1. With $\mathbf{r} = \frac{1}{3}t^3\,\mathbf{i} + \frac{1}{\sqrt{2}}t^2\,\mathbf{j} + t\mathbf{k}$ we have $\dot{\mathbf{r}} = t^2\,\mathbf{i} + \sqrt{2}t\mathbf{j} + \mathbf{k}$. Hence we obtain $v = |\dot{\mathbf{r}}| = \sqrt{t^4 + 2t^2 + 1} = t^2 + 1$, $a_T = \frac{dv}{dt} = 2t$ and $\ddot{\mathbf{r}} = 2t\mathbf{i} + \sqrt{2}\mathbf{j} + 0\mathbf{k}$. So, for $t = 1$, we have $\mathbf{v} = \dot{\mathbf{r}} = \mathbf{i} + \sqrt{2}\mathbf{j} + \mathbf{k}$ and $v = 1^2 + 1 = 2$, implying

(a) $\mathbf{T} = \hat{\mathbf{v}} = \frac{1}{2}\{\mathbf{i} + \sqrt{2}\mathbf{j} + \mathbf{k}\} = \frac{1}{2}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j} + \frac{1}{2}\mathbf{k}.$

(b) $\ddot{\mathbf{r}} = 2\mathbf{i} + \sqrt{2}\mathbf{j} + 0\mathbf{k}$ and $a_T = 2$. So $\ddot{\mathbf{r}} = a_T\mathbf{T} + a_N\mathbf{N}$ implies $a_N\mathbf{N} = \ddot{\mathbf{r}} - a_T\mathbf{T} = 2\mathbf{i} + \sqrt{2}\mathbf{j} + 0\mathbf{k} - \{\mathbf{i} + \sqrt{2}\mathbf{j} + \mathbf{k}\} = \mathbf{i} + 0\mathbf{j} - \mathbf{k}$. The magnitude of this vector is $a_N = |a_N\mathbf{N}| = \sqrt{1^2 + 0^2 + (-1)^2} = \sqrt{2}$, and its direction is $\mathbf{N} = \widehat{a_N\mathbf{N}} = \frac{1}{\sqrt{2}}\{\mathbf{i} + 0\mathbf{j} - \mathbf{k}\} = \frac{1}{\sqrt{2}}\mathbf{i} + 0\mathbf{j} - \frac{1}{\sqrt{2}}\mathbf{k}.$

(c) Because $v = 2$ when $t = 1$, we deduce from $a_N = v^2\kappa$ that $\kappa = \frac{1}{2\sqrt{2}}.$

(d) $\mathbf{B} = \mathbf{T} \times \mathbf{N} = \frac{1}{2\sqrt{2}}\{\mathbf{i} + \sqrt{2}\mathbf{j} + \mathbf{k}\} \times \{\mathbf{i} + 0\mathbf{j} - \mathbf{k}\} = -\frac{1}{2}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j} - \frac{1}{2}\mathbf{k}\}.$

It is easily verified that $\mathbf{T}\cdot\mathbf{N} = \mathbf{T}\cdot\mathbf{B} = \mathbf{N}\cdot\mathbf{B} = 0$.

2. On C we have $\mathbf{r} = \sin(t)\,\mathbf{i} + \sqrt{1+t}\,\mathbf{j} + \cos(t)\,\mathbf{k}$, implying $x = \sin(t)$, $y = \sqrt{1+t}$, $z = \cos(t)$ and $\frac{d\mathbf{r}}{dt} = \cos(t)\,\mathbf{i} + \frac{1}{2}(1+t)^{-1/2}\,\mathbf{j} - \sin(t)\,\mathbf{k}$. Hence $x^2 + z^2 = 1$ and

$$\mathbf{F}\cdot\frac{d\mathbf{r}}{dt} = z\cos(t) + \frac{y\cdot 1}{2\sqrt{1+t}} - x\{-\sin(t)\} = \cos^2(t) + \tfrac{1}{2} + \sin^2(t) = \tfrac{3}{2}.$$

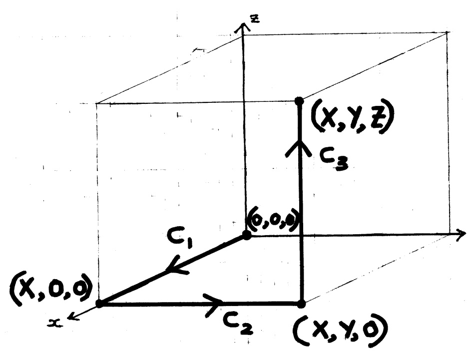
So $\int_C \mathbf{F}\cdot d\mathbf{r} = \int_0^{2\pi} \mathbf{F}\cdot\frac{d\mathbf{r}}{dt}\,dt = \int_0^{2\pi}\frac{3}{2}\,dt = 3\pi.$

3. **(a)** With $\mathbf{F} = (y + ze^x)\,\mathbf{i} + x\,\mathbf{j} + e^x\,\mathbf{k}$ we have $\nabla\times\mathbf{F} =$

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y + ze^x & x & e^x \end{vmatrix} \begin{aligned} &= \left(\tfrac{\partial(e^x)}{\partial y} - \tfrac{\partial x}{\partial z}\right)\mathbf{i} - \left(\tfrac{\partial(e^x)}{\partial x} - \tfrac{\partial(y+ze^x)}{\partial z}\right)\mathbf{j} + \left(\tfrac{\partial(x)}{\partial x} - \tfrac{\partial(y+ze^x)}{\partial y}\right)\mathbf{k} \\ &= (0-0)\mathbf{i} + (e^x - \{0 + e^x\})\mathbf{j} + (1 - \{1+0\})\mathbf{k} \\ &= 0\mathbf{i} + 0\mathbf{j} + 0\mathbf{k} = \mathbf{0}, \end{aligned}$$

implying that $\mathbf{F}$ is conservative.

(b) Because $\nabla\times\mathbf{F} = \mathbf{0}$, there exists a potential $\phi = \phi(x, y, z)$ such that $\mathbf{F} = \nabla\phi$ and $\int_C \mathbf{F}\cdot d\mathbf{r}$ is path-independent. So we can recover ϕ by integrating along any path from $(0, 0, 0)$ to (X, Y, Z), e.g., the curve $C = C_1 \cup C_2 \cup C_3$ consisting of the line segment C_1 between $(0, 0, 0)$ and $(X, 0, 0)$, followed by the line segment C_2 between $(X, 0, 0)$ and $(X, Y, 0)$, followed by the line segment C_3 between $(X, Y, 0)$ and (X, Y, Z).



On C_1 we have $y = 0 = z$, $d\mathbf{r} = dx\mathbf{i}$ and $\mathbf{F} = 0\,\mathbf{i} + x\,\mathbf{j} + e^x\,\mathbf{k}$, implying $\mathbf{F}\cdot d\mathbf{r} = 0$, hence also $\int_{C_1}\mathbf{F}\cdot d\mathbf{r} = 0$. On C_2 we have $d\mathbf{r} = dy\mathbf{j}$ and $\mathbf{F} = (y + 0\cdot e^X)\,\mathbf{i} + X\,\mathbf{j} + e^X\,\mathbf{k}$, implying $\mathbf{F}\cdot d\mathbf{r} = X\,dy$. Thus $\int_{C_2}\mathbf{F}\cdot d\mathbf{r} = \int_0^Y X\,dy = XY$. Finally, along C_3 we have $d\mathbf{r} = dz\mathbf{k}$ and $\mathbf{F} = (Y + ze^X)\,\mathbf{i} + X\,\mathbf{j} + e^X\,\mathbf{k}$, implying $\mathbf{F}\cdot d\mathbf{r} = e^X\,dz$. Thus $\int_{C_3}\mathbf{F}\cdot d\mathbf{r} = \int_0^Z e^X\,dz = e^X Z$. Collecting our results together, we have $\int_C \mathbf{F}\cdot d\mathbf{r} = \int_{C_1}\mathbf{F}\cdot d\mathbf{r} + \int_{C_2}\mathbf{F}\cdot d\mathbf{r} + \int_{C_3}\mathbf{F}\cdot d\mathbf{r} = 0 + XY + e^X Z$, and so $\phi(X, Y, Z) - \phi(0, 0, 0) = \int_C \nabla\phi\cdot d\mathbf{r} = \int_C \mathbf{F}\cdot d\mathbf{r} = XY + e^X Z$. But $\phi(0, 0, 0)$ is just an arbitrary constant that we are free to set equal to zero. Hence $\phi(X, Y, Z) = XY + e^X Z$ or $\phi(x, y, z) = xy + e^x z.$

Alternatively, using the straight-line segment $\mathbf{r} = t(X\mathbf{i} + Y\mathbf{j} + Z\mathbf{k})$, $0 \le t \le 1$ on which $x = tX$, $y = tY$, $z = tZ$ and hence $\frac{d\mathbf{r}}{dt} = X\mathbf{i} + Y\mathbf{j} + Z\mathbf{k}$, so that $\mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = (Yt + Zte^{Xt}) \cdot X + Xt \cdot Y + e^{Xt} \cdot Z = 2XYt + XZte^{Xt} + Ze^{Xt}$, we obtain

$$\phi(X,Y,Z) - \phi(0,0,0) \;=\; \int_C \nabla\phi \cdot \mathbf{dr} \;=\; \int_C \mathbf{F} \cdot \mathbf{dr} \;=\; \int_{t=0}^{t=1} \mathbf{F} \cdot \frac{d\mathbf{r}}{dt}\, dt \;=$$

$$\int_0^1 \{XY \cdot 2t + Z \cdot \{Xte^{Xt} + e^{Xt}\} \;=\; \left\{XY \cdot t^2 + Z \cdot te^{Xt}\right\}\Big|_0^1 \;=\; XY + Ze^X$$

as before.

4. From Lecture 18, S is parameterized in natural coordinates by

$$\mathbf{r} \;=\; \mathbf{a} \;+\; u(\mathbf{b} - \mathbf{a}) \;+\; v(\mathbf{c} - \mathbf{b}), \qquad 0 \le v \le u, \quad 0 \le u \le 1$$

where $\mathbf{a} = 0\mathbf{i} + \mathbf{j} + \mathbf{k}$, $\mathbf{b} = 2\mathbf{i} + 3\mathbf{j} - \mathbf{k}$ and $\mathbf{c} = \mathbf{i} + \mathbf{j} - \mathbf{k}$, i.e., by

$$\begin{aligned} \mathbf{r} \;&=\; \mathbf{j} \;+\; \mathbf{k} \;+\; 2u(\mathbf{i} + \mathbf{j} - \mathbf{k}) \;-\; v(\mathbf{i} + 2\mathbf{j}), \quad 0 \le v \le u,\; 0 \le u \le 1 \\ &=\; (2u - v)\mathbf{i} + (2u - 2v + 1)\mathbf{j} + (1 - 2u)\mathbf{k}, \quad 0 \le v \le u,\; 0 \le u \le 1 \end{aligned}$$

so that $x = 2u - v$, $y = 2u - 2v + 1$, $z = 1 - 2u$, $\mathbf{r}_u = 2(\mathbf{i} + \mathbf{j} - \mathbf{k})$, $\mathbf{r}_v = -\mathbf{i} - 2\mathbf{j} + 0\mathbf{k}$ and $\mathbf{r}_u \times \mathbf{r}_v = 2\{-2\mathbf{i} + \mathbf{j} - \mathbf{k}\}$, which clearly points down, requiring the negative sign to be taken. Then, because

$$\begin{aligned} \mathbf{F} \cdot (\mathbf{r}_u \times \mathbf{r}_v) \;&=\; 2\{-2y + z^2 - x\} \;=\; 2\{-2(2u - 2v + 1) + (1 - 2u)^2 - (2u - v)\} \\ &=\; 2\{4u^2 - 10u + 5v - 1\} \end{aligned}$$

we obtain

$$\begin{aligned} \iint_S \mathbf{F} \cdot \mathbf{dS} \;&=\; -\int_0^1 \int_0^u \mathbf{F} \cdot (\mathbf{r}_u \times \mathbf{r}_v)\, dv\, du \;=\; -2\int_0^1 \int_0^u \{4u^2 - 10u + 5v - 1\}\, dv\, du \\ &=\; -2\int_0^1 \{4u^2 v - 10uv + \tfrac{5}{2}v^2 - v\}\big|_0^u\, du \;=\; -2\int_0^1 \{4u^3 - \tfrac{15}{2}u^2 - u\}\, du \\ &=\; -2\{u^4 - \tfrac{5}{2}u^3 - \tfrac{1}{2}u^2\}\big|_0^1 \;=\; -2(1 - \tfrac{5}{2} - \tfrac{1}{2}\} \;=\; 4. \end{aligned}$$

Alternatively, because every vertex of the triangular region has a nonnegative* first coordinate, we can use the method of Lecture 19 with $\boldsymbol{\sigma} = \mathbf{i}$. The shadow of S is then is the triangle with vertices $(y,z) = (1,1)$, $(y,z) = (3,-1)$ and $(y,z) = (1,-1)$ in the y-z plane . This region is covered by values of z between -1 and $2 - y$ for values of y between 1 and 3. By the method of Lecture 4 or otherwise, S lies in the plane with equation $2x - y + z = 0$ with upward normal

$$\mathbf{n} \;=\; \tfrac{1}{\sqrt{6}}\{2\mathbf{i} - \mathbf{j} + \mathbf{k}\}$$

on which

$$\mathbf{F} \cdot \mathbf{n} \;=\; \tfrac{1}{\sqrt{6}}\{2y - z^2 + x\} \;=\; \tfrac{1}{2\sqrt{6}}\{5y - z - 2z^2\}$$

*In fact, positive, but nonnegative would have done.

(because $x = \frac{1}{2}\{y - z\}$) and $\boldsymbol{\sigma} \cdot \mathbf{n} = \mathbf{i} \cdot \mathbf{n} = \frac{2}{\sqrt{6}}$ implying $|\boldsymbol{\sigma} \cdot \mathbf{n}| = \frac{2}{\sqrt{6}}$ as well. Also, $dS_{\text{shad}} = dydz$. Hence

$$\begin{aligned}
\iint\limits_S \mathbf{F} \cdot \mathbf{dS} &= \iint\limits_S \mathbf{F} \cdot \mathbf{n}\, dS = \iint\limits_{S_{\text{shad}}} \frac{\mathbf{F} \cdot \mathbf{n}}{|\boldsymbol{\sigma} \cdot \mathbf{n}|}\, dS_{\text{shad}} = \tfrac{1}{4} \iint\limits_{S_{\text{shad}}} \{5y - z - 2z^2\}\, dz\, dy \\
&= \tfrac{1}{4} \int_1^3 \int_{-1}^{2-y} \{5y - z - 2z^2\}\, dz\, dy = \tfrac{1}{4} \int_1^3 \{5yz - \tfrac{1}{2}z^2 - \tfrac{2}{3}z^3\}\Big|_{-1}^{2-y}\, dy \\
&= \tfrac{1}{4} \int_1^3 \{\tfrac{2}{3}y^3 - \tfrac{19}{2}y^2 + 25y - \tfrac{15}{2}\}\, dy = \tfrac{1}{4}\Big\{\tfrac{1}{6}y^4 - \tfrac{19}{6}y^3 + \tfrac{25}{2}y^2 - \tfrac{15}{2}y\Big\}\Big|_1^3 \\
&= \tfrac{1}{4}\{18 - 2\} = 4.
\end{aligned}$$