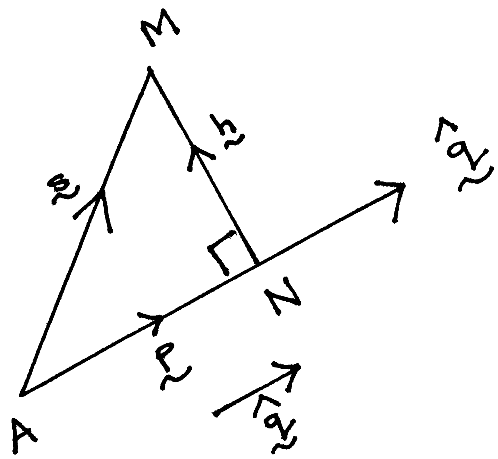
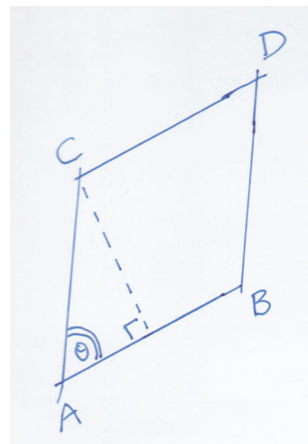
1. Referring to the diagram below, we have $\mathbf{s} = \mathbf{p} + \mathbf{h}$ where $\mathbf{p} = (\mathbf{s} \cdot \hat{\mathbf{q}})\hat{\mathbf{q}}$ and $\mathbf{h} = \mathbf{s} - \mathbf{p}$. Because $\hat{\mathbf{q}} = \{-2\mathbf{i} - \mathbf{j} + 2\mathbf{k}\}/\sqrt{2^2 + 1^2 + (-2)^2} = \{-2\mathbf{i} - \mathbf{j} + 2\mathbf{k}\}/3$, we obtain $\mathbf{s} \cdot \hat{\mathbf{q}} = \{(-3) \times (-2) + (-2) \times (-1) + 2 \times 2\}/3 = 12/3 = 4 \Longrightarrow \mathbf{p} = 4\hat{\mathbf{q}} = 4\{-\frac{2}{3}\mathbf{i} - \frac{1}{3}\mathbf{j} + \frac{2}{3}\mathbf{k}\} = -\frac{8}{3}\mathbf{i} - \frac{4}{3}\mathbf{j} + \frac{8}{3}\mathbf{k} = -\frac{4}{3}\{2\mathbf{i} + \mathbf{j} - 2\mathbf{k}\}$ and $\mathbf{h} = \mathbf{s} - \mathbf{p} = -3\mathbf{i} - 2\mathbf{j} + 2\mathbf{k} + \frac{8}{3}\mathbf{i} + \frac{4}{3}\mathbf{j} - \frac{8}{3}\mathbf{k} = -\frac{1}{3}\mathbf{i} - \frac{2}{3}\mathbf{j} - \frac{2}{3}\mathbf{k} = -\frac{1}{3}\{\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}\}$. So $\mathbf{h} \cdot \mathbf{p} = \left(-\frac{1}{3}\{\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}\}\right) \cdot \left(-\frac{4}{3}\{2\mathbf{i} + \mathbf{j} - 2\mathbf{k}\}\right) = (-\frac{1}{3})(-\frac{4}{3})\{\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}\} \cdot \{2\mathbf{i} + \mathbf{j} - 2\mathbf{k}\}) = \frac{4}{9}\{1 \cdot 2 + 2 \cdot 1 + (2)(-2)\} = 0$—of necessity!





2. **(a)** $\overrightarrow{AB} = \mathbf{b} - \mathbf{a} = 2\mathbf{i} + 3\mathbf{j} - \mathbf{k} - \{\mathbf{i} + \mathbf{j} + 3\mathbf{k}\} = (2-1)\mathbf{i} + (3-1)\mathbf{j} + (-1-3)\mathbf{k} = \mathbf{i} + 2\mathbf{j} - 4\mathbf{k}.$

(b) $\overrightarrow{AC} = \mathbf{c} - \mathbf{a} = 3\mathbf{i} + 5\mathbf{j} + 4\mathbf{k} - \{\mathbf{i} + \mathbf{j} + 3\mathbf{k}\} = (3-1)\mathbf{i} + (5-1)\mathbf{j} + (4-3)\mathbf{k} = 2\mathbf{i} + 4\mathbf{j} + \mathbf{k}.$

(c) If θ is the angle between AB and AC and AB is the base of the parallelogram, then its height is $AC\sin(\theta)$, and so its area is

$$AB\,AC\sin(\theta) \;=\; |\overrightarrow{AB} \times \overrightarrow{AC}|.$$

But

$$\overrightarrow{AB} \times \overrightarrow{AC} \;=\; \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & -4 \\ 2 & 4 & 1 \end{vmatrix}$$

$$= \{2 \cdot 1 - (4)(-4)\}\mathbf{i} - \{1 \cdot 1 - 2 \cdot (-4)\}\mathbf{j} + \{1 \cdot 4 - 2 \cdot 2\}\mathbf{k} \;=\; 9\{2\mathbf{i} - \mathbf{j} + 0\mathbf{k}\}.$$

So the area of the parallelogram is $|\overrightarrow{AB} \times \overrightarrow{AC}| = 9\sqrt{2^2 + (-1)^2 + 0^2} = 9\sqrt{5}$.

(d)

$$\cos(\theta) \;=\; \widehat{\overrightarrow{AB}} \cdot \widehat{\overrightarrow{AC}} \;=\; \frac{\overrightarrow{AB}}{AB} \cdot \frac{\overrightarrow{AC}}{AC}\frac{\{\mathbf{i} + 2\mathbf{j} - 4\mathbf{k}\} \cdot \{2\mathbf{i} + 4\mathbf{j} + \mathbf{k}\}}{\sqrt{1^2 + 2^2 + (-4)^2}\sqrt{2^2 + 4^2 + 1^2}}$$

$$= \frac{1 \cdot 2 + 2 \cdot 4 + (-4) \cdot 1}{\sqrt{21}\sqrt{21}} \;=\; \frac{2}{7} \;\Longrightarrow\; \theta \;=\; \arccos\left(\tfrac{2}{7}\right) \approx 1.28 \text{ radians} \approx 73.4°.$$

Alternatively, since we already know that $AB\,AC\sin(\theta) = 9\sqrt{5}$ and that $AB = AC = \sqrt{21}$, we have $\sin(\theta) = 3\sqrt{5}/7 \Longrightarrow \theta = \arcsin(3\sqrt{5}/7)$, which of course is the same angle.

(e) D has position vector $\mathbf{d} = \overrightarrow{OD} = \overrightarrow{OA} + \overrightarrow{AB} + \overrightarrow{BD} = \overrightarrow{OA} + \overrightarrow{AB} + \overrightarrow{AC} = \mathbf{a} + \mathbf{b} - \mathbf{a} + \mathbf{c} - \mathbf{a} = \mathbf{b} + \mathbf{c} - \mathbf{a} = 4\mathbf{i} + 7\mathbf{j} + 0\mathbf{k}.$

(f) From **(c)** above, we already know that the vector $\mathbf{n} = \frac{1}{9}\overrightarrow{AB} \times \overrightarrow{AC} = 2\mathbf{i} - \mathbf{j} + 0\mathbf{k}$ is perpendicular to Π. We also know that the point A with position vector $\mathbf{a} = \mathbf{i} + \mathbf{j} + 3\mathbf{k}$ lies in the plane. So Π has equation $\mathbf{n} \cdot (\boldsymbol{\rho} - \mathbf{a}) = 0$, implying $2(x-1) - 1 \cdot (y-1) + 0 \cdot (z-3) = 0$ or $2x - y = 1$ (because $\boldsymbol{\rho} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$).

Needless to say, $\mathbf{n} \cdot (\boldsymbol{\rho} - \mathbf{b}) = 0$, $\mathbf{n} \cdot (\boldsymbol{\rho} - \mathbf{c}) = 0$ and $\mathbf{n} \cdot (\boldsymbol{\rho} - \mathbf{d}) = 0$ all yield the same result.

(g) The line through C and D has the same direction as $\overrightarrow{AB} = \mathbf{i} + 2\mathbf{j} - 4\mathbf{k} = \mathbf{w}$, say. Hence the line through C and D has equation $\boldsymbol{\rho} = \mathbf{c} + t\mathbf{w}$, or

$$x = 3 + t, y = 5 + 2t, z = 4 - 4t$$

in parametric form. Equivalently, the line has equation $\boldsymbol{\rho} = \mathbf{d} + \tau\mathbf{w}$, or $x = 4 + \tau, y = 7 + 2\tau, z = -4\tau$ in parametric form. The equations become identical when we set $\tau = t - 1$.

3. The shortest distance between two skew lines lies in a direction perpendicular to both of them. Let M and N be the points of closest approach on L_1 and L_2, respectively. The equations of L_1 can be rewritten in vector form as $\boldsymbol{\rho} = \mathbf{a} + s\mathbf{u}$ where $\mathbf{a} = 9\mathbf{i} - \mathbf{j} + 4\mathbf{k}$ and $\mathbf{u} = 2\mathbf{i} - \mathbf{j} + 2\mathbf{k}$. The equations of L_2 can be rewritten in vector form as $\boldsymbol{\rho} = \mathbf{b} + t\mathbf{v}$ where $\mathbf{b} = 2\mathbf{i} - 3\mathbf{j} + \mathbf{k}$ and $\mathbf{v} = 2\mathbf{i} - 3\mathbf{j} - 2\mathbf{k}$. Because $\mathbf{u}$ is parallel to L_1 and $\mathbf{v}$ is parallel to L_2,

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -1 & 2 \\ 2 & -3 & -2 \end{vmatrix} = 4\{2\mathbf{i} + 2\mathbf{j} - \mathbf{k}\}$$

is perpendicular to both lines and parallel to $\overrightarrow{NM}$. So $\overrightarrow{NM}$ has the direction of $\mathbf{n} = 2\mathbf{i} + 2\mathbf{j} - \mathbf{k}$. Hence the unit vector in the direction of NM is

$$\hat{\mathbf{n}} = \frac{\mathbf{n}}{|\mathbf{n}|} = \frac{2\mathbf{i} + 2\mathbf{j} - \mathbf{k}}{\sqrt{2^2 + 2^2 + (-1)^2}} = \tfrac{1}{3}\{2\mathbf{i} + 2\mathbf{j} - \mathbf{k}\}$$

Setting $s = 0$, we find that the point A with position vector $\mathbf{a}$ is a point on L_1. Likewise setting $t = 0$, the point B with position vector $\mathbf{b}$ is a point on L_2. Let L_3 denote the line through M that is parallel to L_2, let H denote the foot of the perpendicular from B to L_3, and let θ denote the angle between $\overrightarrow{BH}$ and $\overrightarrow{BA} = \mathbf{a} - \mathbf{b} = 7\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$. Then the shortest distance between the lines is

$$\begin{aligned} d &= NM = BH = BA\cos(\theta) = BA.1.\cos(\theta) = \overrightarrow{BA} \cdot \hat{\mathbf{n}} \\ &= \tfrac{1}{3}\{7\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}\} \cdot \{2\mathbf{i} + 2\mathbf{j} - \mathbf{k}\} = \tfrac{1}{3}\{7 \cdot 2 + 2 \cdot 2 + 3 \cdot (-1)\} \\ &= \tfrac{1}{3}\{14 + 4 - 3\} = 5. \end{aligned}$$

That was all you were asked for. Nevertheless, I could have asked to you to determine the points of closest approach. You would then observe that H has position vector $\mathbf{h} = \mathbf{b} + d\hat{\mathbf{n}} = \frac{1}{3}\{16\mathbf{i} + \mathbf{j} - 2\mathbf{k}\}$, so that L_3 has vector equation $\boldsymbol{\rho} = \mathbf{h} + t\mathbf{v}$ or parametric equations $x = \frac{16}{3} + 2t, y = \frac{1}{3} - 3t, z = -\frac{2}{3} - 2t$ and thus meets L_1 where $s = -\frac{25}{12}$ and $t = -\frac{1}{4}$. So M has position vector $\mathbf{m} = \mathbf{h} - \frac{1}{4}\mathbf{v} = \frac{1}{12}\{58\mathbf{i} + 13\mathbf{j} - 2\mathbf{k}\}$, from which N has position vector $\mathbf{m} - d\hat{\mathbf{n}} = \frac{3}{4}\{2\mathbf{i} - 3\mathbf{j} + 2\mathbf{k}\}$.