Series that we know are convergent We are going to use these series for comparison and reference.

The *geometric series*: $\sum_{n=1}^{\infty} c \cdot r^n = \frac{a_1}{1-r}$ for $|r| < 1$.

Determine if the following geometric series are convergent. For each convergent series, determine its sum.

1. $\frac{1}{3} + \frac{1}{6} + \frac{1}{12} + \frac{1}{24} + \frac{1}{48} + \dots$

2. $\sum_{n=1}^{\infty} \frac{1+3^n}{2^n}$

3. $\sum_{n=1}^{\infty} \frac{7^{n+1}}{10^n}$

The *p-series*: $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent if $p > 1$ and divergent if $p \leq 1$.

Use the criterion for p-series to determine if the following series are convergent.

1. $\sum_{n=1}^{\infty} \frac{n+1}{n\sqrt{n}}$.

2. $1 + \frac{1}{2\sqrt{2}} + \frac{1}{3\sqrt{3}} + \frac{1}{4\sqrt{4}} + \dots$

Tests for Convergence or Divergence

The Integral Test: Suppose f is a continuous, positive, decreasing function on $[1, \infty)$ and let $a_n = f(n)$. Then the series $\sum_{n=1}^{\infty} a_n$ is convergent if and only if the improper integral $\int_1^{\infty} f(x)dx$ is convergent:

If $\int_1^{\infty} f(x)dx$ is convergent, then $\sum_{n=1}^{\infty} a_n$ is convergent.

If $\int_1^{\infty} f(x)dx$ is divergent, then $\sum_{n=1}^{\infty} a_n$ is divergent.

Confirm that the series satisfies the conditions of the integral test and then apply the integral test to determine if the series is convergent.

1. $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n^2+1}$

2. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+n^3}$

The Comparison Test: Suppose that $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are series with positive terms.

- If $\sum b_n$ is convergent and $a_n \le b_n$ for all n, then $\sum a_n$ is also convergent.
- If $\sum b_n$ is divergent and $a_n \ge b_n$ for all n, then $\sum a_n$ is also divergent.

Use the comparison test to determine if the series converges.

1. $\displaystyle\sum_{n=1}^{\infty} \frac{n}{2n^3+1}$

2. $\displaystyle\sum_{n=1}^{\infty} \frac{9^n}{3+10^n}$

The Limit Comparison Test: Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms. If

$$\lim_{n\to\infty} \frac{a_n}{b_n} = c$$

where c is a finite number and $c > 0$, then either both series converge or both series diverge.

Use the limit comparison test to determine convergence of the series using the given comparison.

1. Determine if the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2+1}}$ converges by using the comparison test with $b_n = \frac{1}{n}$.

2. Determine if the series $\displaystyle\sum_{n=1}^{\infty} \frac{n+2}{(n+1)^3}$ converges by using the comparison test.

The Alternating Series Test: If the alternating series $\sum_{n=1}^{\infty}(-1)^{n-1}b^n = b_1 - b_2 + b_3 - b_4 + \ldots$ for $b_n > 0$ satisfies both:

1. $b_{n+1} \leq b_n$ for all n.
2. $\lim_{n\to\infty} b_n = 0$

then the series is convergent. Confirm that the series below meet the requirements of the alternating series test and determine if they are convergent.

1. $\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}} - \frac{1}{\sqrt{5}} + \ldots$

2. $\sum_{n=1}^{\infty}(-1)^n \frac{3n-1}{2n+1}$

The Ratio Test: Consider the $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|$.

- If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L < 1$, then the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent, and therefore convergent.
- If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L > 1$ or $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \infty$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.
- If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 1$, then the ratio test is inconclusive.

Determine and simplify the ratio $\frac{|a_{n+1}|}{|a_n|}$ for each of the following series. Then evaluate $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|$.

1. $\sum_{n=1}^{\infty} n\cdot\left(\frac{2}{5}\right)^n$

2. $\sum_{n=1}^{\infty} \frac{10^n}{(n+1)4^{2n+1}}$

The Root Test: Consider $\lim_{n\to\infty} \sqrt[n]{|a_n|}$

- If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L < 1$ then the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent, and therefore convergent.
- If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L > 1$ or $\lim_{n\to\infty} \sqrt[n]{|a_n|} = \infty$ then the series $\sum_{n=1}^{\infty} a_n$ is divergent.
- If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = 1$ then the root test is inconclusive.

Use to root test to determine if each of the following series converges or diverges–or if the root test is inconclusive.

1. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{n^2+1}{2n^2+1}\right)^n$

2. $\displaystyle\sum_{n=1}^{\infty} \left(1+\frac{1}{n}\right)^{n^2}$