

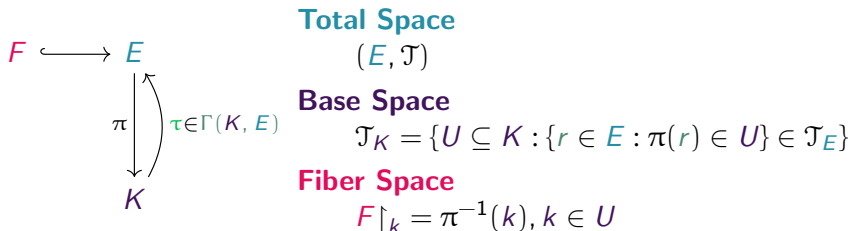
Mathematical Data Abstraction

Fiber Bundles "unified, dimension-independent framework" that expresses data as the mapping between continuity and fields
[butlerVectorBundleClassesForm1992,
butlerVisualizationModelBased1989]

Category Theory Language express constraints in specifications
[wielsManagementEvolvingSpecifications1998]

Sheaves on Bundles "algebraic data structure" for representing data over topological spaces
[ghristElementaryAppliedTopology2014]

Fiber Bundle



Sections

$$\Gamma(U, E|_U) := \{\tau : U \rightarrow E|_U \mid \pi(\tau(k)) = k \text{ for all } k \in U\}$$

Locally Trivial

for every point $k \in K$, there exists an open neighborhood $U \subseteq K$ s.t. there is a homeomorphism $\pi^{-1}(U) \xrightarrow{\varphi} U \times F$

(Globally) Trivial

$$E = K \times F$$

Data Bundle

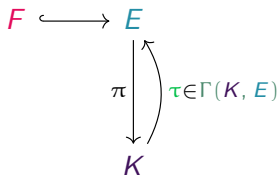
Data E

continuity + fields

Continuity K

how data elements are organized
(topological properties)

[**wilkinsonGrammarGraphics2005**],
index (key) space
[**munznerWhatDataAbstraction2014**])



Fields F

generalization of a schema - named and
typed data fields

[**spivakSIMPLICIALDATABASES**,
spivakDatabasesAreCategories2010]

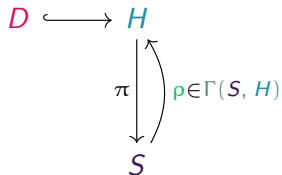
Data

[**butlerVisualizationModelBased1989**,
butlerVectorBundleClassesForm1992]

$$\tau(k) = r, k \in U \subseteq K, r \in F|_k$$

$$r = [\text{field}_1 \text{ value}_1 \quad \text{field}_2 \text{ value}_2 \quad \dots \quad \text{field}_n \text{ value}_n]$$

Graphic Bundle



Graphic H

virtual rendered output

Continuity S

topology of graphic in rendered space

Display D

generalization of renderer specifications,
e.g. $\{xy, rgba\}$, $\{xy, cymk\}$, $\{xyz, rgba\}$

Graphic

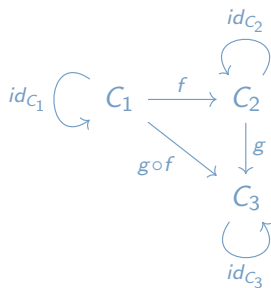
$$\Gamma(W, H|_W) := \{ \rho : W \rightarrow H|_W \mid \pi(\rho(s)) = s \text{ for all } s \in W \}$$

$$\rho(s) = d, \quad s \in W \subseteq S, \quad d \in F|_s$$

$$d = \{x, y, r, g, b\}$$

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1 Artist(data:Data) -> Graphic
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Category \mathcal{C}



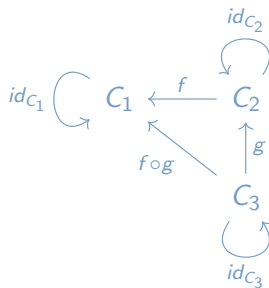
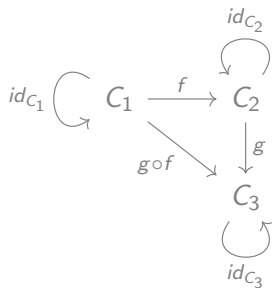
associativity

if $f : C_1 \rightarrow C_2$, $g : C_2 \rightarrow C_3$ and
 $h : C_3 \rightarrow C_4$ then
 $h \circ (g \circ f) = (h \circ g) \circ f$

identity

for every $f : C_1 \rightarrow C_2$ there exists
identity morphisms
 $f \circ id_{C_1} = f = id_{C_2} \circ f$

Opposite Category \mathcal{C}^{op}



Functor $F : \mathcal{C} \rightarrow \mathcal{D}$

composition

$$F(g) \circ F(f) = F(g \circ f)$$

$$\begin{array}{ccc} c & \xrightarrow{F} & F(c) = d \\ f \downarrow & & \downarrow F(f) \\ c' & \xrightarrow{F} & F(c') = d' \end{array}$$

identity

$$F(id_c) = id_{F(c)}$$

Presheaf: $\mathcal{O} : \mathcal{C}^{op} \rightarrow \mathbf{Set}$

$$\begin{array}{ccc}
 F & \hookrightarrow & E \\
 \downarrow \pi & \curvearrowright \tau \in \Gamma(K, E) & \\
 K & &
 \end{array}
 \qquad
 \begin{array}{ccc}
 \mathbf{Set} \ni \Gamma(U_1, E|_{U_1}) & \xleftarrow{\iota^*} & \Gamma(U_2, E|_{U_2}) \\
 \uparrow \mathcal{O}_{K,E} & & \uparrow \mathcal{O}_{K,E} \\
 Ob(\mathcal{K}^{op}) \ni U_1 & \xrightarrow[\iota]{} & U_2
 \end{array}$$

stalk

$$\begin{aligned}
 \mathcal{O}_{K,E}|_k &:= \lim_{U \ni k} \Gamma(U, E|_U) \\
 F_k &\subset \mathcal{O}_{K,E}|_k
 \end{aligned}$$

germ

$$\tau(k) \in \mathcal{O}_{K,E}|_k$$

Sheaves

A sheaf is a presheaf that satisfies the following two axioms [bakerMathsSheaf]

locality

if there exists the open covering $\mathcal{U} = \{U_i\}_{i \in I}$ of U and if $\tau^a, \tau^b \in \mathcal{O}(U_i)$ have the property $\tau^a|_{U_i} = \tau^b|_{U_i}$ for each $U_i \in \mathcal{U}$, then $\tau^a = \tau^b$

gluing

if there exists the open covering $\mathcal{U} = \{U_i\}_{i \in I}$ of U and if $\tau^i \in \mathcal{O}(U_i)$ is given for each $i \in I$ such that $\tau^i|_{U_i \cap U_j} = \tau^j|_{U_i \cap U_j}$ for each pair $U_i, U_j \in \mathcal{U}$, then there exists $\tau \in \mathcal{O}(U)$ such that $\tau|_{U_i} = \tau^i$ for each $i \in I$

Data

$$\begin{array}{c} K \\ \downarrow \mathcal{O}_{K,E} \\ \mathcal{O}_{K,E} \end{array}$$

$$F \hookrightarrow E \xrightarrow{\pi} K$$

$$\mathcal{O}_{K,E} : U \mapsto \Gamma(U, E|_U), U \subset K$$

$$\Gamma(U, E|_U) \ni \tau : U \rightarrow F|_U$$

$$\tau(k) = \{f_0 : v_0, \dots, \}, k \in U$$

Graphic

$$\begin{array}{c} \mathcal{O}_{S,H} \\ \uparrow \\ \mathcal{O}_{S,H} \\ \vdash \\ S \end{array}$$

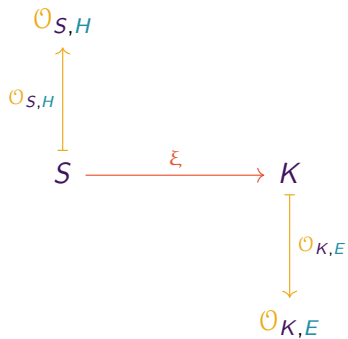
$$D \hookrightarrow H \xrightarrow{\pi} S$$

$$\mathcal{O}_{S,H} : W \mapsto \Gamma(W, E|_W), W \subset S$$

$$\Gamma(W, H|_W) \ni \rho : W \rightarrow D|_W$$

$$\rho(s) = \{d_0, \dots\}, s \in W$$

Functor: $\xi : \mathcal{S} \rightarrow \mathcal{K}$



Pullback: data to region of the visualization

$$\begin{array}{ccc} S & \xrightarrow{\xi} & K \\ \downarrow \text{ } & & \downarrow \text{ } \\ \xi^* \mathcal{O}_{K,E} & \xleftarrow{\xi^*} & \mathcal{O}_{K,E} \end{array}$$

$$\xi^* F \hookrightarrow \xi^* E \xrightarrow{\pi} S$$

$$\xi^* \mathcal{O}_{K,E} : W \mapsto \Gamma(W, \xi^* E|_W), W \subset S$$

$$\Gamma(W, \xi^* E|_W) \ni \xi^* \tau : W \rightarrow \xi^* F|_W$$

$$\xi^* \tau(s) = \tau(\xi(s)) = \tau(k)$$

Pushforward: visualization to index of data

$$\begin{array}{ccc}
 \mathcal{O}_{S,H} & \xrightarrow{\xi_*} & \xi_* \mathcal{O}_{S,H} \\
 \uparrow \mathcal{O}_{S,H} & & \uparrow \\
 S & \xrightarrow{\xi} & K
 \end{array}$$

$$\xi_* D \hookrightarrow \xi_* H \xrightarrow{\pi} K$$

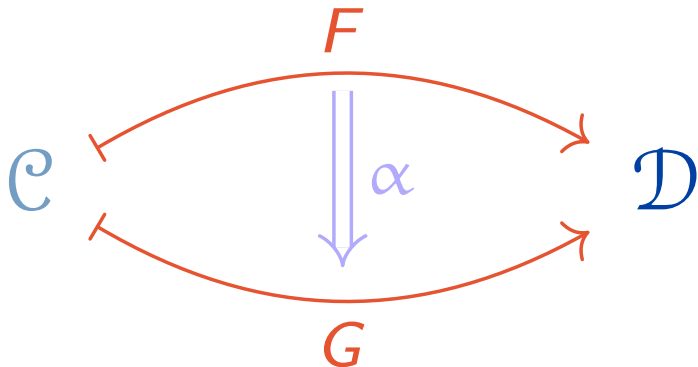
$$\xi_* \mathcal{O}_{S,H} : U \mapsto \Gamma(U, \xi_* H|_U), U \subset K$$

$$\Gamma(U, H|_U) \ni \rho : U \rightarrow D|_U$$

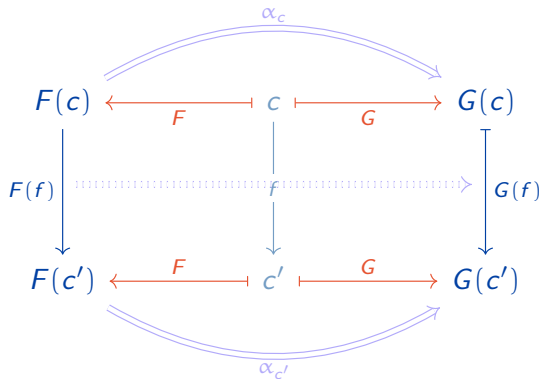
$$\xi_* \rho(s) = \rho|_{\xi^{-1}(k)} = \rho(s) \forall s \in \xi^{-1}(k)$$

$$\mathcal{O}_{K,E} \xrightarrow{A} \mathcal{O}_{S,H}$$

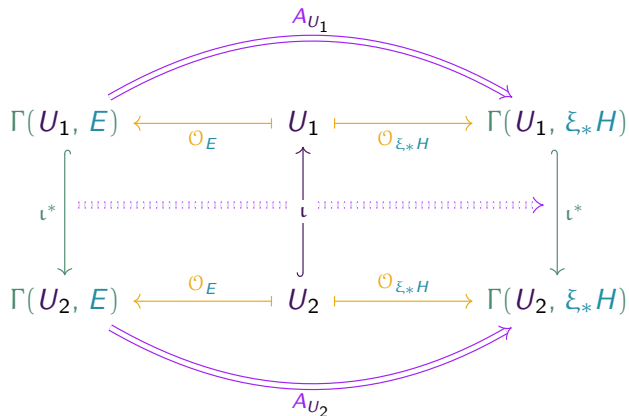
Natural Transformation $\alpha : F \Rightarrow G$



Natural Transformation $\alpha : F \Rightarrow G$



Data Space: $A_U : \mathcal{O}_{K,E} \Rightarrow \xi_* \mathcal{O}_{S,H}$



Display Space: $A_W : \xi^* \mathcal{O}_{K,E} \Rightarrow \mathcal{O}_{S,H}$

$$\begin{array}{ccccc}
 & & A_{W_1} & & \\
 & \swarrow & & \searrow & \\
 \Gamma(W_1, \xi^* E) & \xleftarrow{\mathcal{O}_{\xi^* E}} & W_1 & \xrightarrow{\mathcal{O}_H} & \Gamma(W_1, H) \\
 \downarrow \wr^* & & \uparrow & & \downarrow \wr^* \\
 \Gamma(W_2, \xi^* E) & \xleftarrow{\mathcal{O}_{\xi^* E}} & W_2 & \xrightarrow{\mathcal{O}_H} & \Gamma(W_2, H) \\
 & \swarrow & & \searrow & \\
 & & A_{W_2} & &
 \end{array}$$

Diagram illustrating the Display Space $A_W : \xi^* \mathcal{O}_{K,E} \Rightarrow \mathcal{O}_{S,H}$ for two objects W_1 and W_2 .

The diagram shows a commutative structure involving the following objects and maps:

- Top Row:** $\Gamma(W_1, \xi^* E) \xleftarrow{\mathcal{O}_{\xi^* E}} W_1 \xrightarrow{\mathcal{O}_H} \Gamma(W_1, H)$
- Bottom Row:** $\Gamma(W_2, \xi^* E) \xleftarrow{\mathcal{O}_{\xi^* E}} W_2 \xrightarrow{\mathcal{O}_H} \Gamma(W_2, H)$
- Vertical Maps:**
 - Left: $\Gamma(W_1, \xi^* E) \rightarrow \Gamma(W_2, \xi^* E)$ labeled \wr^*
 - Middle: $W_1 \rightarrow W_2$ (solid arrow pointing down) and $W_2 \rightarrow W_1$ (solid arrow pointing up)
 - Right: $\Gamma(W_1, H) \rightarrow \Gamma(W_2, H)$ labeled \wr^*
- Curved Maps:**
 - Top: $A_{W_1} : \Gamma(W_1, \xi^* E) \rightarrow \Gamma(W_1, H)$
 - Bottom: $A_{W_2} : \Gamma(W_2, \xi^* E) \rightarrow \Gamma(W_2, H)$
- Horizontal Dotted Map:** A dotted arrow from $\Gamma(W_1, \xi^* E)$ to $\Gamma(W_2, \xi^* E)$ labeled \wr^* .

Artist A

$$\begin{array}{ccc}
 \mathcal{O}_{S,H} & \xrightarrow{\xi_*} & \xi_* \mathcal{O}_{S,H} \\
 \uparrow A_W & \nwarrow A & \uparrow A_U \\
 \xi^* \mathcal{O}_{K,E} & \xleftarrow{\xi^*} & \mathcal{O}_{K,E}
 \end{array}$$

$$\text{Nat}_W(\xi^* \mathcal{O}_{K,E}, \mathcal{O}_{S,H}) = \text{Nat}_U(\mathcal{O}_{K,E}, \xi_* \mathcal{O}_{S,H})$$

Artist A

$$\begin{array}{ccc}
 \mathcal{O}_{S,H} & \xrightarrow{\xi_*} & \xi_* \mathcal{O}_{S,H} \\
 \uparrow A_W & \nwarrow A & \uparrow A_U \\
 \xi^* \mathcal{O}_{K,E} & \xleftarrow{\xi^*} & \mathcal{O}_{K,E}
 \end{array}$$

$$\mathcal{O}_{K,E} \xrightarrow{A} \mathcal{O}_{S,H}$$

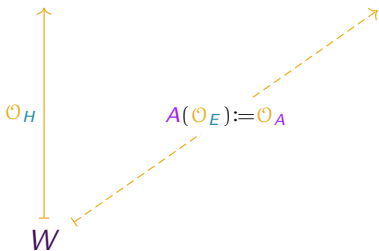
$$(\mathcal{O}_{K,E} \xrightarrow{\xi^*} \xi^* \mathcal{O}_{K,E}) \xrightarrow{A_W} \mathcal{O}_{S,H} \quad (A_W \circ \xi^* = A)$$

$$(\mathcal{O}_{K,E} \xrightarrow{A} \mathcal{O}_{S,H}) \xrightarrow{\xi_*} \xi_* \mathcal{O}_{S,H} \quad (\xi_* \circ A = A_U)$$

$$((\mathcal{O}_{K,E} \xrightarrow{\xi^*} \xi^* \mathcal{O}_{K,E}) \xrightarrow{A_W} \mathcal{O}_{S,H}) \xrightarrow{\xi_*} \xi_* \mathcal{O}_{S,H} \quad (\xi_* \circ A_W \circ \xi^* = A_U)$$

"Valid" viz?

$$\Gamma(W, H \upharpoonright_W) \supset \{ \rho : W \rightarrow H \upharpoonright_W \mid A \circ \phi_\tau \circ \tau = \phi_\rho \circ \rho \}$$



What is ϕ ?

Fiber Bundle Category

object $F \hookrightarrow E \xrightarrow{\pi} K$

morphisms $\phi : E \rightarrow E$

fiber transform equivariance

The fiber F is the sole object of an arbitrary category \mathcal{C} . The bundle morphisms

$$\phi \in \text{Hom}(F, F)$$

are the set of morphisms that also preserve fiber structure.

$$\phi = (\hat{\phi}, \hat{\phi}^*, \tilde{\phi})$$

$$\begin{array}{ccccc}
 \Gamma(U, E|_U) & \xrightarrow{\hat{\phi}^*} & \Gamma(U', \hat{\phi}^* E|_{U'}) & \xrightarrow{\tilde{\phi}} & \Gamma(U', \hat{\phi}^* E|_{U'}) \\
 \uparrow \mathcal{O}_{K,E} & & \uparrow \hat{\phi}^* \mathcal{O}_{K,E} & \nearrow \hat{\phi}^* \mathcal{O}_{K,E} & \\
 U & \xleftarrow{\hat{\phi}} & U' & &
 \end{array}$$

Base K Type stays same: $\hat{\phi} : K \rightarrow K$

Fiber F Type stays the same: $\tilde{\phi} : \hat{\phi}^* E \rightarrow \hat{\phi}^* E$ s.t.
 $\pi(\hat{\phi}^* E) = \pi(\tilde{\phi}(\hat{\phi}^* E))$

Section Transform: $\phi : \tau^a|_U \mapsto \tau^b|_{U'}$

Continuity Equivariance: $\hat{\phi}_S \circ \xi = \xi \circ \hat{\phi}_K$

$$\begin{array}{ccc}
 W & \xrightarrow{\xi} & U \\
 \hat{\phi}_S \downarrow & & \downarrow \hat{\phi}_K \\
 W' & \xrightarrow{\xi} & U'
 \end{array}$$

$$\underbrace{K \times [0, 1]^m}_S \xrightarrow{\xi} K$$

Field Equivariance: $A \circ (\tilde{\Phi}_E \circ \hat{\Phi}_K^*) = (\tilde{\Phi}_H \circ \hat{\Phi}_S^*) \circ A$

$$\begin{array}{ccc}
 \Gamma(U', \hat{\Phi}_K^* E|_{U'}) & \xRightarrow{A} & A[\hat{\Phi}^* \mathcal{O}_{K,E}] \subset \Gamma(W', \hat{\Phi}_S^* H|_{W'}) \\
 \downarrow \tilde{\Phi}_E \circ \hat{\Phi}_K^* & & \downarrow \tilde{\Phi}_H \circ \hat{\Phi}_S^* \\
 \Gamma(U', \tilde{\Phi}_E \hat{\Phi}_K^* E|_{U'}) & \xRightarrow{A} & A[\tilde{\Phi} \hat{\Phi}^* \mathcal{O}_{K,E}] \subset \Gamma(W', \tilde{\Phi}_H \hat{\Phi}_S^* H|_{W'})
 \end{array}$$

Equivariant Artist

$$\begin{array}{ccc} \mathcal{O}_{K,E} & \xRightarrow{A} & \mathcal{O}_A \\ \phi_\tau \downarrow & & \downarrow \phi_\rho \\ \phi_\tau \mathcal{O}_{K,E} & \xRightarrow{A} & \phi_\rho \mathcal{O}_A \end{array}$$

$$A(\tau^a) = A(\tau^b) \not\Rightarrow \tau^a = \tau^b$$

$$A(\tau^a) = A(\tau^b) \Rightarrow A(\phi_\tau(\tau^a)) = A(\phi_\tau(\tau^b))$$

Multiple Fields: $\phi = (\hat{\phi}, \prod_{i=0}^n \tilde{\phi}_i)$

$$\begin{array}{ccc}
 \tilde{\phi}_0 \circ E_0 \cdots \otimes \cdots \tilde{\phi}_i \circ E_i \cdots \otimes \cdots \tilde{\phi}_n \circ E_n & = & \prod_{i=0}^n \tilde{\phi}_i \circ E \\
 \swarrow \pi \quad \downarrow \pi \quad \searrow \pi & & \downarrow \pi \\
 (\hat{\phi} = \hat{\phi}_{i \in [0,n]}) \circ K & = & \hat{\phi} \circ K
 \end{array}$$