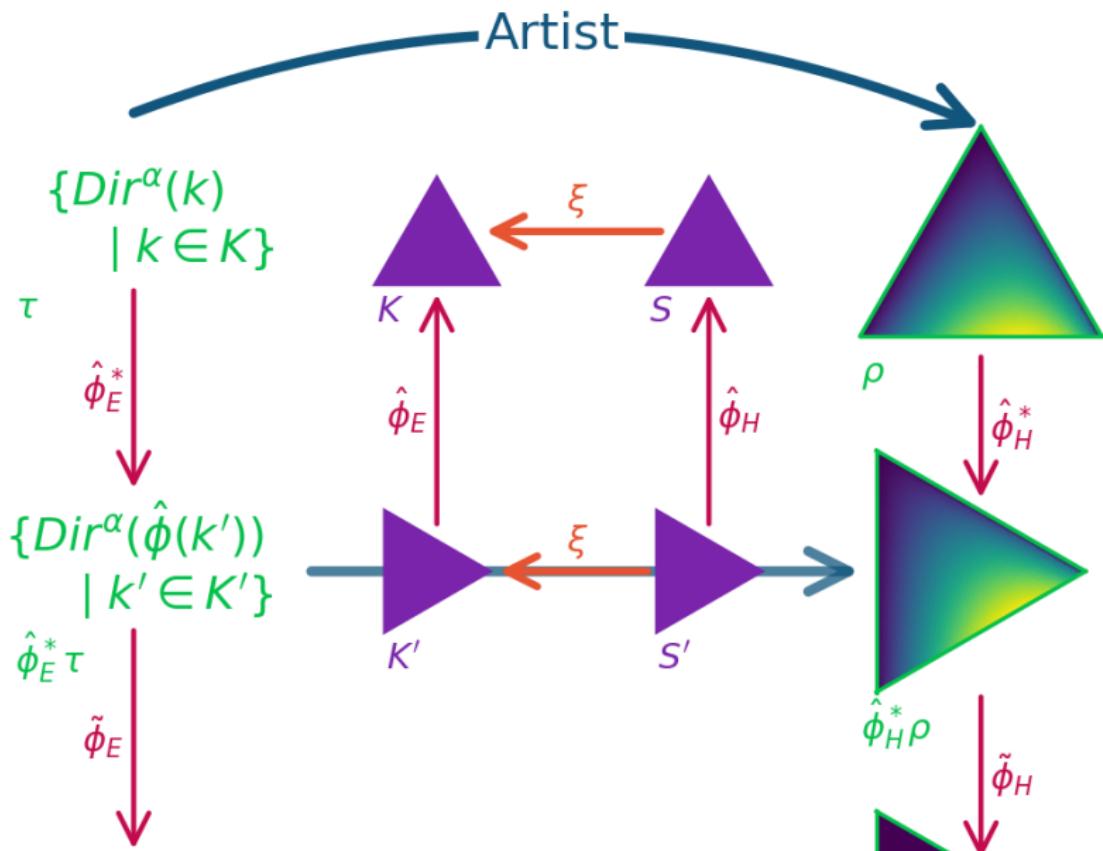
where:

$$Im_A(S, H) \coloneqq \{\rho \mid \exists \tau \in \Gamma(K, E) \text{ s.t. } A(\tau) = \rho, \xi(S) = K\} \quad (15)$$

Equivariant Artist

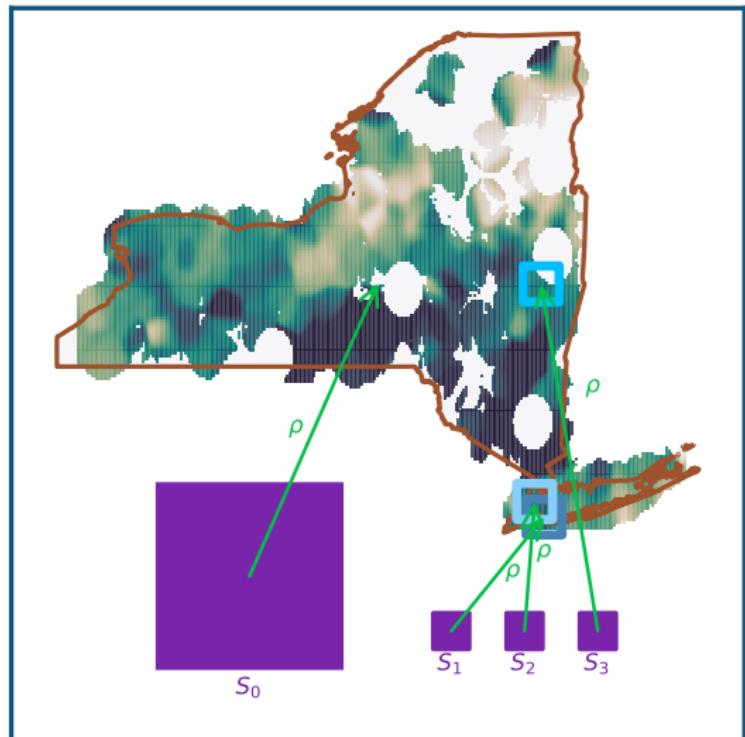
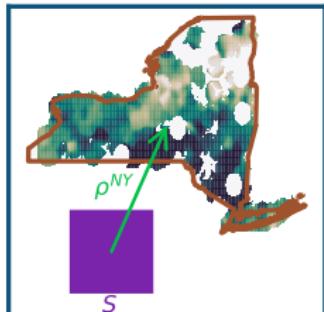
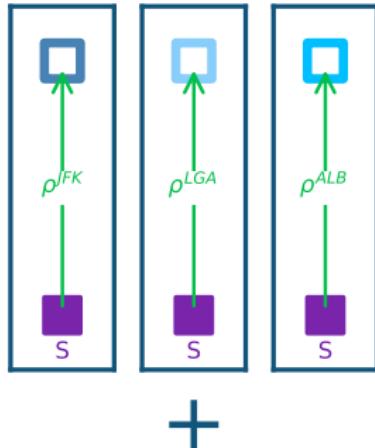
$$\begin{array}{ccccc} \Gamma(K, E) & \xrightarrow{A} & Im_A(S, H) & & \\ \hat{\phi}^*_{E} \downarrow & & & & \hat{\phi}^*_{H} \searrow \\ \Gamma(K', \hat{\phi}^*_{E} E) & & K \xleftarrow{\xi} S & & Im_A(S', \hat{\phi}^*_{H} H) \\ \tilde{\phi}_E \downarrow & & \hat{\phi}_E \uparrow & & \tilde{\phi}_H \downarrow \\ & & K' \xleftarrow{\xi} S' & & \\ \Gamma(K', E') & \xrightarrow{A} & Im_A(S', H') & & \end{array} \quad (16)$$

Equivariant Artist



artist composition

Fiber: $\{x, y, z, r, g, b, \alpha\} \in D$



Animation and Interactivity

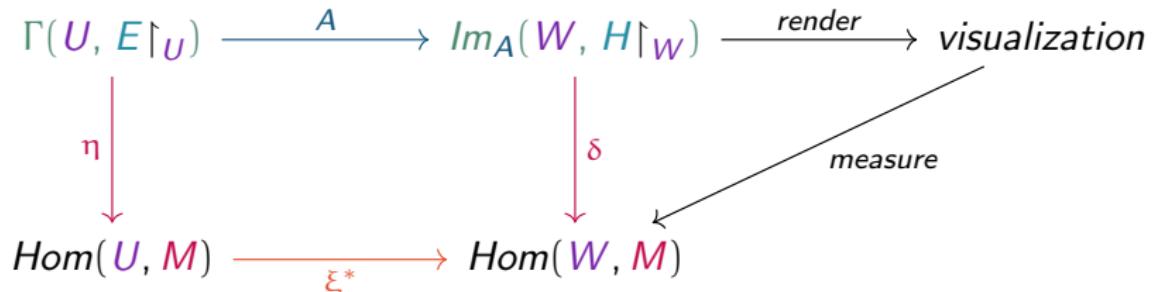
pan, zoom, scroll sheaf: locality + gluing 12

selection and hover pushforward 11, pullback 12

brushing, linking, annotation composition of artists ??

Testing if A is equivariant

M is a (scalar, vector) measurable component (e.g. color, position, shape, texture, rotation,) of the rendered visual element.



$$\text{input } \eta : \tau \mapsto (U \xrightarrow{\eta_\tau} M)$$

$$\text{output } \delta : \rho \mapsto (W \xrightarrow{\delta_\rho} M)$$

$$\eta_\tau(k) = \delta_\rho(s) \text{ for all } \xi(s) = k, k \in K, s \in S$$

Visual Space: Specialized data space

$$P \hookrightarrow V \xrightarrow{\pi} K$$

| | Points | Lines | Areas | Best to show |
|-----------------|--|--|---|---------------------------------|
| Shape |  | <i>possible, but too weird to show</i> | <i>cartogram</i> | <i>qualitative differences</i> |
| Size |  |  | <i>cartogram</i> | <i>quantitative differences</i> |
| Color Hue |  |  |  | <i>qualitative differences</i> |
| Color Value |  |  |  | <i>quantitative differences</i> |
| Color Intensity |  |  |  | <i>qualitative differences</i> |

Encoding

$$\nu : F_k \rightarrow P_k \quad (17)$$

$$F_k \xrightarrow{\nu} P_k := F'_k \xrightarrow{\nu'} P'_k \quad (18)$$

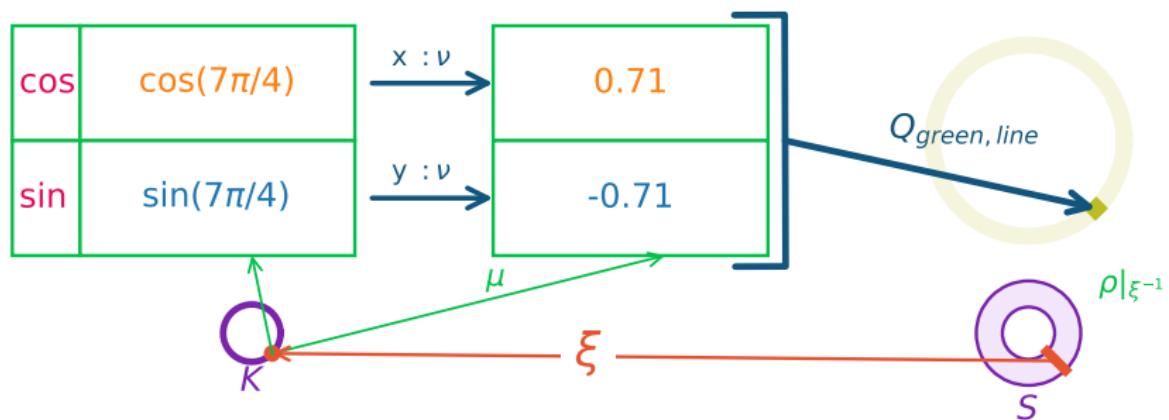
```
graph LR; Fk[F_k] -- "ν" --> Pk[P_k]; Fk -- "ν''" --> Pkp[P'_k]; Fkp[F'_k] -- "ν'" --> Pkp;
```

Composition

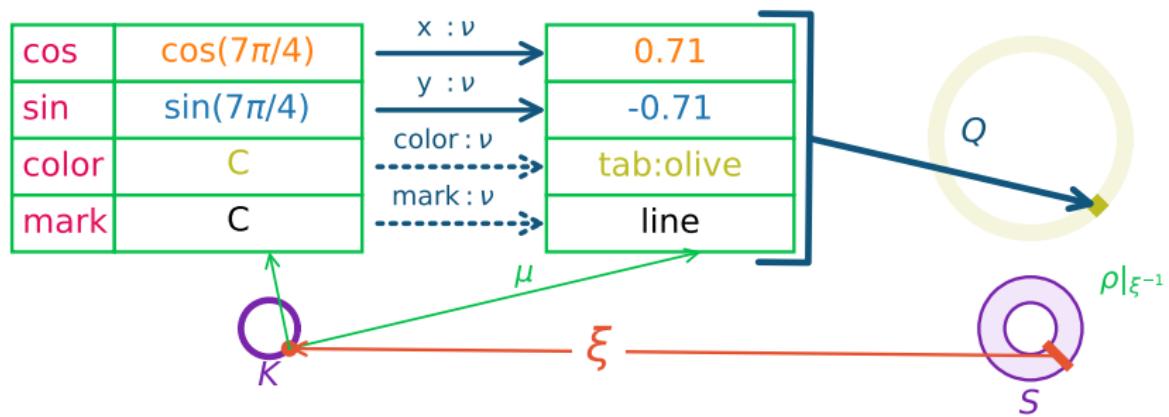
$$Q : \Gamma(K, V) \rightarrow \Gamma(S, H) \quad (19)$$

$$\begin{array}{ccccc} \Gamma(K, V) & \xrightarrow{\gamma} & \Gamma(K, V') & \xrightarrow{Q} & \Gamma(S, H) \\ & \searrow & \downarrow Q' & \nearrow & \\ & & & & \end{array} \quad (20)$$

Construction Stages



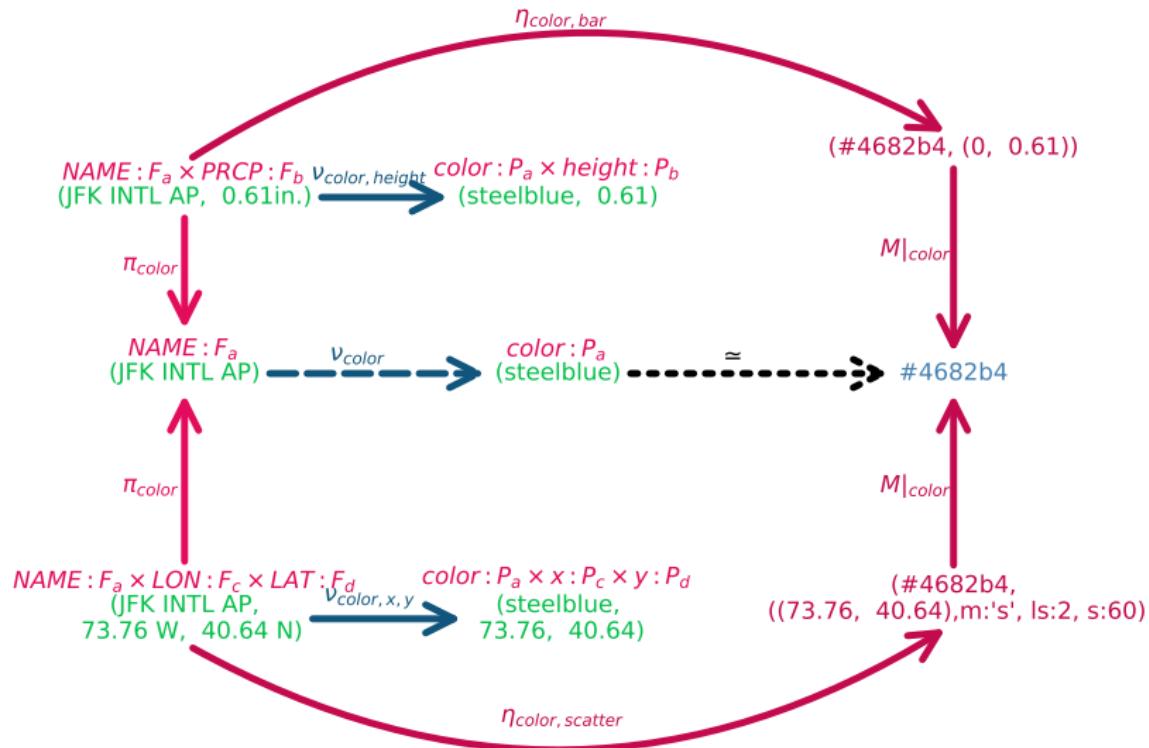
Construction Stages: composition



Verify encoder

$$\begin{array}{ccccc}
 & & \eta_{ab} & & \\
 & F_k^a \times F_k^b & \xrightarrow{\gamma_{ab}} & P_k^a \times P_k^b & M_k^{ab} \\
 \pi_a \downarrow & & & \downarrow \pi_a & \downarrow M \upharpoonright_a \\
 F_k^a & \dashrightarrow^{\gamma_a} & P_k^a & \simeq & M_k^a \\
 \uparrow \pi_a & & \uparrow \pi_a & & \uparrow M \upharpoonright_a \\
 F_k^a \times F_k^c & \xrightarrow{\gamma_{ac}} & P_k^a \times P_k^c & & M_k^{ac} \\
 & & \eta_{ac} & &
 \end{array} \tag{21}$$

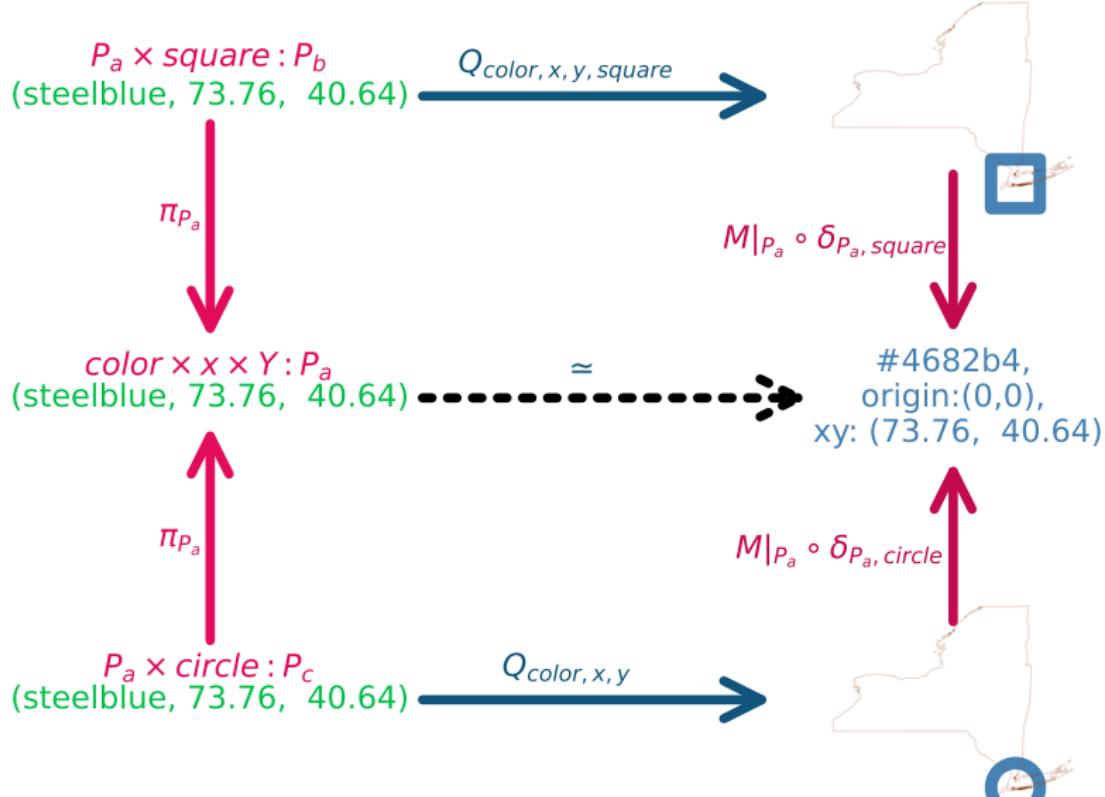
Verify encoder



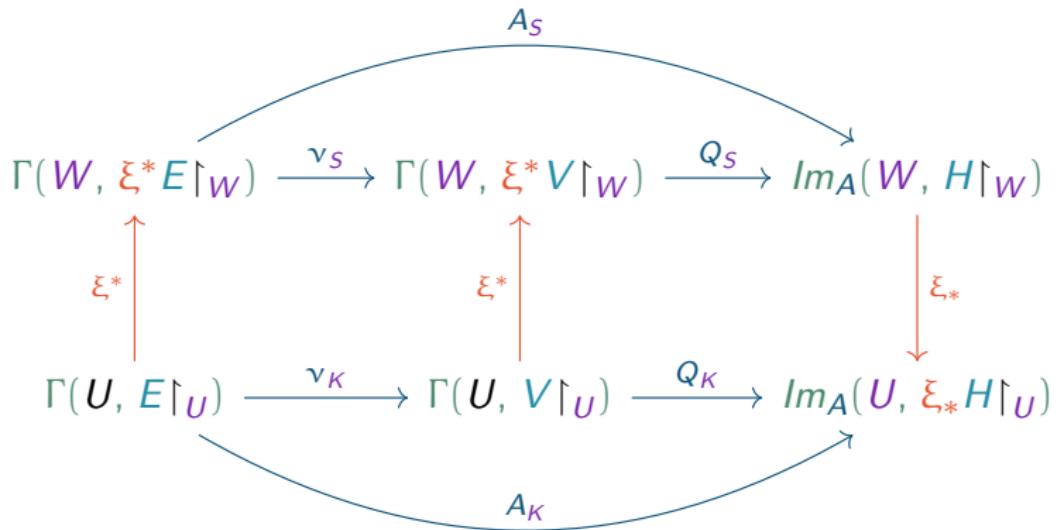
Verify compositor

$$\begin{array}{ccc}
 \Gamma(K, V^a \times V^b) & \xrightarrow[ab]{Q_{ab}} & \text{Im}_{A_{ab}}(S, H) \\
 \pi_a \downarrow & & \downarrow M|_a \circ \delta_{ab} \\
 \Gamma(K, V^a) & \xrightarrow{\simeq} & \text{Hom}(K, M^a) \\
 \uparrow \pi_a & & \uparrow M|_a \circ \delta_{ac} \\
 \Gamma(K, V^a \times V^c) & \xrightarrow[ac]{Q_{ac}} & \text{Im}_{A_{ac}}(S, H)
 \end{array} \tag{22}$$

Verify compositor



Implementation Choices: $A_K = A_S$



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