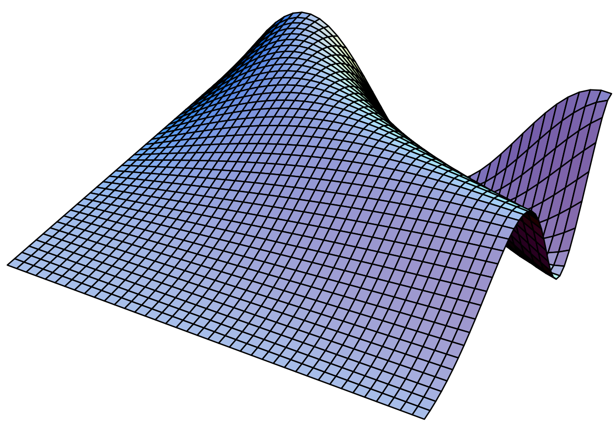
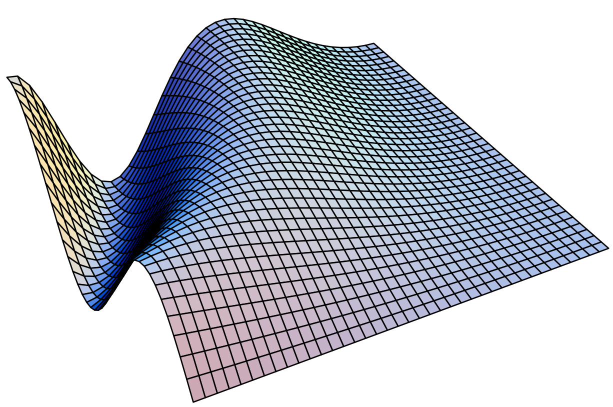
# Cartesian coordinates in three dimensions

Multivariable calculus is the study of functions of more than one variable. In practice, the number of variables in Calculus III is usually only two or three, but in principle there is no limit (for example, the distance between two freely varying points is a function of six variables—see below).

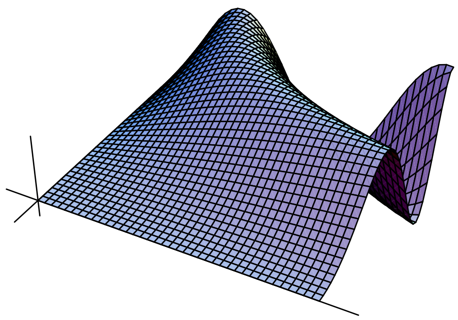
You probably know by now that mathematics has less to say about the truth of a statement than about what that statement would imply if it were true. So, for the sake of argument, I ask you to assume that the earth is essentially flat, extending to infinity in all horizontal directions. In particular, the surface of the ocean (or its average surface, if you are worried about oscillations) is flat, and lies in a well defined plane called sea level. But, of course, even this imaginary earth is not absolutely flat: it rises above sea level to form hills and mountains, and it descends below sea level to form oceans filled with water. Perhaps, for example, there's a place on earth near which the surface looks like this:

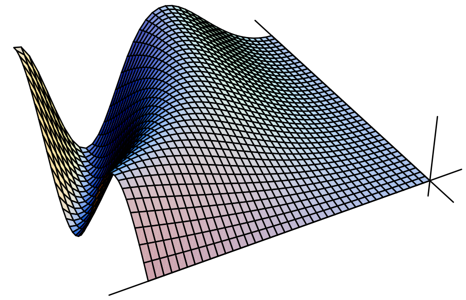


Or, from a different viewpoint, like this:

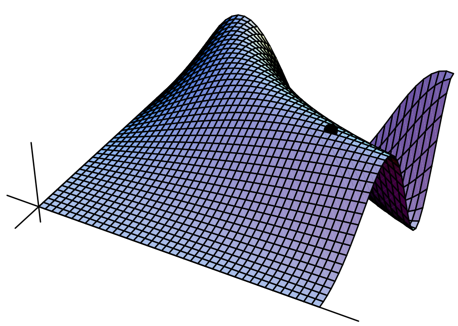


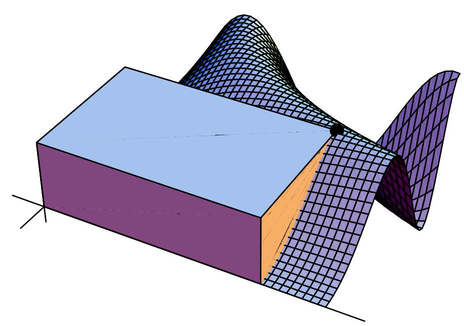
To find our way around on such a surface we need a system of coordinates. We pick an arbitrary point at sea level to be the origin of coordinates O. Through O we draw a horizontal west-east line (the equator), a horizontal south-north line (the prime meridian) and a vertical down-up line; all three lines are infinite and perfectly straight. We call these lines the coordinate axes. Of the following two diagrams, the view on the left is standard, because east points to the right, whereas east points away from you in the view on the right (and in the text, east points at you but to your left):



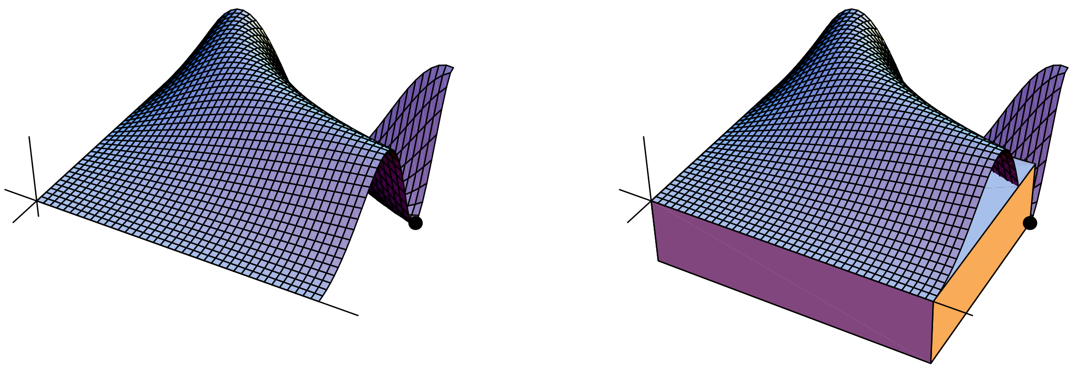


Any point on the surface is at the corner of an imaginary box whose opposite corner is at the origin, and whose sides are parallel to the coordinate axes. The "base" of this box lies in the plane of sea level, the front of the box lies in a vertical plane through the equator, and the left-hand side of the box (not visible) lies in a vertical plane through the prime meridian. The coordinates of the point are the lengths of the sides of the box in the directions of east, north and up, respectively. For example, here is the point with coordinates $\left(\frac{1}{2}\pi, 1, 1\right)$:





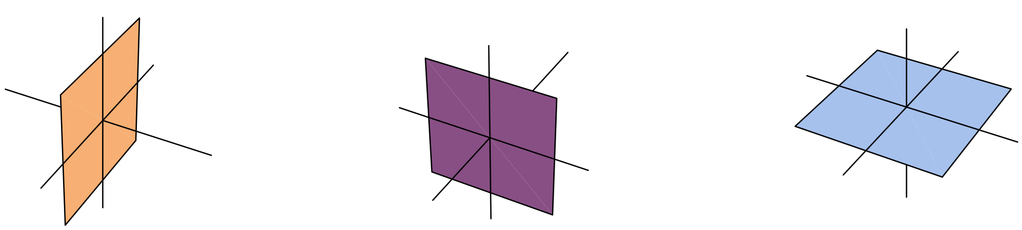
If a point lies below sea level, then the "base" of its box is on top, and the vertical coordinate is negative. For example, here is the point with coordinates $(2, 1.535, -1)$:



The first (east), second (north) and third (up) coordinates of a variable point are usually denoted by $x$, $y$ and $z$, respectively; that is, the point is denoted by $(x, y, z)$. Hence the three coordinate axes are often called the $x$-axis, the $y$-axis and the $z$-axis, respectively.

## The coordinate planes

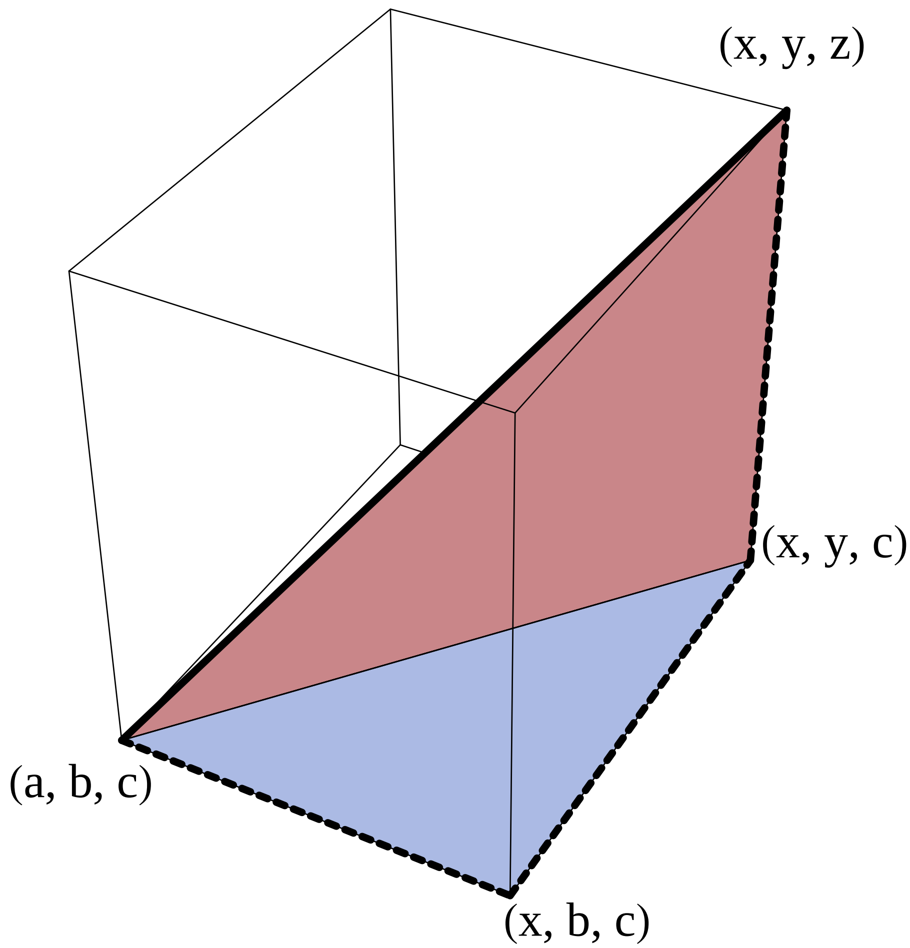
The three coordinate axes are complemented by three coordinate planes. The third coordinate plane is sea level, which is perpendicular to the $z$-axis and contains points of the form $(x, y, 0)$; it is sometimes called $z = 0$, and sometimes the $x$-$y$ plane (and in problems where we know we'll never leave this plane we can even forget about the third coordinate and write the generic point as $(x, y)$). Similarly, the second coordinate plane is perpendicular to the $y$-axis and contains points of the form $(x, 0, z)$; it is sometimes called $y = 0$, and sometimes the $x$-$z$ plane. And finally, the first coordinate plane, which is perpendicular to the $x$-axis and contains points of the form $(0, y, z)$, is sometimes called $x = 0$, and sometimes the $y$-$z$ plane. It may seem strange to call the $x$-$y$ plane the third coordinate plane, because you probably met it first and are therefore inclined to think of it as the first coordinate plane; but the terminology is logically consistent (and if it really bothers you, don't ever use it: just say $z = 0$ or $x$-$y$ plane). Here are the coordinate planes:



### Distance

A fixed point is often denoted by $(a, b, c)$. By the theorem of Pythagoras, the distance as the crow flies—or the worm burrows—between points $(x, y, z)$ and $(a, b, c)$ is

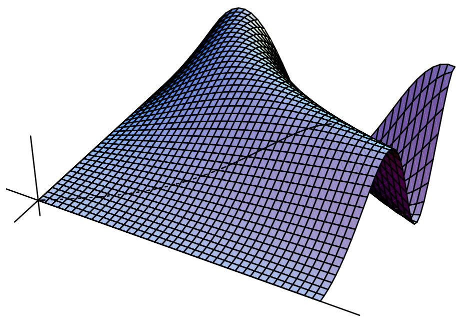
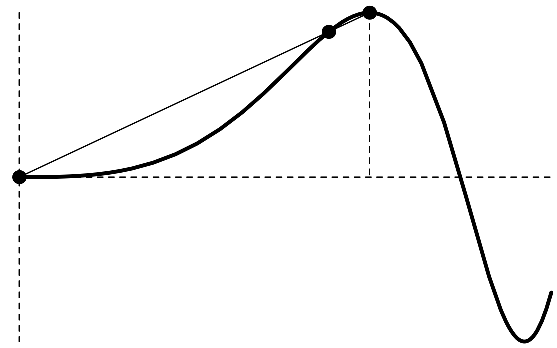
$$\sqrt{(x-a)^2 + (y-b)^2 + (z-c)^2}:$$




But if you are not either a burrowing crow or a flying worm, and are therefore at least sometimes obliged to travel along the surface, the distance in general is longer. For example, for a flying worm, the distance between $(1.571, 1, 1)$ and the origin is

$$\sqrt{(1.571-0)^2 + (1-0)^2 + (1-0)^2} = 2.114.$$

But for a crow it's 2.119, for a non-flying worm it's 2.207, and for the rest of us—e.g., an elephant—it's 2.213. To understand why, it helps to compare the elephant's path in three-dimensional space with the curve of intersection between the surface and a vertical plane through both the origin and $(1.571, 1, 1)$, the middle dot being where the crow starts walking or the non-flying worm starts burrowing:

We conclude our first lecture by finding an expression for the shortest distance between the point $(a, b, c)$ and the $y$-axis, partly because it implies a result we shall need in Lecture 2, but also because it contains an important lesson, namely, that it is often possible to turn a single result into a set of results by invoking symmetry. We proceed as follows. Any point on the $y$-axis has coordinates of the form $(0, y, 0)$. Therefore (from above), the distance between $(a, b, c)$ and an arbitrary point on the $y$-axis must be $\sqrt{(0-a)^2 + (y-b)^2 + (0-c)^2} = \sqrt{a^2 + (y-b)^2 + c^2}$. This expression is clearly smallest when $y = b$. Thus the shortest distance between the point $(a, b, c)$ and the $y$-axis is the distance between $(a, b, c)$ and $(0, b, 0)$, or $\sqrt{a^2 + c^2}$ (which, of course, is the perpendicular distance).

Now the important point. It follows immediately by symmetry that the distance between $(a, b, c)$ and the $x$-axis is $\sqrt{b^2 + c^2}$, and that the distance between $(a, b, c)$ and the $z$-axis is $\sqrt{a^2 + b^2}$: all are special cases of DISTANCE FROM COORDINATE AXIS =

$$\sqrt{\text{SUM OF SQUARES OF OTHER TWO COORDINATES OF POINT}}.$$

It further follows immediately that the distances between $(x, y, z)$ and the $x$-axis, $y$-axis and $z$-axis are $\sqrt{y^2 + z^2}$, $\sqrt{x^2 + z^2}$ and $\sqrt{x^2 + y^2}$, respectively. We will need the last of these results in Lecture 2.

## Problem

Show that the equation of a plane must have the form $G + Bx + Ly + Rz = c$ where $G$, $B$, $L$, $R$ and $c$ are constants.

## Solution

The method that is probably best will emerge in Lecture 4. Meanwhile, however, we can answer another way—which will help to review both your understanding of distance and your algebra skills. Pick any two points that are the same (perpendicular) distance from a plane but on opposite sides of it, say $P_1$ with coordinates $(a_1, b_1, c_1)$ and $P_2$ with coordinates $(a_2, b_2, c_2)$. Let $P$ with coordinates $(x, y, z)$ be any point in the plane. Then, by construction, the distance $PP_1$ must equal the distance $PP_2$ (because if the line $P_1P_2$ intersects the plane at $N$, then $PNP_1$ and $PNP_2$ are congruent right-angled triangles). Hence, from Page 4, $\sqrt{(x-a_1)^2 + (y-b_1)^2 + (z-c_1)^2} = \sqrt{(x-a_2)^2 + (y-b_2)^2 + (z-c_2)^2}$. Squaring yields $(x-a_1)^2 + (y-b_1)^2 + (z-c_1)^2 = (x-a_2)^2 + (y-b_2)^2 + (z-c_2)^2$. Now subtract the right-hand side from the left-hand side and simply, to obtain

$$a_1{}^2 + b_1{}^2 + c_1{}^2 + 2(a_2 - a_1)x + 2(b_2 - b_1)y + 2(c_2 - c_1)z \;=\; a_2{}^2 + b_2{}^2 + c_2{}^2$$

or $G+Bx+Ly+Rz = c$ with $G = a_1{}^2+b_1{}^2+c_1{}^2$, $B = 2(a_2-a_1)$, $L = 2(b_2-b_1)$, $R = 2(c_2-c_1)$ and $c = a_2{}^2 + b_2{}^2 + c_2{}^2$, as required. (For any $G$, $B$, $L$, $R$ and $c$, we can always find $a_1$, $b_1$, $c_1$, $a_2$, $b_2$ and $c_2$ to satisfy these equations.)