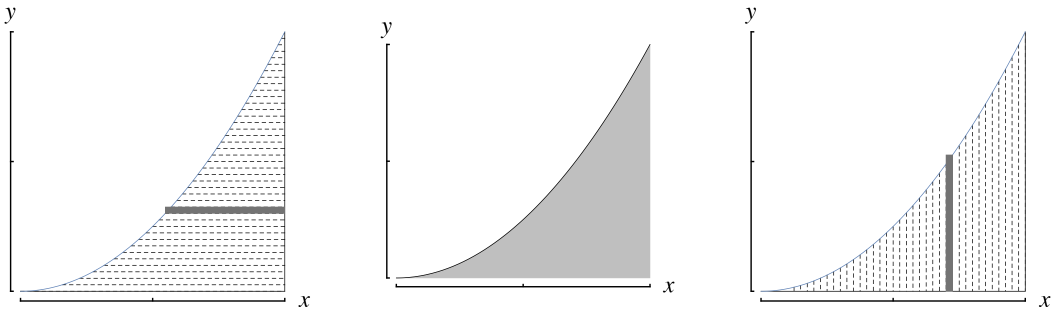
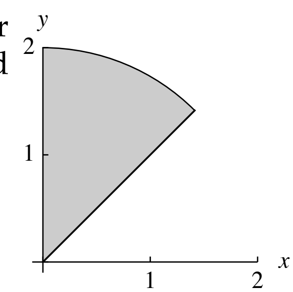


1. **(a)** With respect to Lecture 11, the iterated integral is of Type III: first integrate with respect to x between $x = \sqrt{y}$ and $x = 1$, then with respect to y between $y = 0$ and $y = 1$ (diagram on left). So, if we instead regard it as a Type-I double integral, then the region covered is that which lies to the right of the curve $x = \sqrt{y}$, to the left of the line $x = 1$ and above the line $y = 0$, which implies below $y = 1$ (diagram in middle). The curve $x = \sqrt{y}$ is, of course, the same as the curve $y = x^2$. So, the integral can instead be computed as a Type-II iterated integral, first integrating with respect to y between $y = 0$ and $y = x^2$ and then integrating with respect to x between $x = 0$ and $x = 1$ (diagram on right). Thus

$$\int_0^1 \int_{\sqrt{y}}^1 e^{y/x^2}\, dx\, dy \;=\; \int_0^1 \int_0^{x^2} e^{y/x^2}\, dy\, dx \;=\; \int_0^1 x^2 e^{y/x^2}\Big|_0^{x^2} dx$$

$$= \int_0^1 x^2(e^1 - e^0)\, dx \;=\; (e-1)\int_0^1 x^2\, dx \;=\; (e-1)\cdot \tfrac{1}{3}x^3\Big|_0^1 \;=\; \tfrac{1}{3}(e-1).$$

(b) The region of integration corresponds to $0 \le r \le 2$ for $\frac{1}{4}\pi \le \theta \le \frac{1}{2}\pi$. Hence with $x = r\cos(\theta)$, $y = r\sin(\theta)$ and Jacobian $J = r$, we obtain



$$\iint_D \{\sqrt{x^2+y^2} + y\}\, dxdy \;=\; \iint_D \{r + r\sin(\theta)\}|J|\, drd\theta \;=$$

$$\int_{\frac{1}{4}\pi}^{\frac{1}{2}\pi} \int_0^2 r^2\{1+\sin(\theta)\}\, drd\theta = \int_{\frac{1}{4}\pi}^{\frac{1}{2}\pi} \{1+\sin(\theta)\}\, d\theta \int_0^2 r^2\, dr$$

$$= \{\theta - \cos(\theta)\}\Big|_{\frac{1}{4}\pi}^{\frac{1}{2}\pi} \cdot \tfrac{1}{3}r^3\Big|_0^2 \;=\; \{\tfrac{1}{2}\pi - 0 - (\tfrac{1}{4}\pi - \tfrac{1}{\sqrt{2}})\}\cdot \tfrac{2^3}{3} \;=\; \tfrac{2}{3}(\pi + 2\sqrt{2}).$$

2. **(a)** With $\mathbf{r} = \frac{1}{3}t^3\,\mathbf{i} + \frac{1}{2}t^2\,\mathbf{j} + t\,\mathbf{k}$ we have $\dot{\mathbf{r}} = t^2\,\mathbf{i} + t\,\mathbf{j} + \mathbf{k}$, hence $v = |\dot{\mathbf{r}}| = \sqrt{t^4+t^2+1}$,

$$a_T \;=\; \frac{dv}{dt} \;=\; \tfrac{1}{2}\{t^4+t^2+1\}^{-1/2}\cdot(4t^3+2t) \;=\; \frac{2t^3+t}{\sqrt{t^4+t^2+1}}$$

and $\ddot{\mathbf{r}} = 2t\,\mathbf{i} + \mathbf{j} + 0\cdot\mathbf{k}$. So, for $t = 1$, we have $\dot{\mathbf{r}} = \mathbf{i} + \mathbf{j} + \mathbf{k}$, implying

(i) $\mathbf{T} \;=\; \hat{\dot{\mathbf{r}}} \;=\; \frac{1}{\sqrt{3}}\{\mathbf{i}+\mathbf{j}+\mathbf{k}\}$.

(ii) $\ddot{\mathbf{r}} = 2\mathbf{i} + \mathbf{j} + 0\mathbf{k}$ and $a_T = 3/\sqrt{3} = \sqrt{3}$. So $\ddot{\mathbf{r}} = a_T\mathbf{T} + a_N\mathbf{N}$ implies $a_N\mathbf{N} = \ddot{\mathbf{r}} - a_T\mathbf{T} = \mathbf{i} - \mathbf{k}$. The magnitude of this vector is $a_N = \sqrt{2}$, and its direction is $\mathbf{N} \;=\; \frac{1}{\sqrt{2}}\{\mathbf{i}-\mathbf{k}\}$.

(iii) Because $v = \sqrt{3}$ when $t = 1$, we deduce from $a_N = v^2\kappa$ that $\kappa = \frac{\sqrt{2}}{3}$.

(iv) $\mathbf{B} = \mathbf{T} \times \mathbf{N} = \frac{1}{\sqrt{6}}\{-\mathbf{i} + 2\mathbf{j} - \mathbf{k}\}$.

It is easily verified that $\mathbf{T} \cdot \mathbf{N} = \mathbf{T} \cdot \mathbf{B} = \mathbf{N} \cdot \mathbf{B} = 0$.

(b) On C we have $x = \frac{1}{3}t^3$, $y = \frac{1}{2}t^2$ and $z = t$ with $\frac{d\mathbf{r}}{dt} = t^2\mathbf{i} + t\mathbf{j} + \mathbf{k}$ as above, implying $\mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = y \cdot t^2 + 6x \cdot t + 2z \cdot 1 = \frac{1}{2}t^4 + 2t^4 + 2t = \frac{5}{2}t^4 + 2t$. So $\int_C \mathbf{F} \cdot \mathbf{dr} = \int_0^2 \mathbf{F} \cdot \frac{dr}{dt}\, dt = \int_0^2 \{\frac{5}{2}t^4 + 2t\}\, dt = \{\frac{1}{2}t^5 + t^2\}\Big|_0^2 = 2^4 + 2^2 = 20$. Note that no potential exists because $\nabla \times \mathbf{F} = 0\mathbf{i} + 0\mathbf{j} + 5\mathbf{k} \neq \mathbf{0}$.

3. (a) We have

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z^2 & x & -4y \end{vmatrix} = (-4-0)\mathbf{i} - (0-2z)\mathbf{j} + (1-0)\mathbf{k} = -4\mathbf{i} + 2z\mathbf{j} + \mathbf{k}.$$

S is parameterized in natural coordinates by

$$\mathbf{r} = \mathbf{a} + u(\mathbf{b}-\mathbf{a}) + v(\mathbf{c}-\mathbf{b}), \qquad 0 \le v \le u, \quad 0 \le u \le 1$$

where $\mathbf{a} = \mathbf{i} + 0\mathbf{j} + \mathbf{k}$, $\mathbf{b} = 3\mathbf{i} - \mathbf{j} + 2\mathbf{k}$ and $\mathbf{c} = 5\mathbf{i} + \mathbf{j} + 3\mathbf{k}$, i.e., by

$$\begin{aligned} \mathbf{r} &= \mathbf{i} + 0\mathbf{j} + \mathbf{k} + u(2\mathbf{i} - \mathbf{j} + \mathbf{k}) + v(2\mathbf{i} + 2\mathbf{j} + \mathbf{k}), \quad 0 \le v \le u,\ 0 \le u \le 1 \\ &= (2u + 2v + 1)\mathbf{i} + (-u + 2v)\mathbf{j} + (2 + v)\mathbf{k}, \quad 0 \le v \le u,\ 0 \le u \le 1 \end{aligned}$$

so that $x = 2u + 2v + 1$, $y = -u + 2v$, $z = 2 + v$, $\mathbf{r}_u = 2\mathbf{i} - \mathbf{j} + 0\mathbf{k}$, $\mathbf{r}_v = 2\mathbf{i} + 2\mathbf{j} + \mathbf{k}$ and $\mathbf{r}_u \times \mathbf{r}_v = 3\{-\mathbf{i} + 0\mathbf{j} + 2\mathbf{k}\}$, which is parallel to $(\mathbf{b}-\mathbf{a}) \times (\mathbf{c}-\mathbf{b})$ and hence correctly oriented with respect to S. Then, because

$$\nabla \times \mathbf{F} \cdot (\mathbf{r}_u \times \mathbf{r}_v) = 3\{(-4)\cdot(-1) + (2z)\cdot 0 + 1 \cdot 2\} = 18,$$

from Stokes' theorem we obtain

$$\begin{aligned} \oint_C \mathbf{F} \cdot \mathbf{dr} &= \iint_S \nabla \times \mathbf{F} \cdot \mathbf{dS} = \int_0^1 \int_0^u \nabla \times \mathbf{F} \cdot (\mathbf{r}_u \times \mathbf{r}_v)\, dv\, du \\ &= 18 \int_0^1 \int_0^u dv\, du = 18 \int_0^1 v\Big|_0^u du = 18 \int_0^1 u\, du = 9, \end{aligned}$$

in accordance with $\int_{C_1} \mathbf{F} \cdot \mathbf{dr} = \frac{14}{3}$, $\int_{C_2} \mathbf{F} \cdot \mathbf{dr} = \frac{62}{3}$ and $\int_{C_3} \mathbf{F} \cdot \mathbf{dr} = -\frac{49}{3}$.

(b) We can calculate the area of the triangle either as

$$\begin{aligned} A &= \iint_S dS = \iint_S |\mathbf{r}_u \times \mathbf{r}_v|\, du\, dv \\ &= \int_0^1 \int_0^u 3\sqrt{(-1)^2 + 0^2 + 2^2}\, dv\, du = 3\sqrt{5} \int_0^1 u\, du = \tfrac{3}{2}\sqrt{5} \end{aligned}$$

or as $\frac{1}{2}|(\mathbf{b}-\mathbf{a}) \times (\mathbf{c}-\mathbf{b})|$, which of course yields the same answer.

4. (a) On S we have $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ subject to $x + y + z = 5$ or $z = 5 - x - y$; moreover, in cylindrical polar coordinates with $x = R\cos(\theta)$ and $y = R\sin(\theta)$, $x^2+y^2 \le 9$ corresponds to $0 \le R \le 3, 0 \le \theta \le 2\pi$. So we may parameterize S as $\mathbf{r} = R\cos(\theta)\,\mathbf{i} + R\sin(\theta)\,\mathbf{j} + \{5 - R\cos(\theta) - R\sin(\theta)\}\mathbf{k},\ 0 \le R \le 3, 0 \le \theta \le 2\pi$, obtaining

$$\begin{aligned} \mathbf{r}_R &= \cos(\theta)\,\mathbf{i} + \sin(\theta)\,\mathbf{j} - \{\cos(\theta) + \sin(\theta)\}\mathbf{k} \\ \mathbf{r}_\theta &= R(-\sin(\theta)\,\mathbf{i} + \cos(\theta)\,\mathbf{j} + \{\sin(\theta) - \cos(\theta)\}\mathbf{k}) \end{aligned}$$

with

$$\mathbf{r}_R \times \mathbf{r}_\theta = R\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos(\theta) & \sin(\theta) & -\{\cos(\theta) + \sin(\theta)\} \\ -\sin(\theta) & \cos(\theta) & \sin(\theta) - \cos(\theta) \end{vmatrix} = R(\mathbf{i} + \mathbf{j} + \mathbf{k}),$$

so that $\mathbf{F} \cdot (\mathbf{r}_R \times \mathbf{r}_\theta) = R\{y^2 + z^2 + x^2\}$, with $(\mathbf{r}_R \times \mathbf{r}_\theta) \cdot \mathbf{k}\ (= R) > 0$, as required. Also, $x^2+y^2 = R^2$ and $z^2 = (5-R\{\cos(\theta)+\sin(\theta)\})^2 = 5^2-10R\{\cos(\theta)+\sin(\theta)\}+R^2\{\cos(\theta)+\sin(\theta)\}^2 = 25-10R\{\cos(\theta)+\sin(\theta)\}+R^2\{1+2\sin(\theta)\cos(\theta)\}$. Hence

$$\iint_S \mathbf{F}\cdot\mathbf{dS} = +\iint_S \mathbf{F}\cdot(\mathbf{r}_R \times \mathbf{r}_\theta)\,dR\,d\theta = \int_0^{2\pi}\int_0^3 R\{y^2 + z^2 + x^2\}\,dR\,d\theta$$

$$= \int_0^{2\pi}\int_0^3 R\big(25 + 2R^2 - 10R\{\cos(\theta) + \sin(\theta)\} + 2R^2\sin(\theta)\cos(\theta)\big)\,dR\,d\theta$$

$$= \int_0^{2\pi} d\theta \int_0^3 \{25R + 2R^3\}\,dR \; - \; 10\int_0^{2\pi}\{\cos(\theta) + \sin(\theta)\}\,d\theta \int_0^3 R^2\,dR$$

$$+ \; 2\int_0^{2\pi}\sin(\theta)\cos(\theta)\,d\theta \int_0^3 R^3\,dR = 2\pi\cdot\big\{\tfrac{25}{2}R^2 + \tfrac{1}{2}R^4\big\}\Big|_0^3 - 0 + 0$$

$$= 2\pi\cdot\tfrac{3^2}{2}(25 + 3^2) = 306\pi$$

because $\sin(\theta)$, $\cos(\theta)$ and $\sin(\theta)\cos(\theta)$ all integrate to zero over the interval $[0, 2\pi]$.

Alternatively, from Lecture 18 with the sun way down along the z-axis at infinity, we have $\boldsymbol{\sigma} = \mathbf{k}$; the correctly oriented unit normal to the plane is $\mathbf{n} = (\mathbf{i} + \mathbf{j} + \mathbf{k})/\sqrt{3}$, by inspection, implying $\mathbf{F}\cdot\mathbf{n} = (y^2 + z^2 + x^2)/\sqrt{3}$; and the element of area for S is related to the element for the shadow disk by $dS = dS_{shad}/|\boldsymbol{\sigma}\cdot\mathbf{n}| = \sqrt{3}R dR d\theta$. So

$$\iint_S \mathbf{F}\cdot\mathbf{dS} = \iint_S \mathbf{F}\cdot\mathbf{n}\,dS = \int_0^{2\pi}\int_0^1 \{y^2 + z^2 + x^2\}R\,dR\,d\theta = 306\pi$$

after the same calculations as before.

(b) We have

$$\nabla\times\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 & z^2 & x^2 \end{vmatrix} = (0-2z)\mathbf{i} - (2x-0)\mathbf{j} + (0-2y)\mathbf{k} = -2\{z\mathbf{i} + x\mathbf{j} + y\mathbf{k}\}.$$

So, noting that $x+y+z=5$ on S, from Stokes' theorem we find that the circulation around C in the *counterclockwise* direction when viewed from above is

$$\begin{aligned}
\oint_C \mathbf{F}\cdot\mathbf{dr} &= \iint_S \nabla\times\mathbf{F}\cdot\mathbf{dS} \;=\; +\int_0^{2\pi}\int_0^3 (\nabla\times\mathbf{F})\cdot(\mathbf{r}_R\times\mathbf{r}_\theta)\,dR\,d\theta \\
&= -2\int_0^{2\pi}\int_0^3 \{z\mathbf{i}+x\mathbf{j}+y\mathbf{k}\}\cdot R(\mathbf{i}+\mathbf{j}+\mathbf{k})\,dR\,d\theta \\
&= -2\int_0^{2\pi}\int_0^3 R\{z+x+y\}\,dR\,d\theta \;=\; -2\int_0^{2\pi}\int_0^3 R\cdot 5\,dR\,d\theta \\
&= -5\int_0^{2\pi} d\theta \int_0^3 2R\,dR \;=\; -5\cdot 2\pi\cdot(3^2-0^2) \;=\; -90\pi.
\end{aligned}$$

Alternatively, you can calculate this circulation directly (though it is almost certain that would not want to do so): C has equation $\mathbf{r} = 3\cos(t)\mathbf{i} + 3\sin(t)\mathbf{j} + \{5-3\cos(t)-3\sin(t)\}\mathbf{k}, 0\le t\le 2\pi$, implying

$$\begin{aligned}
\oint_C \mathbf{F}\cdot\mathbf{dr} &= \int_0^{2\pi} \mathbf{F}\cdot\frac{\mathbf{dr}}{dt}\,dt \;=\; \int_0^{2\pi} \mathbf{F}\cdot\Big\{-3\sin(t)\mathbf{i}+3\cos(t)\mathbf{j}+3\{\sin(t)-\cos(t)\}\mathbf{k}\Big\}\,dt \\
&= \int_0^{2\pi}\Big\{-3y^2\sin(t)+3z^2\cos(t)+3x^2\{\sin(t)-\cos(t)\}\Big\}\,dt \\
&= \int_0^{2\pi}\Big\{-3\{3\sin(t)\}^2\sin(t)+3\{5-3\cos(t)-3\sin(t)\}^2\cos(t) \\
&\qquad\qquad + 3\{3\cos(t)\}^2\{\sin(t)-\cos(t)\}\Big\}\,dt \\
&= 3\int_0^{2\pi}\{-15\}\,dt \;+\; 3\int_0^{2\pi}\{34\cos(t)-15\cos(2t)-15\sin(2t)\}\,dt \\
&\qquad + 27\Big\{-\int_0^{2\pi}(1-\sin^2(t))\,d\{\sin(t)\} + \int_0^{2\pi}(1-4\cos^2(t))\,d\{\cos(t)\}\Big\}
\end{aligned}$$

after using $\cos^2(t)+\sin^2(t)=1, 2\cos^2(t)=1+\cos(2t)$ and $2\sin(t)\cos(t)=\sin(2t)$. The first integral yields $3\cdot\{-15\}\cdot 2\pi = -90\pi$, and everything else goes to zero.