The dot product and cross product

Among the many questions we might ask about a pair of vectors are the following: Are they perpendicular? Are they parallel? If not, what is the angle between them (when their tails coincide), and how do we find a vector that's perpendicular to both of them? To answer these and other questions, we introduce two different products for a pair of vectors, namely, the dot or scalar product and the cross or vector product. We will deal with each in turn. But first I want to quickly re-emphasize the point that there often exist several notations for precisely the same quantity, each of them useful in different circumstances. For example, the distance from the origin to the generic point with coordinates (x, y, z) is

$$OP \;=|\,\vec{OP}\,|=\; \left\|\vec{OP}\right\| \;=\; \|\boldsymbol{\rho}\| \;=|\,\boldsymbol{\rho}\,|=\; \rho \tag{1}$$

$(=\sqrt{x^2+y^2+z^2})$.

Now for the dot and cross product.

The dot product

For the sake of simplicity, let's pretend that our vectors are position vectors

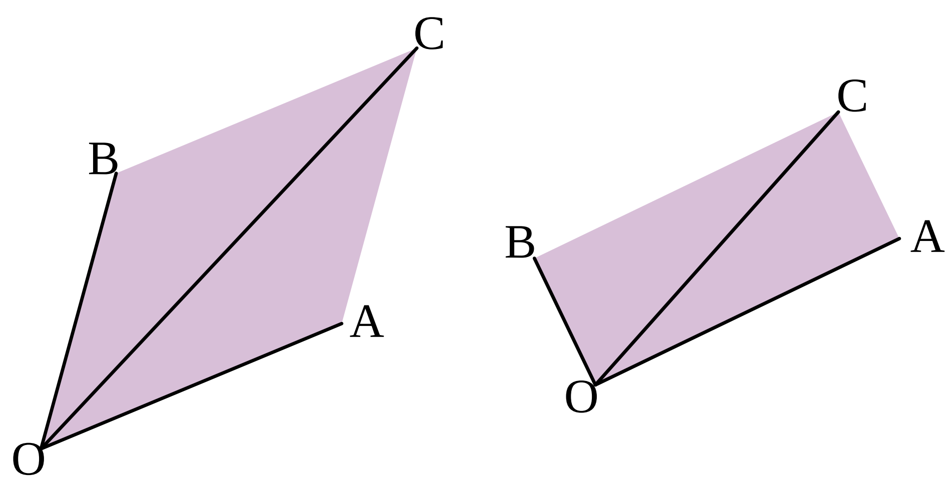
$$\vec{OA} \;=\; \boldsymbol{\rho}_1 \;=\; x_1\mathbf{i}+y_1\mathbf{j}+z_1\mathbf{k}, \tag{2}$$

$$\vec{OB} \;=\; \boldsymbol{\rho}_2 \;=\; x_2\mathbf{i}+y_2\mathbf{j}+z_2\mathbf{k} \tag{3}$$

(although everything will apply to vectors in general) and define

$$\vec{OC} \;=\; \boldsymbol{\rho}_1+\boldsymbol{\rho}_2 \;=\; (x_1+x_2)\mathbf{i}+(y_2+y_1)\mathbf{j}+(z_2+z_1)\mathbf{k}. \tag{4}$$

These three vectors are obliged to lie in a plane, in which they form two sides and a diagonal of the parallelogram $OACB$ (see below). But are $\boldsymbol{\rho}_1$ and $\boldsymbol{\rho}_2$ perpendicular (as in the diagram on the right)?



Well, if they are, then the parallelogram must be a rectangle, in which case, Pythagoras's theorem tells us that $OC^2 = OA^2 + AC^2 = OA^2 + OB^2$ or, on using (2)–(4), that

$$(x_1 + x_2)^2 + (y_2 + y_1)^2 + (z_2 + z_1)^2 \;=\; \{x_1^2 + y_1^2 + z_1^2\} + \{x_2^2 + y_2^2 + z_2^2\}. \tag{5}$$

For example, are $\mathbf{a} = \mathbf{i} - 3\mathbf{j} + 2\mathbf{k}$ and $\mathbf{b} = 4\mathbf{i} - 2\mathbf{j} - 5\mathbf{k}$ perpendicular? Well, yes they are, because $\mathbf{a} + \mathbf{b} = \mathbf{5i} - 5\mathbf{j} - 3\mathbf{k}$, and you can easily verify that $5^2 + (-5)^2 + (-3)^2 = \{1^2 + (-3)^2 + 2^2\} + \{4^2 + (-2)^2 + (-5)^2\}$. But a few moments' thought reveals that this calculation was like using a sledgehammer to burst a soap bubble, because

$$(x_1+x_2)^2+(y_2+y_1)^2+(z_2+z_1)^2-\{x_1^2+y_1^2+z_1^2\}-\{x_2^2+y_2^2+z_2^2\} = 2\{x_1x_2+y_1y_2+z_1z_2\} \tag{6}$$

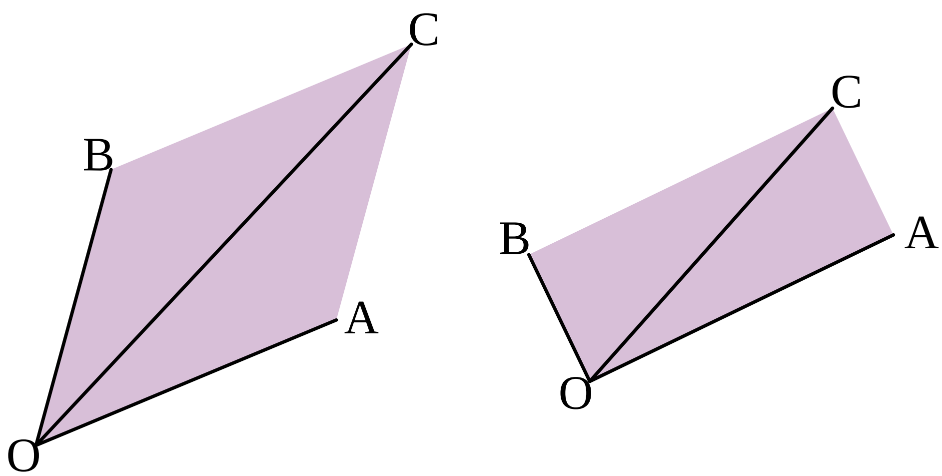
is an identity, that is, it is true for all values of x_1, x_2, y_1, y_2, z_1 and z_2 (as you can easily verify). So (5) reduces to

$$x_1x_2 + y_1y_2 + z_1z_2 \;=\; 0, \tag{7}$$

and it is very much easier to verify that $1 \times 4 + (-3) \times (-2) + 2 \times (-5) = 0$ than to verify (5). The quantity defined by (7) is called the dot product of the vectors $\boldsymbol{\rho}_1$ and $\boldsymbol{\rho}_2$ and is denoted by $\boldsymbol{\rho}_1 \cdot \boldsymbol{\rho}_2$. That is, we define

$$\boldsymbol{\rho}_1 \cdot \boldsymbol{\rho}_2 \;=\; x_1x_2 + y_1y_2 + z_1z_2. \tag{8}$$

Now we have a test for whether $\boldsymbol{\rho}_1$ and $\boldsymbol{\rho}_2$ are perpendicular: do they satisfy $\boldsymbol{\rho}_1 \cdot \boldsymbol{\rho}_2 = 0$?



But suppose that they are not perpendicular (as in the diagram on the left). Might we then not wish to know the angle between them? Let's call it θ; that is, let θ be the smaller of the two possible angles between the vectors when their tails coincide, as indicated in the diagram overleaf. Then, by the cosine rule applied to the triangle OAC, we have

$$\begin{aligned} OC^2 &= OA^2 + AC^2 - 2OA \times AC\cos(\pi - \theta) \\ &= OA^2 + OB^2 + 2OA \times OB\cos(\theta) \end{aligned} \tag{9}$$

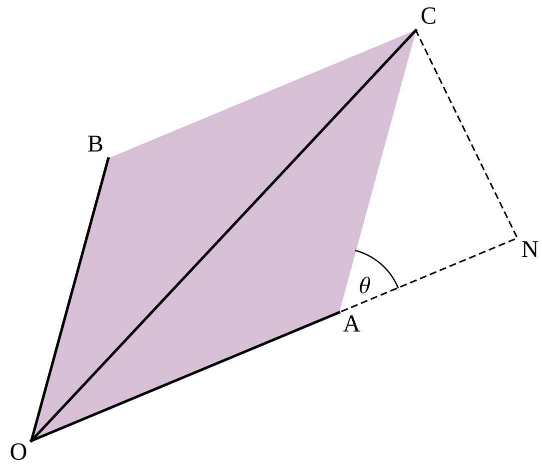
so that[1]

$$2OA\,OB\cos(\theta) \;=\; OC^2 - OA^2 - OB^2. \tag{10}$$

[1]Alternatively, we can apply Pythagoras's theorem directly to the diagram overleaf. We find that $OC^2 = \{OA + AN\}^2 + NC^2 = \{OA + AC\cos(\theta)\}^2 + \{AC\sin(\theta)\}^2$, from which $AC = OB$ implies (10).

From (6) and (8), however, we have $OC^2 - OA^2 - OB^2 = 2\{x_1x_2 + y_1y_2 + z_1z_2\} = 2\boldsymbol{\rho}_1 \cdot \boldsymbol{\rho}_2$, whereas $OA = \|\boldsymbol{\rho}_1\|$ and $OB = \|\boldsymbol{\rho}_2\|$. Thus (10) establishes that $2\,\|\boldsymbol{\rho}_1\|\,\|\boldsymbol{\rho}_2\|\cos(\theta) = 2\boldsymbol{\rho}_1 \cdot \boldsymbol{\rho}_2$ or

$$\boldsymbol{\rho}_1 \cdot \boldsymbol{\rho}_2 \;=\; \|\boldsymbol{\rho}_1\|\,\|\boldsymbol{\rho}_2\|\cos(\theta). \tag{11}$$



In other words, the dot product doesn't just tell us whether $\boldsymbol{\rho}_1$ and $\boldsymbol{\rho}_2$ are perpendicular: it also tells us in general how perpendicular (or unperpendicular) they are. We can rewrite (11) as

$$\cos(\theta) \;=\; \frac{\boldsymbol{\rho}_1 \cdot \boldsymbol{\rho}_2}{\|\boldsymbol{\rho}_1\|\,\|\boldsymbol{\rho}_2\|} \;=\; \left(\frac{\boldsymbol{\rho}_1}{\|\boldsymbol{\rho}_1\|}\right) \cdot \left(\frac{\boldsymbol{\rho}_2}{\|\boldsymbol{\rho}_2\|}\right) \;=\; \hat{\boldsymbol{\rho}}_1 \cdot \hat{\boldsymbol{\rho}}_2. \tag{12}$$

Thus the cosine of the angle between two vectors equals the dot product of the corresponding unit vectors.

The dot product is a versatile mathematical tool. It has lots of nice properties—things like $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$ or $\mathbf{a} \cdot \lambda\, \mathbf{b} = \lambda\, \mathbf{a} \cdot \mathbf{b} = \lambda\mathbf{a} \cdot \mathbf{b}$ where λ is any "scalar" (= number)—and can be used, among other things, to resolve any vector into components along and perpendicular to any given direction, to find the equation of a plane through a given point perpendicular to a given direction, to find the distance between two skew lines, and to obtain neat proofs of theorems in Euclidean geometry. But all of that will emerge through solving problems.

The cross product

The dot product is even versatile enough to help us reach the cross product. Suppose we want to find a vector, say $\mathbf{n} = n_1\,\mathbf{i} + n_2\,\mathbf{j} + n_3\,\mathbf{k}$, that is perpendicular to both $\boldsymbol{\rho}_1$ and $\boldsymbol{\rho}_2$. Then, obviously,

$$\mathbf{n} \cdot \boldsymbol{\rho}_1 \;=\; \mathbf{n} \cdot \boldsymbol{\rho}_2 \;=\; 0. \tag{13}$$

On substitution from (2)-(3), (13) becomes a pair of simultaneous homogeneous equations, $n_1x_1 + n_2y_1 + n_3z_1 = 0 = n_1x_2 + n_2y_3 + n_3z_3$, which yield $n_1 = (y_1z_2 - y_2z_1)n_3/(x_1y_2 - x_2y_1)$ and $n_2 = (x_2z_1 - x_1z_2)n_3/(x_1y_2 - x_2y_1)$ with n_3 arbitrary.[2] On setting $n_3 = x_1y_2 - x_2y_1$

[2]Here we assume $n_3 \neq 0$; but if it's zero we can just relabel and solve in terms of a component that isn't zero, so ultimately no generality is lost.

($\neq 0$), we thus find a vector that is perpendicular to both $\boldsymbol{\rho}_1$ and $\boldsymbol{\rho}_2$, namely,

$$\mathbf{n} \;=\; n_1\,\mathbf{i} + n_2\,\mathbf{j} + n_3\,\mathbf{k} \;=\; (y_1\,z_2 - y_2\,z_1)\,\mathbf{i} + (x_2\,z_1 - x_1\,z_2)\;\mathbf{j} + (x_1\,y_2 - x_2\,y_1)\,\mathbf{k}. \qquad (14)$$

It is easy to verify that (14) satisfies (13). The square of the magnitude of this vector is

$$\left\|\mathbf{n}\right\|^2 \;=\; (y_1\,z_2 - y_2\,z_1)^2 + (x_2\,z_1 - x_1\,z_2)^2 + (x_1\,y_2 - y_1\,x_2)^2. \qquad (15)$$

But from (8) and (11) above we have

$$(x_1^2+y_1^2+z_1^2)(x_2^2+y_2^2+z_2^2)\cos^2(\theta) \;=\; \left\|\boldsymbol{\rho}_1\right\|^2\left\|\boldsymbol{\rho}_2\right\|^2\cos^2(\theta) \;=\; (\boldsymbol{\rho}_1\cdot\boldsymbol{\rho}_2)^2 \;=\; (x_1x_2+y_1y_2+z_1z_2)^2.$$

Thus, after some brute force calculation, we find that (15) yields

$$\begin{aligned}
\left\|\mathbf{n}\right\|^2 &= (y_1\,z_2 - y_2\,z_1)^2 + (x_2\,z_1 - x_1\,z_2)^2 + (x_1\,y_2 - y_1\,x_2)^2\\
&= (x_1^2 + y_1^2 + z_1^2)(x_2^2 + y_2^2 + z_2^2) - (x_1x_2 + y_1y_2 + z_1z_2)^2\\
&= (x_1^2 + y_1^2 + z_1^2)(x_2^2 + y_2^2 + z_2^2)(1 - \cos^2(\theta))\\
&= (x_1^2 + y_1^2 + z_1^2)(x_2^2 + y_2^2 + z_2^2)\sin^2(\theta)\\
&= \left\|\boldsymbol{\rho}_1\right\|^2\left\|\boldsymbol{\rho}_2\right\|^2\sin^2(\theta)
\end{aligned}$$

and hence, on taking square roots, that $\mathbf{n}$ has magnitude

$$\left\|\mathbf{n}\right\| \;=\; \left\|\boldsymbol{\rho}_1\right\|\left\|\boldsymbol{\rho}_2\right\|\sin(\theta).$$

It follows that $\mathbf{n}$ is a vector with magnitude $\left\|\boldsymbol{\rho}_1\right\|\left\|\boldsymbol{\rho}_2\right\|\sin(\theta)$ that is perpendicular to both $\boldsymbol{\rho}_1$ and $\boldsymbol{\rho}_2$. It is called the cross (or vector product) of $\boldsymbol{\rho}_1$ and $\boldsymbol{\rho}_2$, and denoted by $\boldsymbol{\rho}_1 \times \boldsymbol{\rho}_2$. That is,

$$\begin{aligned}
\boldsymbol{\rho}_1 \times \boldsymbol{\rho}_2 &= (y_1\,z_2 - y_2\,z_1)\,\mathbf{i} + (x_2\,z_1 - x_1\,z_2)\;\mathbf{j} + (x_1\,y_2 - y_1\,x_2)\,\mathbf{k}\\
&= \left\|\boldsymbol{\rho}_1\right\|\left\|\boldsymbol{\rho}_2\right\|\sin(\theta)\hat{\mathbf{n}},
\end{aligned} \qquad (16)$$

$\hat{\mathbf{n}}$ being the unit vector in the direction $\mathbf{n} = (y_1\,z_2 - y_2\,z_1)\,\mathbf{i} + (x_2\,z_1 - x_1\,z_2)\;\mathbf{j} + (x_1\,y_2 - y_1\,x_2)\,\mathbf{k}$, defined by (14) above. Note that $\hat{\mathbf{n}}$ always points away from the plane containing $\boldsymbol{\rho}_1$ and $\boldsymbol{\rho}_2$ in the direction determined by the right-hand rule: if $\boldsymbol{\rho}_1$ and $\boldsymbol{\rho}_2$ are spokes on your car's steering wheel, then $\hat{\mathbf{n}}$ is into the dashboard if rotating $\boldsymbol{\rho}_1$ to $\boldsymbol{\rho}_2$'s position means turning the wheel right, and out of the dash-board if it means turning the wheel left. To verify this convention, all you need to is set $x_1 = 1, y_1 = z_1 = 0$ and $x_2 = z_2 = 0, y_2 = 1$ in (16), which then reduces to $\mathbf{i} \times \mathbf{j} = \mathbf{k}$.

The cross product is also a versatile mathematical tool. It can be used, among other things, to find areas of parallelograms or triangles and various equations or intersections of lines and planes—as solving problems will illustrate. The cross product can be used in particular to find the plane through three given points without resorting to solving simultaneous equations, as in the example to follow. Before proceeding, however, two final remarks are in order. First, note that (16) can be written more compactly as

$$\boldsymbol{\rho}_1 \times \boldsymbol{\rho}_2 \;=\; \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k}\\ x_1 & y_1 & z_1\\ x_2 & y_2 & z_2 \end{vmatrix} \;=\; \left\|\boldsymbol{\rho}_1\right\|\left\|\boldsymbol{\rho}_2\right\|\sin(\theta)\hat{\mathbf{n}}. \qquad (17)$$

Second, note that vector products are distributive, i.e.,

$$\mathbf{a} \times (\mathbf{b} + \mathbf{c}) \;=\; \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}, \tag{18}$$

but not commutative, i.e., $\mathbf{a} \times \mathbf{b} \neq \mathbf{b} \times \mathbf{a}$ (on the contrary, it follows immediately from (16) that $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$). The determinant in (17) often yields the easiest method for calculating $\boldsymbol{\rho}_1 \times \boldsymbol{\rho}_2$; however, one can also use the distributive property, as the following example illustrates.

Problem

Find an equation for the plane through the points A with coordinates $(2, 1, -3)$, B with coordinates $(5, -1, 4)$ and C with coordinates $(3, -2, 3)$.

Solution

We have

$$\begin{aligned}
\vec{AB} &= \mathbf{b} - \mathbf{a} = 5\mathbf{i} - \mathbf{j} + 4\mathbf{k} - (2\mathbf{i} + \mathbf{j} - 3\mathbf{k}) = 3\mathbf{i} - 2\mathbf{j} + 7\mathbf{k} \\
\vec{AC} &= \mathbf{c} - \mathbf{a} = 3\mathbf{i} - 2\mathbf{j} + 3\mathbf{k} - (2\mathbf{i} + \mathbf{j} - 3\mathbf{k}) = \mathbf{i} - 3\mathbf{j} + 6\mathbf{k}
\end{aligned}$$

from which

$$\begin{aligned}
\vec{AB} \times \vec{AC} &= (3\mathbf{i} - 2\mathbf{j} + 7\mathbf{k}) \times (\mathbf{i} - 3\mathbf{j} + 6\mathbf{k}) \\
&= 3\mathbf{i} \times (\mathbf{i} - 3\mathbf{j} + 6\mathbf{k}) \\
&\qquad - 2\mathbf{j} \times (\mathbf{i} - 3\mathbf{j} + 6\mathbf{k}) \\
&\qquad\qquad + 7\mathbf{k} \times (\mathbf{i} - 3\mathbf{j} + 6\mathbf{k}) \\
&= 3\mathbf{i} \times \mathbf{i} - 9\mathbf{i} \times \mathbf{j} + 18\mathbf{i} \times \mathbf{k} \\
&\qquad - 2\mathbf{j} \times \mathbf{i} + 6\mathbf{j} \times \mathbf{j} - 12\mathbf{j} \times \mathbf{k} \\
&\qquad\qquad + 7\mathbf{k} \times \mathbf{i} - 21\mathbf{k} \times \mathbf{j} + 42\mathbf{k} \times \mathbf{k} \\
&= \mathbf{0} - 9\mathbf{k} - 18\mathbf{k} \times \mathbf{i} \\
&\qquad - 2(-\mathbf{i} \times \mathbf{j}) + \mathbf{0} - 12\mathbf{i} \\
&\qquad\qquad + 7\mathbf{j} - 21(-\mathbf{j} \times \mathbf{k}) + \mathbf{0} \\
&= -9\mathbf{k} - 18\mathbf{j} \\
&\qquad - 2(-\mathbf{k}) - 12\mathbf{i} \\
&\qquad\qquad + 7\mathbf{j} - 21(-\mathbf{i}) \\
&= -9\mathbf{k} - 18\mathbf{j} + 2\mathbf{k} - 12\mathbf{i} + 7\mathbf{j} + 21\mathbf{i} \\
&= 9\mathbf{i} - 11\mathbf{j} - 7\mathbf{k}
\end{aligned}$$

because $\mathbf{a} \times \mathbf{a} = \mathbf{0}$ for any $\mathbf{a}$. So $\mathbf{n} = 9\mathbf{i} - 11\mathbf{j} - 7\mathbf{k}$ is normal to the plane. But if $\boldsymbol{\rho} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ is the position vector of an arbitrary point in the plane, then $\boldsymbol{\rho} - \mathbf{a}$ must be parallel to the plane (we visualize it as lying in the plane) and so

$$\mathbf{n} \cdot (\boldsymbol{\rho} - \mathbf{a}) = 0$$

implying $\mathbf{n} \cdot \boldsymbol{\rho} - \mathbf{n} \cdot \mathbf{a} = \mathbf{0}$ or

$$\mathbf{n} \cdot \boldsymbol{\rho} = \mathbf{n} \cdot \mathbf{a} = (9\mathbf{i} - 11\mathbf{j} - 7\mathbf{k}) \cdot (2\mathbf{i} + \mathbf{j} - 3\mathbf{k}) = 18 - 11 + 21 = 28.$$

In other words, an equation of the plane through A, B and C is

$$9x - 11y - 7z = 28.$$