The vector and scalar equations of a plane

Suppose that $\mathbf{q} = q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k}$ is the position vector of any fixed point Q in a plane and that $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$ and $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$ are any two independent vectors that are parallel to the plane—independent means not parallel, or $\mathbf{u} \times \mathbf{v} \neq \mathbf{0}$ (where $\mathbf{0}$ is defined as having zero magnitude and hence arbitrary direction). Because $\mathbf{u}$ and $\mathbf{v}$ are not parallel, by drawing lots of lines that are parallel to each, one can cover the plane with two infinite sets of coordinate lines (inclined to one another at an angle $\arccos(\hat{\mathbf{u}} \cdot \hat{\mathbf{v}}) \neq 0$). One can therefore reach any point in the plane by first going from O to Q and then travelling σ units parallel to $\mathbf{u}$ and τ units parallel to $\mathbf{v}$ for suitably chosen (and unique) σ and τ, either of which may be negative. It follows that a vector equation of the plane is

$$\boldsymbol{\rho} \;=\; \mathbf{q} \;+\; \sigma\hat{\mathbf{u}} \;+\; \tau\hat{\mathbf{v}}, \quad -\infty < \sigma, \tau < \infty. \tag{1}$$

But $\hat{\mathbf{u}} = \mathbf{u}/u$ and $\hat{\mathbf{v}} = \mathbf{v}/v$, so that $\sigma\hat{\mathbf{u}} = (\sigma/u)\mathbf{u}$ and $\tau\hat{\mathbf{v}} = (\tau/v)\mathbf{v}$. Thus, rescaling the "oblique coordinates" σ and τ by defining

$$s \;=\; \frac{\sigma}{u}, \qquad t \;=\; \frac{\tau}{v} \tag{2}$$

we find that an equivalent vector equation of the plane is

$$\boldsymbol{\rho} \;=\; \mathbf{q} \;+\; s\mathbf{u} \;+\; t\mathbf{v}, \quad -\infty < s, t < \infty \tag{3}$$

and this is the version we generally prefer to use.

Resolving (3) into components (that is, noting that two vectors are equal if and only if their components are identical), we obtain

$$\begin{aligned} x &= q_1 + su_1 + tv_1 \\ y &= q_2 + su_2 + tv_2 \\ z &= q_3 + su_3 + tv_3. \end{aligned} \tag{4}$$

The first two equations can be solved to yield

$$s \;=\; \frac{q_1v_2 - q_2v_1 - v_2x + v_1y}{u_2v_1 - u_1v_2}, \quad t \;=\; \frac{q_2u_1 - q_1u_2 - u_1y + u_2x}{u_2v_1 - u_1v_2} \tag{5}$$

and substituting from (5) into the last of (4) yields

$$(u_2v_3 - v_2u_3)(x - q_1) + (u_3v_1 - v_3u_1)(y - q_2) + (u_1v_2 - v_1u_2)(z - q_3) \;=\; 0. \tag{6}$$

This is the scalar equation of the plane. But we have derived it by the second worst possible method. Why? Let us set $\mathbf{n} = n_1\mathbf{i} + n_2\mathbf{j} + n_3\mathbf{k}$ and define

$$\mathbf{n} \;=\; \mathbf{u} \times \mathbf{v}, \tag{7}$$

so that $\mathbf{n}$ is normal to the plane. Then we know from Lecture 4 that

$$\mathbf{n} \;=\; \mathbf{u} \times \mathbf{v} \;=\; \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} \;=\; (u_2v_3 - v_2u_3)\,\mathbf{i} + (u_3v_1 - v_3u_1)\,\mathbf{j} + (u_1v_2 - v_1u_2)\,\mathbf{k}, \tag{8}$$

so that (6) reduces to

$$n_1(x - q_1) + n_2(y - q_2) + n_3(z - q_3) \;=\; 0 \tag{9}$$

or, more compactly,

$$\mathbf{n} \cdot (\boldsymbol{\rho} - \mathbf{q}) \;=\; 0. \tag{10}$$

But we could have seen at a glance that (10) must be the equation of the plane, because if P and Q both lie in the plane then $\boldsymbol{\rho} - \mathbf{q}$ must be parallel to the plane, whereas $\mathbf{n}$ is perpendicular to it!

Finally, in case you are still wondering, the absolute worst way to find the scalar equation of a plane is to use the method of the problem at the end of Lecture 1. What it all goes to show is that there good ways and bad ways to go about just about everything, and part of mathematics is discovering more efficient and elegant ways to do things for which we already had a workable but possibly extremely clunky method.

In mathematics, we always prize elegance and efficiency!