§15.3, #26*

Find the volume of the volumetric region E bounded below by the paraboloid $z = x^2 + 3y^2$ and above by the plane $z = x$.

Solution

The plane meets the paraboloid where $x = x^2 + 3y^2 \implies 4x = 4x^2 + 12y^2 \implies 4x^2 - 4x + 12y^2 = 0 \implies 4x^2 - 4x + 112y^2 = 1 \implies (2x-1)^2 + 12y^2 = 1$. This is the equation of an elliptic cylinder whose axis is the line with vector equation $\boldsymbol{\rho} = \frac{1}{2}\mathbf{i} + 0\mathbf{j} + t\mathbf{i}$ (or parametric equations $x = \frac{1}{2}, y = 0, z = t$), every horizontal cross-section being an ellipse with major axis of length $\sqrt{2}$ (parallel to the x-axis) and minor axis of length $1/\sqrt{3}$ (parallel to the y-axis).* Only points above the paraboloid, below the plane and inside this cylinder belong to E. Thus

$$E \;=\; \{(x, y, z) | (2x-1)^2 + 12y^2 \le 1, x^2 + 3y^2 \le z \le x\}. \tag{1}$$

Because

$$(2x-1)^2 + 12y^2 \;\le\; 1, \tag{2}$$

we must in particular have $12y^2 \le 1$, hence

$$-\tfrac{1}{2\sqrt{3}} \;\le\; y \;\le\; \tfrac{1}{2\sqrt{3}}. \tag{3}$$

For y satisfying (3), (2) implies that x is constrained by $(2x-1)^2 \le 1 - 12y^2$ or

$$-\tfrac{1}{2}\{1 - \sqrt{1 - 12y^2}\} \;\le\; x \;\le\; \tfrac{1}{2}\{1 + \sqrt{1 - 12y^2}\}. \tag{4}$$

Hence, from (1)–(4),

$$(x, y, z) \in E \qquad \Longleftrightarrow \qquad \begin{array}{rcccl} x^2 + 3y^2 & \le & z & \le & x \\ -\frac{1}{2}\{1 - \sqrt{1 - 12y^2}\} & \le & x & \le & \frac{1}{2}\{1 + \sqrt{1 - 12y^2}\} \\ -\frac{1}{2\sqrt{3}} & \le & y & \le & \frac{1}{2\sqrt{3}} \end{array} \tag{5}$$

We can now calculate the volume V of E by using Method (ii) of Lecture 11's table with z_L, z_U, x_S, x_R, y_F and y_B defined as follows:

$$\begin{aligned} z_L(x, y) &= x^2 + 3y^2 \\ z_U(x, y) &= x \\ x_S(y) &= -\tfrac{1}{2}\{1 - \sqrt{1 - 12y^2}\} \\ x_R(y) &= -\tfrac{1}{2}\{1 + \sqrt{1 - 12y^2}\} \\ y_F &= -\tfrac{1}{2\sqrt{3}} \\ y_B &= \tfrac{1}{2\sqrt{3}} \end{aligned} \tag{6}$$

*Note that the two surfaces defining E intersect in an ellipse that lies in the plane $z = x$, that is, they intersect in a curve, not in a surface. But this curve of intersection lies entirely on the elliptical cylinder whose equation we have just determined.

Thus

$$
\begin{aligned}
V \;=\; \iiint\limits_E 1\,dV \;&=\; \int\limits_{y_F}^{y_B}\int\limits_{x_S(y)}^{x_R(y)}\int\limits_{z_L(x,y)}^{z_U(x,y)} 1\,dz\,dx\,dy \\
&=\; \int\limits_{-\frac{1}{2\sqrt{3}}}^{\frac{1}{2\sqrt{3}}}\;\int\limits_{\frac{1}{2}\{1-\sqrt{1-12y^2}\}}^{\frac{1}{2}\{1+\sqrt{1-12y^2}\}}\;\int\limits_{x^2+3y^2}^{x} 1\,dz\,dx\,dy \\
&=\; \int\limits_{-\frac{1}{2\sqrt{3}}}^{\frac{1}{2\sqrt{3}}}\;\int\limits_{\frac{1}{2}\{1-\sqrt{1-12y^2}\}}^{\frac{1}{2}\{1+\sqrt{1-12y^2}\}} z\Big|_{x^2+3y^2}^{x}\,dx\,dy \\
&=\; \int\limits_{-\frac{1}{2\sqrt{3}}}^{\frac{1}{2\sqrt{3}}}\;\int\limits_{\frac{1}{2}\{1-\sqrt{1-12y^2}\}}^{\frac{1}{2}\{1+\sqrt{1-12y^2}\}} \{x - x^2 - 3y^2\}\,dx\,dy \\
&=\; \int\limits_{-\frac{1}{2\sqrt{3}}}^{\frac{1}{2\sqrt{3}}} \left\{\tfrac{1}{2}x^2 - \tfrac{1}{3}x^3 - 3y^2x\right\}\Bigg|_{\frac{1}{2}\{1-\sqrt{1-12y^2}\}}^{\frac{1}{2}\{1+\sqrt{1-12y^2}\}}\,dy \\
&=\; \frac{1}{6}\int\limits_{-\frac{1}{2\sqrt{3}}}^{\frac{1}{2\sqrt{3}}} \{1-12y^2\}^{\frac{3}{2}}\,dy
\end{aligned}
\tag{7}
$$

after considerable simplification between the last two lines. We now use the substitution

$$
y \;=\; \tfrac{1}{2\sqrt{3}}\sin(u) \quad\Longrightarrow\quad \frac{dy}{du} \;=\; \tfrac{1}{2\sqrt{3}}\cos(u), \qquad 1-12y^2 \;=\; 1-\sin^2(u) \;=\; \cos^2(u).
$$

Also,

$$
\begin{aligned}
y \;=\; -\tfrac{1}{2\sqrt{3}} \quad&\Longrightarrow\quad u \;=\; -\tfrac{1}{2}\pi \\
y \;=\; \tfrac{1}{2\sqrt{3}} \quad&\Longrightarrow\quad u \;=\; \tfrac{1}{2}\pi
\end{aligned}
$$

So, from (7),

$$
\begin{aligned}
V &= \frac{1}{6} \int\limits_{u=-\frac{1}{2}\pi}^{u=\frac{1}{2}\pi} \{1 - 12y^2\}^{\frac{3}{2}} \frac{dy}{du}\, du \;=\; \frac{1}{6} \int\limits_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \{\cos^2(u)\}^{\frac{3}{2}} \tfrac{1}{2\sqrt{3}} \cos(u)\, du \\
&= \frac{1}{12\sqrt{3}} \int\limits_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \cos^4(u)\ du
\end{aligned}
\tag{8}
$$

We can now use the trigonometric identity $\cos^2(u) = \frac{1}{2}\{1 + \cos(2u)\}$, which immediately also implies $\cos^2(2u) = \frac{1}{2}\{1 + \cos(2 \cdot 2u)\} = \frac{1}{2}\{1 + \cos(4u)\}$. Hence

$$
\begin{aligned}
\cos^4(u) &= \{\cos^2(u)\}^2 \;=\; \left\{\tfrac{1}{2}\{1 + \cos(2u)\}\right\}^2 \;=\; \tfrac{1}{4}\{1 + \cos(2u)\}^2 \\
&= \tfrac{1}{4}\{1 + 2\cos(2u) + \cos^2(2u)\} \\
&= \tfrac{1}{4} + \tfrac{1}{2}\cos(2u) + \tfrac{1}{4}\cos^2(2u) \\
&= \tfrac{1}{4} + \tfrac{1}{2}\cos(2u) + \tfrac{1}{8} + \tfrac{1}{8}\cos(4u) \\
&= \tfrac{3}{8} + \tfrac{1}{2}\cos(2u) + \tfrac{1}{8}\cos(4u).
\end{aligned}
$$

So, from (8),

$$
\begin{aligned}
V &= \frac{1}{12\sqrt{3}} \int\limits_{u\ =\ -\frac{1}{2}\pi}^{u=\frac{1}{2}\pi} \cos^4(u)\ du \;=\; = \frac{1}{12\sqrt{3}} \int\limits_{u\ =\ -\frac{1}{2}\pi}^{u=\frac{1}{2}\pi} \left\{\tfrac{3}{8} + \tfrac{1}{2}\cos(2u) + \tfrac{1}{8}\cos(4u)\right\}\ du \\
&= \frac{1}{12\sqrt{3}} \left\{\tfrac{3}{8}u + \tfrac{1}{4}\sin(2u) + \tfrac{1}{32}\sin(4u)\right\}\Bigg|_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \\
&= \frac{1}{12\sqrt{3}} \left\{\tfrac{3}{8} \cdot \tfrac{1}{2}\pi + \tfrac{1}{4}\sin(\pi) + \tfrac{1}{32}\sin(2\pi) - \left(\tfrac{3}{8} \cdot \left\{-\tfrac{1}{2}\pi\right\} + \tfrac{1}{4}\sin(-\pi) + \tfrac{1}{32}\sin(-2\pi)\right)\right\} \\
&= \frac{\pi}{32\sqrt{3}}.
\end{aligned}
$$