The chain rule and the normal to a surface

Imagine that you are a microscopic creature that leaves a trail of slimy black ooze wherever it goes. You live your life in a very strange world. You are always on the surface $z = f(x, y)$, which is made of transparent material, and there are two ever-shining suns—one directly above, one directly below. So, regardless of whether you are above or below sea level, your trail of slimy ooze makes a shadow on the x-y plane. Furthermore, because your shadow path is directly below (or above = negatively below) your actual path, there is a one-to-one relationship between them: knowing where you are on your shadow path means you know where you are on your actual path, and vice a versa. More precisely, if your shadow coordinates are $(x, y, 0)$, or just (x, y) for short, then your actual coordinates are (x, y, z) where $z = f(x, y)$.

Now suppose that your coordinates are changing with time t according to some known relationship. That is, $x = x(t)$ and $y = y(t)$, implying $z = z(t) = f(x(t), y(t))$. Then, because your shadow path in the x-y plane determines your actual path on the surface, it must also be true that your horizontal motion determines your vertical motion. But how?

Because your eastward displacement is given by x, your eastward (= westward if negative) speed is given by $\frac{dx}{dt}$. Likewise, because your northward displacement is given by y, your northward (= southward if negative) speed is given by $\frac{dy}{dt}$; and because your upward displacement is given by z, your upward (= downward if negative) speed is given by $\frac{dz}{dt}$. Or, in terms of vectors, your displacement from the origin is

$$\boldsymbol{\rho} \;=\; x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k} \;=\; \mathbf{r} + z\,\mathbf{k} \tag{1}$$

(as in Lecture 3), your velocity is

$$\frac{d\boldsymbol{\rho}}{dt} \;=\; \frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j} + \frac{dz}{dt}\mathbf{k} \;=\; \frac{d\mathbf{r}}{dt} + \frac{dz}{dt}\mathbf{k} \tag{2}$$

and your shadow velocity (the velocity of your shadow in the x-y plane) is

$$\mathbf{v} \;=\; \frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j} \;=\; \frac{d\mathbf{r}}{dt}. \tag{3}$$

Now suppose that your shadow is travelling in the direction of $\hat{\mathbf{s}} = \cos(\theta)\mathbf{i} + \sin(\theta)\mathbf{j}$, as in Lecture 7 (so that $\hat{\mathbf{s}} = \hat{\mathbf{v}}$). Then your horizontal distance travelled is denoted by s. Clearly, both s and θ depend on time (because x and y depend on time); i.e., $\theta = \theta(t)$, $s = s(t)$. Because horizontal speed equals magnitude of horizontal velocity, and because any vector equals its magnitude times the unit vector in its direction, we must have

$$\mathbf{v} \;=\; v\,\hat{\mathbf{v}} \;=\; v\,\hat{\mathbf{s}} \;=\; v\{\cos(\theta)\mathbf{i} + \sin(\theta)\mathbf{j}\} \;=\; v\cos(\theta)\mathbf{i} + v\sin(\theta)\mathbf{j}. \tag{4}$$

From (3) and (4) we have

$$\frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j} \;=\; v\cos(\theta)\mathbf{i} + v\sin(\theta)\mathbf{j} \tag{5}$$

(because both are equal to $\mathbf{v}$). Equating components:

$$\frac{dx}{dt} \;=\; v\cos(\theta), \quad \frac{dy}{dt} \;=\; v\sin(\theta). \tag{6}$$

Furthermore, because derivative of horizontal distance travelled equals horizontal speed equals magnitude of horizontal velocity, it follows at once that

$$\frac{ds}{dt} \;=\; v. \tag{7}$$

Let's not lose track of where we are going with all this. We want to know how $\mathbf{v}$ determines $\frac{dz}{dt}$ (see the end of the second paragraph); more briefly, we want to know $\frac{dz}{dt}$. But there's a one-to-one relationship between distance travelled s and time t.[1] So, by the *ordinary* chain rule of Calculus I, if we know $\frac{\partial z}{\partial s} = f_s$ then we can determine $\frac{dz}{dt}$ from

$$\frac{dz}{dt} \;=\; \frac{\partial z}{\partial s}\frac{ds}{dt} \;=\; f_s\,\frac{ds}{dt}. \tag{8}$$

Note that we use $\frac{ds}{dt}$ in preference to $\frac{\partial s}{\partial t}$ here, because t is the only thing that s depends on. But from Lecture 7 we have $f_s \;=\; \cos(\theta)\,f_x + \sin(\theta)\,f_y$ for the directional derivative. Substituting into (8), we obtain

$$\begin{aligned}
\frac{dz}{dt} \;&=\; \{\cos(\theta)\,f_x + \sin(\theta)\,f_y\}\,\frac{ds}{dt} &(9)\\
&=\; f_x\,\frac{ds}{dt}\,\cos(\theta) + f_y\,\frac{ds}{dt}\,\sin(\theta) &(10)\\
&=\; f_x\,v\,\cos(\theta) + f_y\,v\,\sin(\theta) &(11)\\
&=\; f_x\,\frac{dx}{dt} + f_y\,\frac{dy}{dt} &(12)
\end{aligned}$$

on using (6)-(7). Note that we can rewrite this equation as

$$\frac{dz}{dt} \;=\; f_x\,\frac{dx}{dt} + f_y\,\frac{dy}{dt} \;=\; \nabla f \cdot \frac{d\mathbf{r}}{dt} \tag{13}$$

This equation reveals explicitly how horizontal velocity determines vertical velocity (= speed, since there is only one component to vertical velocity). But it's far more general than that. Because we can always pretend that a pair of ordinary functions (here $x = x(t)$ and $y = y(t)$) represents a shadow path, and because any bivariate function has a surface graph, it tells us quite generally how z changes with respect to t if $z = f(x, y)$ and x and y are both functions of t. For that reason it is given a fancy name: it is called the chain rule for functions of two variables. Note that in these more general circumstances there is absolutely no reason why t should stand for time.

Then we notice something else. It wouldn't matter one little bit if x and y both really depended on another variable, say u, that we were holding constant for the time being. So it follows at once that $x = x(t, u)$, $y = y(t, u)$ and $z = f(x, y)$ together imply

$$\frac{\partial z}{\partial t} \;=\; f_x\,\frac{\partial x}{\partial t} + f_y\,\frac{\partial y}{\partial t}. \tag{14a}$$

[1] For example, if the shadow path is defined by $x = \sin(\pi t), y = \cos(\pi t) - 1, 0 \le t \le \frac{1}{2}$, so that it forms part of the circle $x^2 + (y-1)^2 = 1$, then $s = \pi t$, or $t = \pi/s$; you should check that these equations are consistent with (6)-(7).

A moment's thought then suffices to reveal in addition that

$$\frac{\partial z}{\partial u} \;=\; f_x \frac{\partial x}{\partial u} + f_y \frac{\partial y}{\partial u} \tag{14b}$$

(because, in this more general perspective, we are free to pretend that either of the independent variables corresponds to time, and the labels t and u are arbitrary). Equations (14) are a more general version of the chain rule for functions of two variables.

Finally, the chain rule generalizes in the obvious way to a function of three variables, enabling us to show that the gradient of such a function at any point is perpendicular to its level surface through the point. For suppose that the function is $\phi = \phi(x, y, z)$, so that in place of (13) we obtain

$$\frac{d\phi}{dt} \;=\; \phi_x \frac{dx}{dt} + \phi_y \frac{dy}{dt} + \phi_z \frac{dz}{dt} \;=\; \nabla\phi \cdot \frac{d\boldsymbol{\rho}}{dt}, \tag{15}$$

by (2), consider the level surface $\phi(x, y, z) = c$ through the fixed point with position vector $\boldsymbol{\rho}_0 = x_0\mathbf{i} + y_0\mathbf{j} + z_0\mathbf{k}$ (so that, of necessity, $\phi(x_0, y_0, z_0) = c$) and let $\boldsymbol{\rho}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$ be the position vector at time t of a microscopic creature who lives in the surface, implying

$$\phi(x(t), y(t), z(t)) \;=\; c \tag{16}$$

at all possible times. Using the chain rule to differentiate with respect to t:

$$\frac{\partial\phi}{\partial x}\frac{dx}{dt} + \frac{\partial\phi}{\partial y}\frac{dy}{dt} + \frac{\partial\phi}{\partial z}\frac{dz}{dt} \;=\; \frac{dc}{dt} \;=\; 0, \tag{17}$$

implying

$$\left(\frac{\partial\phi}{\partial x}\mathbf{i} + \frac{\partial\phi}{\partial y}\mathbf{j} + \frac{\partial\phi}{\partial z}\mathbf{k}\right) \cdot \left(\frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j} + \frac{dz}{dt}\mathbf{k}\right) \;=\; 0 \tag{18}$$

or

$$\nabla\phi \cdot \frac{d\boldsymbol{\rho}}{dt} \;=\; 0. \tag{19}$$

So the creature's velocity is always perpendicular to the gradient vector, no matter where the creature is at. But the creature lives in the surface. Therefore it always moves parallel to the surface, i.e., parallel to the tangent plane at wherever it happens to be. It follows at once that $\mathbf{n} = \nabla\phi$ must be normal to the tangent plane and hence normal to the surface (because normal to the tangent plane is what normal to the surface means). Note in particular that the surface $z = f(x, y)$, whose equation can be rewritten as $\phi(x, y, z) = c$ with $\phi(x, y, z) = z - f(x, y)$ and $c = 0$, has normal

$$\mathbf{n} \;=\; \nabla\phi \;=\; -f_x\,\mathbf{i} - f_y\,\mathbf{j} + \mathbf{k}, \tag{20}$$

in perfect agreement with Equation (5) of Lecture 7.