Bonus: (*10 pts*) Let $f(t) = t\sin(t)$. Using the formula for $\mathcal{L}\{f''(t)\}$, show that

$$F(s) = \frac{2s}{(s^2+1)^2}.$$

<u>You cannot use the Laplace table!</u> **Hint**: $f'(t) = t\cos(t) + \sin(t)$ and $f''(t) = 2\cos(t) - \underbrace{t\sin(t)}_{\text{f(t)}}$.

SOLUTION:

So, at this point, we know four things:

(i) $f(t) = t\sin(t)$;

(ii) $\mathcal{L}\{f''(t)\} = s^2\mathcal{L}\{f(t)\} - sf(0) - f'(0)$;

(iii) $f'(t) = t\cos(t) + \sin(t)$; and

(iv) $f''(t) = 2\cos(t) - \underbrace{t\sin(t)}_{\text{f(t)}}$

The goal is to find $\mathcal{L}\{f(t)\}$, and to do that, we're going to use (ii) and (iv). From (ii), we have that

$$\mathcal{L}\{f''(t)\} = s^2\mathcal{L}\{f(t)\} - sf(0) - f'(0). \tag{1}$$

Using a combination of (i) and (iii), we have that $f(0) = 0$ and that $f'(0) = 0$; plugging these values into equation (1) yields

$$\mathcal{L}\{f''(t)\} = s^2\mathcal{L}\{f(t)\}. \tag{2}$$

Now, taking the Laplace of (iv) will give <u>another</u> expression for $\mathcal{L}\{f''(t)\}$ which we'll be able to set equal to (2).

Taking the Laplace of (iv) yields the following:

$$\begin{aligned} \mathcal{L}\{f''(t)\} &= \mathcal{L}\{2\cos(t) - t\sin(t)\} \\ &= 2\mathcal{L}\{\cos(t)\} - \mathcal{L}\{t\sin(t)\} \\ &= 2\mathcal{L}\{\cos(t)\} - \mathcal{L}\{f(t)\}, \text{ because } f(t) = t\sin(t) \text{ as noted in (iv) above.} \end{aligned} \tag{3}$$

The expression $2\mathcal{L}\{\cos(t)\}$ from the last line of (3) can be found using a Laplace table and/or using the old method; in particular,

$$2\mathcal{L}\{\cos(t)\} = 2\left(\frac{s}{s^2+1}\right) = \frac{2s}{s^2+1}. \tag{4}$$

Combining (4) with the first+last lines of (3), it follows that

$$\mathcal{L}\{f''(t)\} = \frac{2s}{s^2+1} - \mathcal{L}\{f(t)\}, \tag{5}$$

and setting the RHS of (5) equal to the RHS of (2) [which we can do because both are expressions for $\mathcal{L}\{f''(t)\}$], we have

$$s^2\mathcal{L}\{f(t)\} = \frac{2s}{s^2+1} - \mathcal{L}\{f(t)\}. \tag{6}$$

Now, we solve for $\mathcal{L}\{f(t)\}$ in (6):

$$s^2\mathcal{L}\{f(t)\} = \frac{2s}{s^2+1} - \mathcal{L}\{f(t)\} \iff \mathcal{L}\{f(t)\}\left(s^2+1\right) = \frac{2s}{s^2+1}.$$

Dividing both sides by s^2+1 yields the result:

$$\boxed{\mathcal{L}\{f(t)\} = \frac{2s}{(s^2+1)^2}.}$$