§12.5, #40

This problem involves four planes, three given ones and the plane you are asked to find. Let the three given planes be denoted by Π_1, Π_2 and Π_3 with normals $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$, respectively. Then the equations of the given planes are as follows:

$$\begin{array}{lrcl} \Pi_1: & x + 0y - z &=& 1 \Longrightarrow \mathbf{u} = \mathbf{i} + 0\mathbf{j} - \mathbf{k} \\ \Pi_2: & 0x + y + 2z &=& 3 \Longrightarrow \mathbf{v} = 0\mathbf{i} + \mathbf{j} + 2\mathbf{k} \\ \Pi_3: & x + y - 2z &=& 1 \Longrightarrow \mathbf{w} = \mathbf{i} + \mathbf{j} - 2\mathbf{k} \end{array}$$

Let plain old Π be the plane whose equation we wish to find. We know it contains the line in which Π_1 and Π_2 intersect, which we denote by L . This line is perpendicular to $\mathbf{u}$, by virtue of lying in Π_1. L is also perpendicular to $\mathbf{v}$, by virtue of lying in Π_2. So L, by virtue of being perpendicular to both $\mathbf{u}$ and $\mathbf{v}$, must be parallel to $\mathbf{u} \times \mathbf{v}$. Hence $\mathbf{u} \times \mathbf{v}$ is parallel to Π (because it contains L). But we know that Π is perpendicular to Π_3, and must therefore also be parallel to $\mathbf{w}$. So now we have two directions that are parallel to Π, namely, $\mathbf{u} \times \mathbf{v}$ and $\mathbf{w}$. Hence a normal to Π is the "vector triple product"

$$\mathbf{n} \;=\; (\mathbf{u} \times \mathbf{v}) \times \mathbf{w}$$

We know the equation of a plane if we know both a normal and the position vector $\boldsymbol{\rho}_0$ of any point P_0 in the plane. But now we know a normal. So all we need is a point. Any point on L will do for P_0. By inspection, see that the equations $x - z = 1$ and $y + 2z = 3$ are simultaneously satisfied by $x = 1$, $y = 3$ and $z = 0$. So $\boldsymbol{\rho}_0 = \mathbf{i} + 3\mathbf{j} + 0\mathbf{k}$ will do, and the equation of the plane becomes $\mathbf{n} \cdot (\boldsymbol{\rho} - \boldsymbol{\rho}_0) = 0$, because $\mathbf{n}$ and $\boldsymbol{\rho}_0$ are now known ...

Well, more or less. You certainly know how to calculate $\mathbf{n}$ directly: first calculate the cross product $\mathbf{u} \times \mathbf{v}$, and when you are done, find the cross product of your answer and $\mathbf{w}$. But if you aren't terribly fond of cross products and would much rather calculate dot products—in much the same way that some people love cats but don't much like dogs, or vice versa—then there is a useful little identity that you can employ at this juncture:

$$(\mathbf{u} \times \mathbf{v}) \times \mathbf{w} \;=\; (\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{v} \cdot \mathbf{w})\mathbf{u}$$

This equation is true for *any* $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ whatsoever—the left-hand side is identically equal to the right-hand side, hence the moniker identity. This result is remarkably straightforward to prove—it requires only standard algebraic manipulations—but it is also remarkably tedious, which is why we prefer to believe the result and skip doing the actual proof (which is relegated to Exercise 50 of §12.4 in the text).*

It is easy to see that $\mathbf{u} \cdot \mathbf{w} = 1 + 0 + 2 = 3$ and $\mathbf{v} \cdot \mathbf{w} = 0 + 1 - 4 = -3$. Hence $\mathbf{n} = 3\mathbf{v} - (-3)\mathbf{u} = 3\{\mathbf{i} + \mathbf{j} + \mathbf{k}\}$, from which the equation of Π is readily found to be $x + y + z = 4$.

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*Note, by the way, that the identity is reversed (on both sides of the equals sign, of course) in Theorem 11.6 of §12.4.