Shadow area elements

Suppose that we wish to calculate the flux of a vector field $\mathbf{F}$ across an open planar surface S that lies in an oblique plane, that is, a plane not parallel to a coordinate plane (or, if you prefer, a plane whose normal is not parallel to a coordinate axis). It is easy to find the unit normal $\mathbf{n}$ (by Lecture 4), but what is the natural surface area element dS?

Whatever the oblique planar region, S, it will make a shadow on the x-y plane under a sun whose light shines from the direction of

$$\boldsymbol{\sigma} = \mathbf{k}, \tag{1}$$

i.e., vertically above. Let this shadow region be parameterized in its natural coordinates by

$$\mathbf{r} = \zeta(u,v)\mathbf{i} + \psi(u,v)\mathbf{j}, \qquad a_1 \le u \le a_2, \ b_1 \le v \le b_2 \tag{2}$$

(for example, if the shadow region is a circular disk of radius c with center at the origin, then $u = r$, $v = \theta$, $\zeta(r,\theta) = r\cos(\theta)$, $\psi(r,\theta) = r\sin(\theta)$, $a_1 = 0$, $a_2 = c$, $b_1 = 0$ and $b_2 = 2\pi$). Then, using suffices to denote partial differentiation, we have

$$\mathbf{r}_u = \zeta_u\mathbf{i} + \psi_u\mathbf{j}, \qquad \mathbf{r}_v = \zeta_v\mathbf{i} + \psi_v\mathbf{j} \tag{3}$$

and hence

$$\mathbf{r}_u \times \mathbf{r}_v = (\zeta_u\psi_v - \zeta_v\psi_u)\mathbf{k}, \tag{4}$$

so that (because $||\mathbf{k}|| = 1$) the shadow element of area is

$$dS_{\text{shad}} = ||\mathbf{r}_u \times \mathbf{r}_v||\, du\, dv = |\zeta_u\psi_v - \zeta_v\psi_u|\, du\, dv = |J|\, du\, dv \tag{5}$$

where $|J|$ is the Jacobian determinant (from Lecture 13). Let the plane itself have equation

$$Ax + By + Cz = D. \tag{6}$$

Then, because the point (x,y,z) in the oblique planar region lies directly above its shadow $(x,y,0)$ and $z = (D - Ax - By)/C$, S can be parameterized by

$$\mathbf{r} = \zeta(u,v)\mathbf{i} + \psi(u,v)\mathbf{j} + \Big(\frac{D - A\zeta(u,v) - B\psi(u,v)}{C}\Big)\mathbf{k}, \quad a_1 \le u \le a_2, \ b_1 \le v \le b_2. \tag{7}$$

Simplifying,

$$\mathbf{r} = \zeta(u,v)\Big(\mathbf{i} - \frac{A}{C}\mathbf{k}\Big) + \psi(u,v)\Big(\mathbf{j} - \frac{B}{C}\mathbf{k}\Big) + \frac{D}{C}\mathbf{k}, \quad a_1 \le v \le a_2, b_1 \le v \le b_2. \tag{8}$$

Hence

$$\begin{aligned} \mathbf{r}_u &= \zeta_u\Big(\mathbf{i} - \frac{A}{C}\mathbf{k}\Big) + \psi_u\Big(\mathbf{j} - \frac{B}{C}\mathbf{k}\Big) \\ \mathbf{r}_v &= \zeta_v\Big(\mathbf{i} - \frac{A}{C}\mathbf{k}\Big) + \psi_v\Big(\mathbf{j} - \frac{B}{C}\mathbf{k}\Big) \end{aligned} \tag{9}$$

implying

$$\begin{aligned}
\mathbf{r}_u \times \mathbf{r}_u &= \zeta_u\zeta_v\Big(\mathbf{i} - \frac{A}{C}\mathbf{k}\Big) \times \Big(\mathbf{i} - \frac{A}{C}\mathbf{k}\Big) + \zeta_u\psi_v\Big(\mathbf{i} - \frac{A}{C}\mathbf{k}\Big) \times \Big(\mathbf{j} - \frac{B}{C}\mathbf{k}\Big) \\
&+ \ \psi_u\zeta_v\Big(\mathbf{j} - \frac{B}{C}\mathbf{k}\Big) \times \Big(\mathbf{i} - \frac{A}{C}\mathbf{k}\Big) + \psi_u\psi_v\Big(\mathbf{j} - \frac{B}{C}\mathbf{k}\Big) \times \Big(\mathbf{j} - \frac{B}{C}\mathbf{k}\Big) \\
&= \mathbf{0} + (\zeta_u\psi_v - \psi_u\zeta_v)\Big(\mathbf{i} - \frac{A}{C}\mathbf{k}\Big) \times \Big(\mathbf{j} - \frac{B}{C}\mathbf{k}\Big) + \mathbf{0} \\
&= (\zeta_u\psi_v - \psi_u\zeta_v)\Big(\frac{A}{C}\mathbf{i} + \frac{B}{C}\mathbf{j} + \mathbf{k}\Big) \ = \ \frac{(\zeta_u\psi_v - \psi_u\zeta_v)}{C}(A\mathbf{i} + B\mathbf{j} + C\mathbf{k}),
\end{aligned} \tag{10}$$

from which

$$\begin{aligned}
dS \ = \ ||\mathbf{r}_u \times \mathbf{r}_v||\, du\, dv &= \frac{|\zeta_u\psi_v - \zeta_v\psi_u|}{|C|}\sqrt{A^2 + B^2 + C^2}\, du\, dv \\
&= \frac{\sqrt{A^2 + B^2 + C^2}}{|C|}\, dS_{\text{shad}}
\end{aligned} \tag{11}$$

on using (5). But it follows from (6) that the normal to the oblique planar region is

$$\mathbf{n} \ = \ \pm\frac{A\mathbf{i} + B\mathbf{j} + C\mathbf{k}}{\sqrt{A^2 + B^2 + C^2}}, \tag{12}$$

and hence from (1) that

$$\boldsymbol{\sigma} \cdot \mathbf{n} \ = \ \pm\frac{C}{\sqrt{A^2 + B^2 + C^2}}. \tag{13}$$

Combining (11) and (13), we obtain

$$dS_{\text{shad}} \ = \ \frac{|C|}{\sqrt{A^2 + B^2 + C^2}}\, dS \ = \ |\boldsymbol{\sigma} \cdot \mathbf{n}|\, dS \ = \ |\cos(\phi)|\, dS \tag{14a}$$

where ϕ is the angle between $\boldsymbol{\sigma}$ and $\mathbf{n}$ (the colatitude). Equivalently, and more usefully,

$$dS \ = \ \frac{dS_{\text{shad}}}{|\boldsymbol{\sigma} \cdot \mathbf{n}|} \ = \ \frac{dS_{\text{shad}}}{|\cos(\phi)|} \tag{14b}$$

where $\mathbf{n}$ is the normal to the oblique planar region and $\boldsymbol{\sigma}$ is the direction of the sun (i.e., perpendicular to the plane of the shadow). Thus, for example, the natural area element for the elliptical planar region enclosed by a circular cylinder of radius c whose axis of symmetry is the z-axis (and whose shadow is a circular disk) when the plane with equation (6) slices through the cylinder is

$$dS \ = \ \frac{\sqrt{A^2 + B^2 + C^2}}{|C|} r\, dr\, d\theta. \tag{15}$$

Note that the sun need not be directly overhead. For example, the argument above is virtually unaltered for a shadow on the x-z plane if the sun is in the direction of $\mathbf{j}$: just replace $\boldsymbol{\sigma} = \mathbf{k}$ by $\boldsymbol{\sigma} = \mathbf{j}$ in (1), (4), (7)–(10) and (13)–(14a); replace $\mathbf{i}$ and $\mathbf{j}$ by $\mathbf{k}$ and $\mathbf{i}$ in (2)–(3) and (7)–(10); and replace C by B in the numerator of (13) to obtain $dS = dS_{\text{shad}}/|\boldsymbol{\sigma} \cdot \mathbf{j}|$ or

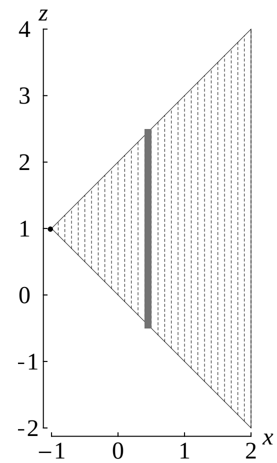
$dS = dS_{\text{shad}}/|\boldsymbol{\sigma}\cdot\mathbf{n}|$ as before (ϕ is no longer the colatitude as in (14b), but it is still the angle between $\boldsymbol{\sigma}$ and $\mathbf{n}$).

To illustrate, let us use this method to calculate the *upward* flux

$$\iint\limits_S \mathbf{F}\cdot\mathbf{dS}$$

of the vector field

$$\mathbf{F} \;=\; y\,\mathbf{i} + z\,\mathbf{j} + x\,\mathbf{k}$$



across the plane triangular surface with vertices at $(-1,0,1)$, $(2,3,4)$ and $(2,1,-2)$. Note that every vertex of the triangle has a nonnegative second component. Thus every vertex lies on or "above" the x-z plane $y=0$ if the sun is in the direction of $\boldsymbol{\sigma} \;=\; \mathbf{j}$. When the sun shines "down" from this direction, the points in the x-z plane directly "below" the vertices $(-1,0,1)$, $(2,3,4)$ and $(2,1,-2)$ are $(-1,0,1)$, $(2,0,4)$ and $(2,0,-2)$, respectively. Hence the shadow in the x-z plane of the original triangle is the triangle on the right with vertices $(x,z)=(-1,1)$, $(x,z)=(2,4)$ and $(x,z)=(2,-2)$, respectively. This region is covered by values of z between $-x$ and $x+2$ for values of x between -1 and 2. By the method of Lecture 4 or otherwise, the original triangle lies in the plane with equation $2x-3y+z+1=0$ with normal

$$\mathbf{n} \;=\; \tfrac{1}{\sqrt{14}}\{2\mathbf{i}-3\mathbf{j}+\mathbf{k}\}$$

on which

$$\mathbf{F}\cdot\mathbf{n} \;=\; \tfrac{1}{\sqrt{14}}\{2y-3z+x\} \;=\; \tfrac{1}{3\sqrt{14}}\{7x-7z+2\}$$

and $\boldsymbol{\sigma}\cdot\mathbf{n} = \mathbf{j}\cdot\mathbf{n} = -\frac{3}{\sqrt{14}}$ implying $|\boldsymbol{\sigma}\cdot\mathbf{n}| = \frac{3}{\sqrt{14}}$. Also, $dS_{\text{shad}} = dxdz$. Hence

$$\begin{aligned}
\iint\limits_S \mathbf{F}\cdot\mathbf{dS} \;&=\; \iint\limits_S \mathbf{F}\cdot\mathbf{n}\,dS \;=\; \iint\limits_{S_{\text{shad}}} \frac{\mathbf{F}\cdot\mathbf{n}}{|\boldsymbol{\sigma}\cdot\mathbf{n}|}\,dS_{\text{shad}} \;=\; \tfrac{1}{9}\iint\limits_{S_{\text{shad}}}\{7x-7z+2\}\,dx\,dz \\
&=\; \tfrac{1}{9}\int_{-1}^{2}\int_{-x}^{x+2}\{7x-7z+2\}\,dz\,dx \;=\; -\tfrac{1}{126}\int_{-1}^{2}(7x-7z+2)^2\Big|_{-x}^{x+2}\,dx \\
&=\; -\tfrac{1}{126}\int_{-1}^{2}\{(-12)^2-2^2(7x+1)^2\}\,dx \;=\; -\tfrac{2}{63}\int_{-1}^{2}\{36-(7x+1)^2\}\,dx \\
&=\; -\tfrac{2}{63}\Big\{36x-\tfrac{1}{21}(7x+1)^3\Big\}\Big|_{-1}^{2} \;=\; -\tfrac{2}{63}\Big\{72-\tfrac{1125}{7}-\big(-36+\tfrac{72}{7}\big)\Big\} \;=\; 2.
\end{aligned}$$

But by no means am I suggesting that the above is an efficient way to calculate the flux in question, and you may agree with me that the method of Lecture 18 is greatly to be preferred.‡

‡From Lecture 18, S is parameterized in natural coordinates by

$$\mathbf{r} \;=\; \mathbf{a} \;+\; u(\mathbf{b}-\mathbf{a}) \;+\; v(\mathbf{c}-\mathbf{b}), \qquad 0\le v\le u, \quad 0\le u\le 1$$

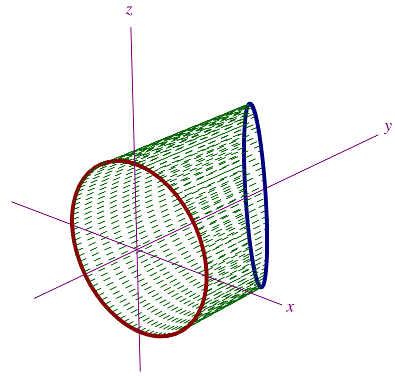
For a second illustration, we calculate the outward flux

$$\iint_S \mathbf{F}\cdot\mathbf{dS}$$

of the vector field

$$\mathbf{F} \;=\; x\,\mathbf{i} + y\,\mathbf{j} + 5\,\mathbf{k}$$

across the boundary of the "sawn-off" cylindrical region bounded by the circular cylinder $x^2+z^2=1$, the "floor" plane $y=0$ and the "roof" plane $x+y=2$.



Let the circular floor (bounded by the red curve), the cylindrical side wall and the sloping elliptical roof (bounded by the blue curve) be denoted by S_1, S_2 and S_3, respectively, so that $S = S_1 \cup S_2 \cup S_3$. Then S_1 is the shadow region for the sloping roof if $\boldsymbol{\sigma} = \mathbf{j}$, that is, if the sun is way out at infinity along the positive y-axis. We can parameterize this disk as

$$\mathbf{r} \;=\; R\sin(\alpha)\mathbf{i} + 0\,\mathbf{j} + R\cos(\alpha)\mathbf{k}, \quad 0 \le R \le 1, \quad 0 \le \alpha \le 2\pi \tag{16}$$

by using "polar " coordinates R and α, where $R = \sqrt{x^2+z^2}$ is the distance of a point $(x,0,z)$ in the shadow region from the y-axis and α is measured from the z-axis, clockwise if looking directly at the sun—which is dangerous, so don't do it—or anti-clockwise if looking directly away from the sun (much safer, so I advise it).§ Clearly, because R and α are polar coordinates, the surface area element for the shadow region is $dS_{\text{shad}} = R\,dR\,d\alpha$.¶ The

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where $\mathbf{a} = -\mathbf{i}+0\mathbf{j}+\mathbf{k}$, $\mathbf{b} = 2\mathbf{i}+3\mathbf{j}+4\mathbf{k}$ and $\mathbf{c} = 2\mathbf{i}+\mathbf{j}-2\mathbf{k}$, i.e., by

$$\mathbf{r} \;=\; -\mathbf{i}+\mathbf{k} \;+\; 3u(\mathbf{i}+\mathbf{j}+\mathbf{k}) \;-\; 2v(\mathbf{j}+3\mathbf{k}),$$
$$0 \le v \le u, \quad 0 \le u \le 1$$

so that $x = 3u-1$, $y = 3u-2v$, $z = 3u-6v+1$ and $\mathbf{r}_u \times \mathbf{r}_v = 6\{-2\mathbf{i}+3\mathbf{j}-\mathbf{k}\}$, which clearly points down, requiring the negative sign to be taken. Then, because

$$\mathbf{F}\cdot(\mathbf{r}_u\times\mathbf{r}_v) \;=\; 6\{-2y+3z-x\} \;=\; 12(2-7v)$$

we obtain

$$\begin{aligned}\iint_S \mathbf{F}\cdot\mathbf{dS} \;&=\; -\int_0^1\!\!\int_0^u \mathbf{F}\cdot(\mathbf{r}_u\times\mathbf{r}_v)\,dv\,du \;=\; -12\int_0^1\!\!\int_0^u (2-7v)\,dv\,du \\ &=\; -12\int_0^1 \{2u - \tfrac{7}{2}u^2\}\,du \;=\; -12\{u^2-\tfrac{7}{6}u^3\}|_0^1 \;=\; -12\{1-\tfrac{7}{6}\} \;=\; 2.\end{aligned}$$

§Thus, in terms of (1) and (2), we have $u = R$, $v = \alpha$, $a_1 = 0 = b_1$, $a_2 = 1$, $b_2 = 2\pi$, $\zeta(u,v) = u\sin(v)$ and $\psi(u,v) = u\cos(v)$ with $\mathbf{j}$ and $\mathbf{k}$ transposed.

¶Or if it is not so clear to you, then it follows directly from (16) because $\mathbf{r}_R = \sin(\alpha)\mathbf{i}+0\,\mathbf{j}+\cos(\alpha)\mathbf{k}$ and $\mathbf{r}_\alpha = R\cos(\alpha)\mathbf{i}+0\,\mathbf{j}-R\sin(\alpha)\mathbf{k}$, implying $\mathbf{r}_R\times\mathbf{r}_\alpha = R\mathbf{j}$ and hence $|\mathbf{r}_R\times\mathbf{r}_\alpha| = |R||\mathbf{j}| = R\cdot 1 = R$.

outward unit normal to the plane $x + y = 2$ is

$$\mathbf{n} \;=\; \tfrac{1}{\sqrt{2}}\{\mathbf{i} + \mathbf{j} + 0\,\mathbf{k}\}, \tag{17}$$

which is therefore also the outward unit normal to S. Hence, with $\boldsymbol{\sigma} = \mathbf{j}$, we obtain

$$dS \;=\; \frac{dS_{\text{shad}}}{|\boldsymbol{\sigma}\cdot\mathbf{n}|} \;=\; \frac{dS_{\text{shad}}}{|\mathbf{j}\cdot\mathbf{n}|} \;=\; \frac{dS_{\text{shad}}}{1/\sqrt{2}} \;=\; \sqrt{2}R\,dR\,d\alpha. \tag{18}$$

Also, on S_3 we have $\mathbf{F}\cdot\mathbf{n} = (x+y)/\sqrt{2} = \sqrt{2}$. Hence

$$\iint\limits_{S_3} \mathbf{F}\cdot\mathbf{dS} \;=\; \iint\limits_{S_3} \mathbf{F}\cdot\mathbf{n}\,dS \;=\; \int\limits_0^{2\pi}\int\limits_0^{1} \sqrt{2}\cdot\sqrt{2}R\,dR\,d\alpha \;=\; 2\pi \tag{19}$$

after a straightforward calculation.

The flux through the floor is clearly zero, because $y = 0$ implies $\mathbf{F} = x\,\mathbf{i} + 0\,\mathbf{j} + 5\,\mathbf{k}$ and the outward normal is $\mathbf{n} = -\mathbf{j}$, so that $\mathbf{F}\cdot\mathbf{n} = 0$. It now remains only to find the flux through the side wall, that is, on the surface S_2 defined by

$$\mathbf{r} \;=\; \sin(\alpha)\mathbf{i} + y\,\mathbf{j} + \cos(\alpha)\mathbf{k}, \quad 0 \le y \le 2 - \sin(\alpha), \quad 0 \le \alpha \le 2\pi. \tag{20}$$

We easily find that $\mathbf{r}_\alpha \times \mathbf{r}_y = \sin(\alpha)\mathbf{i} + \cos(\alpha)\mathbf{k}$ is outward, with $dS_2 = |\mathbf{r}_\alpha \times \mathbf{r}_y|d\alpha\,dy = d\alpha\,dy$. Hence

$$\begin{aligned}
\iint\limits_{S_2} \mathbf{F}\cdot\mathbf{dS} \;&=\; +\iint\limits_{S_2} \mathbf{F}\cdot(\mathbf{r}_\alpha \times \mathbf{r}_y)d\alpha\,dy \\
&=\; \int\limits_0^{2\pi}\int\limits_0^{2-\sin(\alpha)} \{x\sin(\alpha) + 5\cos(\alpha)\}\,dy\,d\alpha \\
&=\; \int\limits_0^{2\pi}\int\limits_0^{2-\sin(\alpha)} \{\sin^2(\alpha) + 5\cos(\alpha)\}\,dy\,d\alpha \;=\; 2\pi
\end{aligned} \tag{21}$$

after another straightforward calculation. Thus

$$\iint\limits_{S} \mathbf{F}\cdot\mathbf{dS} \;=\; \iint\limits_{S_1\cup S_2\cup S_3} \mathbf{F}\cdot\mathbf{dS} \;=\; \iint\limits_{S_1} \mathbf{F}\cdot\mathbf{dS} + \iint\limits_{S_2} \mathbf{F}\cdot\mathbf{dS} + \iint\limits_{S_3} \mathbf{F}\cdot\mathbf{dS} \;=\; 0 + 2\pi + 2\pi = 4\pi. \tag{22}$$

The result we obtained in (14) is a special case of a far more general one. To see why, let us now replace the oblique planar region by an arbitrary open smooth surface S with equation $z = f(x,y)$ or $\phi(x,y,z) = 0$, where we define

$$\phi(x,y,z) \;=\; z - f(x,y). \tag{23}$$

Then with $x = \zeta(u,v)$ and $y = \psi(u,v)$ as in (2), we parameterize S by

$$\mathbf{r} \;=\; \zeta(u,v)\mathbf{i} + \psi(u,v)\mathbf{j} + f(\zeta(u,v),\psi(u,v))\mathbf{k}, \quad a_1 \le u \le a_2,\; b_1 \le v \le b_2 \tag{24}$$

with positive unit normal

$$\mathbf{n} \;=\; \pm\widehat{\nabla\phi} \;=\; \pm\frac{\nabla\phi}{||\nabla\phi||} \;=\; \pm\,\frac{-f_x\mathbf{i}-f_y\mathbf{j}+\mathbf{k}}{||\nabla\phi||}. \tag{25}$$

On using the chain rule, in place of (9) we obtain

$$\begin{aligned}\mathbf{r}_u &= \zeta_u\mathbf{i}+\psi_u\mathbf{j}+\{f_x\zeta_u+f_y\psi_u\}\mathbf{k} \;=\; \zeta_u\Big(\mathbf{i}+f_x\mathbf{k}\Big)+\psi_u\Big(\mathbf{j}+f_y\mathbf{k}\Big)\\ \mathbf{r}_v &= \zeta_v\mathbf{i}+\psi_v\mathbf{j}+\{f_x\zeta_v+f_y\psi_v\}\mathbf{k} \;=\; \zeta_v\Big(\mathbf{i}+f_x\mathbf{k}\Big)+\psi_v\Big(\mathbf{j}+f_y\mathbf{k}\Big)\end{aligned} \tag{26}$$

implying

$$\begin{aligned}\mathbf{r}_u\times\mathbf{r}_u &= \zeta_u\zeta_v\Big(\mathbf{i}+f_x\mathbf{k}\Big)\times\Big(\mathbf{i}+f_x\mathbf{k}\Big)+\zeta_u\psi_v\Big(\mathbf{i}+f_x\mathbf{k}\Big)\times\Big(\mathbf{j}+f_y\mathbf{k}\Big)\\ &\quad+\;\psi_u\zeta_v\Big(\mathbf{j}+f_y\mathbf{k}\Big)\times\Big(\mathbf{i}+f_x\mathbf{k}\Big)+\psi_u\psi_v\Big(\mathbf{j}+f_y\mathbf{k}\Big)\times\Big(\mathbf{j}+f_y\mathbf{k}\Big)\\ &= \mathbf{0}+(\zeta_u\psi_v-\psi_u\zeta_v)\Big(\mathbf{i}+f_x\mathbf{k}\Big)\times\Big(\mathbf{j}+f_y\mathbf{k}\Big)+\mathbf{0}\\ &= (\zeta_u\psi_v-\psi_u\zeta_v)\Big(-f_x\mathbf{i}-f_y\mathbf{j}+\mathbf{k}\Big) \;=\; (\zeta_u\psi_v-\psi_u\zeta_v)\nabla\phi,\end{aligned} \tag{27}$$

from which (11) generalizes to

$$dS \;=\; |\mathbf{r}_u\times\mathbf{r}_v|\,du\,dv \;=\; |\zeta_u\psi_v-\zeta_v\psi_u|||\nabla\phi||\,du\,dv \;=\; ||\nabla\phi||\,dS_{\text{shad}} \tag{28}$$

by (5); and because

$$\boldsymbol{\sigma}\cdot\mathbf{n} \;=\; \pm\frac{1}{||\nabla\phi||} \tag{29}$$

by (1) and (25), we readily recover (14), despite $\mathbf{n}$ now varying over the surface S—whereas when S was planar, $\mathbf{n}$ was fixed.

If this result is so general, why have we focused so strongly on planar regions? The answer is that unless we know both $\mathbf{n}$ and dS_{shad} in advance, calculation of a flux as

$$\iint\limits_S \mathbf{F}\cdot\mathbf{dS} \;=\; \iint\limits_S \frac{\mathbf{F}\cdot\mathbf{n}}{|\boldsymbol{\sigma}\cdot\mathbf{n}|}\,dS_{\text{shad}} \tag{30}$$

is no better and typically worse than integration over S itself, that is, use of

$$\iint\limits_S \mathbf{F}\cdot\mathbf{dS} \;=\; \pm\iint\limits_S \mathbf{F}\cdot(\mathbf{r}_u\times\mathbf{r}_v)\,du\,dv \tag{31}$$

at the outset.[‖] Of course, if u and v are standard coordinates—typically either polar or Cartesian—then we do know dS_{shad} in advance; but we need to know $\mathbf{n}$ as well, and that is why we focus on planar regions, which are easily the most important special case. Nevertheless, there also exists one other example of a surface for which (30) may be a useful result to know, namely, an open surface that is part of a sphere with center at the origin.

[‖]Moreover, it may be no better even for a planar region, as illustrated by the first example above.

Let the sphere in question have radius a. Because the position vector of any point on the sphere has the same direction as the outward normal, we have

$$\mathbf{n} \;=\; \frac{\boldsymbol{\rho}}{a} \;=\; \frac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{a} \tag{32}$$

and hence

$$|\boldsymbol{\sigma} \cdot \mathbf{n}| \;=\; \frac{|z|}{a} \tag{33}$$

by (1). So (30) reduces to

$$\iint\limits_{S} \mathbf{F} \cdot \mathbf{dS} \;=\; \iint\limits_{S} \mathbf{F} \cdot \mathbf{n}\, dS \;=\; \iint\limits_{S} \frac{\mathbf{F} \cdot \boldsymbol{\rho}}{|z|}\, dS_{\text{shad}}. \tag{34}$$

To illustrate, we use cylindrical polar coordinates with $dS_{\text{shad}} = r\,dr\,d\theta$ to calculate the upward flux of the vector field

$$\mathbf{F} \;=\; -y\,\mathbf{i} + yz\,\mathbf{j} + x\,\mathbf{k} \tag{35}$$

through the open spherical cap S with vector equation

$$\boldsymbol{\rho} \;=\; r\cos(\theta)\,\mathbf{i} + r\sin(\theta)\,\mathbf{j} + \sqrt{4 - r^2}\,\mathbf{k},\;\; 0 \le r \le \sqrt{3},\;\; 0 \le \theta \le 2\pi, \tag{36}$$

which forms part of the sphere of radius $a = 2$ that is centered at the origin. Because $z \ge 1$,

$$\begin{aligned} \frac{\mathbf{F} \cdot \boldsymbol{\rho}}{|z|} \;&=\; \frac{-xy + y^2 z + zx}{z} \;=\; -\frac{xy}{z} + y^2 + x \\ &=\; -\frac{r^2}{\sqrt{4 - r^2}}\cos(\theta)\sin(\theta) + r^2\sin^2(\theta) + r\cos(\theta) \end{aligned} \tag{37}$$

by (36). Hence (34) implies

$$\begin{aligned} &\iint\limits_{S} \mathbf{F} \cdot \mathbf{dS} \;=\; \iint\limits_{S} \frac{\mathbf{F} \cdot \boldsymbol{\rho}}{|z|}\, r\,dr\,d\theta \\ &=\; \int_0^{2\pi}\int_0^{\sqrt{3}} \Big\{ -\frac{r^3}{\sqrt{4 - r^2}}\sin(\theta)\cos(\theta) + r^3\sin^2(\theta) + r^2\cos(\theta) \Big\}\, dr\,d\theta \\ &=\; -\int_0^{\sqrt{3}} \frac{r^3}{\sqrt{4 - r^2}}\,dr \int_0^{2\pi} \sin(\theta)\cos(\theta)\,d\theta + \int_0^{\sqrt{3}} r^3\,dr \int_0^{2\pi} \sin^2(\theta)\,d\theta + \int_0^{\sqrt{3}} r^2\,dr \int_0^{2\pi} \cos(\theta)\,d\theta \\ &=\; 0 + \tfrac{1}{4}(\sqrt{3})^4 \cdot \pi + 0 \;=\; \tfrac{9}{4}\pi. \end{aligned} \tag{38}$$

Again, however, we are merely illustrating the shadow technique. By no means are we suggesting that the above is the best way to calculate this particular flux—and it may not be. It may be better to integrate directly over the surface itself, using spherical polars, which are the natural coordinates for any part of a sphere. In spherical polars, (36) becomes

$$\boldsymbol{\rho} \;=\; 2\sin(\phi)\cos(\theta)\,\mathbf{i} + 2\sin(\phi)\sin(\theta)\,\mathbf{j} + 2\cos(\phi)\,\mathbf{k},\;\; 0 \le \phi \le \tfrac{1}{3}\pi,\;\; 0 \le \theta \le 2\pi. \tag{39}$$

The surface area element for a sphere of radius a is $a^2 \sin(\phi)\, d\theta\, d\phi$, and here $a = 2$. So

$$dS \;=\; 4\sin(\phi)\, d\theta\, d\phi. \tag{40}$$

From (32), (35) and (39), we have

$$\begin{aligned}\mathbf{F}\cdot\mathbf{n} &= \mathbf{F}\cdot\frac{\boldsymbol{\rho}}{2} \;=\; \mathbf{F}\cdot\{\sin(\phi)\cos(\theta)\,\mathbf{i} + \sin(\phi)\sin(\theta)\,\mathbf{j} + \cos(\phi)\,\mathbf{k}\}\\ &= -y\sin(\phi)\cos(\theta) + yz\sin(\phi)\sin(\theta) + x\cos(\phi)\\ &= -2\sin^2(\phi)\sin(\theta)\cos(\theta) + 4\sin^2(\phi)\cos(\phi)\sin^2(\theta) + 2\sin(\phi)\cos(\phi)\cos(\theta).\end{aligned}\tag{41}$$

So, from (39)–(41),

$$\begin{aligned}\iint\limits_S \mathbf{F}\cdot\mathbf{n}\, dS &= 4\int_0^{2\pi}\int_0^{\frac{1}{3}\pi}\big\{-2\sin^2(\phi)\sin(\theta)\cos(\theta) + 4\sin^2(\phi)\cos(\phi)\sin^2(\theta)\\ &\qquad\qquad + 2\sin(\phi)\cos(\phi)\cos(\theta)\big\}\sin(\phi)\, d\phi\, d\theta\\ &= -4\int_0^{2\pi}\sin(2\theta)\, d\theta\int_0^{\frac{1}{3}\pi}\sin^3(\phi)\, d\phi + 16\int_0^{2\pi}\sin^2(\theta)\, d\theta\int_0^{\frac{1}{3}\pi}\sin^3(\phi)\cos(\phi)\, d\phi\\ &\qquad + 8\int_0^{2\pi}\cos(\theta)\, d\theta\int_0^{\frac{1}{3}\pi}\sin^2(\phi)\cos(\phi)\, d\phi\\ &= -4\cdot 0\cdot\int_0^{\frac{1}{3}\pi}\sin^3(\phi)\, d\phi + 16\cdot\pi\cdot\tfrac{1}{4}\sin^4(\phi)\Big|_0^{\frac{1}{3}\pi}\\ &\qquad + 8\cdot 0\cdot\int_0^{\frac{1}{3}\pi}\sin^2(\phi)\cos(\phi)\, d\phi\\ &= 4\pi\sin^4\big(\tfrac{1}{3}\pi\big) \;=\; 4\pi\big(\tfrac{\sqrt{3}}{2}\big)^4 \;=\; \tfrac{9}{4}\pi\end{aligned}\tag{42}$$

as before.

Which is the better method here—integration over S itself (where $\rho = a = 2$) using appropriately restricted** spherical polars, or integration over its shadow surface (where $z = 0$) using appropriately restricted cylindrical polars? You can decide!

**To two dimensions.