Mathematical Data Abstraction

Fiber Bundles "unified, dimension-independent framework" that expresses data as the mapping between continuity and fields [**butlerVectorBundleClassesForm1992**, **butlerVisualizationModelBased1989**]

Simplicial Databases systemic way to apply a schema to a fiber, i.e. provides a way to name fields and bind them to types system (e.g. `int`, `float`) [**spivakSIMPLICIALDATABASES**, **spivakDatabasesAreCategories2010**]

Sheaves on Bundles "algebraic data structure" for representing data over topological spaces [**ghristElementaryAppliedTopology2014**]

Fiber Bundles

Base Space $(K, \mathcal{T})$, $k \in U \subset K$

K

Fiber Bundles

F

K

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Fiber Space $F_k = \pi^{-1}(k)$

Fiber Bundles

$$\begin{tikzcd} F & E \\ & K \end{tikzcd}$$

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Total Space $E\restriction_U = K\restriction_U \times F\restriction_U$
for all $U \subset K$

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$$\begin{tikzcd} F \arrow[r, hook] & E \arrow[d, "\pi"'] \\ & K \arrow[u, bend right, "\tau\in\Gamma(K,\, E)"'] \end{tikzcd}$$

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Fiber Space $F_k = \pi^{-1}(k)$

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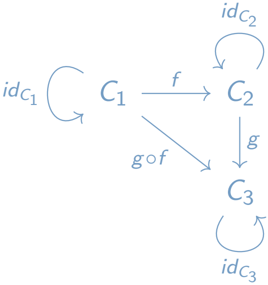
Sections

$$\Gamma(U,\, E\restriction_U) \coloneqq \left\{\tau : U \to E\restriction_U \;\middle|\; \pi(\tau(k)) = k \textit{ for all } k \in U\right\}$$

Category $\mathcal{C}$



associativity

if $f : C_1 \to C_2$, $g : C_2 \to C_3$ and $h : C_3 \to C_4$ then $h \circ (g \circ f) = (h \circ g) \circ f$

identity

for every $f : C_1 \to C_2$ there exists identity morphisms $f \circ id_{C_1} = f = id_{C_2} \circ f$

Category $\mathcal{K}$

$$
\begin{tikzcd}
C_1 \arrow[r, "f"] \arrow[dr, "g\circ f"'] \arrow[loop left, "id_{C_1}"] & C_2 \arrow[d, "g"] \arrow[loop above, "id_{C_2}"] \\
 & C_3 \arrow[loop below, "id_{C_3}"]
\end{tikzcd}
$$

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$$\begin{tikzcd}
C_1 \arrow[loop left, "id_{C_1}"] \arrow[r, "f"] \arrow[dr, "g \circ f"'] & C_2 \arrow[loop above, "id_{C_2}"] \arrow[d, "g"] \\
 & C_3 \arrow[loop below, "id_{C_3}"]
\end{tikzcd}
\qquad
\begin{tikzcd}
U_1 \arrow[loop left, "id_{U_1}"] \arrow[r, "\iota_{1,2}"] \arrow[dr, "\iota_{1,2} \circ \iota_{2,3}"'] & U_2 \arrow[loop above, "id_{U_2}"] \arrow[d, "\iota_{2,3}"] \\
 & U_3 \arrow[loop below, "id_{U_3}"]
\end{tikzcd}$$

Opposite Category $\mathcal{C}^{op}$

$$
\begin{tikzcd}
C_1 \arrow[r, "f"] \arrow[dr, "g \circ f"'] \arrow[loop left, "id_{C_1}"] & C_2 \arrow[d, "g"] \arrow[loop above, "id_{C_2}"] \\
& C_3 \arrow[loop below, "id_{C_3}"]
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
C_1 \arrow[loop left, "id_{C_1}"] & C_2 \arrow[l, "f"'] \arrow[loop above, "id_{C_2}"] \\
& C_3 \arrow[u, "g"'] \arrow[ul, "f \circ g"] \arrow[loop below, "id_{C_3}"]
\end{tikzcd}
$$

$\mathcal{C}^{op}$

$$
\begin{tikzcd}
U_1 \arrow[loop left, "id_{U_1}"] \arrow[r, "\iota_{1,2}"] \arrow[dr, "\iota_{1,2} \circ \iota_{2,3}"'] & U_2 \arrow[loop above, "id_{U_2}"] \arrow[d, "\iota_{2,3}"] \\
& U_3 \arrow[loop below, "id_{U_3}"]
\end{tikzcd}
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$\mathcal{C}^{op}$

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\begin{tikzcd}
U_1 \arrow[loop left, "id_{U_1}"] \arrow[r, "\iota_{1,2}"] \arrow[dr, "\iota_{1,2} \circ \iota_{2,3}"'] & U_2 \arrow[loop above, "id_{U_2}"] \arrow[d, "\iota_{2,3}"] \\
 & U_3 \arrow[loop below, "id_{U_3}"]
\end{tikzcd}
\qquad
\begin{tikzcd}
\Gamma(U_1, E\restriction_{U_1}) & & \Gamma(U_2, E\restriction_{U_2}) \arrow[ll, "\iota^* \restriction_{U_1}"'] \\
 & & \Gamma(U_3, E\restriction_{U_3}) \arrow[u, "\iota^* \restriction_{U_2}"'] \arrow[ull, "\iota^* \restriction_{U_2} \circ \iota^* \restriction_{U_1}"]
\end{tikzcd}
$$

Functor $F : \mathcal{C} \to \mathcal{D}$

$$\begin{tikzcd} c \arrow[d, "f"'] \\ c' \end{tikzcd}$$

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$$\begin{tikzcd} c \arrow[d, "f"'] & d \arrow[d] \\ c' & d' \end{tikzcd}$$

Functor $F : \mathcal{C} \to \mathcal{D}$

$$\begin{tikzcd} c \arrow[r, "F"] \arrow[d, "f"'] & F(c) \arrow[d, "F(f)"] \\ c' \arrow[r, "F"'] & F(c') \end{tikzcd}$$

Functor $F : \mathcal{C} \to \mathcal{D}$

$$\begin{array}{ccc} c & \xrightarrow{F} & F(c) \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle F(f)} \\ c' & \xrightarrow[F]{} & F(c') \end{array}$$

composition

$$F(g) \circ F(f) = F(g \circ f)$$

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composition

$$F(g) \circ F(f) = F(g \circ f)$$

identity

$$F(id_c) = id_{F(c)}$$

Presheaf: $\mathcal{O} : \mathcal{C}^{op} \to \boldsymbol{Set}$

$$\begin{tikzcd} E \arrow[d, "\pi"'] \\ K \end{tikzcd}$$

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$$\begin{tikzcd} E \arrow[d, "\pi"'] \\ K \arrow[u, bend right, "\tau \in \Gamma(K, E)"'] \end{tikzcd}$$

$\Gamma(U_1, E\restriction_{U_1})$

$U_1 \subset K$

Presheaf: $\mathcal{O} : \mathcal{C}^{op} \to \textbf{\textit{Set}}$

$$\begin{tikzcd} E \arrow[d, "\pi"'] \\ K \arrow[u, bend right, "\tau \in \Gamma(K, E)"'] \end{tikzcd}$$

$\Gamma(U_1, E\restriction_{U_1}) \in \textbf{\textit{Set}}$

$U_1 \in Ob(\mathcal{K}^{op})$

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U_1 \in Ob(\mathcal{K}^{op}) \arrow[u, "\mathcal{O}_K"]
\end{tikzcd}
$$

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stalk $\mathcal{O}(K)\restriction_k := \lim\limits_{U \ni k} \Gamma(U, E\restriction_U) \approx F_k$

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\Gamma(U_1,\, E\restriction_{U_1}) & & \Gamma(U_2,\, E\restriction_{U_2}) \arrow[ll, hook', "\iota^*"'] \\
U_1 \arrow[u, "\mathcal{O}_K"] \arrow[rr, hook, "\iota"'] & \arrow[uu, phantom, "\mathcal{O}_K"] & U_2 \arrow[u, "\mathcal{O}_K"']
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locality

$\tau^1, \tau^2 \in \mathcal{O}(U_i)$
$\tau^1\restriction_{U_i} = \tau^2\restriction_{U_i} \implies \tau^1 = \tau^2$

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gluing

$\tau_i \in \mathcal{O}(U_i), \tau_j \in \mathcal{O}(U_j), U_i, U_j \in \mathcal{U}$
$\tau_i\restriction_{U_i \cap U_j} = \tau_j\restriction_{U_i \cap U_j} \implies \tau\restriction_{U_i} = \tau_i$

Morphism of sheaves:

K

Morphism of sheaves:

$S \qquad K$

Morphism of sheaves:

$$S \xrightarrow{\xi} K$$

Morphism of sheaves:

$$\begin{tikzcd} S \arrow[r, "\xi"] & K \arrow[d, "\mathcal{O}_K"] \\ & \mathcal{O}_K(K) \end{tikzcd}$$

Morphism of sheaves: pullback

$$\begin{tikzcd}
S \arrow[r, "\xi"] & K \arrow[d, "\mathcal{O}_K"] \\
(\xi^*\mathcal{O}_K)(S) \arrow[r, "\xi^*"] & \mathcal{O}_K(K)
\end{tikzcd}$$

Morphism of sheaves: pullback

$$\begin{tikzcd}
S \arrow[r, "\xi"] \arrow[d, dashed] & K \arrow[d, "\mathcal{O}_K"] \\
(\xi^* \mathcal{O}_K)(S) \arrow[r, "\xi^*"] & \mathcal{O}_K(K)
\end{tikzcd}$$

Morphism of sheaves: pushforward

$$\begin{tikzcd} \mathcal{O}_S(S) \arrow[r, "\xi_*"] & (\xi_*\mathcal{O}_S)(K) \\ S \arrow[u, "\mathcal{O}_S"] \arrow[r, "\xi"] & K \arrow[u, dashed] \end{tikzcd}$$

Morphism of sheaves: pullback & pushforward

$$
\begin{tikzcd}
\mathcal{O}_S(S) \arrow[r, "\xi_*"] & (\xi_*\mathcal{O}_S)(K) \\
S \arrow[u, "\mathcal{O}_S"] \arrow[r, "\xi"] \arrow[d, dashed] & K \arrow[u, dashed] \arrow[d, "\mathcal{O}_K"] \\
(\xi^*\mathcal{O}_K)(S) \arrow[r, "\xi^*"] & \mathcal{O}_K(K)
\end{tikzcd}
$$

pullback sections

$$\begin{tikzcd} s \arrow[r, "\xi"] \arrow[d, dashed] & k \arrow[d, "\mathcal{O}_K"] \\ \{\xi^*\tau : s \to \mathbf{Set}\} \arrow[r, "\xi^*"] & \{\tau : k \to \mathbf{Set}\} \end{tikzcd}$$

$$\tau^*(s) = \tau(\xi(s)) = \tau(k)$$

pushforward sections

$$
\begin{tikzcd}
\{\rho : s \to \textbf{\textit{Set}}\} \arrow[rr, "\xi_*"] & & (\xi_* \rho : k \to \textbf{\textit{Set}}) \\
s \arrow[u, "\mathcal{O}_s"] \arrow[rr, "\xi"] & & k \arrow[u, dashed]
\end{tikzcd}
$$

$$\rho(s) = \rho_*(k) = \rho(\xi^{-1}(k))$$

pullback & pushforward

$$\begin{tikzcd}
\{\rho : s \to \mathbf{Set}\} \arrow[rr, "\xi_*"] & & (\xi_*\rho : k \to \mathbf{Set}) \\
s \arrow[u, "\mathcal{O}_s"] \arrow[d, dashed] \arrow[rr, "\xi"] & & k \arrow[u, dashed] \arrow[d, "\mathcal{O}_k"] \\
\{\xi^*\tau : s \to \mathbf{Set}\} \arrow[rr, "\xi^*"] & & \{\tau : k \to \mathbf{Set}\}
\end{tikzcd}$$

$$\tau^*(s) = \tau(\xi(s)) = \tau(k)$$

$$\rho(s) = \rho_*(k) = \rho(\xi^{-1}(k))$$

Pullback & Pushforward are Isomorphic

$$\mathcal{O}_S(S) \xrightarrow{\xi_*} (\xi_*\mathcal{O}_S)(K)$$

$$(\xi^*\mathcal{O}_K)(S) \xrightarrow{\xi^*} \mathcal{O}_K(K)$$

Pullback & Pushforward are Isomorphic

$$\begin{tikzcd}
\mathcal{O}_S(S) \arrow[r, "\xi_*"] & (\xi_*\mathcal{O}_S)(K) \\
(\xi^*\mathcal{O}_K)(S) \arrow[u, "hom_S"] \arrow[r, "\xi^*"] & \mathcal{O}_K(K) \arrow[u, "hom_K"']
\end{tikzcd}$$

Pullback & Pushforward are Isomorphic

$$\begin{tikzcd}
\mathcal{O}_S(S) \arrow[r, "\xi_*"] & (\xi_*\mathcal{O}_S)(K) \\
(\xi^*\mathcal{O}_K)(S) \arrow[u, "hom_S"] \arrow[r, "\xi^*"] & \mathcal{O}_K(K) \arrow[u, "hom_K"'] \arrow[ul, dashed, "hom_{K,S}"]
\end{tikzcd}$$

$$hom_S(\xi^*\mathcal{O}_K, \mathcal{O}_S) = hom_K(\mathcal{O}_K, \xi_*\mathcal{O}_S)$$

$A : \mathcal{O}_K \to \mathcal{O}_S$

$$
\begin{tikzcd}
\mathcal{O}_S(S) \arrow[r, "\xi_*"] & (\xi_*\mathcal{O}_S)(K) \\
(\xi^*\mathcal{O}_K)(S) \arrow[u, "A_{\xi_*}"] \arrow[r, "\xi^*"] & \mathcal{O}_K(K) \arrow[u, "A_{\xi^*}"'] \arrow[ul, dashed, "A"']
\end{tikzcd}
$$

$$A_{\xi^*} = A_{\xi_*}$$

$$A : \tau \mapsto \rho$$

$$\begin{tikzcd}
\{\rho : s \to \mathbf{Set}\} \arrow[r, "\xi_*"] & \{\xi_* \rho : k \to \mathbf{Set}\} \\
\{\xi^* \tau : s \to \mathbf{Set}\} \arrow[u, "A_{\xi_*}"] \arrow[r, "\xi^*"] & \{\tau : k \to \mathbf{Set}\} \arrow[u, "A_{\xi^*}"'] \arrow[ul, dashed, "A"]
\end{tikzcd}$$

Natural Transformation $\alpha : F \to G$

$$\begin{tikzcd}
\mathcal{C} \arrow[rr, bend left, "F"] \arrow[rr, bend right, "G"'] \arrow[rr, phantom, "\Downarrow\alpha"] & & \mathcal{D}
\end{tikzcd}$$

Natural Transformation $\alpha : F \to G$

$$\begin{tikzcd} c \arrow[d, "f"] \\ c' \end{tikzcd}$$

Natural Transformation $\alpha : F \to G$

$$\begin{tikzcd}
F(c) \arrow[d, "F(f)"'] & c \arrow[l, "F"] \arrow[d, "f"] \\
F(c') & c' \arrow[l, "F"']
\end{tikzcd}$$

Natural Transformation $\alpha : F \to G$

$$
\begin{tikzcd}
F(c) \arrow[d, "F(f)"'] & c \arrow[l, "F"] \arrow[r, "G"'] \arrow[d, "f"] & G(c) \arrow[d, "G(f)"] \\
F(c') & c' \arrow[l, "F"'] \arrow[r, "G"] & G(c')
\end{tikzcd}
$$

Natural Transformation $\alpha : F \to G$

$$
\begin{tikzcd}
F(c) \arrow[d, "F(f)"'] & c \arrow[l, "F"] \arrow[r, "G"'] \arrow[d, "f"] & G(c) \arrow[d, "G(f)"] \\
F(c') & c' \arrow[l, "F"'] \arrow[r, "G"] & G(c')
\arrow[from=1-1, to=1-3, bend left, "\alpha_c"]
\arrow[from=2-1, to=2-3, bend right, "\alpha_{c'}"']
\end{tikzcd}
$$

Natural Transformation $\alpha : F \to G$

$$
\begin{tikzcd}
F(c) \arrow[r, bend left, "\alpha_c"] \arrow[d, "F(f)"'] \arrow[dr, phantom, "\longrightarrow"] & G(c) \arrow[d, "G(f)"] \\
F(c') \arrow[r, bend right, "\alpha_{c'}"'] & G(c')
\end{tikzcd}
$$

Natural Transformation $\alpha : F \to G$

$$
\begin{tikzcd}
F(c) \arrow[r, "\alpha_c"] \arrow[d, "F(f)"'] & G(c) \arrow[d, "G(f)"] \\
F(c') \arrow[r, "\alpha_{c'}"'] & G(c')
\end{tikzcd}
$$

$$A_{\xi_*} : \mathcal{O}_K \to \xi_* \mathcal{O}_S$$

$$\begin{tikzcd}
\Gamma(U_1, E) \arrow[d, hook, "\iota^*"'] \arrow[rr, bend left, "A_{U_1}"] & U_1 \arrow[l, "\mathcal{O}_E"] \arrow[r, "\mathcal{O}_{\xi_* H}"] & \Gamma(U_1, \xi_* H) \arrow[d, hook, "\iota^*"] \\
\Gamma(U_2, E) \arrow[rr, bend right, "A_{U_2}"'] & U_2 \arrow[u, hook, "\iota"] \arrow[l, "\mathcal{O}_E"'] \arrow[r, "\mathcal{O}_{\xi_* H}"'] & \Gamma(U_2, \xi_* H)
\end{tikzcd} \tag{1}$$

$$A_{\xi^*} : \xi^*\mathcal{O}_K \to \mathcal{O}_S$$

$$\begin{tikzcd}
\Gamma(W_1, \xi^* E) \arrow[d, hook, "\iota^*"'] \arrow[rr, bend left, "A_{W_1}"] & W_1 \arrow[l, "\mathcal{O}_{\xi^* E}"] \arrow[r, "\mathcal{O}_H"'] & \Gamma(W_1, H) \arrow[d, hook, "\iota^*"] \\
\Gamma(W_2, \xi^* E) \arrow[rr, bend right, "A_{W_2}"'] & W_2 \arrow[u, hook, "\iota"] \arrow[l, "\mathcal{O}_{\xi^* E}"'] \arrow[r, "\mathcal{O}_H"] & \Gamma(W_2, H)
\end{tikzcd} \tag{2}$$