§15.9, #39*

Here

$$E = \{(x,y,z) | \sqrt{x^2+y^2} \le z \le \sqrt{2-x^2-y^2}, \quad 0 \le y \le \sqrt{1-x^2}, \quad 0 \le x \le 1\}.$$

Using $z = \rho\cos(\phi)$, $r = \rho\sin(\phi)$, $x = r\cos(\theta)$ and $y = r\sin(\theta)$, we reduce $\sqrt{x^2+y^2} \le z \le \sqrt{2-x^2-y^2}$ to $r \le z$ or $\sin(\phi) \le \cos(\phi)$ or $\tan(\phi) \le 1$ together with $x^2+y^2+z^2 \le 2$ or $\rho^2 \le 2$ or $\rho \le \sqrt{2}$. Likewise, we reduce $0 \le y \le \sqrt{1-x^2}$ to $y \ge 0$ and $x^2+y^2 \le 1$ or $y \ge 0$ and $r \le 1$ or $\sin(\theta) \ge 0$ and $\rho\sin(\phi) \le 1$. Finally, we reduce $0 \le x \le 1$ to $\cos(\theta) \ge 0$ and $\rho\sin(\phi)\cos(\theta) \le 1$. Because $0 \le \phi \le \pi$, $\tan(\phi) \le 1$ implies $0 \le \phi \le \frac{1}{4}\pi$. Also $\sin(\theta) \ge 0$ and $\cos(\theta) \ge 0$ imply $0 \le \theta \le \frac{1}{2}\pi$. So now we have $0 \le \rho \le \sqrt{2}$, $0 \le \phi \le \frac{1}{4}\pi$ and $0 \le \theta \le \frac{1}{2}\pi$. Note that $0 \le \phi \le \frac{1}{4}\pi$ implies $\sin(\phi) \le 1/\sqrt{2}$, which together with $\rho \le \sqrt{2}$ implies $\rho\sin(\phi) \le 1$, which in turn implies $\rho\sin(\phi)\cos(\theta) \le 1$. So all of our inequalites are satisfied. In sum:

$$E = \{(\rho,\phi,\theta) | 0 \le \rho \le \sqrt{2}, 0 \le \phi \le \tfrac{1}{4}\pi, 0 \le \theta \le \tfrac{1}{2}\pi\}.$$

So, not forgetting our Jacobian,

$$\begin{aligned}
I &= \int_0^1 \int_0^{\sqrt{1-x^2}} \int_{\sqrt{x^2+y^2}}^{\sqrt{2-x^2-y^2}} xy\,dz\,dy\,dx \\
&= \int_0^{\frac{1}{2}\pi} \int_0^{\frac{1}{4}\pi} \int_0^{\sqrt{2}} \rho\sin(\phi)\cos(\theta)\cdot\rho\sin(\phi)\sin(\theta)\cdot\rho^2\sin(\phi)\,d\rho\,d\phi\,d\theta \\
&= \int_0^{\frac{1}{2}\pi} \sin(\theta)\cos(\theta)\,d\theta \int_0^{\frac{1}{4}\pi} \sin^3(\phi)\,d\phi \int_0^{\sqrt{2}} \rho^4\,d\rho \\
&= \tfrac{1}{2}\sin^2(\theta)\Big|_0^{\frac{1}{2}\pi} \cdot \left\{\tfrac{1}{3}\cos^3(\phi) - \cos(\phi)\right\}\Big|_0^{\frac{1}{4}\pi} \cdot \tfrac{1}{5}\rho^5\Big|_0^{\sqrt{2}} \\
&= \tfrac{1}{2} \cdot \left\{\tfrac{1}{6\sqrt{2}} - \tfrac{1}{\sqrt{2}} - \left(\tfrac{1}{3} - 1\right)\right\} \cdot \tfrac{4}{5}\sqrt{2} \;=\; \tfrac{4\sqrt{2}}{15} - \tfrac{1}{3}.
\end{aligned}$$