The divergence theorem

Now that we know how to calculate fluxes, let's practice our skills on a cuboid. For the sake of definiteness, let the cuboid have one vertex at the origin and another vertex at $a\,\mathbf{i}+b\,\mathbf{j}+c\,\mathbf{k}$, as depicted in Figure 1. Its surface S can be decomposed into six open surfaces by writing

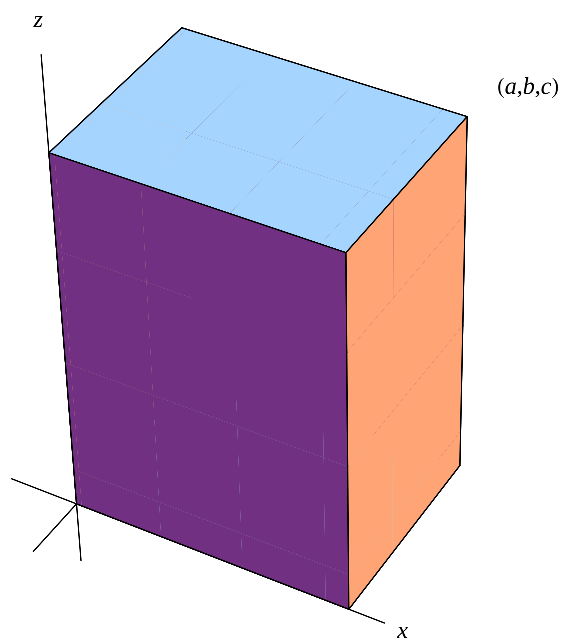


Figure 1: A cuboid

$$S \;=\; S_1 \cup S_2 \cup S_3 \cup S_4 \cup S_5 \cup S_6 \tag{1}$$

where $S_1, \ldots, S_6$ are defined as follows:

S_1:	$0 \le x \le a,\ 0 \le y \le b,\ z = 0$ (bottom)	$\mathbf{n} = -\mathbf{k}$	$dS = dx\,dy$
S_2:	$0 \le x \le a,\ 0 \le y \le b,\ z = c$ (top)	$\mathbf{n} = \mathbf{k}$	$dS = dx\,dy$
S_3:	$0 \le x \le a,\ y = 0,\ 0 \le z \le c$ (front)	$\mathbf{n} = -\mathbf{j}$	$dS = dx\,dz$
S_4:	$0 \le x \le a,\ y = b,\ 0 \le z \le c$ (back)	$\mathbf{n} = \mathbf{j}$	$dS = dx\,dz$
S_5:	$x = 0,\ 0 \le y \le b,\ 0 \le z \le c$ (left side)	$\mathbf{n} = -\mathbf{i}$	$dS = dy\,dz$
S_6:	$x = a,\ 0 \le y \le b,\ 0 \le z \le c$ (right side)	$\mathbf{n} = \mathbf{i}$	$dS = dy\,dz$

We will practice our skills by calculating the flux through S of the vector field

$$\mathbf{F} \;=\; F_1(x,y,z)\,\mathbf{i} + F_2(x,y,z)\,\mathbf{j} + F_3(x,y,z)\,\mathbf{k}. \tag{2}$$

Let's start with the top and the bottom. We have

$$\begin{aligned}
\iint\limits_{S_1} \mathbf{F}\cdot\mathbf{dS} + \iint\limits_{S_2} \mathbf{F}\cdot\mathbf{dS} &= \iint\limits_{S_1} \mathbf{F}\cdot\mathbf{n}\,dS + \iint\limits_{S_2} \mathbf{F}\cdot\mathbf{n}\,dS \\
&= \iint\limits_{S_1} \mathbf{F}\cdot\{-\mathbf{k}\}\,dx\,dy + \iint\limits_{S_2} \mathbf{F}\cdot\mathbf{k}\,dx\,dy \\
&= \iint\limits_{S_1} -F_3(x,y,0)\,dx\,dy + \iint\limits_{S_1} F_3(x,y,c)\,dx\,dy
\end{aligned} \tag{3}$$

because $\mathbf{F}\cdot\mathbf{k} = F_3(x,y,z)$ and $F_3(x,y,c)$ has the same domain as $F_3(x,y,0)$, namely, S_1. So (3) becomes

$$\begin{aligned}
\iint\limits_{S_1} \mathbf{F}\cdot\mathbf{dS} + \iint\limits_{S_2} \mathbf{F}\cdot\mathbf{dS} &= \iint\limits_{S_1} \{F_3(x,y,c) - F_3(x,y,0)\}\,dx\,dy \\
&= \iint\limits_{S_1} F_3(x,y,z)\Big|_{z=0}^{z=c}\,dx\,dy.
\end{aligned} \tag{4}$$

Now, if F_3 were an ordinary function of a single variable x, we would have

$$F_3(z)\Big|_{z=0}^{z=c} = F_3(c) - F_3(0) = \int_0^c F_3'(z)\,dz \tag{5}$$

by the fundamental theorem of calculus. What changes in this equation if F_3 also depends upon x and y, but both are held constant during the (partial) integration? Nothing, except for the notation. So in place of (5) we have

$$F_3(x,y,z)\Big|_{z=0}^{z=c} = F_3(x,y,c) - F_3(x,y,0) = \int_0^c \frac{\partial F_3}{\partial z}(x,y,z)\,dz; \tag{6}$$

and substituting into (4), we obtain

$$\iint\limits_{S_1} \mathbf{F}\cdot\mathbf{dS} + \iint\limits_{S_2} \mathbf{F}\cdot\mathbf{dS} = \iint\limits_{S_1}\int_0^c \frac{\partial F_3}{\partial z}(x,y,z)\,dz\,dx\,dy = \iiint\limits_{\text{CUBOID}} \frac{\partial F_3}{\partial z}\,dV. \tag{7}$$

Similarly, for the front and back:

$$
\begin{aligned}
\iint\limits_{S_3} \mathbf{F}\cdot\mathbf{n}\,dS + \iint\limits_{S_4} \mathbf{F}\cdot\mathbf{n}\,dS \;&=\; \iint\limits_{S_3} \mathbf{F}\cdot\{-\mathbf{j}\}\,dx\,dz + \iint\limits_{S_4} \mathbf{F}\cdot\mathbf{j}\,dx\,dz \\
&=\; \iint\limits_{S_3} -F_2(x,0,z)\,dx\,dz + \iint\limits_{S_3} F_2(x,b,z)\,dx\,dz \\
&=\; \iint\limits_{S_3} \{F_2(x,b,z) - F_2(x,0,z)\}\,dx\,dz \\
&=\; \iint\limits_{S_3} F_2(x,b,z)\Big|_{y=0}^{y=b}\,dx\,dz \\
&=\; \iint\limits_{S_3}\int\limits_0^b \frac{\partial F_2}{\partial y}\,dy\,dx\,dz \;=\; \iiint\limits_{\text{CUBOID}} \frac{\partial F_2}{\partial y}\,dV
\end{aligned}
\tag{8}
$$

because $y = 0$ on S_3 whereas $y = b$ on S_4, and because $F_2(x,b,z)$ has the same domain as $F_2(x,0,z)$. Finally, for the left and right sides:

$$
\begin{aligned}
\iint\limits_{S_5} \mathbf{F}\cdot\mathbf{n}\,dS + \iint\limits_{S_6} \mathbf{F}\cdot\mathbf{n}\,dS \;&=\; \iint\limits_{S_5} \mathbf{F}\cdot\{-\mathbf{i}\}\,dy\,dz + \iint\limits_{S_6} \mathbf{F}\cdot\mathbf{i}\,dy\,dz \\
&=\; \iint\limits_{S_5} -F_1(0,y,z)\,dy\,dz + \iint\limits_{S_5} F_1(a,y,z)\,dy\,dz \\
&=\; \iint\limits_{S_5} \{F_1(a,y,z) - F_1(0,y,z)\}\,dy\,dz \\
&=\; \iint\limits_{S_35} F_1(a,y,z)\Big|_{x=0}^{x=a}\,dy\,dz \\
&=\; \iint\limits_{S_5}\int\limits_0^a \frac{\partial F_1}{\partial x}\,dx\,dy\,dz \;=\; \iiint\limits_{\text{CUBOID}} \frac{\partial F_1}{\partial x}\,dV
\end{aligned}
\tag{9}
$$

because $x = 0$ on S_5 whereas $x = a$ on S_6, and because $F_1(a,y,z)$ has the same domain as $F_1(0,y,z)$. Adding (7)-(9), we obtain

$$
\iint\limits_{S} \mathbf{F}\cdot\mathbf{dS} \;=\; \sum_{i=1}^{6}\iint\limits_{S_i} \mathbf{F}\cdot\mathbf{n}\,dS \;=\; \iiint\limits_{\text{CUBOID}} \left\{\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right\} dV. \tag{10}
$$

The expression in squiggly brackets recurs repeatedly in vector calculus, and for that reason deserves a name. So let us define the *divergence* $\nabla\cdot\mathbf{F}$ of the vector field $\mathbf{F}$ in (2) to be

$$
\nabla\cdot\mathbf{F} \;=\; \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} \tag{11}
$$

Note that $\nabla \cdot \mathbf{F}$ is a *scalar* (associated with a vector). Now (10) becomes

$$\iint\limits_{S} \mathbf{F} \cdot \mathbf{dS} \;=\; \iiint\limits_{\text{CUBOID}} \nabla \cdot \mathbf{F} \, dV. \tag{12}$$

This equation holds for an arbitrary (differentiable) vector field $\mathbf{F}$. So it's already quite a general result, even though it holds only for a particular cuboid.

Note, however, that if we collapse this cuboid down onto $z = 0$ then, in the limit as $c \to 0$ and $S_2 \to S_1$, (4) reduces to

$$\iint\limits_{S_1} \mathbf{F} \cdot \mathbf{n}_{down} \, dS + \iint\limits_{S_1} \mathbf{F} \cdot \mathbf{n}_{up} \, dS \;=\; 0, \tag{13}$$

where $\mathbf{n}_{down} = -\mathbf{k}$ is the downward pointing unit normal to the plane rectangular surface S_1 and $\mathbf{n}_{up} = \mathbf{k}$ is the upward pointing unit normal to the same surface. That is, the upward flux over any plane horizontal rectangular surface exactly cancels the downward flux over the same surface. Now let us take a second cuboid with a vertex at the origin and another vertex at $a\,\mathbf{i} + b\,\mathbf{j} - C\,\mathbf{k}$; by repeating the above analysis, it is easy to show that (12) continues to hold. If we add our two versions of (12), one for each cuboid, then we obtain a new equation whose right-hand side is the integral of $\nabla \cdot \mathbf{F}$ over a larger cuboid with base area ab and height $c + C$; and whose left-hand side is the sum of twelve surface integrals, one for each face of each of the smaller cuboids. But these two cuboids have a horizontal face in common, and so (13) ensures that two of the twelve integrals add up to zero. Of the remaining ten, two correspond to the fluxes of $\mathbf{F}$ over the top and bottom of the larger cuboid; and the remaining eight, when taken in pairs, correspond to the fluxes of $\mathbf{F}$ over the sides of the larger cuboid. We thus determine that (12) must hold for the larger cuboid as well. Continuing in this way, and noting that (13) readily extends to opposite fluxes over a vertical plane surface, on taking the appropriate limit in (8) or (9)—in fact, of course, it extends to opposite fluxes over *any* open surface—it soon becomes clear that we can add as many more cuboids as we please, and (12) will continue to hold (with our collection of cuboids in place of the original CUBOID). The cuboids can be as small as we please, and we can add as many as we please, and the result will continue to hold. But any volume enclosed by a surface can be approximated to any desired accuracy by filling it with zillions and zillions of cuboids. We thus suspect that our result must hold not only for an arbitrary collection of cuboids, but for any volume—which turns out to be quite correct. To be more precise, the flux of a vector field through a piecewise-smooth closed surface equals the integral of the field's divergence over the volume enclosed by the surface.[‡] This result, namely, that

$$\iint\limits_{S} \mathbf{F} \cdot \mathbf{dS} \;=\; \iiint\limits_{V} \nabla \cdot \mathbf{F} \, dV \tag{14}$$

when S encloses V, is known as the divergence theorem.

[‡]A surface is piecewise-smooth if its normal is well defined everywhere, except possibly at a finite number of joins between the open surfaces into which it is decomposed.

In the special case where S encloses an infinitesimal region of volume δV around the point with position vector $x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$, (14) yields $\iint_S \mathbf{F}\cdot\mathbf{dS} \approx \nabla\cdot\mathbf{F}\,\delta V$ or $\nabla\cdot\mathbf{F} \approx \iint_S \mathbf{F}\cdot\mathbf{dS} \div \delta V$, which becomes exact in the limit as the region shrinks to contain only the point:

$$\nabla\cdot\mathbf{F} \;=\; \lim_{\delta V\to 0}\frac{1}{\delta V}\iint\limits_S \mathbf{F}\cdot\mathbf{dS}. \tag{15}$$

Thus, intuitively, we can interpret divergence as local flux per unit volume. Indeed (15) is usually regarded as the definition of the divergence of a vector field, and (11) as merely its Cartesian manifestation.

The divergence theorem often simplifies the calculation of surface integrals. For example, in Lecture 17 we calculated that

$$Q \;=\; \iint\limits_S \{ax\,\mathbf{i}+yz\mathbf{j}+x^2\mathbf{k}\}\cdot\mathbf{dS} \;=\; \tfrac{1}{2}\pi a^2h(h+2a) \tag{16}$$

for a vertical cylinder of height h and radius a with its floor at sea level and its axis aligned along the the z-axis. But the calculation was fairly lengthy. We could have shortened it greatly by using (14) to obtain

$$\begin{aligned}
Q &= \iiint\limits_V \left\{\frac{\partial}{\partial x}(ax)+\frac{\partial}{\partial y}(yz)+\frac{\partial}{\partial z}(x^2)\right\}dV\\
&= \iiint\limits_V \{a+z+0\}\,dV \;=\; a\iiint\limits_V dV + \iiint\limits_V z\,dV\\
&= a\cdot\text{VOLUME} + \iiint\limits_V z|J|\,dr\,d\theta\,dz\\
&= a\cdot\pi a^2h + \int_0^h\int_0^{2\pi}\int_0^a zr\,dr\,d\theta\,dz\\
&= \pi a^3h + \int_0^a r\,dr\int_0^{2\pi}d\theta\int_0^h z\,dz\\
&= \pi a^3h + \tfrac{1}{2}a^2\cdot 2\pi\cdot\tfrac{1}{2}h^2 \;=\; \tfrac{1}{2}\pi a^2h(h+2a)
\end{aligned} \tag{17}$$

in agreement with (16).[§]

[§]We could have shortened the calculation even further by observing that

$$\iiint\limits_V z\,dV \;=\; M\overline{z} \;=\; (\pi a^2h\cdot 1)\tfrac{h}{2}$$

where M denotes the mass of a cylinder of uniform density 1 and $\overline{z}$ denotes the vertical coordinate of its center of mass.

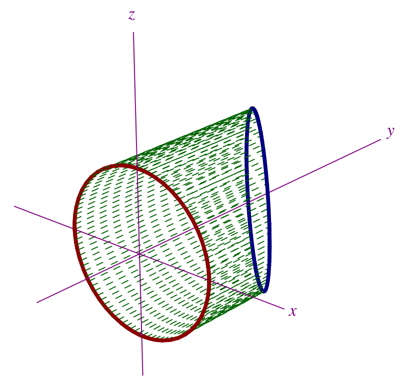
Similarly, in Lecture 19 we calculated that the outward flux

$$\iint\limits_S \mathbf{F} \cdot \mathbf{dS}$$

of the vector field

$$\mathbf{F} \;=\; x\,\mathbf{i} + y\,\mathbf{j} + 5\,\mathbf{k}$$

across the boundary of the "sawn-off" cylindrical region bounded by the circular cylinder $x^2 + z^2 = 1$, the "floor" plane $y = 0$ and the "roof" plane $x + y = 2$ is



$$\iint\limits_S \mathbf{F} \cdot \mathbf{dS} \;=\; 4\pi,$$

and again the calculation was fairly lengthy. Here (11) implies that $\nabla \cdot \mathbf{F} = 1 + 1 + 0 = 2$, and the region enclosed by S is covered by allowing R to range between 0 and 1, α to range between 0 and 2π and y to range between 0 and $2 - x = 2 - R\sin(\alpha)$ in (modified) cylindrical polar coordinates, for which the volume element is $dV = R\,dR\,d\alpha\,dy$,¶ which we will have to reorder as either $dV = R\,dy\,dR\,d\alpha$ or $dV = R\,dy\,d\alpha\,dR$, since y must be the innermost integration variable (because its limits depend on the other two). Hence, on using the divergence theorem, we obtain

$$\begin{aligned}\iint\limits_S \mathbf{F} \cdot \mathbf{dS} \;&=\; \iiint\limits_V \nabla \cdot \mathbf{F}\,dV \;=\; \int\limits_0^1 \int\limits_0^{2\pi} \int\limits_0^{2-R\sin(\alpha)} 2R\,dy\,d\alpha\,dR \;=\; \int\limits_0^1 \int\limits_0^{2\pi} 2Ry\Big|_0^{2-R\sin(\alpha)}\,d\alpha\,dR \\ &=\; \int\limits_0^1 \int\limits_0^{2\pi} 2R\{2 - R\sin(\alpha)\}\,d\alpha\,dR \;=\; 4\int\limits_0^{2\pi} d\alpha \int\limits_0^1 R\,dR - 2\int\limits_0^{2\pi} \sin(\alpha)\,d\alpha \int\limits_0^1 R^2\,dR \\ &=\; 4 \cdot 2\pi \cdot \tfrac{1}{2} - 2 \cdot 0 \cdot \tfrac{1}{3} \;=\; 4\pi\end{aligned}$$

as before, but after a much easier calculation.

On the other hand, the point of Lectures 17 and 19 was to learn about surface integrals—not the divergence theorem—and there is absolutely no doubt, at least in my mind, that doing those calculations the long way round was good for you at the time!

¶If this isn't obvious to you, then by analogy with equation (30) of Lecture 13, set $u = R$, $v = \alpha$ and $w = y$ with $x = R\sin(\alpha)$ and $z = R\cos(\alpha)$ by Lecture 19 to obtain

$$|J(R,\alpha,y)| \;=\; \left\|\begin{matrix} \frac{\partial x}{\partial R} & \frac{\partial y}{\partial R} & \frac{\partial z}{\partial R} \\ \frac{\partial x}{\partial \alpha} & \frac{\partial y}{\partial \alpha} & \frac{\partial z}{\partial \alpha} \\ \frac{\partial x}{\partial y} & \frac{\partial y}{\partial y} & \frac{\partial z}{\partial y} \end{matrix}\right\| \;=\; \left\|\begin{matrix} \sin(\alpha) & 0 & \cos(\alpha) \\ R\cos(\alpha) & 0 & -R\sin(\alpha) \\ 0 & 1 & 0 \end{matrix}\right\| \;=\; R,$$

implying $dV = |J|\,dR\,d\alpha\,dy = R\,dR\,d\alpha\,dy$.