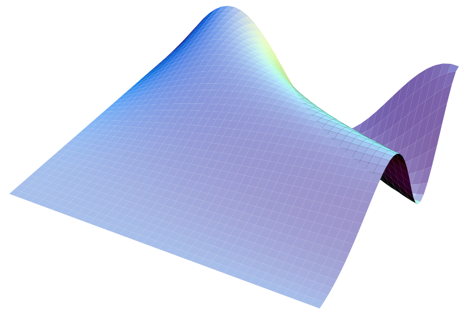
Surfaces as Graphs. Contour Maps



An important property of our surface from Lecture 1 (above) is that there are no overhangs: a vertical line can intersect the surface only once. The surface can therefore be regarded as the graph of a function of two variables, say f, implicitly defined by stipulating that $f(x, y) = z$ if (x, y, z) lies on the surface. The domain of this function is the square $[0, 1] \times [0, 1] = \{(x, z)|0 \leq x \leq 2, 0 \leq y \leq 2\}$, and its range is the interval $[-1, 1] = \{z| - 1 \leq z \leq 1\}$. Thus, for example, because $(1.57, 1, 1)$ lies on the surface, we have $f(1.571, 1) = 1$; and because $(2, 1.535, -1)$ lies on the surface, we have $f(2, 1.535) = -1$. We can easily produce a large table of values of z associated with particular (x, y) pairs; and this table can be used in place of the graph to define the function (at least in principle, and at least approximately). Here are some of the entries in that table, at 49 points that are uniformly distributed over the domain (with x increasing to the right from 0 to 2 and y increasing toward the top of the page from 0 to 2, both in increments of $\frac{1}{3}$):

0.0	0.972	0.457	-0.757	-0.813	0.374	0.989
0.0	0.799	0.961	0.356	-0.533	-0.997	-0.665
0.0	0.559	0.927	0.979	0.697	0.178	-0.402
0.0	0.327	0.618	0.841	0.972	0.995	0.909
0.0	0.148	0.292	0.43	0.559	0.675	0.776
0.0	0.037	0.074	0.111	0.148	0.184	0.22
0.0	0.0	0.0	0.0	0.0	0.0	0.0

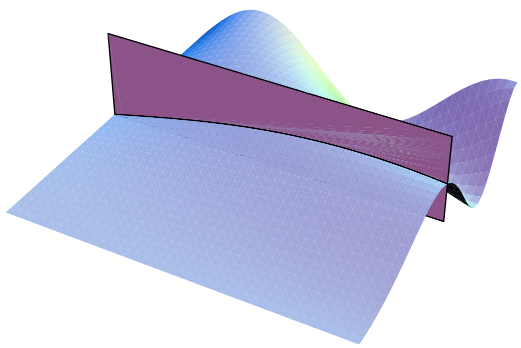
Thus, for example, $f(2/3, 4/3) = 0.927$, and $f(1, 5/3) = 0.356$. But the simplest way to define this particular function is in terms of a formula, namely, $f(x, y) = \sin(xy^2)$. In fact, a more accurate version of the table above is:

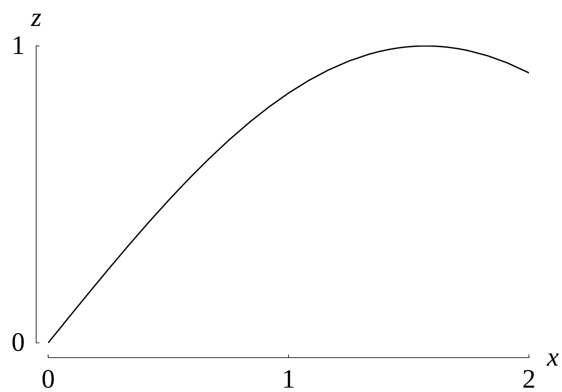
0	$\sin\left(\frac{4}{3}\right)$	$\sin\left(\frac{8}{3}\right)$	$\sin(4)$	$\sin\left(\frac{16}{3}\right)$	$\sin\left(\frac{20}{3}\right)$	$\sin(8)$
0	$\sin\left(\frac{25}{27}\right)$	$\sin\left(\frac{50}{27}\right)$	$\sin\left(\frac{25}{9}\right)$	$\sin\left(\frac{100}{27}\right)$	$\sin\left(\frac{125}{27}\right)$	$\sin\left(\frac{50}{9}\right)$
0	$\sin\left(\frac{16}{27}\right)$	$\sin\left(\frac{32}{27}\right)$	$\sin\left(\frac{16}{9}\right)$	$\sin\left(\frac{64}{27}\right)$	$\sin\left(\frac{80}{27}\right)$	$\sin\left(\frac{32}{9}\right)$
0	$\sin\left(\frac{1}{3}\right)$	$\sin\left(\frac{2}{3}\right)$	$\sin(1)$	$\sin\left(\frac{4}{3}\right)$	$\sin\left(\frac{5}{3}\right)$	$\sin(2)$
0	$\sin\left(\frac{4}{27}\right)$	$\sin\left(\frac{8}{27}\right)$	$\sin\left(\frac{4}{9}\right)$	$\sin\left(\frac{16}{27}\right)$	$\sin\left(\frac{20}{27}\right)$	$\sin\left(\frac{8}{9}\right)$
0	$\sin\left(\frac{1}{27}\right)$	$\sin\left(\frac{2}{27}\right)$	$\sin\left(\frac{1}{9}\right)$	$\sin\left(\frac{4}{27}\right)$	$\sin\left(\frac{5}{27}\right)$	$\sin\left(\frac{2}{9}\right)$
0	0	0	0	0	0	0

Although the graph of a function of two variables may be the most natural way to visualize it, for some purposes it is useful to plot curves in which its graph intersects vertical or horizontal planes. First we deal with vertical planes, then we deal with horizontal ones.

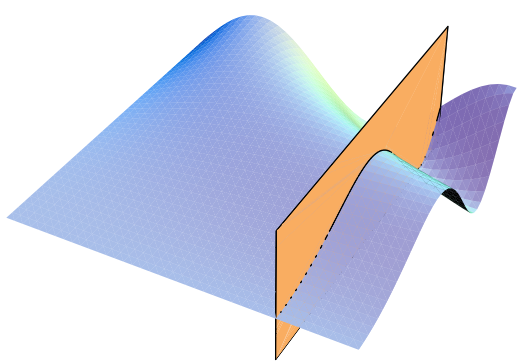
Vertical cross sections

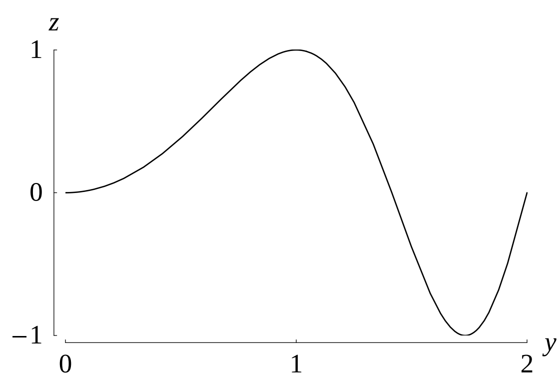
If the plane is vertical, then the curve of intersection with the surface yields the graph of an ordinary function of a single variable. For example, our surface $z = \sin(xy^2)$ intersects the vertical plane $y = 1$ in the ordinary graph $z = \sin(x)$:



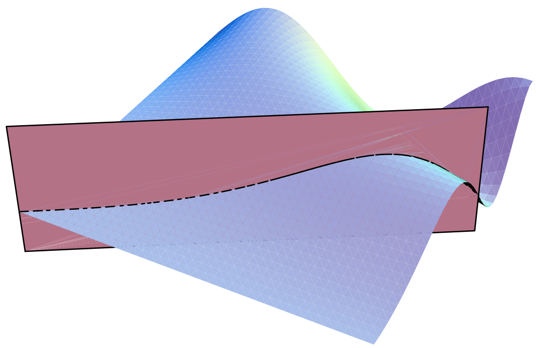


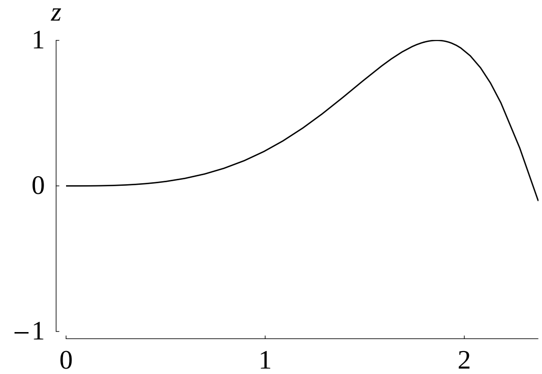
The same surface intersects the vertical plane $x = \frac{1}{2}\pi$ in the ordinary graph $z = \sin\left(\frac{1}{2}\pi y^2\right)$:





But we get the graph of an ordinary function even if the vertical plane of section isn't parallel to a coordinate plane. For example, the elephant path in Lecture 1 is part of the curve in which the surface $z = \sin(xy^2)$ intersects the vertical plane through $(0, 0, 0)$ and $\left(\frac{1}{2}\pi, 1, 1\right)$:

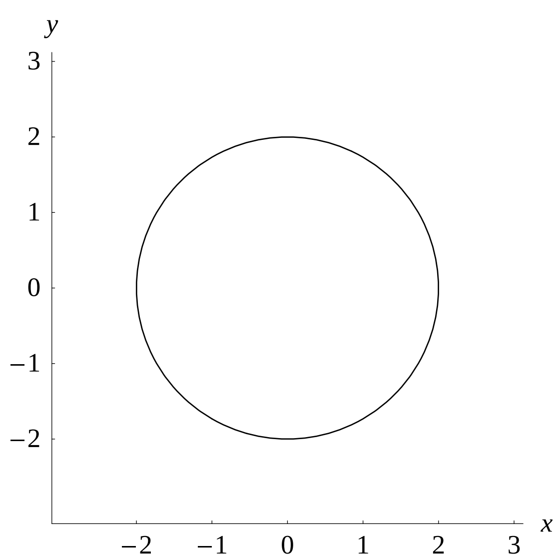


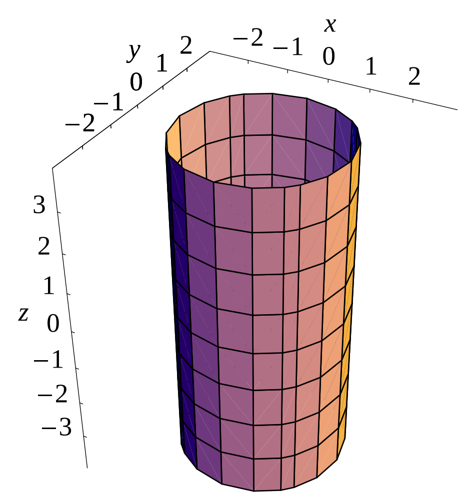


We can think of taking a vertical section as turning a surface into a curve. But it is also possible to turn a curve into a surface by either of two important methods, namely, translation and rotation. First we deal with translation.

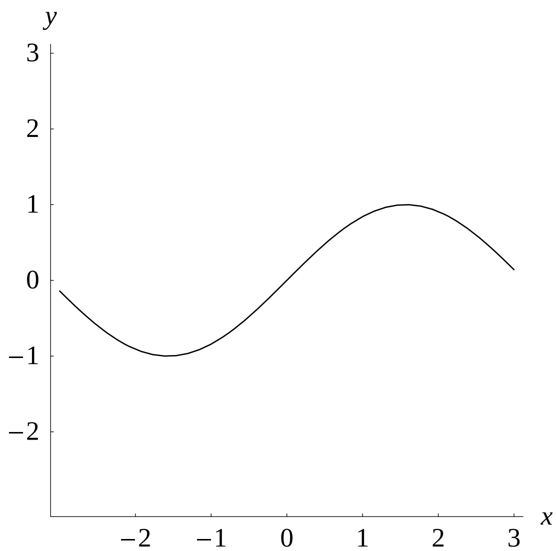
Translational symmetry

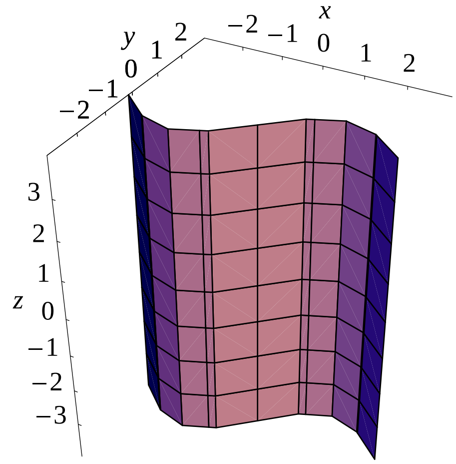
For the sake of argument (and without loss of generality), let's suppose that the curve lies in the x-y plane. Then to turn it into a surface, we can move it perpendicularly to the x-y plane toward infinity in both directions. What we get is a surface with translational symmetry: every horizontal cross section is the same. Moreover, the equation of the surface is identical to the equation of the curve: whether it's a one-dimensional curve in two-dimensional space or a two-dimensional surface in three-dimensional space depends only on how we think about it, and not on its equation. For example, here are the curve and surface with equation $x^2 + y^2 = 4$:





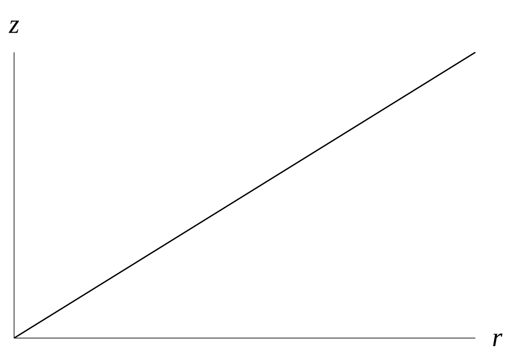
And here are the curve and surface with equation $y = \sin(x)$:

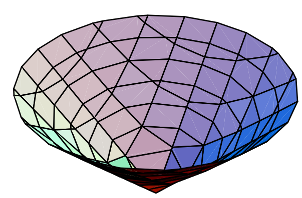




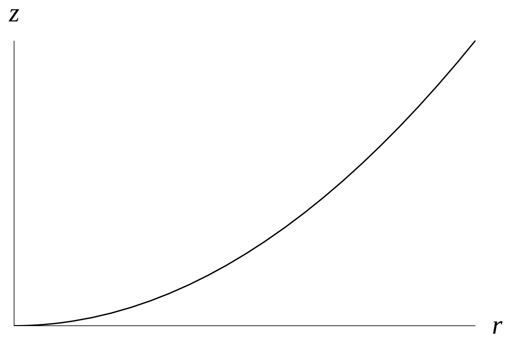
Rotational symmetry

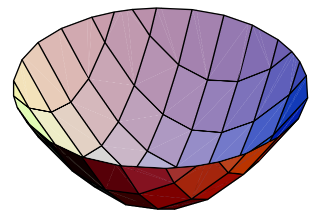
Now we deal with rotation. For the sake of argument (and again without loss of generality), let's suppose that the axis of rotation is the z-axis, and that $z = F(r)$ is the graph of an ordinary function with domain $[0, R] = \{r|0 \leq r \leq R\}$. Then to turn this curve into a surface, we can rotate it once around the z-axis to obtain a surface of revolution. This surface is the graph of a function of two variables, say f, whose domain is a disc of radius R. Moreover, because the height of the surface depends only on the (shortest) distance from the z-axis, we have $z = f(x, y) = F(r) = F(\sqrt{x^2 + y^2})$, by the result from the end of Lecture 1. Conversely, whenever f depends only on $\sqrt{x^2 + y^2}$, its graph is a surface of revolution. For example, $z = F(r) = r$ or $z = f(x, y) = \sqrt{x^2 + y^2}$ is the equation of a cone:



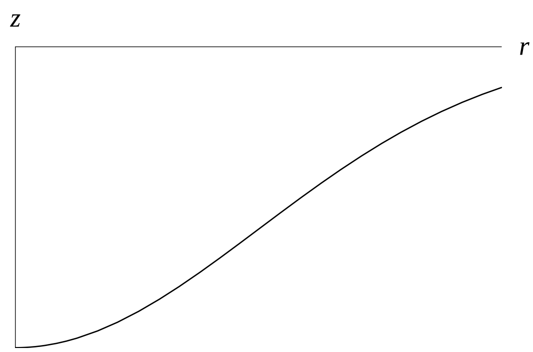


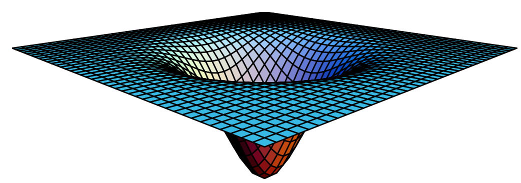
And $z = F(r) = r^2$ or $z = f(x, y) = x^2 + y^2$ is the equation of a paraboloid:



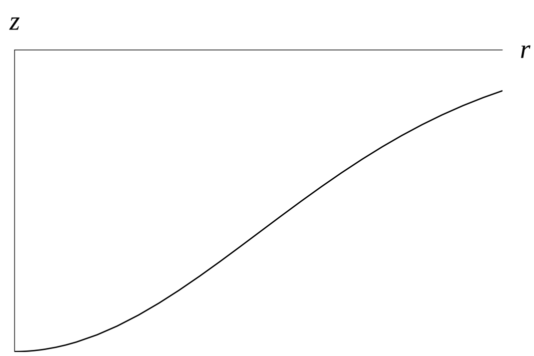


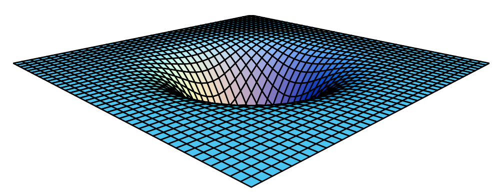
Similarly, for $z = F(r) = -e^{-r^2}$ or $z = f(x, y) = -e^{-x^2-y^2}$ we have:





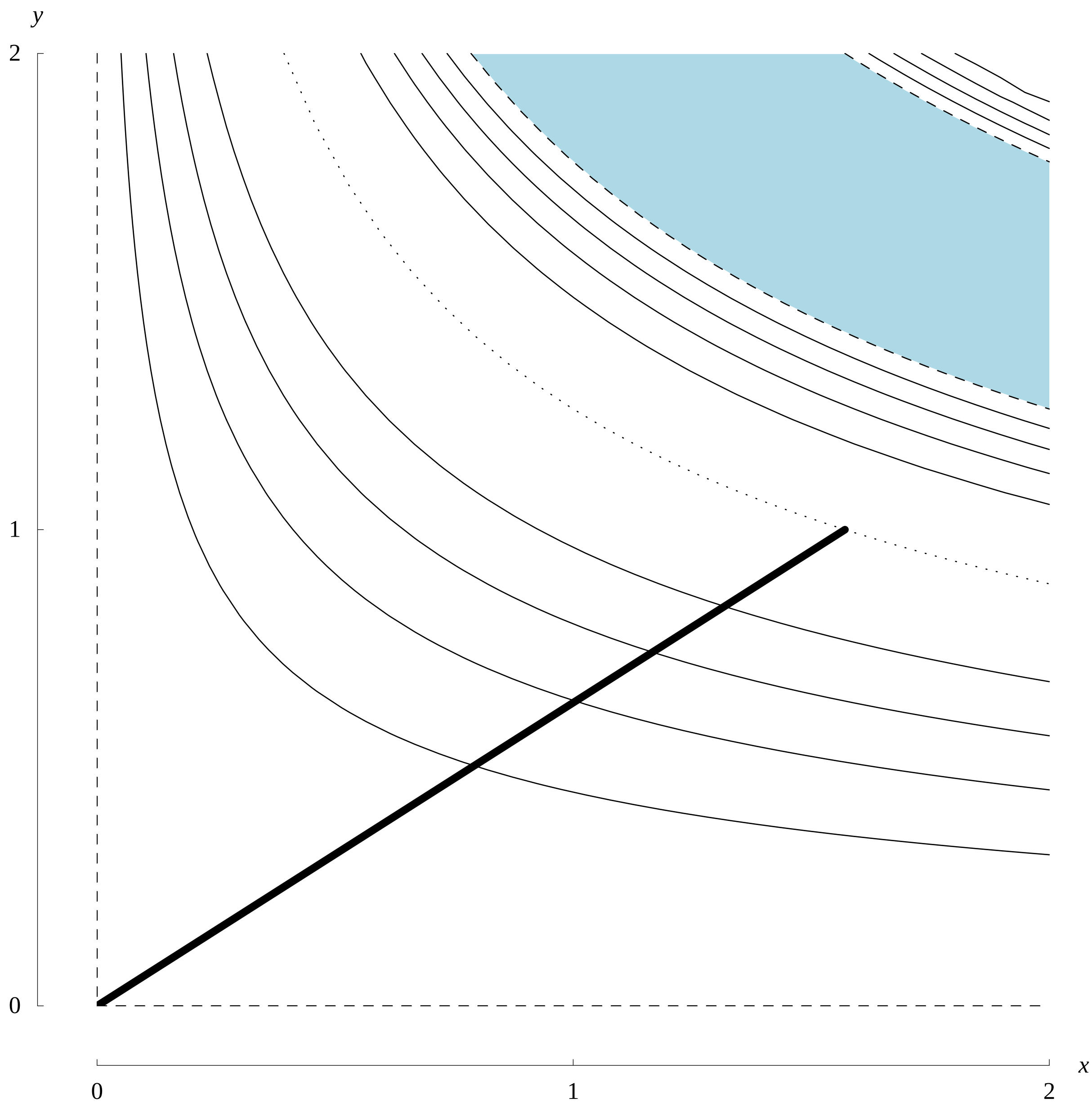
We can even tilt it up a bit for you:





Horizontal cross sections

When the planes are horizontal (parallel to sea level), the curves of intersection are known as level curves or contours, and the resulting plot is called a contour map. Contours join points of equal height above (or depth below) sea level. For example, here is a contour map for $f(x, y) = \sin(xy^2)$:



Joined in this diagram (except where they lie on separate arcs) are the (x, y) pairs for which $f(x, y)$ equals 0, 0.2, 0.4, 0.6, 0.8 and 1; the points where $f(x, y) = 0$ are shown dashed, and those where $f(x, y) = 1$ are shown dotted. Also shown is the path our elephant took to reach the summit (as seen from a helicopter flying overhead). Notice how the contours are close together where the surface is steeper and further apart where it rises or falls more gently. Also notice that there is a region in which no contours have been drawn; it corresponds to (x, y) pairs for which $f(x, y) < 0$. Suppose we filled this trough with water up to sea level. What would be the (flat) surface area of the water? (This is basically a Calculus II problem, and so it will serve as an excellent review exercise.)