Vector Spaces II

Recently, I mentioned that I'd begin using the phrase "vector space" but wouldn't formally define it in class. To alleviate the burden caused, I've decided to give some handouts on the topic instead.

First, recall the definition:

Definition 1: A *vector space* consists of

- a nonempty set V of objects (called *vectors*); and
- a pair of operations called *addition* (denoted "+") and *scalar multiplication* (denoted by juxtaposition, i.e. $c\mathbf{u}$ for the scalar multiple of $\mathbf{u}$ by c)

which together will always satisfy the following ten axioms (for all scalars c and d and for all vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ in V):

Recall the subspace axioms!

(i) $\mathbf{u}+\mathbf{v} \in V$ } We can add things!

(ii) $c\mathbf{u} \in V$ } We can scalar multiply

(iii) there exists a zero vector $\mathbf{0} \in V$ which satisfies $\mathbf{u}+\mathbf{0}=\mathbf{u}$

(iv) $\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}$

(v) $\mathbf{u}+(\mathbf{v}+\mathbf{w})=(\mathbf{u}+\mathbf{v})+\mathbf{w}$

(vi) for each $\mathbf{u} \in V$, there exists a vector $-\mathbf{u} \in V$ such that $\mathbf{u}+(-\mathbf{u})=\mathbf{0}$

(vii) $c(\mathbf{u}+\mathbf{v})=c\mathbf{u}+c\mathbf{v}$

(viii) $(c+d)\mathbf{u}=c\mathbf{u}+d\mathbf{u}$

(ix) $c(d\mathbf{u})=(cd)\mathbf{u}$

(x) $1\mathbf{u}=\mathbf{u}$

Addition and scalar multiplication work the way we want them to!

As mentioend in class one way to think about vector spaces is to think about it as a *collection of things* which we can *add and scalar multiply* and for which *addition and scalar multiplication "work the way they do for real numbers."* Moreover—even without a thorough explanation—we can always understand vector spaces simply by employing the following mantra:

When you see "...n-dimensional vector space V..." just replace it with "...$\mathbb{R}^n$ as a subspace of $\mathbb{R}^n$..."!

Even so, the purpose of this handout (and its peers) is to provide the aforementioned "thorough explanation."

Example 1: $V = \mathbb{R}^n$

If our mantra tells us that every n-dimensional vector space can be thought of as $\mathbb{R}^n$, then it would be truly unfortunate for $\mathbb{R}^n$ not to be an n-dimensional vector space. Unsurprisingly, this *isn't* the case, and we're going to prove it here!

To do this, we first outline what our *vectors*, *addition*, and *scalar multiplication* are; then, we show that these three objects satisfy the ten axioms given in **Definition 1**.

Outline of our Objects

<u>Vectors:</u>

Our "vectors" are elements $\mathbf{u} \in V = \mathbb{R}^n$ of the form $\mathbf{u} = \begin{pmatrix} u_1, & \cdots, & u_n \end{pmatrix}^\mathsf{T}$ for $u_1, \ldots, u_n \in \mathbb{R}$.

<u>Addition:</u>

Addition of our vectors is done component-wise: If $\mathbf{u} = \begin{pmatrix} u_1, & \cdots, & u_n \end{pmatrix}^\mathsf{T}$ and $\mathbf{v} = \begin{pmatrix} v_1, & \cdots, & v_n \end{pmatrix}^\mathsf{T}$ are elements of V, then

$$\mathbf{u} + \mathbf{v} = \begin{pmatrix} u_1 + v_1, & \cdots, & u_n + v_n \end{pmatrix}^\mathsf{T}.$$

<u>Scalar Multiplication:</u>

Scalar multiplication for our vectors is *also* done component-wise: If $\mathbf{u} = \begin{pmatrix} u_1, & \cdots, & u_n \end{pmatrix}^\mathsf{T} \in V$ and $c \in \mathbb{R}$ is a real scalar, then

$$c\mathbf{u} = \begin{pmatrix} cu_1, & \cdots, & cu_n \end{pmatrix}^\mathsf{T}.$$

Axiom Verification

Just a word of warning:

Many of the following verifications will seem like you're doing nothing. The reason? <u>Everything we've ever done in $\mathbb{R}^n$ has used the fact that it's a vector space!</u>

So for most of what follows, your reaction will be, *Uh, DUH!* and...I get that! However, to verify that something is a vector space, there really isn't any way around doing the boring tedium.

Axiom (i): $\mathbf{u} + \mathbf{v} \overset{?}{\in} V = \mathbb{R}^n$

Note that in the definition of component-wise addition in $\mathbb{R}^n$,

$$\mathbf{u} + \mathbf{v} = \begin{pmatrix} \underbrace{u_1 + v_1}_{\text{adding two real numbers}}, & \cdots, & \underbrace{u_n + v_n}_{\text{adding two real numbers}} \end{pmatrix}^\mathsf{T}.$$

Because adding two real numbers always yields a real number, the result is a n-tuple of real numbers, i.e. $\mathbf{u} + \mathbf{v} \in V$!

Axiom (ii): $c\mathbf{u} \overset{?}{\in} V$

As above, we note that, in the definition of component-wise scalar multiplication in $\mathbb{R}^n$,

$$c\mathbf{u} = \Bigg(\underbrace{cu_1}_{\text{multiplying two real numbers}}, \quad \cdots, \quad \underbrace{cu_n}_{\text{multiplying two real numbers}} \Bigg)^{\mathsf{T}}.$$

Analogously to the above, *multiplying* two real numbers always yields a real number; hence, the result is a n-tuple of real numbers, and so $c\mathbf{u} \in V$!