The flux of a vector through a triangular surface

Suppose that we wish to calculate the flux Q of a vector field $\mathbf{F}$ across a (planar) triangular surface S with vertices at A, B and C (Figure 1). Then we already know from Lecture 17 that

$$Q = \iint_S \mathbf{F} \cdot \mathbf{dS} = \iint_S \mathbf{F} \cdot \mathbf{n}\, dS = \pm \iint_S \mathbf{F} \cdot (\mathbf{r}_u \times \mathbf{r}_v)\, du\, dv \tag{1}$$

where u and v are the triangular surface's natural coordinates. Let $\mathbf{r}$ be the position vector of an arbitrary point on S. Then, from Figure 1,

$$\begin{aligned} \mathbf{r} &= \overrightarrow{OA} + \overrightarrow{AM} + \overrightarrow{MP} \\ &= \overrightarrow{OA} + u\, \overrightarrow{AB} + v\, \overrightarrow{BC} \\ &= \mathbf{a} + u(\mathbf{b} - \mathbf{a}) + v(\mathbf{c} - \mathbf{b}) \end{aligned} \tag{2}$$

where A is the vertex at which we enter the triangle, M is a fraction u of the way down the side AB from the initial vertex, P is a fraction v of the way down the line segment through M that is parallel to the side *opposite* the vertex at which we enter the triangle, u increases parallel to AB and v increases parallel to BC. Note that u may take any value between 0 (initial vertex) and 1 (side BC), but v is constrained to satisfy

$$v \le u \tag{3}$$

because points where $v > u$ lie outside S, on the other side of AC from P. Hence S is parameterized in natural coordinates by

$$\mathbf{r} = \mathbf{a} + u(\mathbf{b} - \mathbf{a}) + v(\mathbf{c} - \mathbf{b}), \qquad 0 \le v \le u, \quad 0 \le u \le 1. \tag{4}$$

Let us assume that in Figure 1 we are looking down on S and that $\mathbf{n}$ points up, so that Q is the upward flux. Then because $\mathbf{r}_u \times \mathbf{r}_v = (\mathbf{b} - \mathbf{a}) \times (\mathbf{c} - \mathbf{b})$, which points down, it follows from (4) that

$$Q = -\int_0^1 \int_0^u \mathbf{F} \cdot (\mathbf{r}_u \times \mathbf{r}_v)\, dv\, du. \tag{5}$$

Suppose, for example, that we wish to determine the upward flux of

$$\mathbf{F} = xy\mathbf{i} + z\mathbf{j} + (y - xz)\mathbf{k} \tag{6}$$

through the triangular planar surface with vertices at $(1, -1, 2)$, $(1, 0, 3)$ and $(2, 1, -2)$. Then S is parameterized by

$$\begin{gathered} \mathbf{r} = \mathbf{i} - \mathbf{j} + 2\mathbf{k} + u(\mathbf{j} + \mathbf{k}) + v(\mathbf{i} + \mathbf{j} - 5\mathbf{k}), \\ 0 \le v \le u, \quad 0 \le u \le 1 \end{gathered}$$

with $\mathbf{r}_u \times \mathbf{r}_v = -6\mathbf{i} + \mathbf{j} - \mathbf{k}$, which clearly points down, requiring the negative sign to be taken. Then, because

$$\begin{aligned} \mathbf{F} \cdot (\mathbf{r}_u \times \mathbf{r}_v) &= -6xy + z - y + xz = -(6x + 1)y + (x + 1)z \\ &= -\{6(1 + v) + 1\}(-1 + u + v) + \{(1 + v) + 1\}(2 + u - 5v) \end{aligned} \tag{7}$$

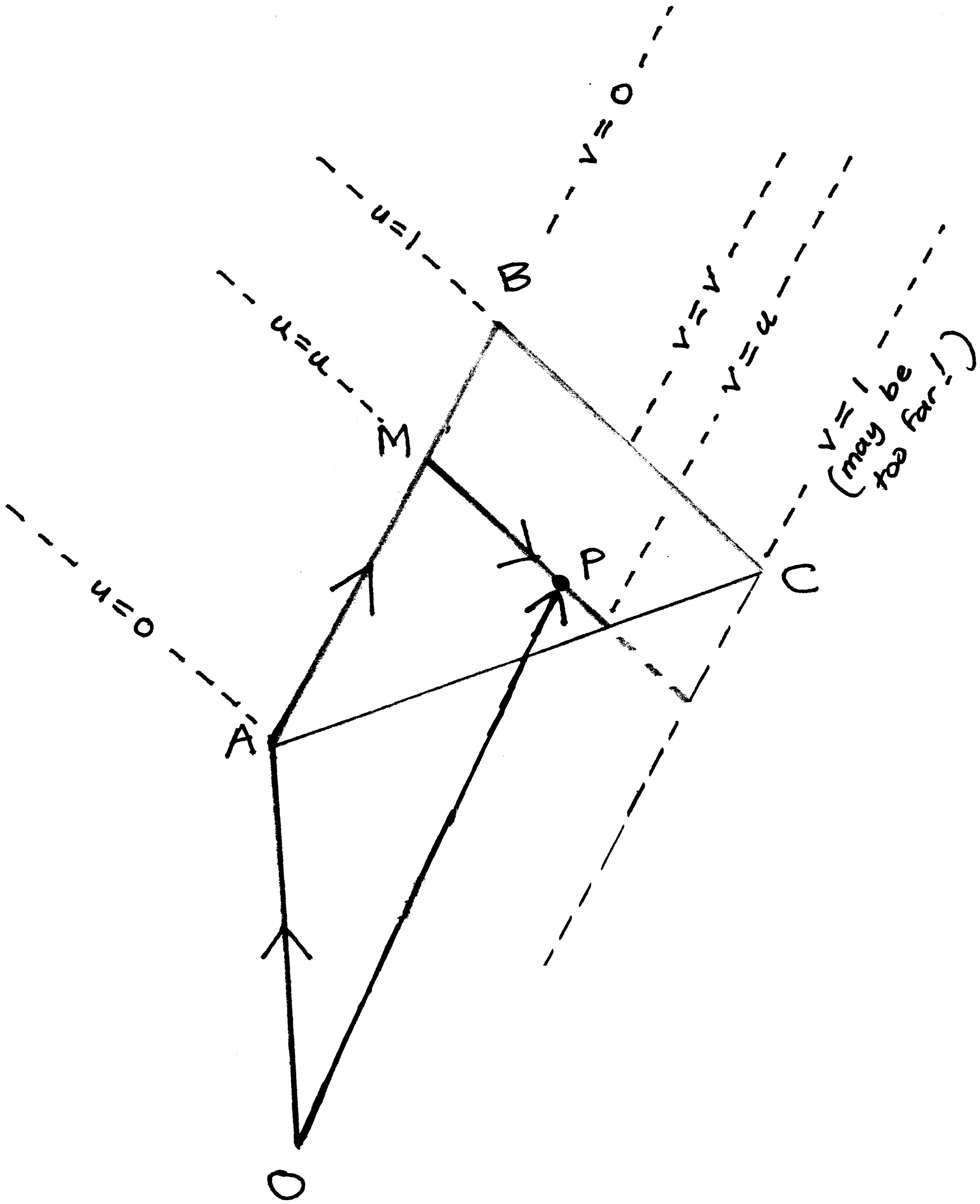

Figure 1: An open triangular surface

it follows from (5) that

$$
\begin{aligned}
Q &= -\int_0^1 \int_0^u \{11 - 9v - 11v^2 - 5(1+v)u\}\, dv\, du \\
&= -\int_0^1 \{11u - \tfrac{19}{2}u^2 - \tfrac{37}{6}u^3\}\, du \;=\; -\tfrac{19}{24}.
\end{aligned}
\tag{8}
$$

The upward flux is negative, so the flux is actually downward.

For completeness, we note that the parallelogram with vertices A, B, C and D with position vector $\mathbf{d} = \mathbf{a} - \mathbf{b} + \mathbf{c}$ is parameterized in natural coordinates by

$$
\mathbf{r} \;=\; \mathbf{a} \;+\; u(\mathbf{b} - \mathbf{a}) \;+\; v(\mathbf{c} - \mathbf{b}), \qquad 0 \le u \le 1, \quad 0 \le v \le 1, \tag{9}
$$

i.e., by substituting $v \le 1$ for $v \le u$ in (3). For example, the flux of $\mathbf{F}$ defined by (6) through the planar surface bounded by the parallelogram with vertices at $(1, -1, 2)$, $(1, 0, 3)$ and $(2, 1, -2)$ and $(2, 0, -3)$ is

$$
\begin{aligned}
Q &= -\int_0^1 \int_0^1 \{11 - 9v - 11v^2 - 5(1+v)u\}\, dv\, du \\
&= -\int_0^1 \{\tfrac{17}{6} - \tfrac{15}{2}u\}\, du \;=\; \tfrac{11}{12}
\end{aligned}
\tag{10}
$$

(so that the flux through the triangle with vertices at D, C and A is $\frac{41}{24}$).