# Mathematical Data Abstraction

**Fiber Bundles** "unified, dimension-independent framework" that expresses data as the mapping between continuity and fields [**butlerVectorBundleClassesForm1992**, **butlerVisualizationModelBased1989**]

**Category Theory Language** express constraints in specifications [**wielsManagementEvolvingSpecifications1998**]

**Sheaves on Bundles** "algebraic data structure" for representing data over topological spaces [**ghristElementaryAppliedTopology2014**]

# Fiber Bundle

$$\begin{tikzcd} F \arrow[r, hook] & E \arrow[d, "\pi"', shift right] \\ & K \arrow[u, bend right, "\tau\in\Gamma(K,\,E)"'] \end{tikzcd}$$

**Total Space**

$(E, \mathcal{T})$

**Base Space**

$\mathcal{T}_K = \{U \subseteq K : \{r \in E : \pi(r) \in U\} \in \mathcal{T}_E\}$

**Fiber Space**

$F\restriction_k = \pi^{-1}(k), k \in U$

**Sections**

$\Gamma(U, E\restriction_U) \coloneqq \left\{\tau : U \to E\restriction_U \;\middle|\; \pi(\tau(k)) = k \textit{ for all } k \in U\right\}$

**Locally Trivial**

for every point $k \in K$, there exists an open neighborhood $k \in U \subseteq K$ s.t. there is a homeomorphism $\pi^{-1}(U) \xrightarrow{\varphi} U \times F$

**(Globally) Trivial**

$E = K \times F$

# Data Bundle

$$\begin{tikzcd} F \arrow[r, hook] & E \arrow[d, "\pi"'] \\ & K \arrow[u, bend right, "\tau \in \Gamma(K, E)"'] \end{tikzcd}$$

**Data** $E$
: continuity + fields

**Continuity** $K$
: how data elements are organized (topological properties) [**wilkinsonGrammarGraphics2005**], index (key) space [**munznerWhatDataAbstraction2014**])

**Fields** $F$
: generalization of a schema - named and typed date fields [**spivakSIMPLICIALDATABASES**, **spivakDatabasesAreCategories2010**]

**Data** [**butlerVisualizationModelBased1989**, **butlerVectorBundleClassesForm1992**]

$\tau(k) = r, \; k \in U \subseteq K, \; r \in F\restriction_k$

$r = \{field : value \quad field : value \quad field : value\}$

# Graphic Bundle

$$\begin{tikzcd} D \arrow[r, hook] & H \arrow[d, "\pi"'] \\ & S \arrow[u, bend right, "\rho \in \Gamma(S, H)"'] \end{tikzcd}$$

**Graphic $H$**
: virtual rendered output

**Continuity $S$**
: topology of graphic in rendered space

**Display $D$**
: generalization of renderer specifications, e.g. {xy,rgba}, {xy, cymk}, {xyz, rgba}

**Graphic**

$$\Gamma(W, H\restriction_W) \coloneqq \left\{\rho : W \to H\restriction_W \;\middle|\; \pi(\rho(s)) = s \textit{ for all } s \in W\right\}$$

$$\rho(s) = d,\ s \in W \subseteq S,\ d \in F\restriction_s$$

$$d = \{x, y, r, g, b\}$$

```
Artist(data:Data) -> Graphic
```

# Category $\mathcal{C}$

$$
\begin{tikzcd}
C_1 \arrow[r, "f"] \arrow[dr, "g\circ f"'] \arrow[loop left, "id_{C_1}"] & C_2 \arrow[d, "g"] \arrow[loop above, "id_{C_2}"] \\
 & C_3 \arrow[loop below, "id_{C_3}"]
\end{tikzcd}
$$

**associativity**

if $f : C_1 \to C_2$, $g : C_2 \to C_3$ and $h : C_3 \to C_4$ then $h \circ (g \circ f) = (h \circ g) \circ f$

**identity**

for every $f : C_1 \to C_2$ there exists identity morphisms $f \circ id_{C_1} = f = id_{C_2} \circ f$

# Opposite Category $\mathcal{C}^{op}$

$$
\begin{tikzcd}
C_1 \arrow[loop left, "id_{C_1}"] \arrow[r, "f"] \arrow[dr, "g \circ f"'] & C_2 \arrow[loop above, "id_{C_2}"] \arrow[d, "g"] \\
& C_3 \arrow[loop below, "id_{C_3}"]
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
C_1 \arrow[loop left, "id_{C_1}"] & C_2 \arrow[loop above, "id_{C_2}"] \arrow[l, "f"'] \\
& C_3 \arrow[loop below, "id_{C_3}"] \arrow[u, "g"'] \arrow[ul, "f \circ g"]
\end{tikzcd}
$$

# Functor $F : \mathcal{C} \to \mathcal{D}$

composition

$$\begin{tikzcd} c \arrow[r, mapsto, "F"] \arrow[d, "f"'] & F(c) = d \arrow[d, "F(f)"] \\ c' \arrow[r, mapsto, "F"'] & F(c') = d' \end{tikzcd}$$

$$F(g) \circ F(f) = F(g \circ f)$$

identity

$$F(id_c) = id_{F(c)}$$

# Presheaf: $\mathcal{O} : \mathcal{C}^{op} \to \mathbf{Set}$

$$\begin{tikzcd} F \arrow[r, hook] & E \arrow[d, "\pi"'] \\ & K \arrow[u, bend right, "\tau \in \Gamma(K, E)"'] \end{tikzcd}$$

$$\begin{tikzcd} \mathbf{Set} \ni \Gamma(U_1, E\restriction_{U_1}) & & \Gamma(U_2, E\restriction_{U_2}) \arrow[ll, hook', "\iota^*"'] \\ Ob(\mathcal{K}^{op}) \ni U_1 \arrow[u, mapsto, "\mathcal{O}_{K,E}"] \arrow[rr, hook, "\iota"'] & {} \arrow[u, mapsto, "\mathcal{O}_{K,E}"] & U_2 \arrow[u, mapsto, "\mathcal{O}_{K,E}"'] \end{tikzcd}$$

stalk

$$\mathcal{O}_{K,E}\restriction_k \coloneqq \lim_{U \ni k} \Gamma(U, E\restriction_U)$$

$$F_k \subset \mathcal{O}_{K,E}\restriction_k$$

germ

$$\tau(k) \in \mathcal{O}_{K,E}\restriction_k$$

# Sheaves

A sheaf is a presheaf that satisfies the following two axioms[**bakerMathsSheaf**]

## locality

if there exists the open covering $\mathcal{U} = \{U_i\}_{i \in I}$ of $U$ and if $\tau^a, \tau^b \in \mathcal{O}(U_i)$ have the property $\tau^a\restriction_{U_i} = \tau^b\restriction_{U_i}$ for each $U_i \in \mathcal{U}$, then $\tau^a = \tau^b$

## gluing

if there exists the open covering $\mathcal{U} = \{U_i\}_{i \in I}$ of $U$ and if $\tau^i \in \mathcal{O}(U_i)$ is given for each $i \in I$ such that $\tau^i\restriction_{U_i \cap U_j} = \tau^j\restriction_{U_i \cap U_j}$ for each pair $U_i, U_j \in \mathcal{U}$, then there exists $\tau \in \mathcal{O}(U)$ such that $\tau\restriction_{U_i} = \tau^i$ for each $i \in I$

# Data

$$\begin{array}{c} K \\ \Big\downarrow{\scriptstyle \mathcal{O}_{K,E}} \\ \mathcal{O}_{K,E} \end{array}$$

$F \hookrightarrow E \xrightarrow{\pi} K$

$\mathcal{O}_{K,E} : U \mapsto \Gamma(U, E\restriction_U), U \subset K$

$\Gamma(U, E\restriction_U) \ni \tau : U \to F\restriction_U$

$\tau(k) = \{f_0 : v_0, \cdots ,\}, k \in U$

# Graphic

$$
\begin{tikzcd}
\mathcal{O}_{S,H} \\
S \arrow[u, mapsto, "\mathcal{O}_{S,H}"]
\end{tikzcd}
$$

$$D \hookrightarrow H \xrightarrow{\pi} S$$

$$\mathcal{O}_{S,H} : W \mapsto \Gamma(W, E\restriction_W), W \subset S$$

$$\Gamma(W, H\restriction_W) \ni \rho : W \to D\restriction_W$$

$$\rho(s) = \{d_0, \cdots\}, s \in W$$

# Functor: $\xi : \mathcal{S} \to \mathcal{K}$

$$
\begin{tikzcd}
\mathcal{O}_{S,H} & \\
S \arrow[u, mapsto, "\mathcal{O}_{S,H}"] \arrow[r, "\xi"] & K \arrow[d, mapsto, "\mathcal{O}_{K,E}"] \\
& \mathcal{O}_{K,E}
\end{tikzcd}
$$

# Pullback: data to region of the visualization

$$
\begin{tikzcd}
S \arrow[r, "\xi"] \arrow[d, dashed, mapsto] & K \arrow[d, mapsto, "\mathcal{O}_{K,E}"] \\
\xi^*\mathcal{O}_{K,E} & \mathcal{O}_{K,E} \arrow[l, "\xi^*"]
\end{tikzcd}
$$

$$\xi^* F \hookrightarrow \xi^* E \xrightarrow{\pi} S$$

$$\xi^*\mathcal{O}_{K,E} : W \mapsto \Gamma(W, \xi^* E\restriction_W), W \subset S$$

$$\Gamma(W, \xi^* E\restriction_W) \ni \xi^*\tau : W \to \xi^* F\restriction_W$$

$$\xi^*\tau(s) = \tau(\xi(s)) = \tau(k)$$

# Pushforward: visualization to index of data

$$\begin{tikzcd} \mathcal{O}_{S,H} \arrow[r, "\xi_*"] & \xi_*\mathcal{O}_{S,H} \\ S \arrow[u, mapsto, "\mathcal{O}_{S,H}"] \arrow[r, "\xi"] & K \arrow[u, mapsto, dashed] \end{tikzcd}$$

$$\xi_* D \hookrightarrow \xi_* H \xrightarrow{\pi} K$$

$$\xi_* \mathcal{O}_{S,H} : U \mapsto \Gamma(U, \xi_* H\restriction_U), U \subset K$$

$$\Gamma(U, H\restriction_U) \ni \rho : U \to D\restriction_U$$

$$\xi_* \rho(s) = \rho\restriction_{\xi^{-1}(k)} = \rho(s) \forall s \in \xi^{-1}(k)$$

$$\mathcal{O}_{K,E} \xrightarrow{A} \mathcal{O}_{S,H}$$

# Natural Transformation $\alpha : F \Rightarrow G$

$$\begin{tikzcd}[column sep=huge]
\mathcal{C} \arrow[r, bend left=50, maps to, "F"{name=U}] \arrow[r, bend right=50, maps to, "G"'{name=L}] & \mathcal{D}
\arrow[Rightarrow, from=U, to=L, "\alpha"]
\end{tikzcd}$$

# Natural Transformation $\alpha : F \Rightarrow G$

$$
\begin{tikzcd}
F(c) \arrow[d, "F(f)"'] \arrow[rr, Rightarrow, bend left, "\alpha_c"] & c \arrow[l, mapsto, "F"] \arrow[r, mapsto, "G"'] \arrow[d, "f"] & G(c) \arrow[d, "G(f)"] \\
F(c') \arrow[rr, Rightarrow, bend right, "\alpha_{c'}"'] & c' \arrow[l, mapsto, "F"'] \arrow[r, mapsto, "G"] & G(c')
\end{tikzcd}
$$

# Data Space: $A_U : \mathcal{O}_{K,E} \Rightarrow \xi_* \mathcal{O}_{S,H}$

$$
\begin{tikzcd}
\Gamma(U_1, E) \arrow[d, hook, "\iota^*"'] \arrow[rr, Rightarrow, bend left, "A_{U_1}"] & U_1 \arrow[l, mapsto, "\mathcal{O}_E"'] \arrow[r, mapsto, "\mathcal{O}_{\xi_* H}"'] & \Gamma(U_1, \xi_* H) \arrow[d, hook, "\iota^*"] \\
\Gamma(U_2, E) \arrow[rr, Rightarrow, bend right, "A_{U_2}"'] & U_2 \arrow[u, hook, "\iota"] \arrow[l, mapsto, "\mathcal{O}_E"] \arrow[r, mapsto, "\mathcal{O}_{\xi_* H}"] & \Gamma(U_2, \xi_* H)
\end{tikzcd}
$$

# Display Space: $A_W : \xi^*\mathcal{O}_{K,E} \Rightarrow \mathcal{O}_{S,H}$

$$
\begin{tikzcd}[column sep=large, row sep=huge]
\Gamma(W_1, \xi^*E) \arrow[d, hook, "\iota^*"'] \arrow[rr, Rightarrow, bend left, "A_{W_1}"] & W_1 \arrow[l, mapsto, "\mathcal{O}_{\xi^*E}"] \arrow[r, mapsto, "\mathcal{O}_H"'] & \Gamma(W_1, H) \arrow[d, hook, "\iota^*"] \\
\Gamma(W_2, \xi^*E) \arrow[rr, Rightarrow, bend right, "A_{W_2}"'] & W_2 \arrow[u, hook, "\iota"] \arrow[l, mapsto, "\mathcal{O}_{\xi^*E}"'] \arrow[r, mapsto, "\mathcal{O}_H"] & \Gamma(W_2, H)
\end{tikzcd}
$$

# Artist $A$

$$\begin{tikzcd}
\mathcal{O}_{S,H} \arrow[rr, "\xi_*"] & & \xi_*\mathcal{O}_{S,H} \\
\\
\xi^*\mathcal{O}_{K,E} \arrow[uu, Rightarrow, "A_W"] & & \mathcal{O}_{K,E} \arrow[ll, "\xi^*"] \arrow[uu, Rightarrow, "A_U"'] \arrow[uull, Rightarrow, dashed, "A"]
\end{tikzcd}$$

$$Nat_W(\xi^*\mathcal{O}_{K,E}, \mathcal{O}_{S,H}) = Nat_U(\mathcal{O}_{K,E}, \xi_*\mathcal{O}_{S,H})$$

# Artist $A$

$$
\begin{tikzcd}
\mathcal{O}_{S,H} \arrow[r, "\xi_*"] & \xi_*\mathcal{O}_{S,H} \\
\xi^*\mathcal{O}_{K,E} \arrow[u, Rightarrow, "A_W"] & \mathcal{O}_{K,E} \arrow[l, "\xi^*"'] \arrow[u, Rightarrow, "A_U"'] \arrow[ul, Rightarrow, dashed, "A" description]
\end{tikzcd}
$$

$$\mathcal{O}_{K,E} \xrightarrow{A} \mathcal{O}_{S,H}$$

$$\left(\mathcal{O}_{K,E} \xmapsto{\xi^*} \xi^*\mathcal{O}_{K,E}\right) \xRightarrow{A_W} \mathcal{O}_{S,H} \qquad (A_W \circ \xi^* = A)$$

$$\left(\mathcal{O}_{K,E} \xRightarrow{A} \mathcal{O}_{S,H}\right) \xmapsto{\xi_*} \xi_*\mathcal{O}_{S,H} \qquad (\xi_* \circ A = A_U)$$

$$\left(\left(\mathcal{O}_{K,E} \xmapsto{\xi^*} \xi^*\mathcal{O}_{K,E}\right) \xRightarrow{A_W} \mathcal{O}_{S,H}\right) \xmapsto{\xi_*} \xi_*\mathcal{O}_{S,H} \qquad (\xi_* \circ A_W \circ \xi^* = A_U)$$

# "Valid" viz?

$$\begin{tikzcd}
\Gamma(W, H\restriction_W) & \supset & \left\{\rho : W \to H\restriction_W \;\middle|\; A \circ \phi_\tau \circ \tau = \phi_\rho \circ \rho\right\} \\
W \arrow[u, mapsto, "\mathcal{O}_H"] \arrow[urr, dashed, mapsto, "A(\mathcal{O}_E) := \mathcal{O}_A" description]
\end{tikzcd}$$

# What is $\phi$?

## Fiber Bundle Category

**object** $F \hookrightarrow E \xrightarrow{\pi} K$

**morphisms** $\phi : E \to E$

## fiber transform equivariance

The fiber $F$ is the sole object of an arbitrary category $\mathcal{C}$. The bundle morphisms

$$\phi \in Hom(F, F)$$

are the set of morphisms that also preserve fiber structure.

# $\phi = (\hat{\phi}, \hat{\phi}^*, \tilde{\phi})$

$$\begin{tikzcd}
\Gamma(U, E\upharpoonright_U) \arrow[r, "\hat{\phi}^*"] & \Gamma(U', \hat{\phi}^* E\upharpoonright_{U'}) \arrow[r, "\tilde{\phi}"] & \Gamma(U', \hat{\phi}^* E\upharpoonright_{U'}) \\
U \arrow[u, mapsto, "\mathcal{O}_{K,E}"] & U' \arrow[l, "\hat{\phi}"] \arrow[u, mapsto, "\hat{\phi}^* \mathcal{O}_{K,E}"'] \arrow[ur, mapsto, "\hat{\phi}^* \mathcal{O}_{K,E}"'] &
\end{tikzcd}$$

Base $K$ Type stays same: $\hat{\phi} : K \to K$

Fiber $F$ Type stays the same: $\tilde{\phi} : \hat{\phi}^* E \to \hat{\phi}^* E$ s.t. $\pi(\hat{\phi}^* E) = \pi(\tilde{\phi}(\hat{\phi}^* E))$

Section Transform: $\phi : \tau^a\upharpoonright_U \mapsto \tau^b\upharpoonright_{U'}$

# Continuity Equivariance: $\hat{\phi}_K \circ \xi = \xi \circ \hat{\phi}_S$

$$
\begin{tikzcd}
W \arrow[r, "\xi"] \arrow[d, dashed, "\hat{\phi}_S"'] & U \arrow[d, "\hat{\phi}_K"] \\
W' \arrow[r, "\xi"'] & U'
\end{tikzcd}
$$

$$\underbrace{K \times [0,1]^m}_{S} \overset{\xi}{\longmapsto} K$$

# Field Equivariance: $A \circ (\tilde{\phi}_E \circ \hat{\phi}^*{}_K) = (\tilde{\phi}_H \circ \hat{\phi}^*{}_S) \circ A$

$$\begin{tikzcd}
\Gamma(U', \hat{\phi}^*{}_K E\restriction_{U'}) \arrow[r, Rightarrow, "A"] \arrow[d, "\tilde{\phi}_E \circ \hat{\phi}^*{}_K"'] & A[\hat{\phi}^*\mathcal{O}_{K,E}] \subset \Gamma(W', \hat{\phi}^*{}_S H\restriction_{W'}) \arrow[d, dashed, "\tilde{\phi}_H \circ \hat{\phi}^*{}_S"] \\
\Gamma(U', \tilde{\phi}_E \hat{\phi}^*{}_K E\restriction_{U'}) \arrow[r, Rightarrow, "A"] & A[\tilde{\phi}\hat{\phi}^*\mathcal{O}_{K,E}] \subset \Gamma(W', \tilde{\phi}_H \hat{\phi}^*{}_S H\restriction_{W'})
\end{tikzcd}$$

# Equivariant Artist

$$\begin{tikzcd}
\mathcal{O}_{K,E} \arrow[r, Rightarrow, "A"] \arrow[d, "\phi_\tau"'] & \mathcal{O}_A \arrow[d, dashed, "\phi_\rho"] \\
\phi_\tau \mathcal{O}_{K,E} \arrow[r, Rightarrow, "A"] & \phi_\rho \mathcal{O}_A
\end{tikzcd}$$

$$A(\tau^a) = A(\tau^b) \;\not\Longrightarrow\; \tau^a = \tau^b$$

$$A(\tau^a) = A(\tau^b) \;\Longrightarrow\; A(\phi_\tau(\tau^a)) = A(\phi_\tau(\tau^b))$$

# Multiple Fields: $\phi = (\hat{\phi}, \hat{\phi}^*, \prod_{i=0}^{n} \tilde{\phi}_i)$

$$
\begin{tikzcd}
\tilde{\phi}_0 \circ \hat{\phi}^* E_0 \cdots \otimes \cdots \arrow[dr, "\pi"] & \tilde{\phi}_i \circ \hat{\phi}^* E_i \arrow[d, "\pi"] & \cdots \otimes \cdots \tilde{\phi}_n \circ \hat{\phi}^* E_n \arrow[dl, "\pi"] & = & \prod_{i=0}^{n} \tilde{\phi}_i \circ \hat{\phi}^* E \arrow[d, "\pi"] \\
& (\hat{\phi} = \hat{\phi}_{i \in [0,n]}) \circ K & & = & \hat{\phi} \circ K
\end{tikzcd}
$$

## Add New Field to E

**construct** $\odot : F_{field} \times K \rightarrow E_{field}$

**concatenate** $\otimes : E_{field} \times E \mapsto E$ s.t. $\pi(E_{field}) = \pi(E)$

# Field Level Equivariance

$$\oslash : A \times F \to \textit{visual encoding}$$

$$\begin{tikzcd}
 & \mathcal{O}_{K,E} \arrow[r, "A"] & \mathcal{O}_{A} \arrow[dr, "\oslash F_{field}"] & \\
F_{field} \arrow[ur, "\otimes_{E} \circ \odot_{K}"] \arrow[dr, "\otimes_{E'} \circ \odot_{K'}"'] & & & \textit{visual encoding} \\
 & \mathcal{O}_{K',E'} \arrow[r, "A"'] & \mathcal{O}'_{A} \arrow[ur, "\oslash F_{field}"'] &
\end{tikzcd}$$