1. (a) C has equation $\mathbf{r} = 2\cos(t)\mathbf{i} + 2\sin(t)\mathbf{j} + \{2 - \cos(t) - \sin(t)\}\mathbf{k}, 0 \le t \le 2\pi$, implying

$$\begin{aligned}
\oint_C \mathbf{F}\cdot d\mathbf{r} &= \int_0^{2\pi} \mathbf{F}\cdot\tfrac{d\mathbf{r}}{dt}\,dt = \int_0^{2\pi} \mathbf{F}\cdot\left\{-2\sin(t)\mathbf{i} + 2\cos(t)\mathbf{j} + \{\sin(t)-\cos(t)\}\mathbf{k}\right\}dt \\
&= \int_0^{2\pi}\left\{-2z\sin(t) + 2x\cos(t) + y\{\sin(t)-\cos(t)\}\right\}dt \\
&= \int_0^{2\pi}\left\{-2\{2-\cos(t)-\sin(t)\}\sin(t) + 2\cdot 2\cos(t)\cdot\cos(t)\right. \\
&\qquad \left. +\ 2\sin(t)\cdot\{\sin(t)-\cos(t)\}\right\}dt = \int_0^{2\pi}\{4 - 4\sin(t)\}\,dt \\
&= \{4t + 4\cos(t)\}\Big|_0^{2\pi} = 4\{2\pi+1\} - 4\{0+1\} = 8\pi.
\end{aligned}$$

(on using $\cos^2(t)+\sin^2(t)=1$).

(b) We have

$$\nabla\times\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z & x & y \end{vmatrix} = (1-0)\mathbf{i} - (0-1)\mathbf{j} + (1-0)\mathbf{k} = \mathbf{i}+\mathbf{j}+\mathbf{k}.$$

The unit upward normal to the plane with equation $x+y+2z=4$, and hence also to the planar elliptical disk $S = S_3$, is $\mathbf{n} = (\mathbf{i}+\mathbf{j}+2\mathbf{k})/\sqrt{6}$. This oblique planar region is bounded by the curve C where the plane meets the cylinder. So when the sun is directly overhead in the direction of $\boldsymbol{\sigma} = \mathbf{k}$, the shadow region in the plane $z=0$ is a circular disk of radius 2, corresponding to $x^2+y^2 \le 4$ in Cartesian coordinates or $0 \le R \le 2, 0 \le \theta \le 2\pi$ in cylindrical polars—which are clearly the best coordinates to use. Hence $dS_{\text{shad}} = R\,dR\,d\theta$ and we obtain

$$\iint_S \nabla\times\mathbf{F}\cdot d\mathbf{S} = \iint_S \nabla\times\mathbf{F}\cdot\mathbf{n}\,dS = \iint_{S_{\text{shad}}} \frac{\nabla\times\mathbf{F}\cdot\mathbf{n}}{|\boldsymbol{\sigma}\cdot\mathbf{n}|}\,dS_{\text{shad}} = \iint_{S_{\text{shad}}} \frac{\frac{1\cdot1+1\cdot1+1\cdot2}{\sqrt{6}}}{\frac{2}{\sqrt{6}}}\,dS_{\text{shad}}$$

$$= \int_0^{2\pi}\int_0^2 2R\,dR\,d\theta = \int_0^{2\pi} d\theta \int_0^2 2R\,dR = 2\pi\cdot R^2\Big|_0^2 = 2\pi\cdot 4 = 8\pi.$$

Alternatively, with $x+y+2z=4$, for $0 \le R \le 2, 0 \le \theta \le 2\pi$ we can parameterize S as $\mathbf{r} = R\cos(\theta)\mathbf{i} + R\sin(\theta)\mathbf{j} + \frac{1}{2}\{4 - R\cos(\theta) - R\sin(\theta)\}\mathbf{k} = R\cos(\theta)\{\mathbf{i}-\frac{1}{2}\mathbf{k}\} + R\sin(\theta)\{\mathbf{j}-\frac{1}{2}\mathbf{k}\} + 2\mathbf{k}$, so that $\mathbf{r}_R = \cos(\theta)\{\mathbf{i}-\frac{1}{2}\mathbf{k}\} + \sin(\theta)\{\mathbf{j}-\frac{1}{2}\mathbf{k}\}$ and $\mathbf{r}_\theta = -R\sin(\theta)\{\mathbf{i}-\frac{1}{2}\mathbf{k}\} + R\cos(\theta)\{\mathbf{j}-\frac{1}{2}\mathbf{k}\}$, implying (after rearrangement and simplification) that

$$\mathbf{r}_r\times\mathbf{r}_\theta = R\{\cos^2(\theta)+\sin^2(\theta)\}\{\mathbf{i}-\tfrac{1}{2}\mathbf{k}\}\times\{\mathbf{j}-\tfrac{1}{2}\mathbf{k}\} = \tfrac{1}{2}R\mathbf{i} + \tfrac{1}{2}R\mathbf{j} + R\mathbf{k},$$

which clearly points upward as required, and yields

$$\iint_S \nabla\times\mathbf{F}\cdot d\mathbf{S} = +\iint_S \nabla\times\mathbf{F}\cdot(\mathbf{r}_r\times\mathbf{r}_\theta)\,dR\,d\theta = \int_0^{2\pi}\int_0^2\{1\cdot\tfrac{1}{2}R + 1\cdot\tfrac{1}{2}R + 1\cdot R\}\,dR\,d\theta,$$

which is what we had before.

Either way, it is clear that $\oint_C \mathbf{F}\cdot d\mathbf{r} = \iint_{S_3}\nabla\times\mathbf{F}\cdot d\mathbf{S}$.

2. (a) $\nabla \cdot \mathbf{F} = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z) = 1 + 1 + 1 = 3$. In cylindrical polar coordinates, the volumetric region E extends from $z = 0$ to $z = 2 - \frac{1}{2}(x + y) = 2 - \frac{1}{2}R\{\cos(\theta) + \sin(\theta)\}$ for values of R between 0 and 2 for values of θ between 0 and 2π. Hence, because the Jacobian determinant for cylindrical polars is $|J| = R$, or—equivalently—the volume element is $dV = R\,dR\,d\theta\,dz$, we obtain

$$\begin{aligned}
\iiint\limits_E \nabla \cdot \mathbf{F}\,dV &= \iiint\limits_E \nabla \cdot \mathbf{F}\,|J|\,dR\,d\theta\,dz = \int_0^{2\pi}\int_0^{2}\int_0^{2-\frac{1}{2}R\{\cos(\theta)+\sin(\theta)} 3R\,dz\,dR\,d\theta \\
&= \int_0^{2\pi}\int_0^{2} 3Rz\big|_0^{2-\frac{1}{2}R\{\cos(\theta)+\sin(\theta)}\,dR\,d\theta \\
&= \int_0^{2\pi}\int_0^{2} 3R\{2 - \tfrac{1}{2}R\{\cos(\theta) + \sin(\theta) - 0\}\,dR\,d\theta \\
&= \int_0^{2\pi}\int_0^{2} \{6R - \tfrac{3}{2}R^2\cos(\theta) - \tfrac{3}{2}R^2\sin(\theta)\}\,dR\,d\theta \\
&= 3\int_0^2 2R\,dR\int_0^{2\pi} d\theta - \tfrac{3}{2}\int_0^2 R^2\,dR\int_0^{2\pi}\{\cos(\theta) + \sin(\theta)\}\,d\theta \\
&= 3 \cdot 2^2 \cdot 2\pi - \tfrac{3}{2}\int_0^2 R^2\,dR \cdot 0 = 24\pi - 0 = 24\pi.
\end{aligned}$$

(b) On S_1 where $z = 0$, we have $\mathbf{F} = x\,\mathbf{i} + y\,\mathbf{j} + 0\,\mathbf{k}$ and the outward unit normal is $\mathbf{n} = -\mathbf{k}$ (down), implying $\mathbf{F} \cdot \mathbf{n} = 0$. Hence

$$\iint\limits_{S_1} \mathbf{F} \cdot \mathbf{dS} = \iint\limits_{S_1} \mathbf{F} \cdot \mathbf{n}\,dS = \iint\limits_{S_1} 0\,dS = 0.$$

On S_2 where $R = 2$ for $0 \le z \le 2 - \cos(\theta) - \sin(\theta)$ and $0 \le \theta \le 2\pi$, we have $\mathbf{r} = 2\cos(\theta)\mathbf{i} + 2\sin(\theta)\mathbf{j} + z\,\mathbf{k}$, implying $\mathbf{r}_\theta = -2\sin(\theta)\,\mathbf{i} + 2\cos(\theta)\,\mathbf{j} + 0\,\mathbf{k}$ and $\mathbf{r}_z = 0\,\mathbf{i} + 0\,\mathbf{j} + \mathbf{k}$, so that $\mathbf{r}_\theta \times \mathbf{r}_z = -2\sin(\theta)\,\mathbf{i} \times \mathbf{k} + 2\cos(\theta)\,\mathbf{j} \times \mathbf{k} = 2\sin(\theta)\,\mathbf{k} \times \mathbf{i} + 2\cos(\theta)\,\mathbf{j} \times \mathbf{k} = 2\sin(\theta)\,\mathbf{j} + 2\cos(\theta)\,\mathbf{i} = 2\{\cos(\theta)\,\mathbf{i} + \sin(\theta)\,\mathbf{j} + 0\,\mathbf{k}\}$, implying $\mathbf{n} = \cos(\theta)\,\mathbf{i} + \sin(\theta)\,\mathbf{j} + 0\,\mathbf{k}$ (directly outward away from the axis of symmetry of the cylinder). Hence either because $\mathbf{F} \cdot (\mathbf{r}_\theta \times \mathbf{r}_z) = x \cdot 2\cos(\theta) + y \cdot 2\sin(\theta) + z \cdot 0 = 2\cos(\theta) \cdot 2\cos(\theta) + 2\sin(\theta) \cdot 2\sin(\theta) = 4\cos^2(\theta) + 4\sin^2(\theta) = 4$ or because $dS = 2\,d\theta\,dz$ by Equation (22) of Lecture 17 with $a = 2$ and $\mathbf{F} \cdot \mathbf{n} = x \cdot \cos(\theta) + y \cdot \sin(\theta) + z \cdot 0 = 2\cos^2(\theta) + 2\sin^2(\theta) = 2$, we obtain

$$\begin{aligned}
\iint\limits_{S_2} \mathbf{F} \cdot \mathbf{dS} &= \iint\limits_{S_2} \mathbf{F} \cdot \mathbf{n}\,dS = +\iint\limits_{S_2} \mathbf{F} \cdot (\mathbf{r}_\theta \times \mathbf{r}_z)\,d\theta\,dz \\
&= 4\int_0^{2\pi}\int_0^{2-\cos(\theta)-\sin(\theta)} dz\,d\theta = 4\int_0^{2\pi}\{2 - \cos(\theta) - \sin(\theta)\}\,d\theta \\
&= 4\{2\theta - \sin(\theta) + \cos(\theta)\}\big|_0^{2\pi} = 4(4\pi - 0) = 16\pi.
\end{aligned}$$

For S_3 we can recycle some–but only some, absolutely not all—of what we already know from Question 1(b). To start with, $\mathbf{n} = (\mathbf{i} + \mathbf{j} + 2\mathbf{k})/\sqrt{6}$, $\boldsymbol{\sigma} = \mathbf{k}$, $0 \leq R \leq 2, 0 \leq \theta \leq 2\pi$ and $dS_{\text{shad}} = R\,dR\,d\theta$. Because $x + y + 2z = 4$ on S_3, we obtain

$$\iint\limits_{S_3} \mathbf{F} \cdot \mathbf{dS} = \iint\limits_{S_3} \mathbf{F} \cdot \mathbf{n}\,dS = \iint\limits_{S_{\text{shad}}} \frac{\mathbf{F} \cdot \mathbf{n}}{|\boldsymbol{\sigma} \cdot \mathbf{n}|}\,dS_{\text{shad}} = \iint\limits_{S_{\text{shad}}} \frac{\frac{x \cdot 1 + y \cdot 1 + z \cdot 2}{\sqrt{6}}}{\frac{2}{\sqrt{6}}}\,dS_{\text{shad}}$$

$$= \int_0^{2\pi}\int_0^2 2R\,dR\,d\theta \;=\; 8\pi$$

again by Question 1(b)—we have already calculated that integral once, no point in doing it again![*] Hence

$$\iint\limits_{S} \mathbf{F} \cdot \mathbf{dS} \;=\; \iint\limits_{S_1 \cup S_2 \cup S_3} \mathbf{F} \cdot \mathbf{dS} \;=\; \sum_{n=1}^{3} \iint\limits_{S_n} \mathbf{F} \cdot \mathbf{dS} \;=\; 0 + 16\pi + 8\pi \;=\; 24\pi.$$

Clearly $\iint\limits_{S} \mathbf{F} \cdot \mathbf{dS} = \iiint\limits_{E} \nabla \cdot \mathbf{F}\,dV$.

[*]It is extremely important to note, however, that we do not already know the answer to this problem from Question 1, because the surface integral in Question 1 is for the curl of a different vector. It just so happens that the same double integral ultimately arises in both cases—but we need a fresh calculation to actually reach the point where we have demonstrated that the same integral arises again before we can recycle our answer from Question 1.