Motion and properties of space curves

Imagine that you are a daredevil pilot buzzing around in space, looping the loop or whatever it is that daredevils like to do, and that a dandy little onboard computer continuously records the coordinates, the velocity and the acceleration of (the center of mass of) your aircraft at every instant t from the moment you leave the airport at $t = 0$ to the moment you return at $t = t_f$. Let us assume, for the sake of simplicity, that the earth is flat; and let $x(t)$, $y(t)$ and $z(t)$ be, respectively, your displacement east of the airport (which is negative if you are west of it), your distance north of the airport (which is negative if you are south of it), and your altitude. These are all things your computer knows. Therefore, to each $t \in [0, t_f]$, your computer is able to assign a unique, time-dependent position vector

$$\boldsymbol{\rho} \;=\; \boldsymbol{\rho}(t) \;=\; x(t)\mathbf{i} \;+\; y(t)\mathbf{j} \;+\; z(t)\mathbf{k}. \tag{1}$$

This rule defines a vector-valued function. It also determines your flight path—a one-dimensional curve lying in three-dimensional space, which is often called a *space curve*. We assume throughout that x, y and z are all (at least twice) differentiable functions of t and that the curve has a well defined tangent everywhere, which is equivalent to saying that you always know in which direction you are headed, which is equivalent to saying that you have a well defined velocity everywhere, which is guaranteed as long as $\boldsymbol{\rho}'(t) \neq \mathbf{0}$ for all $t \in (0, t_f)$, where the prime denotes differentiation with respect to t. Such a space curve is said to be *smooth*.

The vector $\boldsymbol{\rho} = \boldsymbol{\rho}(t)$ determines not only your position or displacement from the origin but also both your velocity

$$\mathbf{v}(t) \;=\; \frac{d\boldsymbol{\rho}}{dt} \;=\; \frac{dx}{dt}\mathbf{i} \;+\; \frac{dy}{dt}\mathbf{j} \;+\; \frac{dz}{dt}\mathbf{k} \tag{2}$$

and your acceleration

$$\mathbf{a}(t) \;=\; \frac{d\mathbf{v}}{dt} \;=\; \frac{d^2\boldsymbol{\rho}}{dt^2} \;=\; \frac{d^2x}{dt^2}\mathbf{i} \;+\; \frac{d^2y}{dt^2}\mathbf{j} \;+\; \frac{d^2z}{dt^2}\mathbf{k}. \tag{3}$$

Suppose, e.g., you are flying a *helix* or corkscrew inscribed on a cylinder of radius A whose axis of symmetry rises vertically from the origin:

$$\boldsymbol{\rho}(t) \;=\; A\cos(t)\mathbf{i} \;+\; A\sin(t)\mathbf{j} \;+\; Bt\mathbf{k} \tag{4}$$

(which means, incidentally, that the airport must be at $\boldsymbol{\rho}(0) = A\mathbf{i}$); this helix is said to have *pitch* $2\pi B$ because every 2π units of time your shadow on flat earth describes a circle of radius A while your altitude increases by $2\pi B$. Then your velocity and acceleration are

$$\mathbf{v}(t) \;=\; -A\sin(t)\mathbf{i} \;+\; A\cos(t)\mathbf{j} \;+\; B\mathbf{k} \tag{5}$$

and

$$\mathbf{a}(t) \;=\; -A\cos(t)\mathbf{i} \;-\; A\sin(t)\mathbf{j}, \tag{6}$$

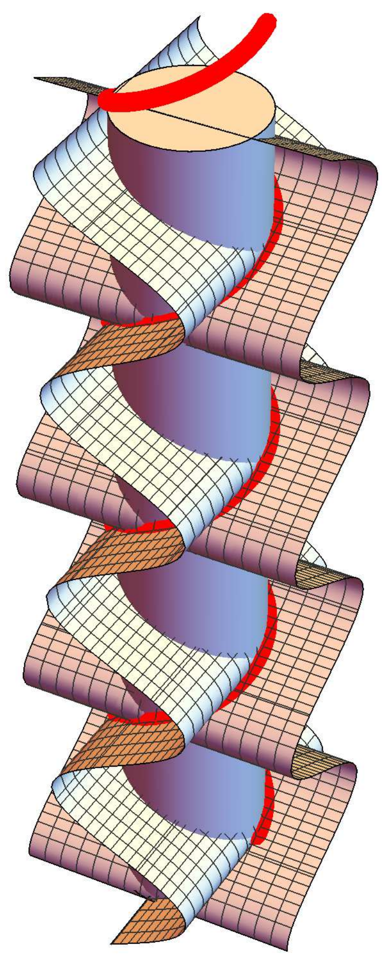
respectively, by (2)-(3). Furthermore, your direction of motion is just the unit vector $\hat{\mathbf{v}}$ in the direction of your velocity, and so from (6) we have*

$$\hat{\mathbf{v}}(t) \;=\; \frac{-A\sin(t)\mathbf{i} \;+\; A\cos(t)\mathbf{j} \;+\; B\mathbf{k}}{\sqrt{A^2+B^2}}. \tag{7}$$

Note that because $x = A\cos(t)$, $y = A\sin(t)$, $z = Bt$ and hence

$$x^2+y^2 \;=\; A^2, \qquad x \;=\; A\cos\Big(\frac{z}{B}\Big), \qquad y \;=\; A\sin\Big(\frac{z}{B}\Big) \tag{8}$$

by (1) and (4), your flight path must lie where two sinusoidal surfaces intersect both each other and a vertical cylinder, as illustrated by the diagram below, where the helix is red.



*Because the direction of motion at any point on a space curve is also the direction of the tangent line at that point, we will eventually denote $\hat{\mathbf{v}}(t)$ by $\mathbf{T}(t)$ instead, where $\mathbf{T}$ stands for unit tangent vector; see (16).

Even though time is the most natural variable to use for parameterizing your space curve, nobody insists that you have to use time—you could, for example, instead use the distance you have travelled since leaving the airport to arrive where you currently are. If this distance is denoted by s (for arc length, because it is measured along your space curve), then your speed is

$$\frac{ds}{dt} \;=\; v \;=\; |\mathbf{v}| \;=\; \sqrt{A^2+B^2} \tag{9}$$

from (5). Hence, integrating, $s = \sqrt{A^2+B^2}\,t +$ a constant. But at time $t=0$ you are at the airport, which means that your distance travelled is precisely zero, or $s=0$. It follows that the constant is zero, and the parameters s and t are related by $s=\sqrt{A^2+B^2}\,t$ or

$$t \;=\; \frac{s}{\sqrt{A^2+B^2}}. \tag{10}$$

More generally, speed will depend upon time: $v = |\mathbf{v}| = |\mathbf{v}(t)| = |\boldsymbol{\rho}'(t)|$. Then by integration we find that $\frac{ds}{dt} = v$ yields

$$s \;=\; \int_0^t v(\tau)\,d\tau \;=\; \int_0^t |\boldsymbol{\rho}'(\tau)|\,d\tau \;=\; \int_0^t \sqrt{\{x'(\tau)\}^2+\{y'(\tau)\}^2+\{z'(\tau)\}^2}\,d\tau, \tag{11}$$

which (2) reduces to $s=\sqrt{A^2+B^2}\,t$, or (10), in the special case where (4) holds.

From (4) and (10), another parametric form of your flight path is

$$\boldsymbol{\rho} = \boldsymbol{\rho}(s) = A\cos\Big(\frac{s}{\sqrt{A^2+B^2}}\Big)\mathbf{i} + A\sin\Big(\frac{s}{\sqrt{A^2+B^2}}\Big)\mathbf{j} + \frac{Bs}{\sqrt{A^2+B^2}}\mathbf{k} \tag{12}$$

for which

$$\frac{d\boldsymbol{\rho}}{ds} = \frac{1}{\sqrt{A^2+B^2}}\bigg(-A\sin\Big(\frac{s}{\sqrt{A^2+B^2}}\Big)\mathbf{i} + A\cos\Big(\frac{s}{\sqrt{A^2+B^2}}\Big)\mathbf{j} + B\mathbf{k}\bigg). \tag{13}$$

Compare (13) with (7), using (10). What do you notice? I'll answer for you:

$$\frac{d\boldsymbol{\rho}}{ds} \;=\; \hat{\mathbf{v}}. \tag{14}$$

This is a general property: whenever arc length is used as the parameter for your space curve, you can determine your direction by differentiating $\boldsymbol{\rho}$ with respect to s—there is no need to convert your answer to a unit vector, because it already is one. It isn't hard to see why: by the chain rule,

$$\frac{d\boldsymbol{\rho}}{ds} \;=\; \frac{d\boldsymbol{\rho}}{dt}\frac{dt}{ds} \;=\; \frac{\frac{d\boldsymbol{\rho}}{dt}}{\frac{ds}{dt}} \;=\; \frac{\mathbf{v}}{v} \;=\; \hat{\mathbf{v}}. \tag{15}$$

The upshot is that the unit tangent vector to the curve $\boldsymbol{\rho} = \boldsymbol{\rho}(s)$ where s denotes arc length is always its derivative, which we henceforth denote by

$$\mathbf{T} \;=\; \frac{d\boldsymbol{\rho}}{ds} \tag{16}$$

in place of $\hat{\mathbf{v}}$. Note that $\mathbf{T}$ is like $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ in the sense that despite being a unit vector, it never wears a hat. Note also that $\mathbf{T}$ can vary dramatically with s (by virtue of pointing in different directions at different points of its space curve), even though its magnitude is always precisely 1.

Time-dependent vectors satisfy useful rules for the derivative of a sum, multiple, scalar product or vector product of vectors, namely,

$$\frac{d}{dt}(\mathbf{u}+\mathbf{v}) \;=\; \frac{d\mathbf{u}}{dt} \;+\; \frac{d\mathbf{v}}{dt} \tag{17}$$

$$\frac{d}{dt}(\lambda\mathbf{u}) \;=\; \frac{d\lambda}{dt}\mathbf{u} \;+\; \lambda\frac{d\mathbf{u}}{dt} \tag{18}$$

$$\frac{d}{dt}(\mathbf{u}\cdot\mathbf{v}) \;=\; \frac{d\mathbf{u}}{dt}\cdot\mathbf{v} \;+\; \mathbf{u}\cdot\frac{d\mathbf{v}}{dt} \tag{19}$$

and

$$\frac{d}{dt}(\mathbf{u}\times\mathbf{v}) \;=\; \frac{d\mathbf{u}}{dt}\times\mathbf{v} \;+\; \mathbf{u}\times\frac{d\mathbf{v}}{dt} \tag{20}$$

where λ is an ordinary (scalar) function and $\mathbf{v}$ is any time-dependent vector (not necessarily velocity). Each of these rules is straightforward to prove by writing $\mathbf{u}$ and $\mathbf{v}$ in terms of their components, i.e., by writing

$$\mathbf{u} \;=\; u_1\mathbf{i} \;+\; u_2\mathbf{j} \;+\; u_3\mathbf{k}, \qquad \mathbf{v} \;=\; v_1\mathbf{i} \;+\; v_2\mathbf{j} \;+\; v_3\mathbf{k} \tag{21}$$

and using known results for derivatives of ordinary functions—although the proof of (20) becomes quite tedious this way.[†]

It follows from (19) that the derivative of a unit vector is always perpendicular to it. For if $\hat{\mathbf{n}}$ is the unit vector, then (19) with $\mathbf{u}=\mathbf{v}=\hat{\mathbf{n}}$ implies

$$\frac{d\hat{\mathbf{n}}}{dt}\cdot\hat{\mathbf{n}} \;=\; \frac{1}{2}\Big(\frac{d\hat{\mathbf{n}}}{dt}\cdot\hat{\mathbf{n}}+\hat{\mathbf{n}}\cdot\frac{d\hat{\mathbf{n}}}{dt}\Big) \;=\; \frac{1}{2}\frac{d(\hat{\mathbf{n}}\cdot\hat{\mathbf{n}})}{dt} \;=\; \frac{1}{2}\frac{d(1)}{dt} \;=\; 0. \tag{22}$$

In particular, with $t=s$ and $\hat{\mathbf{n}}=\mathbf{T}$ we obtain

$$\frac{d\mathbf{T}}{ds}\cdot\mathbf{T} \;=\; 0. \tag{23}$$

[†]However, there exists a much neater way to establish (20). For any permutation ijk of the integers 1, 2 and 3 beginning with 123, define

$$\epsilon_{ijk} \;=\; \begin{cases} 1 & \text{if } ijk \text{ is an even permuation of } 123 \\ -1 & \text{if } ijk \text{ is an odd permutation of } 123. \end{cases}$$

For example, $\epsilon_{132}=-1$ (because going from 123 to 132 requires an odd number of pairwise transpositions), $\epsilon_{312}=1$ (because from 123 to 312 requires an even number of pairwise transpositions), and so on. Furthermore, observe "Einstein's summation convention," namely, that repeated indices are automatically summed over all allowable values. So, for example, u_iv_i is understood to mean $u_1v_1+u_2v_2+u_3v_3$ and $\epsilon_{1jk}u_jv_k$ is understood to mean $\epsilon_{123}u_2v_3+\epsilon_{132}u_3v_2=u_2v_3-u_3v_2$ (because $1jk$ must be a permutation of 123, and so the only allowable values for the indices are $j=2$ and $k=3$ or $j=3$ and $k=2$). Then, for $i=1,2,3$, the i-th component of

$$\mathbf{u}\times\mathbf{v} \;=\; (u_2\,v_3-v_2\,u_3)\,\mathbf{i}+(v_1\,u_3-u_1\,v_3)\,\mathbf{j}+(u_1\,v_2-u_2\,v_1)\,\mathbf{k}$$

is $\epsilon_{ijk}u_jv_k$ (check it out). Because the j-th component of $\frac{d\mathbf{u}}{dt}$ is $\frac{du_j}{dt}$, it follows at once that the i-th component of $\frac{d\mathbf{u}}{dt}\times\mathbf{v}$ must be $\epsilon_{ijk}\frac{du_j}{dt}v_k$. Similarly, because the k-th component of $\frac{d\mathbf{v}}{dt}$ is $\frac{dv_k}{dt}$, it follows at once that the i-th component of $\mathbf{u}\times\frac{d\mathbf{v}}{dt}$ must be $\epsilon_{ijk}u_j\frac{dv_k}{dt}$. Adding, we deduce that the i-th component of $\frac{d\mathbf{u}}{dt}\times\mathbf{v}+\mathbf{u}\times\frac{d\mathbf{v}}{dt}$ must be $\epsilon_{ijk}\frac{du_j}{dt}v_k+\epsilon_{ijk}u_j\frac{dv_k}{dt}=\epsilon_{ijk}\big(\frac{du_j}{dt}v_k+u_j\frac{dv_k}{dt}\big)=\epsilon_{ijk}\frac{d}{dt}\big(u_jv_k\big)=\frac{d}{dt}\big(\epsilon_{ijk}u_jv_k\big)$, which is the derivative of the i-th component of $\mathbf{u}\times\mathbf{v}$. In other words, (20) must hold.

That is, $\frac{d\mathbf{T}}{ds}$ must be normal to the direction of motion. But it doesn't have to be a unit vector. So let $\mathbf{N}$ be the unit vector in the direction of $\frac{d\mathbf{T}}{ds}$, that is, define

$$\mathbf{N} \;=\; \widehat{\frac{d\mathbf{T}}{ds}} \tag{24}$$

so that $\mathbf{N}$ must be parallel to $\frac{d\mathbf{T}}{ds}$. Then there must exist $\kappa > 0$ such that

$$\frac{d\mathbf{T}}{ds} \;=\; \kappa\mathbf{N}. \tag{25}$$

We refer to κ as the *curvature* of the space curve. For example, by (16) and (13), for the helix (4) we have

$$\begin{aligned}
\kappa\mathbf{N} \;&=\; \frac{d\mathbf{T}}{ds} \;=\; \frac{d^2\boldsymbol{\rho}}{ds^2} \\
&=\; \frac{1}{\sqrt{A^2+B^2}}\frac{d}{ds}\bigg(-A\sin\Big(\frac{s}{\sqrt{A^2+B^2}}\Big)\mathbf{i} + A\cos\Big(\frac{s}{\sqrt{A^2+B^2}}\Big)\mathbf{j} + B\mathbf{k}\bigg) \\
&=\; \frac{A}{A^2+B^2}\bigg(-\sin\Big(\frac{s}{\sqrt{A^2+B^2}}\Big)\mathbf{i} - \cos\Big(\frac{s}{\sqrt{A^2+B^2}}\Big)\mathbf{j} + 0\mathbf{k}\bigg)
\end{aligned} \tag{26}$$

so that, by inspection, the curvature κ is given by

$$\kappa \;=\; \frac{|A|}{A^2+B^2} \tag{27}$$

while the unit vector $\mathbf{N}$ is given by

$$\mathbf{N} \;=\; -\sin\Big(\frac{s}{\sqrt{A^2+B^2}}\Big)\mathbf{i} - \cos\Big(\frac{s}{\sqrt{A^2+B^2}}\Big)\mathbf{j} + 0\mathbf{k} \tag{28}$$

if $A > 0$ or the negative of (28) if $A < 0$. Note that, because $\mathbf{T}$ and $\mathbf{N}$ depend upon s, usually so will κ. In this regard the helix is an exception: its curvature is constant.

Why do we call κ the curvature? Because $\mathbf{T}$ is a unit vector, it changes only by virtue of changing its direction, never by virtue of changing its magnitude. So a large value of $|\frac{d\mathbf{T}}{ds}|$ must mean that the curve is changing its direction rapidly—i.e., has high curvature. In general, because $|\mathbf{N}| = 1$, it follows from (25) that

$$\kappa \;=\; \Big|\frac{d\mathbf{T}}{ds}\Big|, \tag{29}$$

of which (27) is a special case.

To recapitulate: we have now identified unit vectors $\mathbf{T}$ and $\mathbf{N}$ that are, respectively, tangential and normal to our space curve. But $\mathbf{N}$ is not the only unit vector that is normal to our curve. Another is

$$\mathbf{B} \;=\; \mathbf{T}\times\mathbf{N}. \tag{30}$$

We distinguish between $\mathbf{N}$ and $\mathbf{B}$ by referring to $\mathbf{N}$ as the *principal normal* and to $\mathbf{B}$ as the *binormal*. Note that $\mathbf{T}$, $\mathbf{N}$ and $\mathbf{B}$ form a triad of mutually orthogonal vectors with $\mathbf{T}\times\mathbf{N} = \mathbf{B}$, $\mathbf{N}\times\mathbf{B} = \mathbf{T}$ and $\mathbf{B}\times\mathbf{T} = \mathbf{N}$ at every point of the curve. In this sense, $\mathbf{T}$, $\mathbf{N}$

and $\mathbf{B}$ are very like $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$. But in another sense, these two triads are very different: $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ are absolutely constant vectors, whereas $\mathbf{T}$, $\mathbf{N}$ and $\mathbf{B}$ determine a local orthogonal coordinate system that keeps changing its orientation as its (local) origin of coordinates moves along the curve.

Because (22) implies that the derivative of a unit vector is always perpendicular to it, and because $\mathbf{T}$, $\mathbf{N}$ and $\mathbf{B}$ are mutually orthogonal, the vector $\frac{d\mathbf{N}}{ds}$ must lie in the same plane as $\mathbf{T}$ and $\mathbf{B}$. Therefore, there must exist scalars $\chi(s)$ and $\tau(s)$ such that

$$\frac{d\mathbf{N}}{ds} = \chi(s)\mathbf{T} + \tau(s)\mathbf{B}, \tag{31}$$

implying

$$\mathbf{T} \times \frac{d\mathbf{N}}{ds} = \chi(s)\mathbf{T} \times \mathbf{T} + \tau(s)\mathbf{T} \times \mathbf{B} = \tau(s)\{-\mathbf{N}\} \tag{32}$$

because $\mathbf{T} \times \mathbf{T} = \mathbf{0}$ and $\mathbf{B} \times \mathbf{T} = \mathbf{N}$.[‡] Hence

$$\begin{aligned} \frac{d\mathbf{B}}{ds} &= \frac{d(\mathbf{T} \times \mathbf{N})}{ds} = \frac{d\mathbf{T}}{ds} \times \mathbf{N} + \mathbf{T} \times \frac{d\mathbf{N}}{ds} = \kappa(s)\mathbf{N} \times \mathbf{N} + \mathbf{T} \times \frac{d\mathbf{N}}{ds} \\ &= -\tau(s)\mathbf{N} \end{aligned} \tag{33}$$

by (20), (25) and (32) because $\mathbf{N} \times \mathbf{N} = \mathbf{0}$. We call $\tau(s)$ the *torsion*: it is a measure of a curve's (clockwise) twistiness as s increases along it. From (20), (25) and (33) we now obtain

$$\begin{aligned} \frac{d\mathbf{N}}{ds} &= \frac{d(\mathbf{B} \times \mathbf{T})}{ds} = \frac{d\mathbf{B}}{ds} \times \mathbf{T} + \mathbf{B} \times \frac{d\mathbf{T}}{ds} \\ &= -\tau(s)\mathbf{N} \times \mathbf{T} + \mathbf{B} \times \kappa(s)\mathbf{N} = -\tau(s)\{-\mathbf{B}\} + \kappa(s)\{-\mathbf{T}\} \\ &= -\kappa(s)\mathbf{T} + \tau(s)\mathbf{B}, \end{aligned} \tag{34}$$

implying that $\chi(s) = -\kappa(s)$ in (31). Equations (25), (34) and (33) are known collectively as the Serret-Frenet formulae.

The most useful thing about this local coordinate system is that it enables you to resolve your acceleration into tangential and normal components at any point along your flight path. For differentiating

$$\mathbf{v} = v\mathbf{T} \tag{35}$$

with the help of (18) and using the chain rule yields

$$\begin{aligned} \mathbf{a} = \frac{d\mathbf{v}}{dt} &= \frac{dv}{dt}\mathbf{T} + v\frac{d\mathbf{T}}{dt} \\ &= \frac{dv}{dt}\mathbf{T} + v\frac{d\mathbf{T}}{ds}\frac{ds}{dt} \\ &= \frac{dv}{dt}\mathbf{T} + v^2\frac{d\mathbf{T}}{ds} \\ &= \frac{dv}{dt}\mathbf{T} + v^2\kappa\mathbf{N} + 0 \cdot \mathbf{B} \end{aligned} \tag{36}$$

[‡]Or, if you prefer, you can use the identity for a vector triple product to obtain $\mathbf{T} \times \mathbf{B} = \mathbf{T} \times (\mathbf{T} \times \mathbf{N}) = (\mathbf{T} \cdot \mathbf{N})\mathbf{T} - (\mathbf{T} \cdot \mathbf{T})\mathbf{N} = 0 \cdot \mathbf{T} - 1 \cdot \mathbf{N} = -\mathbf{N}$.

by (25). Your acceleration therefore has only two (potentially) nonzero components in the local orthogonal coordinate system, namely, a tangential component

$$a_T = \frac{dv}{dt} = \frac{d^2s}{dt^2} \tag{37}$$

and a normal component

$$a_N = v^2\kappa. \tag{38}$$

(≥ 0). Now you understand why $\mathbf{N}$ is called the principal normal. It is because you have precisely zero acceleration in the direction of $\mathbf{B}$: the plane containing the vectors $\mathbf{T}$ and $\mathbf{N}$—the *osculating plane*—is also your plane of motion at any particular instant. Note that a_N is intrinsically nonnegative, whereas a_T may be negative. Note also that because the normal component of acceleration is the centripetal acceleration v^2/r whenever your space curve is a circle of radius r, the quantity $1/\kappa$—the reciprocal of the curvature—is known as the *radius of curvature*.

Finally, another expression for curvature now follows easily from above. If $\mathbf{T}$ is constant (meaning that the space curve is a straight line, because its direction never changes) then $\kappa = 0$ by (29). Otherwise, by (30)-(36), and using an overdot in place of a prime to denote differentiation with respect to time t, we obtain

$$\dot{\boldsymbol{\rho}} \times \ddot{\boldsymbol{\rho}} = \mathbf{v} \times \dot{\mathbf{v}} = v\mathbf{T} \times \big(a_T\,\mathbf{T} + a_N\,\mathbf{N}\big) = va_N\,\mathbf{B} \tag{39}$$

because $\mathbf{T} \times \mathbf{T} = \mathbf{0}$. Hence, taking the magnitude of both sides, $|\dot{\boldsymbol{\rho}} \times \ddot{\boldsymbol{\rho}}| = va_N \cdot 1 = v^3\kappa$ or

$$\kappa = \frac{|\dot{\boldsymbol{\rho}} \times \ddot{\boldsymbol{\rho}}|}{v^3} = \frac{|\dot{\boldsymbol{\rho}} \times \ddot{\boldsymbol{\rho}}|}{|\dot{\boldsymbol{\rho}}|^3}. \tag{40}$$

Problem

For the three-dimensional motion defined by

$$\boldsymbol{\rho} \;=\; t\mathbf{i} \;+\; \cos^2(t)\mathbf{j} \;+\; \sin^2(t)\mathbf{k}$$

find the unit tangent vector $\mathbf{T}$, the principal unit normal vector $\mathbf{N}$, the curvature κ and the binormal vector $\mathbf{B}$ at the moment when $t = \frac{1}{6}\pi$. Check that $\mathbf{T}$, $\mathbf{N}$ and $\mathbf{B}$ are mutually orthogonal.

Solution

Again using an overdot to denote differentiation with respect to time have

$$\dot{\boldsymbol{\rho}} = \mathbf{i} - 2\cos(t)\sin(t)\mathbf{j} + 2\sin(t)\cos(t)\mathbf{k} = \mathbf{i} - \sin(2t)\mathbf{j} + \sin(2t)\mathbf{k} = \mathbf{i} + \sin(2t)\{-\mathbf{j} + \mathbf{k}\},$$

hence $v = |\dot{\boldsymbol{\rho}}| = \sqrt{1 + 2\sin^2(2t)}$,

$$a_T \;=\; \frac{dv}{dt} \;=\; \frac{2\sin(4t)}{\sqrt{1 + 2\sin^2(2t)}}$$

and $\ddot{\boldsymbol{\rho}} = 0 \cdot \mathbf{i} + 2\cos(2t)\{-\mathbf{j} + \mathbf{k}\}$. So, for $t = \frac{1}{6}\pi$, we have $\dot{\boldsymbol{\rho}} = \mathbf{i} + \frac{1}{2}\sqrt{3}\{-\mathbf{j} + \mathbf{k}\}$, implying

$$\mathbf{T} \;=\; \hat{\boldsymbol{\rho}} \;=\; \sqrt{\tfrac{2}{5}}\mathbf{i} + \sqrt{\tfrac{3}{10}}\{-\mathbf{j} + \mathbf{k}\},$$

together with $\ddot{\boldsymbol{\rho}} = -\mathbf{j} + \mathbf{k}$ and $a_T = \sqrt{\frac{6}{5}}$. So $\ddot{\boldsymbol{\rho}} = a_T\mathbf{T} + a_N\mathbf{N}$ implies $a_N\mathbf{N} = \ddot{\boldsymbol{\rho}} - a_T\mathbf{T} = -\frac{2}{5}\sqrt{3}\mathbf{i} + \frac{2}{5}\{-\mathbf{j} + \mathbf{k}\} = \frac{2}{5}\{-\sqrt{3}\mathbf{i} - \mathbf{j} + \mathbf{k}\}$ after simplification. The magnitude of this vector is $a_N = 2/\sqrt{5}$, and its direction is

$$\mathbf{N} \;=\; \tfrac{1}{\sqrt{5}}\{-\sqrt{3}\mathbf{i} - \mathbf{j} + \mathbf{k}\}.$$

Because $v = \sqrt{\frac{5}{2}}$ when $t = \frac{1}{6}\pi$, we now deduce from $a_N = v^2\kappa$ that

$$\kappa \;=\; \tfrac{4}{5\sqrt{5}}.$$

Finally,

$$\mathbf{B} \;=\; \mathbf{T} \times \mathbf{N} \;=\; 0\,\mathbf{i} - \tfrac{1}{\sqrt{2}}\{\mathbf{j} + \mathbf{k}\},$$

and it is easily verified that $\mathbf{T} \cdot \mathbf{N} = \mathbf{T} \cdot \mathbf{B} = \mathbf{N} \cdot \mathbf{B} = 0$.