We have to extremize $f(x,y,z) = z$

subject to

$$g(x,y,z) \triangleq 4x - 3y + 8z = 5 \qquad (d)$$

and $$h(x,y,z) \triangleq x^2+y^2 - z^2 = 0 \qquad (e)$$

Lagrange says there exist λ, μ such that $\nabla f = \lambda \nabla g + \mu \nabla h$ or

$$\left.\begin{aligned} f_x &= \lambda g_x + \mu h_x \\ f_y &= \lambda g_y + \mu h_y \\ f_z &= \lambda g_z + \mu h_z \end{aligned}\right\} \Rightarrow \begin{aligned} 0 &= 4\lambda + 2\mu x \\ 0 &= -3\lambda + 2\mu y \\ 1 &= 8\lambda - 2\mu z \end{aligned} \Rightarrow \begin{aligned} x &= \frac{-2\lambda}{\mu} && (a) \\ y &= \frac{3\lambda}{2\mu} && (b) \\ z &= \frac{8\lambda - 1}{2\mu} && (c) \end{aligned}$$

We cannot have $\mu = 0$ because then this would imply $\lambda = 0$ as well, contradicting this,

Substitute (a), (b) and (c) into (d). Get $\mu = \dfrac{39\lambda - 8}{10}$

" " " " " " (e). Get

$$\left(\frac{-2\lambda}{\mu}\right)^2 + \left(\frac{3\lambda}{2\mu}\right)^2 = \left(\frac{8\lambda-1}{2\mu}\right)^2.$$ Multiply by $4\mu^2$.

Get $16\lambda^2 + 9\lambda^2 = (8\lambda - 1)^2$

$\Rightarrow \quad 25\lambda^2 = (8\lambda-1)^2$ (by this)

$\Rightarrow$ either $5\lambda = 8\lambda - 1 \Rightarrow \lambda = 1/3 \Rightarrow \mu = 1/2$

or $5\lambda = 1 - 8\lambda \Rightarrow \lambda = 1/13 \Rightarrow \mu = -\frac{1}{2}$

In the first case we get $x = -4/3,\ y = 1,\ z = \frac{5}{3}$ by (a)–(c)

" " " " " " $x = 4/13,\ y = \frac{-3}{13},\ z = \frac{5}{13}$ "

So the highest point of the ellipse is $(x,y,z) = \left(-4/3, 1, \frac{5}{3}\right)$ where $f\left(-4/3, 1, \frac{5}{3}\right) = \frac{5}{3}$

and " lowest " " " " " $(x,y,z) = \left(4/13, \frac{-3}{13}, \frac{5}{13}\right)$

where $f\left(\frac{-4}{13}, \frac{-3}{13}, \frac{5}{13}\right) = \frac{5}{13}$