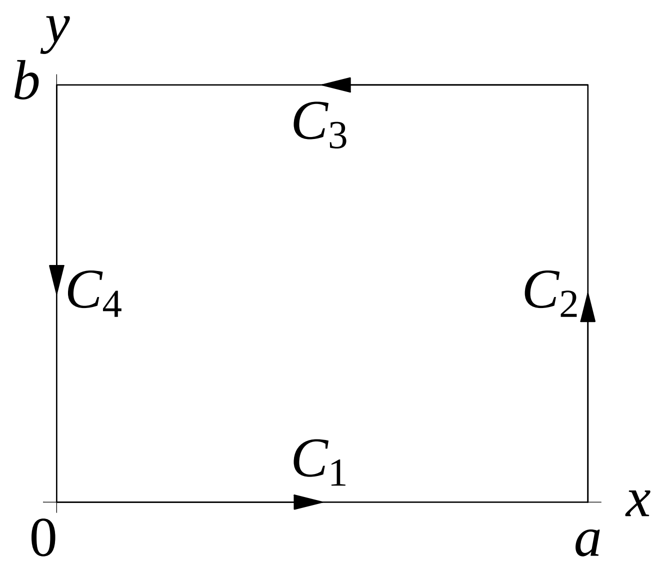

Figure 1: A rectangle, traversed counterclockwise

Stokes' theorem

Now that we know how to calculate both fluxes and line integrals, let's practice our skills by calculating the circulation of the vector field

$$\mathbf{F} \;=\; \mathbf{F}(x,y,z) \;=\; F_1(x,y,z)\,\mathbf{i} + F_2(x,y,z)\,\mathbf{j} + F_3(x,y,z)\,\mathbf{k} \tag{1}$$

around the rectangle C depicted in Figure 1; for the sake of simplicity, we have chosen this rectangle to lie in a horizontal plane at height $\hat{z}$ above (or below) sea level, with vertices at $0\,\mathbf{i} + 0\,\mathbf{j} + \hat{z}\mathbf{k}$, $a\,\mathbf{i} + 0\,\mathbf{j} + \hat{z}\mathbf{k}$, $a\,\mathbf{i} + b\,\mathbf{j} + \hat{z}\mathbf{k}$ and $0\,\mathbf{i} + b\,\mathbf{j} + \hat{z}\mathbf{k}$.* We decompose C into four straight line segments by writing

$$C \;=\; C_1 \cup C_2 \cup C_3 \cup C_4, \tag{2}$$

where $C_1, \ldots, C_4$ and their associated values of $\mathbf{F}$, $\mathbf{dr} = dx\,\mathbf{i} + dy\,\mathbf{j} + dz\,\mathbf{k}$ and $\mathbf{F}\cdot\mathbf{dr}$ are as follows:

$$\begin{array}{llllll}
C_1: & 0 \le x \le a,\;\; y = 0,\; z = \hat{z} & \mathbf{dr} = dx\,\mathbf{i} & \mathbf{F} = \mathbf{F}(x,0,\hat{z}) & \mathbf{F}\cdot\mathbf{dr} = F_1(x,0,\hat{z})\,dx \\
C_2: & x = a,\;\; 0 \le y \le b,\; z = \hat{z} & \mathbf{dr} = dy\,\mathbf{j} & \mathbf{F} = \mathbf{F}(a,y,\hat{z}) & \mathbf{F}\cdot\mathbf{dr} = F_2(a,y,\hat{z})\,dy \\
C_3: & a \ge x \ge 0,\;\; y = b,\; z = \hat{z} & \mathbf{dr} = dx\,\mathbf{i} & \mathbf{F} = \mathbf{F}(x,b,\hat{z}) & \mathbf{F}\cdot\mathbf{dr} = F_1(x,b,\hat{z})\,dx \\
C_4: & x = 0,\;\; b \ge y \ge 0,\; z = \hat{z} & \mathbf{dr} = dy\,\mathbf{j} & \mathbf{F} = \mathbf{F}(0,y,\hat{z}) & \mathbf{F}\cdot\mathbf{dr} = F_2(0,y,\hat{z})\,dy
\end{array}$$

Thus

*Translating the rectangle to any other location in the plane $z = \hat{z}$ would not in any way affect the result we are about to obtain—it would just make the algebra a bit messier. Also note that the hat on $\hat{z}$ does not in any way signify a connection with unit vectors.

$$
\begin{aligned}
\oint_C \mathbf{F}\cdot\mathbf{dr} &= \int_{C_1} \mathbf{F}\cdot\mathbf{dr} + \int_{C_2} \mathbf{F}\cdot\mathbf{dr} + \int_{C_3} \mathbf{F}\cdot\mathbf{dr} + \int_{C_4} \mathbf{F}\cdot\mathbf{dr} \\
&= \int_{x=0}^{x=a} F_1(x,0,\hat{z})\,dx + \int_{y=0}^{y=b} F_2(a,y,\hat{z})\,dy \\
&\qquad + \int_{x=a}^{x=0} F_1(x,b,\hat{z})\,dx + \int_{y=b}^{y=0} F_2(0,y,\hat{z})\,dy \\
&= \int_{x=0}^{x=a} F_1(x,0,\hat{z})\,dx + \int_{y=0}^{y=b} F_2(a,y,\hat{z})\,dy \\
&\qquad - \int_{x=0}^{x=a} F_1(x,b,\hat{z})\,dx - \int_{y=0}^{y=b} F_2(0,y,\hat{z})\,dy \\
&= \int_{y=0}^{y=b} \{F_2(a,y,\hat{z}) - F_2(0,y,\hat{z})\} dy \\
&\qquad - \int_{x=0}^{x=a} \{F_1(x,b,\hat{z}) - F_1(x,0,\hat{z})\}\,dx \\
&= \int_{y=0}^{y=b} F_2(x,y,\hat{z})\Big|_{x=0}^{x=a} dy \quad - \quad \int_{x=0}^{x=a} F_1(x,y,\hat{z})\Big|_{y=0}^{y=b} dx \\
&= \int_0^b \left\{ \int_0^a \frac{\partial F_2}{\partial x}(x,y,\hat{z})\,dx \right\} dy - \int_0^a \left\{ \int_0^b \frac{\partial F_1}{\partial y}(x,y,\hat{z})\,dy \right\} dx \\
&= \int_{x=0}^{x=a}\int_{y=0}^{y=b} \left\{ \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right\} dy\,dx \;=\; \iint_{R_1} \left\{ \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right\} dS
\end{aligned}
\tag{3}
$$

where $dS = dx\,dy$ is the element of surface area and R_1 denotes the flat, rectangular region in Figure 1. But the integrand in this double integral is simply the third component of $\nabla \times \mathbf{F}$. Thus (3) implies that

$$
\oint_C \mathbf{F}\cdot\mathbf{dr} \;=\; \iint_{R_1} (\nabla\times\mathbf{F})\cdot\mathbf{k}\,dS \;=\; \iint_{R_1} (\nabla\times\mathbf{F})\cdot\mathbf{dS} \tag{4}
$$

provided that $\mathbf{n} = \mathbf{k}$ (as opposed to $-\mathbf{k}$). That is, for the special case of a flat rectangular open surface—but for an arbitrary vector field $\mathbf{F}$—the circulation of $\mathbf{F}$ around the boundary of the surface equals the flux through the surface of the curl of $\mathbf{F}$.

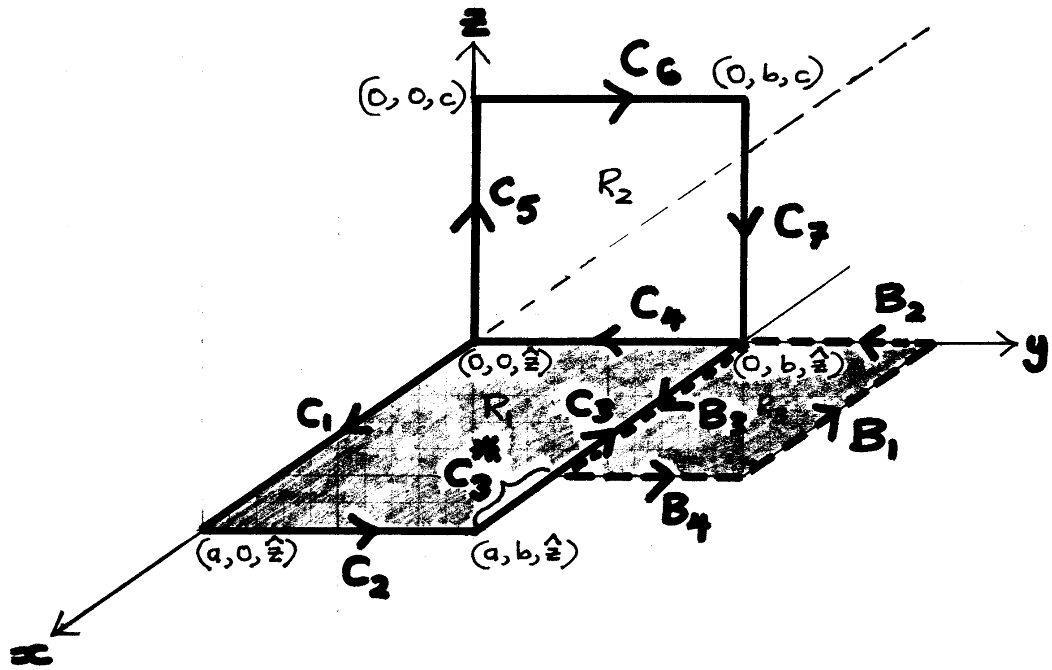

Figure 2: The old rectangle accompanied by two new ones

To obtain the equivalent result for a vertical plane, let us also calculate the circulation of **F** around the boundary of the vertical rectangular region R_2 in Figure 2. Now

$$C \;=\; C_4 \cup C_5 \cup C_6 \cup C_7, \tag{5}$$

where $C_4, \ldots, C_7$ and their associated values of **F**, **dr** and **F** $\cdot$ **dr** are as follows:

$$\begin{array}{lllll}
C_4: & x=0,\; b \ge y \ge 0,\; z=\hat{z} & \mathbf{dr}=dy\mathbf{j} & \mathbf{F}=\mathbf{F}(0,y,\hat{z}) & \mathbf{F}\cdot\mathbf{dr}=F_2(0,y,\hat{z})\,dy \\
C_5: & x=0,\; y=0,\; \hat{z} \le z \le c & \mathbf{dr}=dz\mathbf{k} & \mathbf{F}=\mathbf{F}(0,0,z) & \mathbf{F}\cdot\mathbf{dr}=F_3(0,0,z)\,dz \\
C_6: & x=0,\; 0 \le y \le b,\; z=c & \mathbf{dr}=dy\mathbf{j} & \mathbf{F}=\mathbf{F}(0,y,c) & \mathbf{F}\cdot\mathbf{dr}=F_2(0,y,c)\,dy \\
C_7: & x=0,\; y=b,\; c \ge z \ge \hat{z} & \mathbf{dr}=dz\mathbf{k} & \mathbf{F}=\mathbf{F}(0,b,z) & \mathbf{F}\cdot\mathbf{dr}=F_3(0,b,z)\,dz
\end{array}$$

Thus, by analogy with our previous calculation, we obtain

$$
\begin{aligned}
\oint_C \mathbf{F}\cdot d\mathbf{r} &= \int_{C_4} \mathbf{F}\cdot d\mathbf{r} + \int_{C_5} \mathbf{F}\cdot d\mathbf{r} + \int_{C_6} \mathbf{F}\cdot d\mathbf{r} + \int_{C_7} \mathbf{F}\cdot d\mathbf{r} \\
&= \int_{y=b}^{y=0} F_2(0,y,\hat{z})\,dy + \int_{z=\hat{z}}^{z=c} F_3(0,0,z)\,dz \\
&\qquad + \int_{y=0}^{y=b} F_2(0,y,c)\,dy + \int_{z=c}^{z=\hat{z}} F_3(0,b,z)\,dy \\
&= \int_{y=0}^{y=b} \{F_2(0,y,c) - F_2(0,y,\hat{z})\} dy \\
&\qquad - \int_{z=\hat{z}}^{z=c} \{F_3(0,b,z) - F_3(0,0,z)\}\,dz \qquad (6) \\
&= \int_{y=0}^{y=b} F_2(0,y,z)\Big|_{z=\hat{z}}^{z=c}\,dy \quad - \int_{z=\hat{z}}^{z=c} F_3(0,y,z)\Big|_{y=0}^{y=b}\,dz \\
&= \int_0^b \left\{ \int_{\hat{z}}^{c} \frac{\partial F_2}{\partial z}(0,y,z)\,dz \right\} dy - \int_{\hat{z}}^{c} \left\{ \int_0^b \frac{\partial F_3}{\partial y}(0,y,z)\,dy \right\} dz \\
&= \int_{y=0}^{y=b}\int_{z=\hat{z}}^{z=c} \left\{ \frac{\partial F_2}{\partial z} - \frac{\partial F_3}{\partial y} \right\} dz\,dy \;=\; \iint_{R_2} \left\{ \frac{\partial F_2}{\partial z} - \frac{\partial F_3}{\partial y} \right\} dS \\
&= \iint_{R_2} (\nabla\times\mathbf{F})\cdot\{-\mathbf{i}\}\,dS \;=\; \iint_{R_2} (\nabla\times\mathbf{F})\cdot d\mathbf{S}
\end{aligned}
$$

where $dS = dy\,dz$ is the element of surface area, provided that $\mathbf{n} = -\mathbf{i}$ (as opposed to $\mathbf{i}$). Comparing (4) with (6), we see that, if C is the boundary of R, then

$$
\oint_C \mathbf{F}\cdot d\mathbf{r} \;=\; \iint_R (\nabla\times\mathbf{F})\cdot\mathbf{n}\,dS \;=\; \iint_R (\nabla\times\mathbf{F})\cdot d\mathbf{S}, \qquad (7)
$$

holds for both $R = R_1$ and $R = R_2$, provided that $\mathbf{n}$ points upwards in the first case and along the negative x-axis in the second case; in other words, provided you choose the positive side of the surface so that it would always be on your left if you walked around the boundary in the positive direction along C (that is, in the direction of the arrows). Thus (7) holds for an arbitrary rectangle parallel to two of the three coordinate planes, and an analogous calculation readily shows that it also holds for a rectangle that is parallel to the third. Note that any of these rectangles may be as small as we please (because a, b, c and $\hat{z}$, etc., are arbitrary).

What happens if we traverse the boundary of R_2 in the opposite direction? Then we have to reverse the direction of each of the line segments: using $-C_n$ to denote C_n traversed backwards, our boundary becomes

$$C \;=\; -C_4 \cup -C_7 \cup -C_6 \cup -C_5, \tag{8}$$

in place of (5). We must also, of course, reverse the direction of $\mathbf{n}$ from $\mathbf{n} = -\mathbf{i}$ to $\mathbf{n} = \mathbf{i}$, in order to keep the positive side of the surface on our left as we traverse C. But subject to this reinterpretation, (7) continues to hold.† That is, we have both

$$\oint\limits_{C_1\cup C_2\cup C_3\cup C_4} \mathbf{F}\cdot\mathbf{dr} \;=\; \iint\limits_{R_1} (\nabla\times\mathbf{F})\cdot\mathbf{dS} \tag{9}$$

and

$$\oint\limits_{-C_4\cup -C_7\cup -C_6\cup -C_5} \mathbf{F}\cdot\mathbf{dr} \;=\; \iint\limits_{R_2} (\nabla\times\mathbf{F})\cdot\mathbf{dS} \tag{10}$$

where $\mathbf{dS} = \mathbf{n}\,dS$. Adding, we obtain

$$\oint\limits_{\substack{C_1\cup C_2\cup C_3\\ \cup C_4\cup -C_4\\ \cup -C_7\cup -C_6\cup -C_5}} \mathbf{F}\cdot\mathbf{dr} \;=\; \iint\limits_{R_1} (\nabla\times\mathbf{F})\cdot\mathbf{dS} + \iint\limits_{R_2} (\nabla\times\mathbf{F})\cdot\mathbf{dS}. \tag{11}$$

But if C_n is parameterized by values of t between $t = a$ and $t = b$, then

$$\int\limits_{-C_n} \mathbf{F}\cdot\mathbf{dr} \;=\; \int\limits_{t=b}^{t=a} \mathbf{F}\cdot\frac{d\mathbf{r}}{dt}\,dt \;=\; -\int\limits_{t=a}^{t=b} \mathbf{F}\cdot\frac{d\mathbf{r}}{dt}\,dt \;=\; -\int\limits_{C_n} \mathbf{F}\cdot\mathbf{dr}, \tag{12}$$

and so

$$\int\limits_{-C_n\cup C_n} \mathbf{F}\cdot\mathbf{dr} \;=\; \int\limits_{-C_n} \mathbf{F}\cdot\mathbf{dr} + \int\limits_{C_n} \mathbf{F}\cdot\mathbf{dr} \;=\; -\int\limits_{C_n} \mathbf{F}\cdot\mathbf{dr} + \int\limits_{C_n} \mathbf{F}\cdot\mathbf{dr} \;=\; 0. \tag{13}$$

In particular, setting $n = 4$ in (13), we find that C_4 and $-C_4$ cancel each other out in (11), which therefore reduces to

$$\oint\limits_{\substack{C_1\cup C_2\cup C_3\\ \cup -C_7\cup -C_6\cup -C_5}} \mathbf{F}\cdot\mathbf{dr} \;=\; \iint\limits_{R_1} (\nabla\times\mathbf{F})\cdot\mathbf{dS} + \iint\limits_{R_2} (\nabla\times\mathbf{F})\cdot\mathbf{dS} \tag{14}$$

or—reproducing (7) for ease of reference—

$$\oint\limits_{C} \mathbf{F}\cdot\mathbf{dr} \;=\; \iint\limits_{R} (\nabla\times\mathbf{F})\cdot\mathbf{dS}, \tag{15}$$

†You might perhaps argue that we should replace R_2 by $-R_2$, because what used to be the positive side of the surface is now its negative side, and vice versa. But most mathematicians would probably regard this as overkill, because you always know which is the positive side of the surface from the orientation of its boundary.

where $R = R_1 \cup R_2$ and

$$C = C_1 \cup C_2 \cup C_3 \cup -C_7 \cup -C_6 \cup -C_5. \tag{16}$$

Careful inspection of Figure 2 reveals the significance of this result: (7) or (15) holds not only for $R = R_1$ and $R = R_2$, but also for $R = R_1 \cup R_2$, provided only that the positive side of the compound surface is always on our left as we traverse the boundary.

We now begin to suspect that (7) or (15) is a much more general result, because we can use the above method to extend R indefinitely, one rectangle at a time. For example, an almost trivial modification of the argument that produced (4) yields

$$\oint_{B_1 \cup B_2 \cup B_3 \cup B_4} \mathbf{F} \cdot \mathbf{dr} = \iint_{R_3} (\nabla \times \mathbf{F}) \cdot \mathbf{dS} \tag{17}$$

where $B_1, \ldots B_4$ are line segments defined by Figure 2 and R_3 is the rectangle they enclose. Adding (15) and (17) yields (15) again, but now with with $R = R_1 \cup R_2 \cup R_3$ and

$$C = C_1 \cup C_2 \cup C_3^* \cup B_4 \cup B_1 \cup B_2 \cup C_4, \tag{18}$$

where C_3^* is defined by Figure 2, and we have used the fact that part of the contribution from C_3 is cancelled by the contribution from B_3 when (15) and (17) are added.

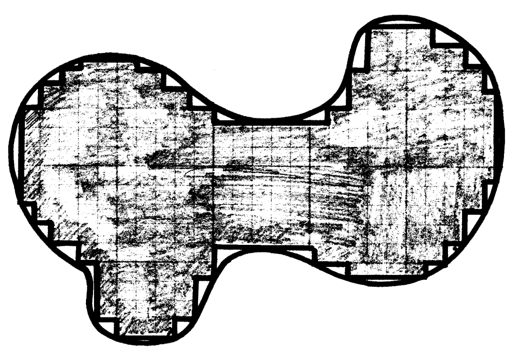

Figure 3: A piecewise-rectangular approximation of a flat, two-dimensional region.

There is no limit to how many rectangles we can add, and we can make them as small as we please. Thus, for example, the result of (15) must also hold for the piecewise-rectangular region shaded in Figure 3. But there isn't much white space between this region and the outer curve, and we can keep adding smaller and smaller rectangles to it, and every time we do so, (15) will continue to hold. In the limit, as the number of such rectangles approaches ∞ (and their size approaches zero), we find that the shading extends all the way to the outer curve. But the result of (15) must still continue to hold. In this limited generality—i.e., to an arbitrary planar region—it is known as Green's theorem.

But the result is even more general than that: it can be shown that

$$\oint_C \mathbf{F} \cdot \mathbf{dr} \;=\; \iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{dS} \tag{19}$$

for an arbitrary vector field $\mathbf{F}$ on an arbitrary open surface S bounded by a closed curve C, provided only that $\mathbf{F}$ is smooth, S is piecewise-smooth (i.e., has a finite number of joins at which its normal $\mathbf{n}$ is undefined), and C is traversed with the positive side of S on its left. In its fullest generality, the result is known as Stokes' theorem. We don't have time for a proof of this theorem in all its glory, but Figure 4 should suffice to convince you that it must hold in general if it holds for an arbitrary (horizontal or vertical) flat rectangular region parallel to one of the coordinate planes—which it does, as we have demonstrated. Briefly: any open surface in space can be approximated by a very large collection of very small rectangular regions parallel to one or other of the coordinate planes; and in the limit as the number of such rectangular regions approaches infinity and their sizes all approach zero, the result that emerges is Stokes' theorem.

From Stokes's theorem, i.e., from (19), we see at once that the circulation of $\mathbf{F}$ around any closed curve must be zero if $\nabla \times \mathbf{F} = \mathbf{0}$. Hence any such vector field must be irrotational. This clinches the result we had to take for granted in earlier lectures. In the special case where C encloses an infinitesimal open surface of area δS around the point with position vector $x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, (19) yields $\oint_C \mathbf{F} \cdot \mathbf{dr} \approx (\nabla \times \mathbf{F}) \cdot \mathbf{n}\delta S$ or $(\nabla \times \mathbf{F}) \cdot \mathbf{n} \approx \oint_C \mathbf{F} \cdot \mathbf{dr} \div \delta S$, which becomes exact in the limit as the surface shrinks to contain only the point:

$$(\nabla \times \mathbf{F}) \cdot \mathbf{n} \;=\; \lim_{\delta S \to 0} \frac{1}{\delta S} \oint_C \mathbf{F} \cdot \mathbf{dr}. \tag{20}$$

If α is the angle between $\mathbf{n}$ and $\nabla \times \mathbf{F}$, then the left-hand side of (20) is $|\nabla \times \mathbf{F}||\mathbf{n}| \cos(\alpha)$, which—for any given $\mathbf{r}$—is maximized by choosing $\alpha = 0$, in which case we obtain

$$|\nabla \times \mathbf{F}| \;=\; \lim_{\delta S \to 0} \frac{1}{\delta S} \oint_C \mathbf{F} \cdot \mathbf{dr}, \quad \mathbf{n} \;=\; \frac{\nabla \times \mathbf{F}}{|\nabla \times \mathbf{F}|}. \tag{21}$$

Thus, intuitively, we can interpret $|\nabla \times \mathbf{F}|$ as the local strength of the maximum circulation around any closed curve, and $\nabla \times \mathbf{F}$ itself as yielding the local axis around which the circulation is greatest. Indeed (21) is usually regarded as yielding the definition of the curl of a vector field, with our earlier definition being regarded as merely its Cartesian manifestation.

Stokes' theorem is initially a surprising result, because it says that the flux of $\nabla \times \mathbf{F}$ over an open surface is completely independent of the values $\mathbf{F}$ takes on that surface—it depends only on the values $\mathbf{F}$ takes on the boundary. It means, for example, that the flux of $\nabla \times \mathbf{F}$ over the left-hand surface in Figure 5 is identical to its flux over the right-hand surface—even though one is dome-shaped and the other is bowl-shaped—simply because they share a common boundary along the circle $\mathbf{r} = 2\cos(\theta)\,\mathbf{i} + 2\sin(\theta)\,\mathbf{j} + 0\,\mathbf{k}$ (or, if you prefer, $x^2 + y^2 = 4, z = 0$). Well, that is surprising, isn't it? But it's less surprising when you consider that placing the left-hand open surface neatly on top of the right-hand open surface yields a closed surface, provided only that we change the direction of the lower open-surface

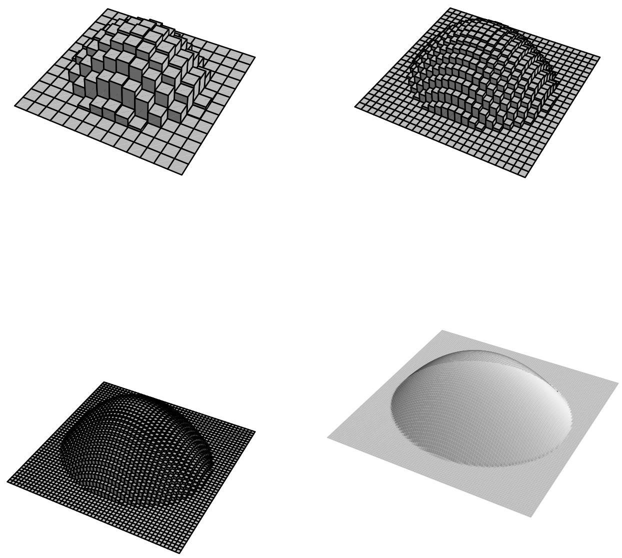

Figure 4: Piecewise-rectangular approximation of arbitrary open surface.

normal $\mathbf{n}$ to make the closed-surface normal point outward all the way round. We can then apply the divergence theorem to the volume V enclosed by $S = S_1 \cup S_2$:

$$\begin{aligned}\iint\limits_{S_1} (\nabla \times \mathbf{F}) \cdot \mathbf{n} dS + \iint\limits_{S_2} (\nabla \times \mathbf{F}) \cdot \{\text{-}\mathbf{n}\} dS &= \iint\limits_{S} (\nabla \times \mathbf{F}) \cdot \mathbf{dS} \\ &= \iiint\limits_{V} \nabla \cdot (\nabla \times \mathbf{F})\, dV.\end{aligned} \tag{22}$$

But $\nabla \times \mathbf{F} = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\mathbf{i} + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\mathbf{j} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\mathbf{k}$, implying

$$\begin{aligned}\nabla \cdot (\nabla \times \mathbf{F}) &= \frac{\partial}{\partial x}\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right) + \frac{\partial}{\partial y}\left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right) + \frac{\partial}{\partial z}\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) \\ &= \frac{\partial^2 F_3}{\partial x \partial y} - \frac{\partial^2 F_2}{\partial x \partial z} + \frac{\partial^2 F_1}{\partial y \partial z} - \frac{\partial^2 F_3}{\partial y \partial x} + \frac{\partial^2 F_2}{\partial z \partial x} - \frac{\partial^2 F_1}{\partial z \partial y} \;=\; 0\end{aligned} \tag{23}$$

(by virtue of our standard assumption that continuous second partial derivatives exist): intuitively, a vector field that represents pure rotationality must have zero divergence because, well, if you're just spinning around then you can't be flowing out as well. So (22) reduces to

$$\iint\limits_{S_1} (\nabla \times \mathbf{F}) \cdot \mathbf{n} dS \;+ \iint\limits_{S_2} (\nabla \times \mathbf{F}) \cdot \{\text{-}\mathbf{n}\} dS \;=\; 0, \tag{24}$$

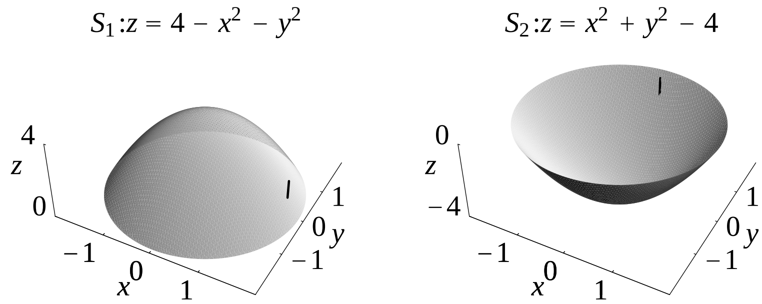

Figure 5: Open surfaces such that $\iint\limits_{S_1} (\nabla \times \mathbf{F}) \cdot \mathbf{dS} = \iint\limits_{S_2} (\nabla \times \mathbf{F}) \cdot \mathbf{dS}$

which immediately implies that

$$\iint\limits_{S_1} (\nabla \times \mathbf{F}) \cdot \mathbf{n} dS \; - \iint\limits_{S_2} (\nabla \times \mathbf{F}) \cdot \mathbf{n} dS \;=\; 0 \tag{25}$$

or, which is exactly the same thing, that

$$\iint\limits_{S_1} (\nabla \times \mathbf{F}) \cdot \mathbf{dS} \;=\; \iint\limits_{S_2} (\nabla \times \mathbf{F}) \cdot \mathbf{dS} \tag{26}$$

Now we can understand why the flux of $\nabla \times \mathbf{F}$ over the left-hand surface in Figure 5 must be identical to its flux over the right-hand surface: these two fluxes have to be the same, because the curl has no divergence between the surfaces that could make the fluxes differ.