1. **(a)** Here $\nabla f = f_x\mathbf{i} + f_y\mathbf{j} = (-6y+3x^2)\mathbf{i} + (3y^2-6x)\mathbf{j} = 3\{(x^2-2y)\mathbf{i} + (y^2-2x)\mathbf{j}\}$. At a critical point, $\nabla f = 0$ or $f_x = 0 = f_y$. That is, $x^2 = 2y$ and $y^2 = 2x$, implying $x^4 = 4y^2 = 8x$ or $x(x-2)(\{x+1\}^2+3) = 0$. Because the quadratic factor is strictly positive, either $x = 0$ implying $y = 0$ or $x = 2$ implying $y = \frac{1}{2}x^2 = \frac{1}{2}\cdot 2^2 = 2$. So the critical points are $(0,0)$ and $(2,2)$. The discriminant is

$$D = f_{xx}f_{yy} - f_{xy}^{\,2} \;=\; (6x-0)(6y-0) - (-6)^2 = 36(xy-1).$$

Because $D < 0$ for $x = 0 = y$ and $D > 0$ for $x = 2 = y$, $(0,0)$ is a saddle point, whereas $(2,2)$ is a local extremum; and because $f_{xx}(2,2) = 12 = f_{yy}(2,2)$ is positive, $(2,2)$ is a local minimizer.

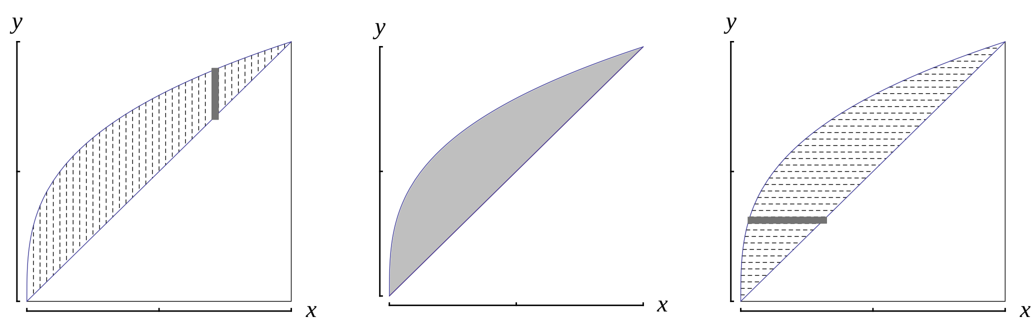
(b) From above, $\nabla f = 3\{(x^2-2y)\mathbf{i} + (y^2-2x)\mathbf{j}\} \Longrightarrow \nabla f(\mathbf{r}_0) = 3\{(5^2 - 2(-3))\mathbf{i} + ((-3)^2 - 2\cdot 5)\mathbf{j}\} = 3(31\mathbf{i} - \mathbf{j})$. Also, $\hat{\mathbf{s}} = \{12\mathbf{i} - 5\mathbf{j}\}/\sqrt{12^2 + (-5)^2} = \frac{1}{13}\{12\mathbf{i} - 5\mathbf{j}\}$. Hence $\left.\frac{\partial f}{\partial s}\right|_{\mathbf{r}=\mathbf{r}_0} = \hat{\mathbf{s}}\cdot\nabla f(\mathbf{r}_0) = \frac{3}{13}\{12\mathbf{i} - 5\mathbf{j}\}\cdot\{31\mathbf{i} - \mathbf{j}\} = \frac{3}{13}\{12\cdot 31 + (-5)(-1)\} = \frac{3\cdot 377}{13} = 87.$

2. With respect to Lecture 11, the iterated integral is of Type II: first integrate with respect to y between $y = x$ and $y = \sqrt[3]{x}$, then integrate with respect to x between $x = 0$ and $x = 1$ (diagram on left). So, if we instead regard it as a Type-I double integral, then the region covered is that which is shaded in the middle diagram. The curves $y = x$ and $y = \sqrt[3]{x}$ correspond to the curves $x = y$ and $x = y^3$. So the integral can instead be computed as a Type-III iterated integral, first integrating with respect to x between $x = y^3$ and $x = y$ and then integrating with respect to y between $y = 0$ and $y = 1$ (diagram on right). Thus

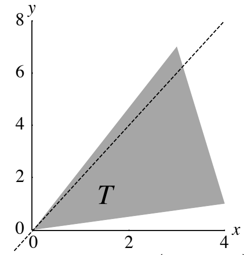
$$\int_0^1\int_x^{\sqrt[3]{x}} \sin(x/y)\,dy\,dx \;=\; \int_0^1\int_{y^3}^{y} \sin(x/y)\,dx\,dy \;=\; \int_0^1 -y\cos(x/y)\Big|_{y^3}^{y}\,dy$$

$$= \int_0^1 \{-y\cos(1) + y\cos(y^2)\}\,dy \;=\; \{-\tfrac{1}{2}y^2\cos(1) + \tfrac{1}{2}\sin(y^2)\}\Big|_0^1$$

$$= \{-\tfrac{1}{2}\cdot 1^2\cos(1) + \tfrac{1}{2}\sin(1^2) + \tfrac{1}{2}\cdot 0^2\cos(1) - \tfrac{1}{2}\sin(0^2)\} \;=\; \tfrac{1}{2}\{\sin(1) - \cos(1)\}.$$



3. **(a)** $y - 2x$ is positive above the line $y = 2x$ (shown dashed) and negative below it. Hence the subset of T on which $y - 2x$ is negative has a much greater area than the subset of T on which $y - 2x$ is positive. We therefore expect I_3 to be negative.



(c) The triangle T contains points with position vectors $u(4\mathbf{i} + \mathbf{j}) + v(-\mathbf{i} + 6\mathbf{j}) = (4u - v)\mathbf{i} + (u + 6v)\mathbf{j}$ for $0 \le v \le u \le 1$. So, with $\mathbf{r} = x\mathbf{i} + y\mathbf{j}$, we obtain the transformation

$$x = 4u - v, \quad y = u + 6v$$

with Jacobian determinant

$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 4 & 1 \\ -1 & 6 \end{vmatrix} = 4 \cdot 6 - (-1) \cdot 1 = 25 \implies |J| = |25| = 25.$$

Note that $y - 2x$ becomes $-7u + 8v$. Hence

$$I_3 = \iint_T \{y - 2x\}\, dA = \int_0^1 \int_0^u \{-7u + 8v\}|J|\, dv\, du$$

$$= 25 \int_0^1 \int_0^u \{-7u + 8v\}\, dv\, du = 25 \int_0^1 \{-7uv + 4v^2\}\Big|_0^u\, du$$

$$= 25 \int_0^1 \{-7u^2 + 4u^2 - 0\}\, du = -25u^3\Big|_0^1 = -25(1^3 - 0^3) = -25,$$

which is negative as expected.

4. If ρ, ϕ, θ denote spherical polars and r, θ, z denote cylindrical polars with x, y, z denoting Cartesians, then $x^2 + y^2 + z^2 = r^2 + z^2 = \rho^2$, $r = \rho\sin(\phi)$ and $z = \rho\cos(\phi)$, implying that $\sqrt{x^2 + y^2} = r = \rho\sin(\phi)$. Because the sphere is where $x^2 + y^2 + z^2 = 16$ or $\rho = 4$ and the cone is where $z = r$ or $\rho\cos(\phi) = \rho\sin(\phi) \implies \tan(\phi) = 1 \implies \phi = \frac{1}{4}\pi$, E is where $0 \le \rho \le 4$, $0 \le \phi \le \frac{1}{4}\pi$ and $0 \le \theta \le 2\pi$. Also, $J = \rho^2\sin(\phi)$ for spherical polars. So

$$I = \iiint_E \sqrt{x^2 + y^2}\, dV = \int_0^{2\pi} \int_0^{\frac{1}{4}\pi} \int_0^4 \rho\sin(\phi)|J|\, d\rho\, d\phi\, d\theta = \int_0^{2\pi} \int_0^{\frac{1}{4}\pi} \int_0^4 \rho^3 \sin^2(\phi)\, d\rho\, d\phi\, d\theta$$

$$= \int_0^{2\pi} 1\, d\theta \int_0^{\frac{1}{4}\pi} \sin^2(\phi)\, d\phi \int_0^4 \rho^3\, d\rho = 2\pi \int_0^{\frac{1}{4}\pi} \tfrac{1}{2}\{1 - \cos(2\phi)\}d\phi \cdot \tfrac{1}{4}\rho^4\Big|_0^4$$

$$= \pi\{\phi - \tfrac{1}{2}\sin(2\phi)\}\Big|_0^{\frac{1}{4}\pi} \cdot 4^3 = \pi\{\tfrac{1}{4}\pi - \tfrac{1}{2}\sin(\tfrac{1}{2}\pi) - 0\} \cdot 4^3$$

$$= \pi\{\pi - 2\} \cdot 4^2 = 16\pi(\pi - 2) \approx 57.38.$$