The Demise of Trig Substitutions?

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A West Texas Surprise

Having taught Calculus II for more years than we would like to count, we recently were stunned to learn an aspect of trig substitutions that has apparently gone unnoticed by calculus book authors for over three decades (maybe longer, but that's the oldest calculus book we could locate).

While we are not absolutely certain that these authors, and others, were unaware of it, or rather conveniently chose to overlook it, we do know that having it brought to light by one of our students was an unsettling revelation.

The first author, in the process of marking wrong this student's answers to *some* of the trig substitution integrals, suddenly noticed that the answers were correct even though there was not a trig function in sight (the absence of which accounted for marking them wrong).

Examination of the student's work, for the computation of $\int x^3\sqrt{4-x^2}\,dx$ showed that, indeed, by letting $u^2 = 4 - x^2$, one is led to a solution somewhat more quickly that the customary trig substitution $x = 2\sin\theta$. The odd power 3 on the x was the key to this working and you can quickly determine that such a substitution always works in the odd power case.[1]

After recovering from our shock, we did what any mathematician would do: search for a simple, non-trig substitution in the *even* power case. Over the course of several days we

[1] The student, Jaymison King, said he was just "fiddling around" and found that this technique works for the odd power case. However, another student, later caught using an internet tutor, employed the same technique, so we suspect Jaymison did not discover this on his own.

found some possibilities, but they were too complicated, and for a while we were convinced that only about half of the trig substitutions could be done in another way. But finally, what now seems the most obvious thing to do presented itself.

$\mathbf{u^2}$-substitutions

There are four types of integrals that we can effectively compute using trig substitutions. They are,

$$\text{(a)} \int x^n R^k \, dx \quad \text{(b)} \int \frac{x^n}{R^k} \, dx \quad \text{(c)} \int \frac{R^k}{x^n} \, dx, \quad \text{(d)} \int \frac{1}{x^n R^k} \, dx \tag{1}$$

where $n \geq 0, k > 0$ are integers, k is odd, and

$$R = (a^2 - x^2)^{1/2}, (a^2 + x^2)^{1/2}, \quad \text{or } (x^2 - a^2)^{1/2} \tag{2}$$

In all cases, these can be computed (sometimes more quickly) using what, for lack of better name, we call u^2-substitutions:

$$u^2 = R^2 \quad (n \text{ odd}) \tag{3}$$

$$u^2 = \frac{R^2}{x^2} \quad (n \text{ even}) \tag{4}$$

Before discussing these substitutions from a rigorous point of view, it is motivationally best to show how you might present this to your students.

Pedagogy

Notice the substitution in (3), and in (4), is the same form regardless of which of the radicals R in (2) occurs. So just remember to use the form in (3) when n is odd and the form in (4) when n is even. Since the process is the same for all three kinds of radicals R and all four types of integrals (1), we just describe what to do for the $R = (a^2 - x^2)^{1/2}$

CASE I (n Odd): In this case the substitution has the form

$$u^2 = a^2 - x^2 \tag{5}$$

Substituting this into the radical $R^k = (a^2 - x^2)^{k/2}$ gives u^k. Differentiating each side of (5) gives

$$2u \, du = -2x \, dx, \quad \text{so} \quad -\frac{u}{x} \, du = dx \tag{6}$$

Substitute this in place of dx and then combine the $1/x$ in it with x^n. This results in an even power of x, say x^{2m}. So rewrite (5) as $x^2 = a^2 - u^2$ and use this to get $x^{2m} = (a^2 - u^2)^m$. Now the integral should be entirely in terms of u. After integrating with respect to u, transform the result back to x by using (5) in the form

$$u = (a^2 - x^2)^{1/2} \tag{7}$$

Here are two examples.

Example 1: $\int x^3(4 - x^2)^{1/2}\, dx$,
Let

$$u^2 = 4 - x^2, \quad \text{so that} \quad u\,du = -x\,dx, \quad \text{i.e.,} \quad -\frac{u}{x}\,du = dx$$

Note that $x^2 = 4 - u^2$. Then

$$\begin{aligned}
\int x^3(4-x^2)^{1/2}\,dx &= \int x^3 u\,(-\frac{u}{x}\,du) = -\int x^2u^2\,du = -\int (4-u^2)u^2du \\
&= \int (u^4 - 4u^2)\,du = \frac{1}{5}u^5 - \frac{4}{3}u^3 + C \\
&= \frac{1}{5}(4-x^2)^{5/2} - \frac{4}{3}(4-x^2)^{3/2} + C
\end{aligned}$$

Example 2: $\int \frac{x^3}{(9-x^2)^{5/2}}\,dx$,
Let

$$u^2 = 9 - x^2, \quad \text{so that} \quad u\,du = -x\,dx, \quad \text{i.e.,} \quad -\frac{u}{x}\,du = dx$$

Note that $x^2 = 9 - u^2$. Then

$$\begin{aligned}
\int \frac{x^3}{(9-x^2)^{5/2}}\,dx &= \int \frac{x^3}{u^5}\,(-\frac{u}{x}\,du) = -\int \frac{x^2}{u^4}\,du = -\int \frac{9-u^2}{u^4}\,du \\
&= \int (u^{-2} - 9u^{-4})\,du = -u^{-1} + \frac{9}{3}u^{-3} + C \\
&= -\frac{1}{(9-x^2)^{1/2}} + \frac{3}{(9-x^2)^{3/2}} + C
\end{aligned}$$

CASE II (n Even): In this case the substitution has the form

$$u^2 = \frac{a^2 - x^2}{x^2} = a^2x^{-2} - 1 \tag{8}$$

We also write this as

$$x^2u^2 = a^2 - x^2 \tag{9}$$

Substituting this into the radical $R^k = (a^2 - x^2)^{k/2}$ gives $x^k u^k$. Differentiating each side of (8) gives

$$2u\,du = -2a^2 x^{-3}\,dx, \quad \text{so} \quad -\frac{x^3 u}{a^2}\,du = dx \tag{10}$$

Substitute this in place of dx and then combine the x^3 in it with the other powers of x, i.e., x^n and x^k. This results in an even power of x, say x^{2m}. So rewrite (8) as

$$x^2 = \frac{a^2}{1+u^2}, \tag{11}$$

and use this to get $x^{2m} = a^{2m}/(1+u^2)^m$. Now the integral should be entirely in terms of u. After integrating with respect to u, transform the result back to x by using (8) in the form

$$u = \frac{(a^2 - x^2)^{1/2}}{x} \tag{12}$$

Here are two examples.

Example 3: $\displaystyle\int \frac{(16-x^2)^{1/2}}{x^4}\,dx,$

Let

$$u^2 = \frac{16-x^2}{x^2} = 16x^{-2} - 1, \quad \text{so that} \quad u\,du = -16x^{-3}\,dx \quad \text{and} \quad x^2 = \frac{16}{1+u^2}$$

Note also that $x^2u^2 = 16 - x^2$. Then

$$\begin{aligned}\int \frac{(16-x^2)^{1/2}}{x^4}\,dx &= \int \frac{xu}{x^4}\,(-\frac{x^3u}{16}\,du) = -\frac{1}{16}\int u^2\,du \\ &= -\frac{1}{48}u^3 + C = -\frac{1}{48}\cdot\frac{(16-x^2)^{3/2}}{x^3} + C\end{aligned}$$

Example 4: $\displaystyle\int \frac{1}{x^4(9-x^2)^{1/2}}\,dx,$

Let

$$u^2 = \frac{9-x^2}{x^2} = 9x^{-2} - 1, \quad \text{so that} \quad u\,du = -9x^{-3}\,dx \quad \text{and} \quad x^2 = \frac{9}{1+u^2}$$

Note also that $x^2u^2 = 9 - x^2$. Then

$$\begin{aligned}\int \frac{1}{x^4(9-x^2)^{1/2}}\,dx &= \int \frac{1}{x^4(xu)}\,(-\frac{x^3u}{9}\,du) = -\frac{1}{9}\int \frac{1}{x^2}\,du \\ &= -\frac{1}{9}\int \frac{1+u^2}{9}\,du = -\frac{1}{81}\int (1+u^2)\,du \\ &= -\frac{1}{81}\,u - \frac{1}{243}\,u^3 + C = -\frac{1}{81}\frac{(9-x^2)^{1/2}}{x} - \frac{1}{243}\frac{(9-x^2)^{3/2}}{x^3} + C\end{aligned}$$

Some Mathematical Details

To justify the heuristic manipulations used above in the u^2-substitutions, we consider the mathematical basis for substitution techniques, a.k.a, the change of variables formula. For functions f of a single variable, the indefinite integral change of variables formula is

$$\int f(x)dx = \int f(\phi(u))\phi'(u)du = G(u) + C = G(\phi^{-1}(x)) + C, \tag{13}$$

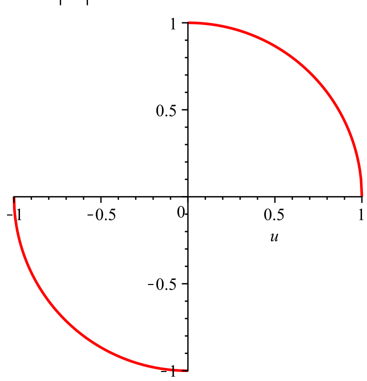
where $x = \phi(u)$ is the change of variables. Here we assume f is continuous on an open interval $I = (a, b)$, and that $\phi : J = (c, d) \to I$ is a continuously differentiable bijection from J to I. Formula (13) says that if G is an antiderivative of $(f \circ \phi)\,\phi'$ on J, then $G \circ \phi^{-1}$ is an antiderivative of f on I. This result holds also in the more general case when I and J are unions of disjoint intervals

In doing u^2-substitutions, either $u^2 = R^2$ or $u^2 = R^2/x^2$ and there are three different kinds of radicals R. This means that if you solve these equations for x, getting $x = \phi(u)$, there will be six different functions ϕ to deal with.

NOTE: Without loss of generality, we can assume $a = 1$. Then the six change of variables functions ϕ are given in the following tables.

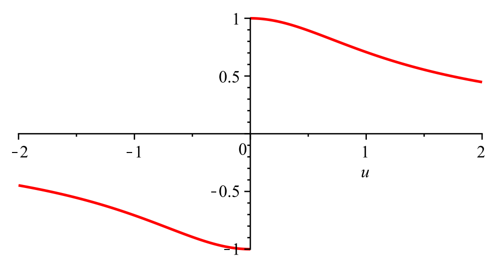
Sine Radical: $\mathbf{R = \sqrt{1 - x^2}}$

$x = \phi(u) = \dfrac{u}{|u|}\sqrt{1-u^2}$



Odd Case: $u^2 = 1 - x^2$

$x = \phi(u) = \dfrac{u}{|u|}\dfrac{1}{\sqrt{1+u^2}}$



Even Case: $u^2 = \dfrac{1-x^2}{x^2}$

Figure 1: In odd case (left side) $I = J = (-1, 0) \cup (0, 1)$, while in the even case (right side) $I = (-1, 0) \cup (0, 1)$ and $J = (-\infty, 0) \cup (0, \infty)$.

Tangent Radical: $\mathbf{R} = \sqrt{\mathbf{1 + x^2}}$

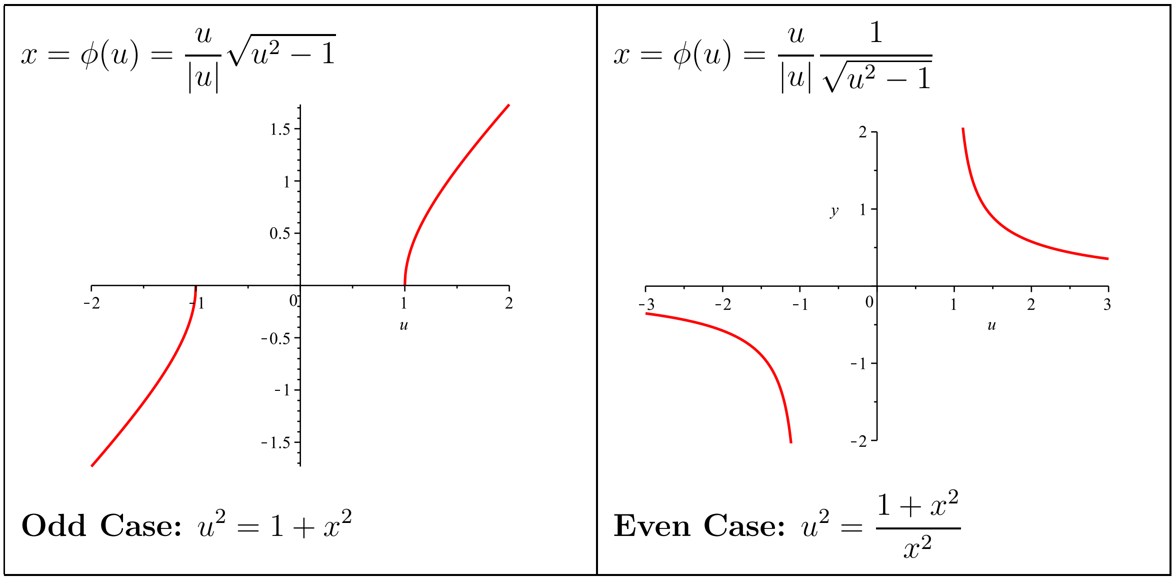


Figure 2: In odd case (left side) $I = (-\infty, 0) \cup (0, \infty)$, $J = (-\infty, -1) \cup (1, \infty)$, while in the even case (right side) $I = (-\infty, 0) \cup (0, \infty)$ and $J = (-\infty, -1) \cup (1, \infty)$.

Secant Radical: $\mathbf{R} = \sqrt{\mathbf{x^2 - 1}}$

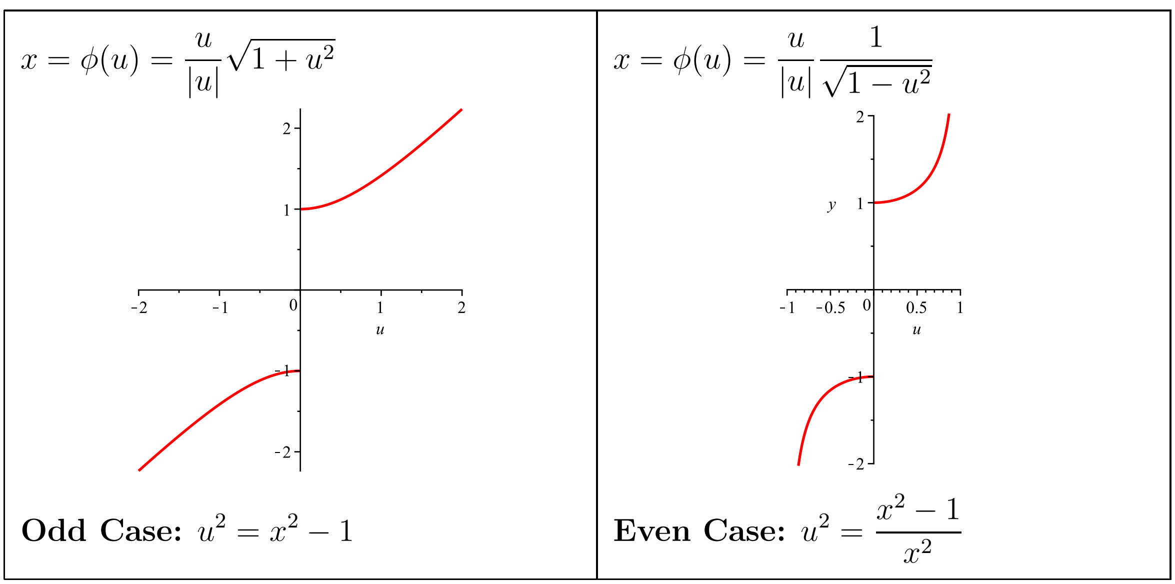


Figure 3: In odd case (left side) $I = (-\infty, -1) \cup (1, \infty)$, $J = (-\infty, 0) \cup (0, \infty)$, while in the even case (right side) $I = (-\infty, -1) \cup (1, \infty)$ and $J = (-1, 0) \cup (0, 1)$.

Using these maps ϕ and the change of variables formula (13), one can transform each of the four types of trig substitution integrals (a)–(d) into ones that are specific types of rational functions in the variable u. While the method used here in rewriting the integrands is different than the pedagogical method discussed in the last section, the resulting integrals are the same. With either method, it is not too hard to verify that the results are the integrals shown in Table 1 below.

Type	n odd	n even
(a) $\int x^n R^k\, dx$	$-\int (1-u^2)^{(n-1)/2}\, u^{k+1}\, du$	$-\int \frac{u^{k+1}}{(1+u^2)^{(n+k+3)/2}}\, du$
(b) $\int \frac{x^n}{R^k}\, dx$	$-\int \frac{(1-u^2)^{(n-1)/2}}{u^{k-1}}\, du$	$-\int \frac{1}{u^{k-1}\,(1+u^2)^{(n-k+3)/2}}\, du$
(c) $\int \frac{R^k}{x^n}\, dx$	$-\int \frac{u^{k+1}}{(1-u^2)^{(n+1)/2}}\, du$	$-\int \frac{u^{k+1}}{(1+u^2)^{(k-n+3)/2}}\, du$
(d) $\int \frac{1}{x^n R^k}\, dx$	$-\int \frac{1}{u^{k-1}\,(1-u^2)^{(n+1)/2}}\, du$	$-\int \frac{(1+u^2)^{(n+k-3)/2}}{u^{k-1}}\, du$

Table 1 (Sine Radical): $R = \sqrt{1-x^2}$

We briefly indicate how, in the table, each of the integrals involving u can be computed. Types (a)-odd, (b)-odd, and (d)-even are the easy ones. Using the binomial theorem and basic algebra, their integrands can be expressed as linear combinations of integral power functions in u. The same is true for type (b)-even when $k \geq n+3$ and for type (c)-even when $n \geq k+3$. They are easy for the same reason.

The remaining types are the hard ones. But they can be computed by first using one of the following reduction formulas:

Two Reduction Formulas: For $p > 1$ an integer and s any number:

$$\int \frac{u^s}{(1+u^2)^p}\, du = \frac{-1}{2(p-1)} \cdot \frac{u^{s-1}}{(1+u^2)^{p-1}} + \frac{(s-1)}{2(p-1)} \int \frac{u^{s-2}}{(1+u^2)^{p-1}}\, du \tag{14}$$

$$\int \frac{u^s}{(1-u^2)^p}\, du = \frac{1}{2(p-1)} \cdot \frac{u^{s-1}}{(1-u^2)^{p-1}} - \frac{(s-1)}{2(p-1)} \int \frac{u^{s-2}}{(1-u^2)^{p-1}}\, du \tag{15}$$

Repeated use of the formulas, starting with $p > 1$ and s an even integer will eventually yield an integral where $p = 1$ and s is still an even integer. Thus, we are left with an integrand of one of the following two types.

Two Identities: For $m \geq 1$ and integer:

$$\frac{1}{u^{2m}(1+u^2)} = \frac{1}{u^{2m}} - \frac{1}{u^{2m-2}} + \cdots + (-1)^{m-1}\frac{1}{u^2} + (-1)^m \frac{1}{1+u^2} \tag{16}$$

$$\frac{1}{u^{2m}(1-u^2)} = \frac{1}{u^{2m}} + \frac{1}{u^{2m-2}} + \cdots + \frac{1}{u^2} + \frac{1}{1-u^2} \tag{17}$$

These identities allow us to easily compute the integral with these types of integrands.

An Unexpected Consequence

The discussion in the previous section shows that u^2-substitutions are just as effective as trig substitutions in computing integrals of types (a)–(d). By way of comparison, note that trig substitutions give trig integrals and the easy trig integrals are the ones that can be done by a u-substitution. That's two substitutions in a row, which can be combined into one substitution. This is then a u^2-substitution for the odd n case. The hard trig integrals are the ones that require rewriting the integrand as a power of a single trig function and then using a reduction formula repeatedly. Formulas (14)–(15) and identities (16)–(17) are the analogs of this for u^2-substitutions.

Additionally, one sees that by using u^2-substitutions, the resulting integrals in u, when computed involve only linear combinations of integral power functions $p(u)$, the functions $\ln|1+u|, \ln|1-u|$, and the inverse tangent function $\tan^{-1} u$. Thus, we don't need the inverse sine or the inverse secant functions to compute integrals of types (a)–(d). That is an unexpected consequence.

The following two examples illustrate this.

Example 5 (No Inverse Secant): $\displaystyle\int \frac{1}{x\sqrt{x^2-1}}\,dx,$

Let

$$u^2 = x^2 - 1, \quad \text{so that} \quad u\,du = x\,dx \quad \text{i.e.,} \quad \frac{u}{x}\,du = dx$$

Note that $x^2 = 1 + u^2$. Then

$$\begin{aligned} \int \frac{1}{x\sqrt{x^2-1}}\,dx &= \int \frac{1}{xu}\left(\frac{u}{x}\,du\right) = \int \frac{1}{x^2}\,du = \int \frac{1}{1+u^2}\,du \\ &= \tan^{-1} u + C = \tan^{-1}(\sqrt{x^2-1}) + C \end{aligned}$$

Example 6 (No Inverse Sine): $\displaystyle\int \frac{1}{\sqrt{1-x^2}}\,dx,$

Let

$$u^2 = \frac{1-x^2}{x^2} = x^{-2} - 1, \quad \text{so that} \;\; u\,du = -x^{-3}\,dx \;\; \text{and} \;\; x^2 = \frac{1}{1+u^2}$$

Note also that $x^2u^2 = 1 - x^2$. Then

$$\begin{aligned} \int \frac{1}{\sqrt{1-x^2}}\,dx &= \int \frac{1}{xu}\,(-x^3 u\,du) = -\int x^2\,du = -\int \frac{1}{1+u^2}\,du \\ &= -\tan^{-1} u + C = -\tan^{-1}\left(\frac{\sqrt{1-x^2}}{x}\right) + C \end{aligned}$$

Of course the facts that

$$\tan^{-1}\left(\frac{\sqrt{1-x^2}}{x}\right) = \cos^{-1} x + C = -\sin^{-1} x + K \tag{18}$$

$$\tan^{-1}(\sqrt{x^2-1}) = \sec^{-1} x + C = -\csc^{-1} x + K \tag{19}$$

are perhaps well-known and can be easily verified by differentiation and using the equal derivatives theorem. (Or one can use geometry.) The graphs of these alternatives to the inverse sine/cosine and inverse secant/cosecant are shown in Figure 4 below.

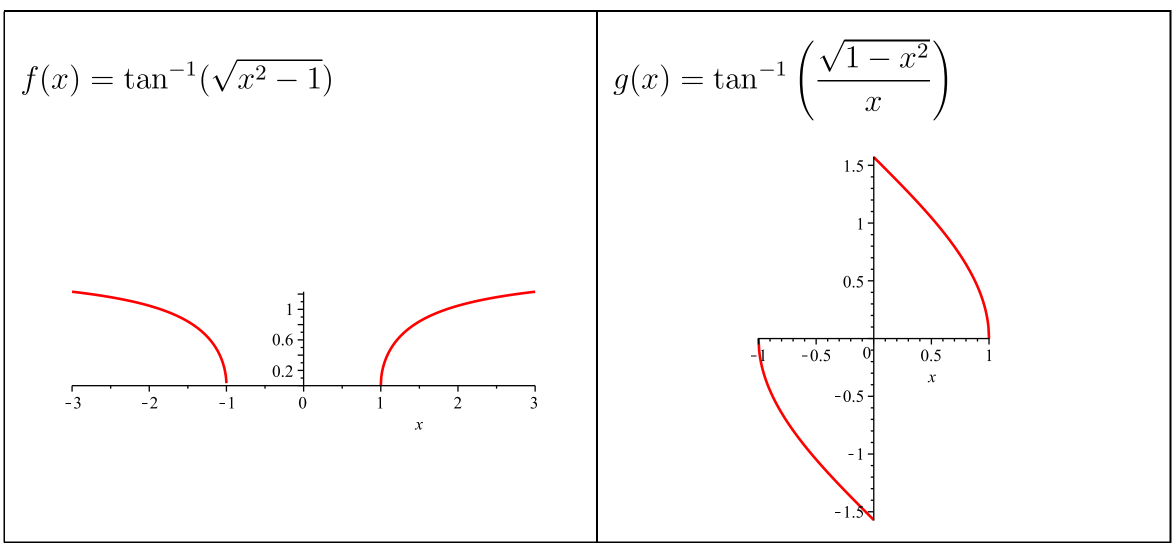


Figure 4: An alternative to $\sec^{-1} x$ (on the left) and $\sin^{-1} x$ (on the right).

We stress that there is no advocacy here for doing away with the inverse sine and secant functions (although, arguably, the inverse tangent function is one of our nicest functions).