Double integrals

Suppose that you wish to find the mass M of a rectangular plate of constant or "uniform" density (= mass per unit area) f, which you happen to know. Then all you have to do is multiply the area of the plate by its density: immediately, you know its mass. For the sake of definiteness, suppose that the plate occupies the rectangular region or "domain"

$$D = \{(x,y)|0 \le x \le a, 0 \le y \le b\} \tag{1}$$

having sides of length a, b with vertices at $(0,0)$, $(a,0)$, (a,b) and $(0,b)$, and that the plate has uniform density 2 kilograms per square meter. Then because $f = 2$ and the plate has area ab, its mass is just $M = f \cdot ab = 2ab$.

But what if the density isn't uniform? Again, for the sake of definiteness, let us suppose that the density varies over the rectangular domain D according to

$$f(x,y) = x + y. \tag{2}$$

Then we can't just multiply density by area because the density isn't constant. However, we can approximate the mass by chopping D into a large number mn of equal rectangles of horizontal dimension

$$\Delta x = \frac{a}{m} \tag{3}$$

and vertical dimension

$$\Delta y = \frac{b}{n} \tag{4}$$

Let D_{ij} denote the rectangle with vertices at $((i-1)\Delta x, (j-1)\Delta x)$, $(i\Delta x, (j-1)\Delta x)$, $(i\Delta x, j\Delta x)$ and $((i-1)\Delta x, j\Delta x)$. Then, if m and n are both large and hence the area $\Delta x \Delta y$ of D_{ij} is small, the density will be "almost constant" over it—the rectangle just isn't big enough for f to vary by much. And since its value is pretty much the same everywhere in this tiny rectangle, it shouldn't much matter where we choose to evaluate f. Thus M, which is the sum of the masses of all the rectangles, should be well approximated by

$$\begin{aligned} &\sum_{i=1}^{m}\sum_{j=1}^{n}\left\{\begin{matrix}\text{A value of } x+y \text{ in}\\ \text{the rectangle } D_{ij}\end{matrix}\right\} \cdot \left\{\begin{matrix}\text{Area of the}\\ \text{rectangle } D_{ij}\end{matrix}\right\} \\ &= \sum_{i=1}^{m}\sum_{j=1}^{n}\left\{\begin{matrix}\text{A value of } x+y \text{ in}\\ \text{the rectangle } D_{ij}\end{matrix}\right\} \cdot \Delta x \Delta y. \end{aligned} \tag{5}$$

Now,

$$M \approx \sum_{i=1}^{m}\sum_{j=1}^{n}\left\{\begin{matrix}\text{A value of } x+y \text{ in}\\ \text{the rectangle } D_{ij}\end{matrix}\right\} \cdot \Delta x \Delta y$$

is not terribly precise for mathematical purposes, but because $x+y$ is least in D_{ij} at the lower left-hand corner $(x,y) = ((i-1)\Delta x, (j-1)\Delta x)$ and greatest in D_{ij} at the upper right-hand corner $(x,y) = (i\Delta x, j\Delta x)$, we must have

$$L \le M \le U \tag{6}$$

where

$$L = \sum_{i=1}^{m}\sum_{j=1}^{n}\{(i-1)\Delta x + (j-1)\Delta x\}\Delta x\Delta y \tag{7}$$

(sum of all lower left-hand corner values) and

$$U = \sum_{i=1}^{m}\sum_{j=1}^{n}\{i\Delta x + j\Delta x\}\Delta x\Delta y. \tag{8}$$

(sum of all upper right-hand corner values). Inequalities (6) are valid for any value of m or n, however large.

Recall from yesteryear that the sum of the first m natural numbers is $\frac{1}{2}m(m+1)$. In summation notation, and emphasizing that the symbol used for summation index is arbitrary:

$$\sum_{i=1}^{m} i = \sum_{j=1}^{m} j = \sum_{k=1}^{m} k = \tfrac{1}{2}m(m+1). \tag{9}$$

Thus, because Δx and Δy are independent of i and j, from (7) we obtain

$$\begin{aligned}
L &= \sum_{i=1}^{m}\sum_{j=1}^{n}\{(i-1)\Delta x + (j-1)\Delta y\}\Delta x\Delta y \\
&= \Delta x\Delta y\sum_{i=1}^{m}\sum_{j=1}^{n}\{(i-1)\Delta x + (j-1)\Delta y\} \\
&= \Delta x\Delta y\sum_{i=1}^{m}\left\{\sum_{j=1}^{n}\{(i-1)\Delta x + (j-1)\Delta y\}\right\} \\
&= \Delta x\Delta y\sum_{i=1}^{m}\left\{\sum_{j=1}^{n}(i-1)\Delta x + \sum_{j=1}^{n}(j-1)\Delta y\right\} \\
&= \Delta x\Delta y\sum_{i=1}^{m}\left\{(i-1)\Delta x\sum_{j=1}^{n}1 + \Delta y\sum_{j=1}^{n}(j-1)\right\} \\
&= \Delta x\Delta y\sum_{i=1}^{m}\left\{(i-1)\Delta x\cdot n + \Delta y\Bigl(\sum_{j=1}^{n}j - \sum_{j=1}^{n}1\Bigr)\right\} \\
&= \Delta x\Delta y\sum_{i=1}^{m}\left\{n\Delta x\,(i-1) + \Delta y\Bigl(\tfrac{1}{2}n(n+1) - n\Bigr)\right\} \\
&= \Delta x\Delta y\sum_{i=1}^{m}\left\{n\Delta x\,(i-1) + \tfrac{1}{2}n(n-1)\Delta y\right\}
\end{aligned} \tag{10}$$

after using (9) with $m = n$ and simplifying. Now, because Δx, Δy and n are all independent of i, we further obtain

$$
\begin{aligned}
L &= \Delta x \Delta y \left\{ \sum_{i=1}^{m} n\Delta x\,(i-1) + \sum_{i=1}^{m} \left\{ \tfrac{1}{2} n(n-1)\Delta y \right\} \right\} \\
&= \Delta x \Delta y \left\{ n\Delta x \sum_{i=1}^{m} (i-1) + \left\{ \tfrac{1}{2} n(n-1)\Delta y \right\} \sum_{i=1}^{m} 1 \right\} \\
&= \Delta x \Delta y \left\{ n\Delta x \left(\sum_{i=1}^{m} i - \sum_{i=1}^{m} 1 \right) + \tfrac{1}{2} n(n-1)\Delta y \sum_{i=1}^{m} 1 \right\} \\
&= \Delta x \Delta y \left\{ n\Delta x \left(\tfrac{1}{2} m(m+1) - m \right) + \tfrac{1}{2} n(n-1)\Delta y \cdot m \right\} \\
&= \tfrac{1}{2} \Delta x \Delta y \left\{ (m-1)\Delta x + (n-1)\Delta y \right\} mn \\
&= \tfrac{1}{2} ab \left\{ a\left(1 - \frac{1}{m}\right) + b\left(1 - \frac{1}{n}\right) \right\}
\end{aligned}
\tag{11}
$$

after using (3)–(4) and simplifying. Similarly, from (8) we obtain

$$
\begin{aligned}
U &= \sum_{i=1}^{m} \sum_{j=1}^{n} \{ i\Delta x + j\Delta y \} \Delta x \Delta y \\
&= \Delta x \Delta y \sum_{i=1}^{m} \left\{ \sum_{j=1}^{n} i\Delta x + \sum_{j=1}^{n} j\Delta y \right\} \\
&= \Delta x \Delta y \sum_{i=1}^{m} \left\{ i\Delta x \sum_{j=1}^{n} 1 + \Delta y \sum_{j=1}^{n} j \right\} \\
&= \Delta x \Delta y \sum_{i=1}^{m} \left\{ i\Delta x \cdot n + \Delta y \cdot \tfrac{1}{2} n(n+1) \right\} \\
&= \Delta x \Delta y \left\{ \sum_{i=1}^{m} n\Delta x\, i + \sum_{i=1}^{m} \left\{ \tfrac{1}{2} n(n+1)\Delta y \right\} \right\} \\
&= \Delta x \Delta y \left\{ n\Delta x \sum_{i=1}^{m} i + \tfrac{1}{2} n(n+1)\Delta y \sum_{i=1}^{m} 1 \right\} \\
&= \Delta x \Delta y \left\{ n\Delta x \cdot \tfrac{1}{2} m(m+1) + \tfrac{1}{2} n(n+1)\Delta y \cdot m \right\} \\
&= \tfrac{1}{2} \Delta x \Delta y \left\{ (m+1)\Delta x + (n+1)\Delta y \right\} mn \\
&= \tfrac{1}{2} ab \left\{ a\left(1 + \frac{1}{m}\right) + b\left(1 + \frac{1}{n}\right) \right\}
\end{aligned}
\tag{12}
$$

after using (9) and simplifying. Hence, from (6),

$$
\tfrac{1}{2} ab \left\{ a\left(1 - \frac{1}{m}\right) + b\left(1 - \frac{1}{n}\right) \right\} \;\le\; M \;\le\; \tfrac{1}{2} ab \left\{ a\left(1 + \frac{1}{m}\right) + b\left(1 + \frac{1}{n}\right) \right\}. \tag{13}
$$

Now take the limit of (13) as both $m \to \infty$ and $n \to \infty$, so that the number of rectangles becomes infinite. We obtain

$$\tfrac{1}{2}ab\{a(1-0)+b(1-0)\} \;\le\; M \;\le\; \tfrac{1}{2}ab\{a(1+0)+b(1+0)\}$$

or

$$\tfrac{1}{2}ab(a+b) \;\le\; M \;\le\; \tfrac{1}{2}ab(a+b), \tag{14}$$

implying

$$M \;=\; \tfrac{1}{2}ab(a+b). \tag{15}$$

This is the *exact* value of the total mass of the plate with non-uniform density.

Our analysis has wider implications. From (5), what we have shown is that

$$\lim_{\substack{m\to\infty\\ n\to\infty}} \sum_{i=1}^{m}\sum_{j=1}^{n} \left\{ \begin{matrix} \text{A value of } x+y \text{ in} \\ \text{the rectangle } D_{ij} \end{matrix} \right\} \cdot \Delta x \Delta y$$

is independent of which value of $x+y$ in the rectangle D_{ij} we choose. But in the above expression, we can replace $x+y$ by any other continuous function f of x and y, and the double limit is still independent of which $f(x,y)$ in the rectangle D_{ij} we choose. Essentially the only difference is that the maximum and minimum of f over the rectangle D_{ij} need not occur at the upper right-hand and bottom left-hand corners, respectively—but they still occur somewhere, because f is continuous, and that is all we need for the result. We write

$$\lim_{\substack{m\to\infty\\ n\to\infty}} \sum_{i=1}^{m}\sum_{j=1}^{n} \left\{ \begin{matrix} \text{A value of } f(x,y) \text{ in} \\ \text{the rectangle } D_{ij} \end{matrix} \right\} \cdot \Delta x \Delta y \;=\; \iint_D f(x,y)\,dA \tag{16}$$

and call (16) the double integral of f over the domain or region D.

Now, to calculate (15) we performed four operations: we summed twice (with respect to i and j) and then took two limits (with respect to m and n). As long as we sum over i before taking a limit with respect to m, and likewise sum over j before taking a limit with respect to n, however, the order in which we perform these operations is immaterial. There are therefore three ways we can do it:

I: Sum with respect to i and j, then take limits with respect to m and n

II Sum with respect to j, take limit with respect to n, then sum with respect to i, and finally take limit with respect to m.

III: Sum with respect to i, take limit with respect to m, then sum with respect to j, and finally take limit with respect to n

Method I, which we used to obtain (15), can be regarded as the fundamental meaning of double integration. But Methods II and III will always give the same result. For example, if we take the limit as $n \to \infty$ in the last line of (10)—having already summed over j—then, on substituting from (4), we first obtain

$$
\begin{aligned}
&\lim_{n\to\infty} \Delta x \Delta y \sum_{i=1}^{m} \Big\{ n\Delta x\,(i-1) + \tfrac{1}{2} n(n-1)\Delta y \Big\} \\
&\quad = \lim_{n\to\infty} b\Delta x \sum_{i=1}^{m} \Big\{ \Delta x\,(i-1) + \tfrac{1}{2}\Big(1-\frac{1}{n}\Big) b \Big\} \;=\; b\Delta x \sum_{i=1}^{m} \Big\{ \Delta x\,(i-1) + \tfrac{1}{2} b \Big\}.
\end{aligned}
\tag{17}
$$

If we now sum over m, then (17) becomes

$$
b\Delta x \Big\{ \tfrac{1}{2} m(m-1)\Delta x + \tfrac{1}{2} bm \Big\} \;=\; ab\Big\{ \tfrac{1}{2}\Big(1-\frac{1}{m}\Big) a + \tfrac{1}{2} b \Big\} \tag{18}
$$

on using (3), and taking the limit as $m \to \infty$ reproduces (15). Likewise for Method III.

Summing over both i and j and taking limits as both $m \to \infty$ and $n \to \infty$ means calculating a double integral. Summing over j and taking a limit as $n \to \infty$ means integrating with respect to y. Summing over i and taking a limit as $m \to \infty$ means integrating with respect to x. Thus Method II and Method III are two different ways of using iterated integration for calculating a double integral, and they always yield the same result (or should).

For Method II, one first holds x constant and performs a "partial" integration with respect to y along a vertical strip from the lowest y value along that strip—y_L, say—to the uppermost y value along that very same strip—y_U, say.* Having done the partial integration with respect to y, one then integrates horizontally with respect to x across all of the (vertical) strips from the leftmost x value—x_L, say—to the rightmost x value—x_R, say.† Method II is thus equivalent to integrating first with respect to y (the "inner" integration), and then with respect to x (the "outer" integration). In this way we compute the iterated integral

$$
\int_{x=x_L}^{x=x_R} \Big\{ \int_{y=y_L}^{y=y_U} f(x,y)\,dy \Big\}\,dx.
$$

For f defined by (2), we obtain

$$
\begin{aligned}
\int_{x=x_L}^{x=x_R} \Big\{ \int_{y=y_L}^{y=y_U} f(x,y)\,dy \Big\}\,dx &= \int_{x=0}^{x=a} \Big\{ \int_{y=0}^{y=b} \{x+y\}\,dy \Big\}\,dx \;=\; \int_{0}^{a} \Big\{ \big\{ xy + \tfrac{1}{2} y^2 \big\}\big|_0^b \Big\}\,dx \\
&= \int_{0}^{a} \Big\{ x\cdot b + \tfrac{1}{2} b^2 - \big\{ x\cdot 0 + \tfrac{1}{2} 0^2 \big\} \Big\}\,dx \\
&= \int_{0}^{a} \Big\{ b\,x + \tfrac{1}{2}\,b^2 \Big\}\,dx \;=\; \big\{ \tfrac{1}{2}\,bx^2 + \tfrac{1}{2} b^2 x \big\}\big|_0^a \\
&= \tfrac{1}{2}\,ba^2 + \tfrac{1}{2} b^2 a - \{ \tfrac{1}{2}\,b\cdot 0^2 + \tfrac{1}{2} b^2\cdot 0 \} \;=\; \tfrac{1}{2} ab(a+b)
\end{aligned}
\tag{19}
$$

*For D defined by (1), we have $y_L = 0$ and $y_U = b$, but either y_L or y_U may depend on x for another domain.

†For D defined by (1), we have $x_L = 0$ and $x_R = a$.

agreeing, of course, with (15).

For Method III, one first holds y constant and performs a partial integration with respect to x along a horizontal strip from the leftmost x value along that strip—x_L, say—to the rightmost x value along that very same strip—x_R, say.‡ Having done the partial integration with respect to x, one then integrates vertically with respect to y across all of the (horizontal) strips from the lowest y value—y_L, say—to the uppermost y value—y_U, say.§ Method III is thus equivalent to integrating first with respect to x (the "inner" integration), and then with respect to y (the "outer" integration). In this way we compute the iterated integral

$$\int_{y=y_L}^{y=y_U} \left\{ \int_{x=x_L}^{x=x_R} f(x,y)\,dx \right\} dy.$$

For f defined by (2), we obtain

$$\begin{aligned}
\int_{y=y_L}^{y=y_U} \left\{ \int_{x=x_L}^{x=x_R} f(x,y)\,dx \right\} dy \;&=\; \int_{y=0}^{y=b} \left\{ \int_{x=0}^{x=a} \{x+y\}\,dx \right\} dy \;=\; \int_{0}^{b} \left\{ \left\{\tfrac{1}{2}x^2 + yx\right\}\Big|_0^a \right\} dy \\
&=\; \int_0^b \left\{ \tfrac{1}{2}a^2 + ay - \left\{\tfrac{1}{2}0^2 + y\cdot 0\right\} \right\} dy \\
&=\; \int_0^b \left\{ \tfrac{1}{2}a^2 + ay \right\} dy \;=\; \left\{\tfrac{1}{2}\,a^2 y + \tfrac{1}{2}ay^2\right\}\Big|_0^b \\
&=\; \tfrac{1}{2}a^2 b + \tfrac{1}{2}ab^2 - \left\{\tfrac{1}{2}\,a^2\cdot 0 + \tfrac{1}{2}a\cdot 0^2\right\} \;=\; \tfrac{1}{2}ab(a+b)
\end{aligned} \tag{20}$$

again in line with (15). The upshot is that for D defined by (1), we have just confirmed that

$$\iint_D \{x+y\}\,dA \;=\; \int_0^a \left\{ \int_0^b \{x+y\}\,dy \right\} dx \;=\; \int_0^b \left\{ \int_0^a \{x+y\}\,dx \right\} dy \;=\; \tfrac{1}{2}ab(a+b). \tag{21}$$

Methods I, II and III do indeed all yield the same result—as, of course, they must.

Everything generalizes in the obvious way. We don't have to chop D into lots of equal rectangles—we can chop it up any way we like, as long as we cover it completely. Moreover, D does not have to be a rectangle—it can be a triangular region, a circular disk, or any other region we care to think of. But only in principle does Method I generalize in this way.

In practice, however, Method I is never used—we always use either Method II or Method III. Sometimes the choice between these two is a matter of indifference, as illustrated by a comparison of (19) with (20). At other times, one way is easier than the other, either because an anti-derivative is known with respect to x but not with respect to y, or vice versa, or because variable limits of integration are easier to deal with by one of the two methods.

These points are most readily appreciated through examples. There follow two.

‡For D defined by (1), we have $x_L = 0$ and $x_R = a$, but either x_L or x_R may depend on y for another domain.

§For D defined by (1), we have $y_L = 0$ and $y_U = b$.

Problem

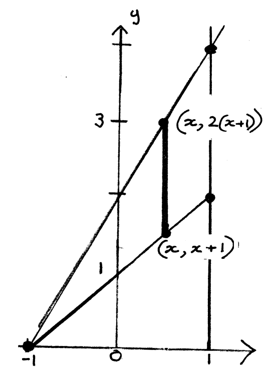
Evaluate

$$I \;=\; \iint\limits_{T} \{x + y^2\}\, dA$$

where T is a triangular region with vertices at $(-1, 0)$, $(1, 2)$ and $(1, 4)$.

Solution

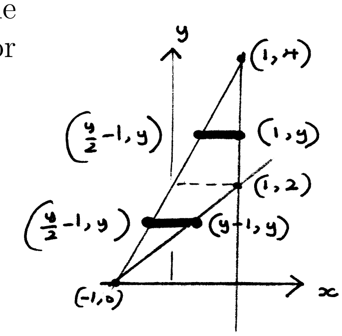
A line with slope m through $(-1, 0)$ has equation $y = m(x+1)$. So the upper and lower boundaries of the triangular region are $y = 2(x + 1)$ and $y = x + 1$, respectively. Hence, by Method II:



$$\begin{aligned}
I \;&=\; \iint\limits_{\substack{x+1\le y\le 2(x+1)\\ -1\le x\le 1}} \{x + y^2\}\, dA \;=\; \int_{-1}^{1}\int_{x+1}^{2(x+1)} \{x + y^2\}\, dy\, dx \\
&=\; \int_{-1}^{1} \left\{xy + \tfrac{1}{3}y^3\right\}\Big|_{x+1}^{2(x+1)} dx \;=\; \int_{-1}^{1} \left\{x(1 + x) + \tfrac{7}{3}(x + 1)^3\right\} dx \\
&=\; \int_{-1}^{1} x\, dx + \int_{-1}^{1} x^2\, dx + \tfrac{7}{3}\int_{-1}^{1} (x + 1)^3\, dx \\
&=\; 0 + 2\int_{0}^{1} x^2\, dx + \tfrac{7}{12}(x + 1)^4\Big|_{-1}^{1} \;=\; \tfrac{2}{3} + \tfrac{7}{12}(2^4 - 0) \;=\; 10
\end{aligned}$$

because x is odd and x^2 is even.

Alternatively, by Method III (which is less appealing because it will mean two integrations instead of one): Now the left-hand boundary of the integration region is always $x = \frac{1}{2}y - 1$ but the right-hand boundary is $x = y - 1$ for $0 \le y \le 2$ but $x = 1$ for $2 \le y \le 4$ (hence the need for two integrations). We obtain



$$\begin{aligned}
I \;&=\; \int_{0}^{2}\int_{\frac{1}{2}y-1}^{y-1} \{x + y^2\}\, dx\, dy \;+\; \int_{2}^{4}\int_{\frac{1}{2}y-1}^{1} \{x + y^2\}\, dx\, dy \\
&=\; \int_{0}^{2} \left\{\tfrac{1}{2}(x + y^2)^2\right\}\Big|_{\frac{1}{2}y-1}^{y-1} dy \;+\; \int_{2}^{4} \left\{\tfrac{1}{2}(x + y^2)^2\right\}\Big|_{\frac{1}{2}y-1}^{1} dy \\
&=\; \tfrac{1}{2}\int_{0}^{2} \left\{y^3 + \tfrac{3}{4}y^2 - y\right\} dy \;+\; \tfrac{1}{2}\int_{2}^{4} \left\{y + \tfrac{15}{4}y^2 - y^3\right\} dy \\
&=\; \tfrac{1}{8}\left\{y^4 + y^3 - 2y^2\right\}\Big|_{0}^{2} \;+\; \tfrac{1}{8}\left\{2y^2 + 5y^3 - y^4\right\}\Big|_{2}^{4} \\
&=\; \tfrac{1}{8}(16 - 0) \;+\; \tfrac{1}{8}(96 - 32) \;=\; 2 + 8 \;=\; 10.
\end{aligned}$$

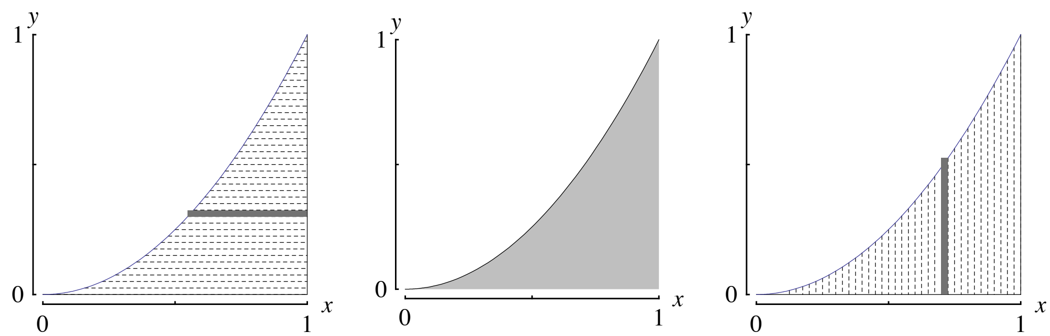
Problem

Calculate

$$\int_0^1 \int_{\sqrt{y}}^1 y x^{-3} e^{x^2}\, dx\, dy$$

by first reversing the order of integration.

Solution



With respect to Lecture 11, the iterated integral is of Type III: first integrate with respect to x between $x = \sqrt{y}$ and $x = 1$, then integrate with respect to y between $y = 0$ and $y = 1$ (diagram on left). So, if we instead regard it as a Type-I double integral, then the region covered is that which lies to the right of the curve $x = \sqrt{y}$, to the left of the line $x = 1$ and above the line $y = 0$, which implies below $y = 1$ (diagram in middle). The curve $x = \sqrt{y}$ is, of course, the same as the curve $y = x^2$. So, the integral can instead be computed as a Type-III iterated integral, first integrating with respect to y between $y = 0$ and $y = x^2$ and then integrating with respect to x between $x = 0$ and $x = 1$ (diagram on right). Thus

$$\int_0^1 \int_{\sqrt{y}}^1 y x^{-3} e^{x^2}\, dx\, dy \;=\; \int_0^1 \int_0^{x^2} x^{-3} e^{x^2} y\, dy\, dx \;=\; \int_0^1 \tfrac{1}{2} x^{-3} e^{x^2} y^2 \Big|_0^{x^2}\, dx$$

$$= \int_0^1 \tfrac{1}{2} x^{-3} e^{x^2} (x^4 - 0^2)\, dx \;=\; \tfrac{1}{2} \int_0^1 x e^{x^2}\, dx \;=\; \tfrac{1}{4} \int_{0^2}^{1^2} e^u\, du \;=\; \tfrac{1}{4} e^u \Big|_0^1 \;=\; \tfrac{1}{4}(e - 1)$$

on using the substitution $u = x^2$ (which implies $du = 2x\, dx$ or $x\, dx = \frac{1}{2} du$).