Functions of Three Variables

A quantity (dependent variable) that depends on several other quantities (independent variables) changes (= increases or decreases) linearly when a change in one of the independent variables causes precisely the same change in the dependent variable at every magnitude of that quantity. For example, suppose that your grocery bill depends, among other things, on the amounts you buy of barley, lentils and rye flour (items that can at least somewhat realistically be purchased in virtually any quantity). Suppose, to be precise, that you purchase x kilograms of barley at B dollars per kilo, y kilograms of lentils at L dollars per kilo and z kilograms of rye flour at R dollars per kilo. Then increasing your order of barley by Δx kilos will increase your grocery bill by $B\Delta x$ dollars regardless of how much food you have ordered already (assuming, for the sake of argument, that there are no discounts). Likewise, increasing your order of lentils by Δy kilos or your order of rye flour by Δz kilos will increase your grocery bill by $L\Delta y$ or $R\Delta z$ dollars, respectively, regardless of how much barley or anything else you have already ordered. In fact, if G denotes the cost of the items you have already ordered elsewhere, then your grocery bill depends on x, y and z according to

$$g(x, y, z) \;=\; G + Bx + Ly + Rz. \tag{1}$$

You should check that, e.g., increasing the lentil allowance by Δy increases the bill by

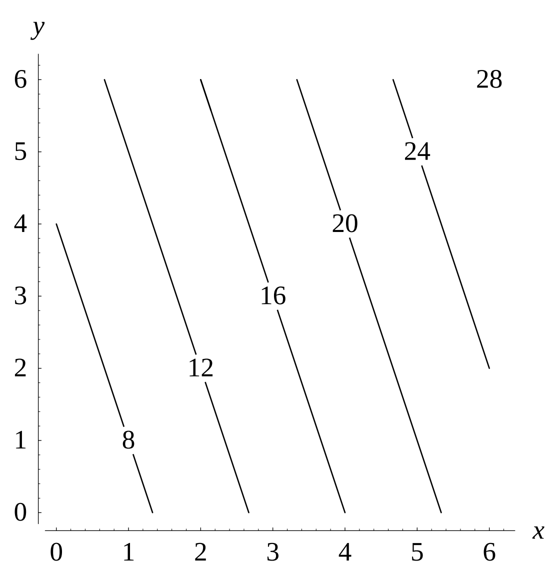
$$\begin{aligned} g(x, y + \Delta y, z) - g(x, y, z) \;&=\; G + Bx + L(y + \Delta y) + Rz - (G + Bx + Ly + Rz) \\ &=\; L\Delta y, \end{aligned} \tag{2}$$

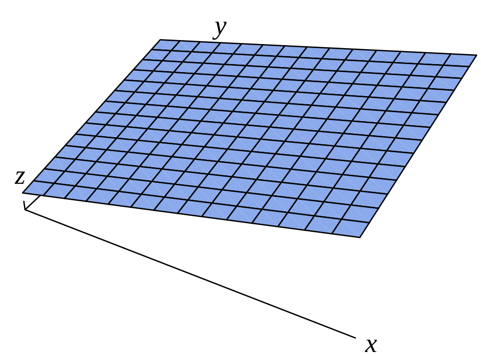
and similarly for the other independent variables.

But now we have a problem that we wouldn't have if you didn't like rye flour. Then your food bill would be a function of only two independent variables, x and y, namely

$$f(x, y) \;=\; G + Bx + Ly, \tag{3}$$

and you could easily plot a contour map or draw a surface graph; for example, here are a contour map and a surface graph for $G = 4$, $B = 3$ and $L = 1$.

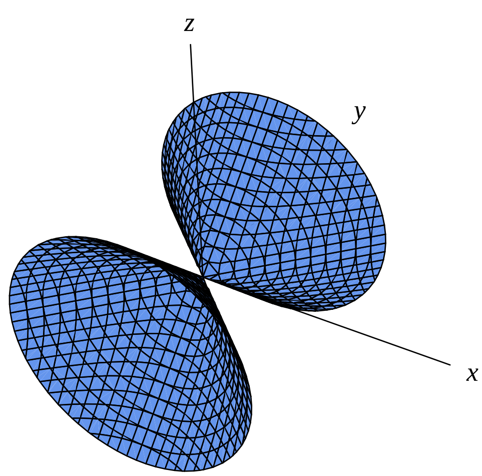




The problem when you do indeed like rye flour is that it is difficult to visualize the graph of a function of three (or more) variables. But if we are mainly interested in whether it increases or decreases in a certain direction, then we can still obtain the information we need by plotting its contours. However, these contours are no longer level curves; rather, they are level surfaces.

To understand the difference between a level curve and a level surface, let's compare f with g. A particular contour of f is the set of all (x, y) for which $f(x, y) = c$; this is a curve, more precisely, the straight line $G + Bx + Ly = c$ or $y = (c - G - Bx)/L$. A particular contour of g is the set of all (x, y, z) for which $g(x, y, z) = c$; this is a surface, more precisely, the plane with equation $G+Bx+Ly+Rz = c$ or $z = (c-G-Bx-Ly)/R$. How do we know that that's a plane? Because with $\boldsymbol{\rho} = x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$, $\mathbf{a} = (c-G)\mathbf{k}/R$ and $\mathbf{n} = B\mathbf{i}+L\mathbf{j}+R\mathbf{k}$ the equation $G+Bx+Ly+Rz = c$ becomes $\mathbf{n}\cdot(\boldsymbol{\rho}-\mathbf{a}) = 0$, which tells us that $\boldsymbol{\rho}$ is an arbitrary point on a plane perpendicular to $\hat{\mathbf{n}}$ through the point $(0, 0, (c - G)/R)$. See Lecture 4.

Of course, level surfaces are not in general planes; for example, here is the zero level surface of the function g defined by $g(x, y, z) = x^2 - y^2 + z^2$ (in other words, here is the surface $x^2 - y^2 + z^2 = 0$):



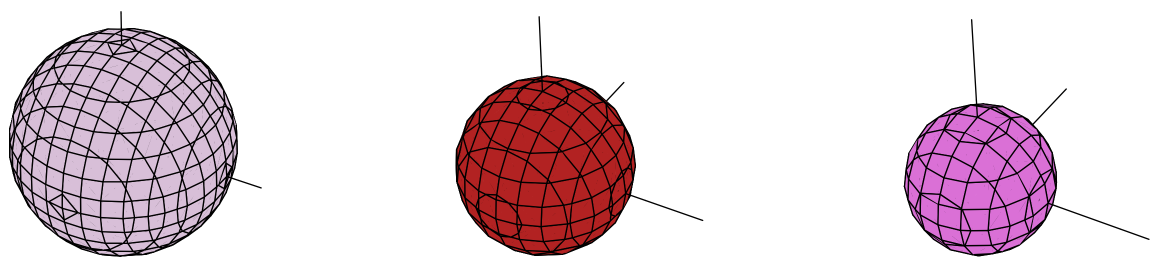
Moreover, level surfaces are not in general surface graphs: you can readily see that a vertical line can intersect the above surface more than once. Nevertheless, level surfaces and surface graphs still share the property that their intersection with any plane—in particular, with any coordinate plane—is a curve. For example, the intersection of the above surface with a plane perpendicular to the y-axis is a circle; and intersection of the surface with a plane perpendicular to the x-axis or z-axis is a hyperbola.

Level surfaces can provide useful information about how a function of three variables, say g, behaves. By plotting contours $g(x, y, z) = \alpha$ and $g(x, y, z) = \beta$ where $\beta > \alpha$, we can tell that the function increases in the direction of going from the first of these two level surfaces to the second. In principle, this ruse works for any function of three variables, regardless of whether that function is linear or nonlinear; in practice, however, it often works best if the function has spherical symmetry.

For example, some level surfaces of

$$g(x, y, z) \;=\; e^{-(x^2+y^2+z^2)} \tag{4}$$

are these:



The figure shows the level surface $g(x, y, z) = c$ for (from left to right) $c = \frac{1}{10}$, $c = \frac{1}{3}$ and $c = \frac{1}{2}$. You immediately discover the unsurprising fact that $g(x, y, z)$ increases as you approach the origin of coordinates (or, which is precisely the same thing, decreases as you move away from it). This fact is unsurprising because there is a sense in which you are only plotting a function of a single variable, namely, $e^{-\rho^2}$, where $\rho = \sqrt{x^2 + y^2 + z^2}$ denotes distance as the (hypothetical) crow flies from the origin. And you know very well that $e^{-\rho^2}$ decreases with ρ.