Curl: a measure of rotationality

If you know that a vector field, say $\mathbf{F}$, is a potential field[1]—i.e., if you know that there exists a scalar field $\phi = \phi(\mathbf{r})$ such that

$$\mathbf{F} \;=\; \nabla\phi, \tag{1}$$

then you can easily recover the unknown potential ϕ from the known field $\mathbf{F}$ by using a result from the end of the previous lecture, namely, that

$$\int\limits_C \nabla\phi \cdot \mathbf{dr} \;=\; \phi(\mathbf{R}) - \phi(\mathbf{r}_0), \tag{2}$$

for any curve C between $\mathbf{r}_0 = x_0\mathbf{i} + y_0\mathbf{j} + z_0\mathbf{k}$ and $\mathbf{R} = X\mathbf{i} + Y\mathbf{j} + Z\mathbf{k}$. This potential is not unique in the strictest sense: if ϕ satisfies (1), then so does ϕ + constant for any constant value. Thus there are infinitely many potentials associated with each conservative vector field, just as there are infinitely many antiderivatives associated with every ordinary function, but all of them differ by a constant. Consequently, no generality is lost by setting $\mathbf{r}_0 = \mathbf{0}$ and $\phi(\mathbf{r}_0) = \phi(\mathbf{0}) = \alpha$ in (2) and associating $\mathbf{F}$ with the potential defined by

$$\phi(\mathbf{R}) \;=\; \alpha + \int\limits_C \mathbf{F} \cdot \mathbf{dr}, \tag{3}$$

where C is any curve between $\mathbf{0}$ and the variable endpoint $\mathbf{R}$. Because C is arbitrary, we are free to choose any such curve, and it is probably most instructive to choose the curve in Figure 1 consisting of the line segment C_1 between $(0, 0, 0)$ and $(X, 0, 0)$, followed by the line segment C_2 between $(X, 0, 0)$ and $(X, Y, 0)$, followed by the line segment C_3 between $(X, Y, 0)$ and (X, Y, Z);[2] and because α is arbitrary, we can choose it to yield the simplest possible form.

Suppose, for example, that an irrotational vector field is defined by

$$\mathbf{F} \;=\; yz^2\{yz\,\mathbf{i} + 2xz\,\mathbf{j} + 3xy\,\mathbf{k}\} - e^{-xy}(y\,\mathbf{i} + x\,\mathbf{j}). \tag{4}$$

Then on C_1 we have $\mathbf{dr} = dx\mathbf{i}$ and $\mathbf{F} = -x\mathbf{j}$, implying $\mathbf{F} \cdot \mathbf{dr} = 0$, and so $\int_{C_1} \mathbf{F} \cdot \mathbf{dr} = 0$ as well. On C_2 we have $\mathbf{dr} = dy\mathbf{j}$ and $\mathbf{F} = -e^{-Xy}(y\,\mathbf{i} + X\,\mathbf{j})$, implying $\mathbf{F} \cdot \mathbf{dr} = -e^{-Xy}X\,dy$. Thus

$$\int_{C_2} \mathbf{F} \cdot \mathbf{dr} \;=\; -\int_{y=0}^{y=Y} e^{-Xy}X\,dy \;=\; e^{-Xy}\Big|_0^Y \;=\; e^{-XY} - e^0 \;=\; e^{-XY} - 1. \tag{5}$$

Finally, along C_3 we have $\mathbf{dr} = dz\mathbf{k}$ and $\mathbf{F} = Yz^2\{Yz\,\mathbf{i} + 2Xz\,\mathbf{j} + 3XY\,\mathbf{k}\} - e^{-XY}(Y\,\mathbf{i} + X\,\mathbf{j})$, implying $\mathbf{F} \cdot \mathbf{dr} = 3XY^2z^2\,dz$. Thus

$$\int_{C_3} \mathbf{F} \cdot \mathbf{dr} \;=\; \int_{z=0}^{z=Z} 3XY^2z^2\,dz \;=\; XY^2z^3\Big|_0^Z \;=\; XY^2Z^3. \tag{6}$$

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[1]Or a gradient field, conservative field, path-independent field or irrotational field, because the potential field is a creature of many disguises!

[2]Nevertheless, we are by no means obliged to choose this curve—see the problem at the end of the lecture.

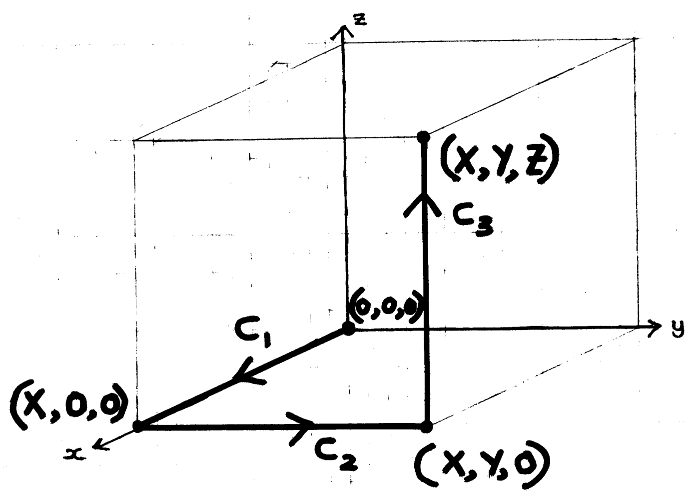

Figure 1: The curve $C = C_1 \cup C_2 \cup C_3$

Collecting our results together, we have

$$\int_C \mathbf{F} \cdot \mathbf{dr} \;=\; \int_{C_1} \mathbf{F} \cdot \mathbf{dr} + \int_{C_2} \mathbf{F} \cdot \mathbf{dr} + \int_{C_3} \mathbf{F} \cdot \mathbf{dr} \;=\; e^{-XY} - 1 + XY^2Z^3, \tag{7}$$

so that (3) and (7) imply

$$\phi(\mathbf{R}) \;=\; \alpha - 1 + e^{-XY} + XY^2Z^3. \tag{8}$$

We now choose $\alpha = 1$ and replace $\mathbf{R}$ by $\mathbf{r}$ to yield the potential

$$\phi(\mathbf{r}) \;=\; e^{-xy} + xy^2z^3. \tag{9}$$

You can easily verify that (9) satisfies (1).

All of the above is conditional, however, on knowing that $\mathbf{F} = F_1\,\mathbf{i} + F_2\,\mathbf{j} + F_3\,\mathbf{k}$ is irrotational to begin with. But how can we tell? Let's rewrite (1) in terms of its three components:

$$F_1 \;=\; \frac{\partial \phi}{\partial x}, \quad F_2 \;=\; \frac{\partial \phi}{\partial y}, \quad F_3 \;=\; \frac{\partial \phi}{\partial z}. \tag{10}$$

Under our differentiability assumptions, we always have

$$\frac{\partial^2 \phi}{\partial y \partial z} - \frac{\partial^2 \phi}{\partial z \partial y} \;=\; 0, \quad \frac{\partial^2 \phi}{\partial z \partial x} - \frac{\partial^2 \phi}{\partial x \partial z} \;=\; 0, \quad \frac{\partial^2 \phi}{\partial x \partial y} - \frac{\partial^2 \phi}{\partial y \partial x} \;=\; 0. \tag{11}$$

Hence if $\mathbf{F}$ is irrotational, that is, if ϕ exists such that $\mathbf{F}$ is defined by (11), then

$$\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z} \;=\; 0, \quad \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x} \;=\; 0, \quad \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \;=\; 0. \tag{12}$$

If the three left-hand sides of (12) were the components of a vector, then that vector would have to be the zero vector for $\mathbf{F}$ to be irrotational; and the greater the difference between the magnitude of this vector and zero, the less irrotational $\mathbf{F}$ would be. The vector is therefore a measure of the rotationality of $\mathbf{F}$. We call it the curl of $\mathbf{F}$. That is, we define

$$\operatorname{curl}(\mathbf{F}) \;=\; \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\mathbf{i} + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\mathbf{j} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\mathbf{k}. \tag{13}$$

We can recast this vector in the form of a determinant as

$$\operatorname{curl}(\mathbf{F}) \;=\; \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}, \tag{14}$$

where we interpret the partial derivatives in the second row as operating on the elements in the third row whenever a "product" between them appears. We can think of (14) as the vector product of the gradient operator

$$\nabla \;=\; \mathbf{i}\,\frac{\partial}{\partial x} + \mathbf{j}\,\frac{\partial}{\partial y} + \mathbf{k}\,\frac{\partial}{\partial z} \tag{15}$$

and $\mathbf{F}$. Thus an alternative notation for $\operatorname{curl}(\mathbf{F})$ is $\nabla \times \mathbf{F}$.

By construction,

$$\nabla \times \nabla\phi \;=\; \mathbf{0}. \tag{16}$$

That is, any irrotational field has zero curl. The converse happens also to be true: a field with zero curl is irrotational.[3] So now we can know in advance with whether a vector field $\mathbf{F}$ is irrotational: all we have to do is find its curl (and see whether it is the zero vector).

Consider, for example, the field defined by

$$\mathbf{F} = F_1\,\mathbf{i} + F_2\,\mathbf{j} + F_3\,\mathbf{k} \;=\; -\frac{y}{x^2+y^2}\,\mathbf{i} + \frac{x}{x^2+y^2}\,\mathbf{j} + 0\,\mathbf{k}. \tag{17}$$

Substituting into (13), we have

$$\nabla \times \mathbf{F} \;=\; 0\,\mathbf{i} + 0\,\mathbf{j} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\mathbf{k} \;=\; \mathbf{0}, \tag{18}$$

because $\frac{\partial F_2}{\partial x} = \frac{y^2-x^2}{(x^2+y^2)^2} = \frac{\partial F_1}{\partial y}$. Thus our theory says that $\mathbf{F}$ is irrotational, which means that its circulation around any closed curve is zero.

Let's check to see if the theory works by calculating the circulation of $\mathbf{F}$ around a circle of radius ϵ that lies in the x-y plane and is centered at the origin, i.e., around the curve C with equation $x^2+y^2=\epsilon^2, z=0$ or

$$\mathbf{r} \;=\; \epsilon\{\cos(\theta)\,\mathbf{i} + \sin(\theta)\,\mathbf{j}\} + 0\,\mathbf{k}, \quad 0 \le \theta \le 2\pi. \tag{19}$$

On C, we have $x=\epsilon\cos(\theta)$, $y=\epsilon\sin(\theta)$ and hence

$$\mathbf{F} = \tfrac{-\epsilon\sin(\theta)}{\{\epsilon\sin(\theta)\}^2+\{\epsilon\cos(\theta)\}^2}\,\mathbf{i} + \tfrac{\epsilon\cos(\theta)}{\{\epsilon\sin(\theta)\}^2+\{\epsilon\cos(\theta)\}^2}\,\mathbf{j} = \tfrac{1}{\epsilon}\{-\sin(\theta)\,\mathbf{i} + \cos(\theta)\,\mathbf{j}\} \tag{20}$$

$$\tfrac{d\mathbf{r}}{d\theta} \;=\; \epsilon\{-\sin(\theta)\,\mathbf{i} + \cos(\theta)\,\mathbf{j}\}, \quad 0 \le \theta \le 2\pi \tag{21}$$

[3]See Lecture 21.

so that $\mathbf{F} \cdot \frac{d\mathbf{r}}{d\theta} = \{-\sin(\theta)\}^2 + \{\cos(\theta)\}^2 = 1$. Thus

$$\oint_C \mathbf{F} \cdot \mathbf{dr} \;=\; \int_{\theta=0}^{\theta=2\pi} \mathbf{F} \cdot \frac{d\mathbf{r}}{d\theta}\, d\theta \;=\; \int_0^{2\pi} 1\, d\theta \;=\; \theta\Big|_0^{2\pi} \;=\; 2\pi. \tag{22}$$

Needless to say, 2π does not equal zero. So where did our theory go wrong? Isn't $\nabla \times \mathbf{F} = \mathbf{0}$ supposed to mean that $\mathbf{F}$ is irrotational and hence the gradient of a potential?

The answer is that ever since Calculus I, nothing about functions has ever meant anything except on a given domain. For example, in casual conversation we may say that f defined by $f(x) = \frac{1}{x}$ is a decreasing function, but what we actually mean is that f decreases on $(-\infty, 0) \cup (0, \infty)$: it can't decrease on $(-\infty, \infty)$, because it doesn't exist on $(-\infty, \infty)$, because it isn't defined where $x = 0$. Likewise, when we say that $\nabla \times \mathbf{F} = \mathbf{0}$ makes $\mathbf{F}$ irrotational, what we actually mean is that its circulation is zero around any closed contour surrounding a region in which the potential actually exists. You can readily show that $\mathbf{F}$ defined by (17) satisfies $\mathbf{F} = \nabla\phi$ with

$$\phi \;=\; \arctan\left(\frac{y}{x}\right), \tag{23}$$

But as soon as you write it down, you can see a problem with this potential: it does not exist where $\mathbf{r} = \mathbf{0}$. It can be shown that everything is fine and dandy as long as C is a curve that does not contain the origin: then the circulation of $\mathbf{F}$ around C is precisely zero. But our circle does contain the origin, where $\mathbf{F} = \nabla\phi$ fails to be satisfied because neither ϕ nor $\mathbf{F}$ is defined at that point.

Finally, note the important point that (22) is completely independent of the value of ϵ. We can make our circle as small as we please, and the circulation of $\mathbf{F}$ around it will still be zero: intuitively, as $\epsilon \to 0$, the length of the contour gets smaller and smaller but the strength of the field gets bigger and bigger in such a way that the circulation remains constant. Now imagine covering the whole of two dimensional space with zillions and zillions of very tiny closed curves, one of which is a circle of infinitesimal radius, centered at the origin. Although $\mathbf{F} = \nabla\phi$ fails to be satisfied inside this contour, it is satisfied inside every one of the zillions and zillions of other contours, so that the circulation of $\mathbf{F}$ is zero round every one of them. The only infinitesimal curve with a non-zero circulation is our itsy-witsy circle at the origin, whose circulation remains 2π no matter how small we make it. What this means is that all of $\mathbf{F}$'s rotationality is concentrated at the origin: everywhere else the field is irrotational. Thus we can think of the origin as a point source of rotationality in an otherwise irrotational field. We'll return to this point in our discussion of Stokes' theorem.

Problem

Recover the potential (9) by using the straight line from $\mathbf{0}$ to $\mathbf{R}$ in place of the curve C in Figure 1.

Solution

In class on March 18.