Vectors

Displacement vectors come in two categories: position vectors and free vectors. We'll deal with each in turn.

Position vectors

We can represent going from $(0, 0, 0)$ to (x, y, z) either as a single mathematical object—a vector—or as a collection of three objects, the vector's components. Let's call the vector $\boldsymbol{\rho}$. Then, according to the first view, $\boldsymbol{\rho}$ is a matter of travelling a distance $\rho = \sqrt{x^2 + y^2 + z^2}$ in the direction of from $(0, 0, 0)$ to (x, y, z). We can express our ideas more compactly if we use the idea of a unit vector. A unit vector in a given direction is the vector of length 1 that points in precisely the same direction. We usually denote it by putting a hat over the original vector. Thus the unit vector in the direction of $\boldsymbol{\rho}$, that is, in the direction of from $(0, 0, 0)$ to (x, y, z), is denoted by $\hat{\boldsymbol{\rho}}$. There are special notations for the unit vectors in the directions of from $(0, 0, 0)$ to $(1, 0, 0)$, from $(0, 0, 0)$ to $(0, 1, 0)$ and from $(0, 0, 0)$ to $(0, 0, 1)$: we denote them by $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$, respectively. Notice that they don't have hats because the hats aren't necessary—not because the sun doesn't shine, but because the distance from $(0, 0, 0)$ to $(1, 0, 0)$ or from $(0, 0, 0)$ to $(0, 1, 0)$ or from $(0, 0, 0)$ to $(0, 0, 1)$ is 1 to begin with.

Armed with this notation, we can now be more precise about how we represent going from $(0, 0, 0)$ to (x, y, z) either as a single object or as a collection of three. According to the first perspective, which is essentially geometric—especially when we think of $\boldsymbol{\rho}$ as an arrow with its tail at $(0, 0, 0)$ and its head at (x, y, z)—we stretch or shrink the unit vector $\hat{\boldsymbol{\rho}}$ by factor

$$\rho \;=\; \sqrt{x^2 + y^2 + z^2} \tag{1}$$

until it takes us precisely to where we want to go, namely, $\boldsymbol{\rho}$. So

$$\boldsymbol{\rho} \;=\; \rho\hat{\boldsymbol{\rho}}. \tag{2}$$

According to the second perspective, which is essentially algebraic, we think of going from $(0, 0, 0)$ to (x, y, z) as knowing the components x, y and z. That is, we imagine travelling a distance x in the direction of going from $(0, 0, 0)$ to $(x, 0, 0)$, followed by a further distance y in the direction of going from $(x, 0, 0)$ to $(x, y, 0)$, followed by a further distance z in the direction of going from $(x, y, 0)$ to (x, y, z). But the direction of going from $(x, 0, 0)$ to $(x, y, 0)$ is exactly the same thing as the direction of going from $(0, 0, 0)$ to $(0, 1, 0)$, because these two directions are parallel; and if two directions are parallel, then they must be *exactly* the same, because there is nothing more to going in a direction than the way you are headed (east is east and west is west, regardless of whether you start in Paris or Hong Kong). Similarly, the direction of going from $(x, y, 0)$ to (x, y, z) is exactly the same thing as the direction of going from $(0, 0, 0)$ to $(0, 0, 1)$. Therefore, in other words, we have imagined travelling a distance x in the direction $\mathbf{i}$, followed by a further distance y in the direction $\mathbf{j}$, followed by a further distance z in the direction $\mathbf{k}$. But we have to end up in the same place as before. So

$$\boldsymbol{\rho} \;=\; x\mathbf{i} + y\mathbf{j} + z\mathbf{k}. \tag{3}$$

(If it helps, think of the right-hand side as going x units east followed by y units north followed by z units vertically upward, where $x < 0$ is actually west, $y < 0$ is actually south, and $z < 0$ is actually vertically downward.) Thus, from (2),

$$\rho\hat{\boldsymbol{\rho}} \;=\; x\mathbf{i} + y\mathbf{j} + z\mathbf{k}, \tag{4}$$

implying

$$\begin{aligned}\hat{\boldsymbol{\rho}} \;&=\; \frac{x}{\rho}\mathbf{i} + \frac{y}{\rho}\mathbf{j} + \frac{z}{\rho}\mathbf{k} \\ &=\; \frac{x}{\sqrt{x^2+y^2+z^2}}\,\mathbf{i} + \frac{y}{\sqrt{x^2+y^2+z^2}}\,\mathbf{j} + \frac{z}{\sqrt{x^2+y^2+z^2}}\,\mathbf{k}.\end{aligned} \tag{5}$$

What does this mean? It means that if we know the components of our position vector (namely, x, y and z) then we can easily find both its magnitude (namely, ρ, which in this case means distance as the hypothetical crow flies) and its direction (namely, $\hat{\boldsymbol{\rho}}$) by using (1) and (5). Conversely, if we know both magnitude and direction (i.e., both ρ and $\hat{\boldsymbol{\rho}}$), then (5) implies that we also know x/ρ, y/ρ and z/ρ, and so we easily recover the components x, y and z on multiplying by ρ. For example, it follows from above that a displacement of

$$\mathbf{a} \;=\; 5\mathbf{i} - 3\mathbf{j} + 4\mathbf{k} \tag{6}$$

is equivalent to travelling a distance[1] $a = \sqrt{5^2 + (-3)^2 + 4^2} = 5\sqrt{2}$ in the direction

$$\hat{\mathbf{a}} \;=\; \tfrac{5}{5\sqrt{2}}\,\mathbf{i} - \tfrac{3}{5\sqrt{2}}\,\mathbf{j} + \tfrac{4}{5\sqrt{2}}\,\mathbf{k} \;=\; \tfrac{1}{\sqrt{2}}\,\mathbf{i} - \tfrac{3}{5\sqrt{2}}\,\mathbf{j} + \tfrac{4}{5\sqrt{2}}\,\mathbf{k}, \tag{7}$$

whereas a displacement of magnitude 10 in the direction

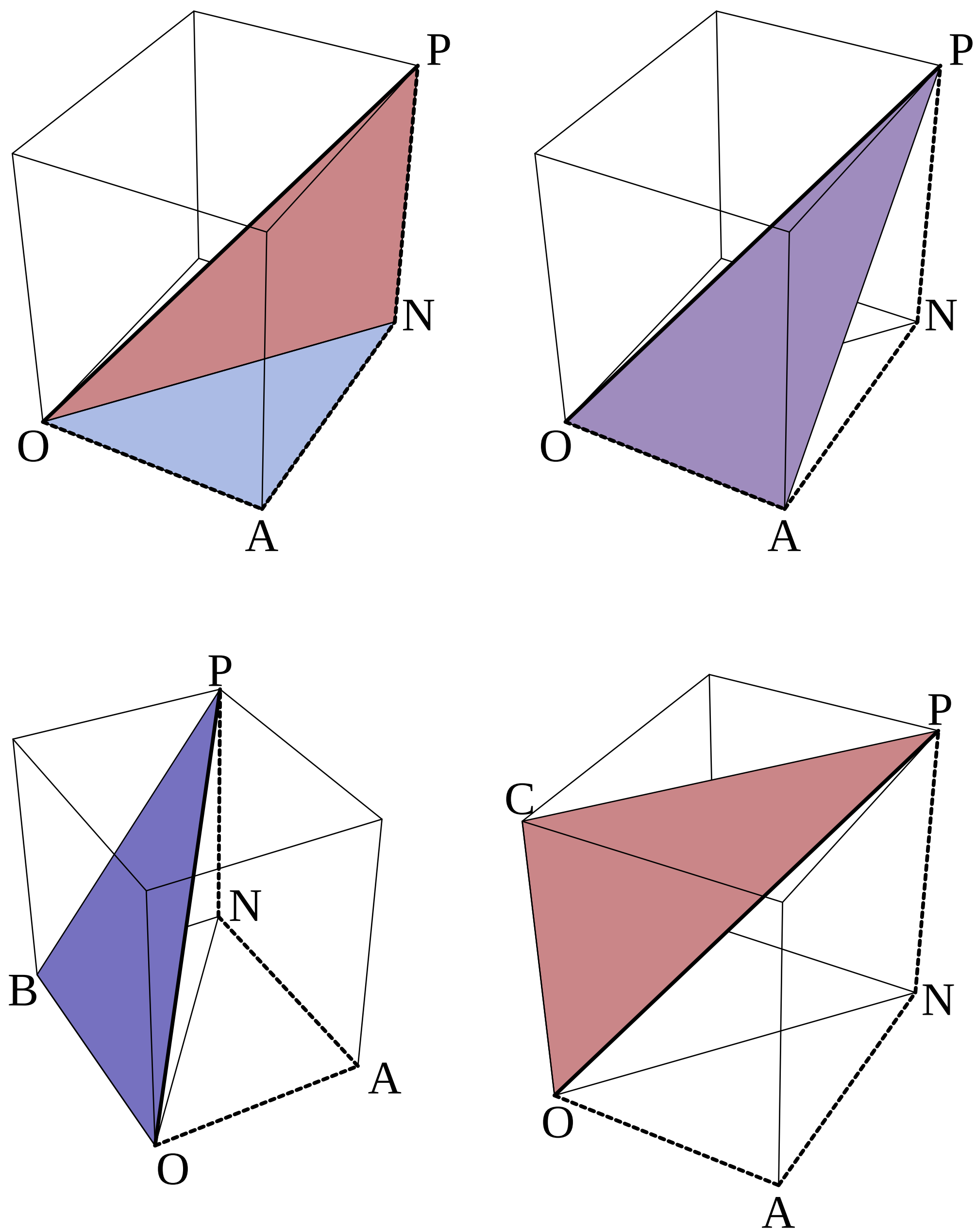
$$\hat{\mathbf{b}} \;=\; \tfrac{2}{3}\mathbf{i} - \tfrac{1}{3}\mathbf{j} + \tfrac{2}{3}\mathbf{k} \tag{8}$$

has eastward, northward and upward components of $10 \times \frac{2}{3}$, $10 \times \frac{-1}{3}$ and $10 \times \frac{2}{3}$, or $\frac{20}{3}$, $-\frac{10}{3}$ and $\frac{20}{3}$, respectively.

Does it seem strange to give directions in terms of unit vectors? Are you more familiar with directions in terms of angles? Well, we can fix that. Let the angles our vector $\boldsymbol{\rho}$ makes with the coordinate axes be α, β and γ, respectively, and take a look at the figure overleaf, in which the coordinates of the points labelled O, A, B, C, N and P are $(0,0,0)$, $(x,0,0)$, $(0,y,0)$, $(0,0,z)$, $(x,y,0)$ and (x,y,z), respectively.[2] In the figure $\boldsymbol{\rho}$ is represented by a directed line segment from the origin O with coordinates $(0,0,0)$ to the generic point P with coordinates (x,y,z):

[1]Note the consistency of our notation: the magnitude of the vector represented by the bold-faced letter $\mathbf{a}$ is denoted by the plain-faced letter a. Other commonly used notations for the magnitude of $\mathbf{a}$ are $|\mathbf{a}|$ and $||\mathbf{a}||$. Thus a, $|\mathbf{a}|$ and $||\mathbf{a}||$ are all precisely the same thing. Likewise, another commonly used notation for $x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ is $\langle x, y, z\rangle$ . . . which I don't especially like, and therefore will tend not to use.

[2]Bear in mind that the third (bottom left) panel's orientation differs from that of the others. The first panel should look familiar: it is virtually identical to Lecture 1's distance formula diagram.



$$\boldsymbol{\rho} = \overrightarrow{OP}, \tag{9}$$

with corresponding magnitude $OP = \rho$. Similarly, $\overrightarrow{OA} = x\mathbf{i}$, $\overrightarrow{OB} = y\mathbf{j}$ and $\overrightarrow{OC} = z\mathbf{k}$, with corresponding magnitudes

$$OA = x, \quad OB = y, \quad OC = z, \tag{10}$$

so that (3) becomes $\overrightarrow{OP} = \overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC}$. All of the colored triangles on Page 3 are right-angled. So, on using (10) together with $OP = \rho$, $\angle AOP = \alpha$, $\angle BOP = \beta$ and $\angle COP = \gamma$, we have

$$\cos(\alpha) \;=\; \frac{x}{\rho}, \quad \cos(\beta) \;=\; \frac{y}{\rho}, \quad \cos(\gamma) \;=\; \frac{z}{\rho}. \tag{11}$$

It follows immediately from (5) above that

$$\hat{\boldsymbol{\rho}} \;=\; \cos(\alpha)\mathbf{i} + \cos(\beta)\mathbf{j} + \cos(\gamma)\mathbf{k}. \tag{12}$$

So if we know the direction in terms of angles, we also know the unit vector, by (12); and conversely, if we know the unit vector, we can get the direction in terms of angles from (11). For example, a crow that takes off at 45° to both the x-axis and the z-axis (and therefore, inevitably at 90° to the y-axis) is flying in the direction

$$\tfrac{1}{\sqrt{2}}\mathbf{i} + 0\,\mathbf{j} + \tfrac{1}{\sqrt{2}}\mathbf{k} \;=\; \tfrac{1}{\sqrt{2}}(\mathbf{i} + \mathbf{k}). \tag{13}$$

Conversely, a crow that flies in the direction $\hat{\mathbf{b}}$ defined by (8) is rising in a direction of somewhere between south and east. More precisely, it rises at an angle of $\arccos(2/3) = 48.19°$ both to the east and to the vertical, but at an angle of $\arccos(-1/3) = 109.5°$ to the north—or, which of course is exactly the same thing, an angle of 70.5° to the south. The numbers $\cos(\alpha)$, $\cos(\beta)$ and $\cos(\gamma)$ are called *direction cosines,* and clearly satisfy

$$\cos^2(\alpha) + \cos^2(\beta) + \cos^2(\gamma) \;=\; 1. \tag{14}$$

Don't they? Why?

Before proceeding, note that going from $(0,0)$ to (x,y) in two dimensions is frequently represented by

$$\mathbf{r} \;=\; x\mathbf{i} + y\mathbf{j}, \tag{15}$$

an inherently two-dimensional position vector with magnitude

$$r \;=\; \sqrt{x^2 + y^2}. \tag{16}$$

It follows at once from (4) and (15) that

$$\boldsymbol{\rho} \;=\; \mathbf{r} + z\mathbf{k}, \tag{17}$$

and from (1) and (16) that that

$$\rho \;=\; \sqrt{r^2 + z^2}. \tag{18}$$

Free vectors

The difference between a position vector and a free vector is that the former has its tail firmly rooted at the origin of coordinates, whereas the second does not. For example, suppose that one hypothetical crow has flown the displacement

$$\boldsymbol{\rho}_1 \;=\; x_1\mathbf{i} + y_1\mathbf{j} + z_1\mathbf{k}, \tag{19}$$

i.e., from $(0,0,0)$ to (x_1, y_1, z_1), while another equally hypothetical crow has flown to

$$\boldsymbol{\rho}_2 \;=\; x_2\mathbf{i} + y_2\mathbf{j} + z_2\mathbf{k}, \tag{20}$$

i.e., from $(0, 0, 0)$ to (x_2, y_2, z_2); both of these vectors correspond to arrows with tails at $(0, 0, 0)$, but the first has its head at (x_1, y_1, z_1) whereas the second's head is at (x_2, y_2, z_2). What is now the displacement of the second crow relative to the first? It is

$$\boldsymbol{\rho}_{12} \;=\; \boldsymbol{\rho}_2 - \boldsymbol{\rho}_1 \;=\; (x_2 - x_1)\mathbf{i} + (y_2 - y_1)\mathbf{j} + (z_2 - z_1)\mathbf{k}, \tag{21}$$

i.e., the vector that represents a displacement of magnitude

$$d_{12} \;=\; \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2} \tag{22}$$

in the direction of from (x_1, y_1, z_1) to (x_2, y_2, z_2) or

$$\hat{\boldsymbol{\rho}}_{12} \;=\; \frac{x_2 - x_1}{d_{12}}\mathbf{i} + \frac{y_2 - y_1}{d_{12}}\mathbf{j} + \frac{z_2 - z_1}{d_{12}}\mathbf{k}. \tag{23}$$

Similarly, the displacement of the first crow relative to the second is

$$\begin{aligned} \boldsymbol{\rho}_{21} &= \boldsymbol{\rho}_1 - \boldsymbol{\rho}_2 \;=\; (x_1 - x_2)\mathbf{i} + (y_1 - y_2)\mathbf{j} + (z_1 - z_2)\mathbf{k} \\ &= -(\boldsymbol{\rho}_2 - \boldsymbol{\rho}_1) \\ &= -\boldsymbol{\rho}_{12}, \end{aligned} \tag{24}$$

i.e., the vector that represents a displacement of magnitude d_{12} in the direction of from (x_2, y_2, z_2) to (x_1, y_1, z_1) or $\hat{\boldsymbol{\rho}}_{21} = -\hat{\boldsymbol{\rho}}_{12}$, where $\hat{\boldsymbol{\rho}}_{12}$ is defined by (23): to reverse a direction, just multiply by -1.

The vectors $\boldsymbol{\rho}_{12}$ and $\boldsymbol{\rho}_{21}$ are no longer position vectors like $\boldsymbol{\rho}_1$ and $\boldsymbol{\rho}_2$. Why? Because their tails are no longer at $(0, 0, 0)$. In fact, it doesn't really matter where their tails are:[3] the relative displacement of the two crows will always be precisely the same, regardless of where you put the origin of coordinates (and hence, regardless of what are the two position vectors, even though both would change whenever you moved the origin). We call such vectors free vectors—or simply vectors (free being implied in much the same way that distance implies shortest distance). In every other respect, however, free vectors are just like position vectors. In fact, we often draw them with their tails at the origin of coordinates, not because that's where they are, but because everything has to be somewhere, and if it doesn't matter where then it might as well be there.

Vectors in general

Anything that has both magnitude and direction (e.g., force, momentum or velocity) can be represented as a vector, and behaves in all important respects like a displacement vector—although it can't actually be a position vector, of course, both because its magnitude isn't a distance (e.g., if the vector's a velocity, then its magnitude is a speed) and its tail is not at the origin.

[3]By contrast, if you're a crow, it probably does matter where your tail is.

The vector equation of a line

You have known for ages that $y = mx + b$ is the equation of a line in the x-y plane with slope m and y-intercept b. You think of the line as a locus of points (x, y). But you can also think of it as a locus of position vectors $\mathbf{r} = x\mathbf{i} + y\mathbf{j}$. How do you get from the origin O to an arbitrary point P on this line? One way is to go straight there, that is, to follow $\overrightarrow{OP}$. Another way is to go from O to B and then from B to P, that is, to follow the position vector $\overrightarrow{OB}$ and then add the displacement $\overrightarrow{BP}$ to reach the point whose position vector is $\overrightarrow{OB} + \overrightarrow{BP} = \overrightarrow{OP}$. Let $\overrightarrow{OB} = \mathbf{b} = b\mathbf{j}$; and let $\overrightarrow{BP} = \mathbf{s}$ (with magnitude $BP = s$), so that

$$\mathbf{s} \;=\; BP\hat{\mathbf{s}} \;=\; s\hat{\mathbf{s}} \;=\; s\{\cos(\alpha)\mathbf{i} + \cos(\beta)\mathbf{j}\} \;=\; s\{\cos(\alpha)\mathbf{i} + \sin(\alpha)\mathbf{j}\}. \tag{25}$$

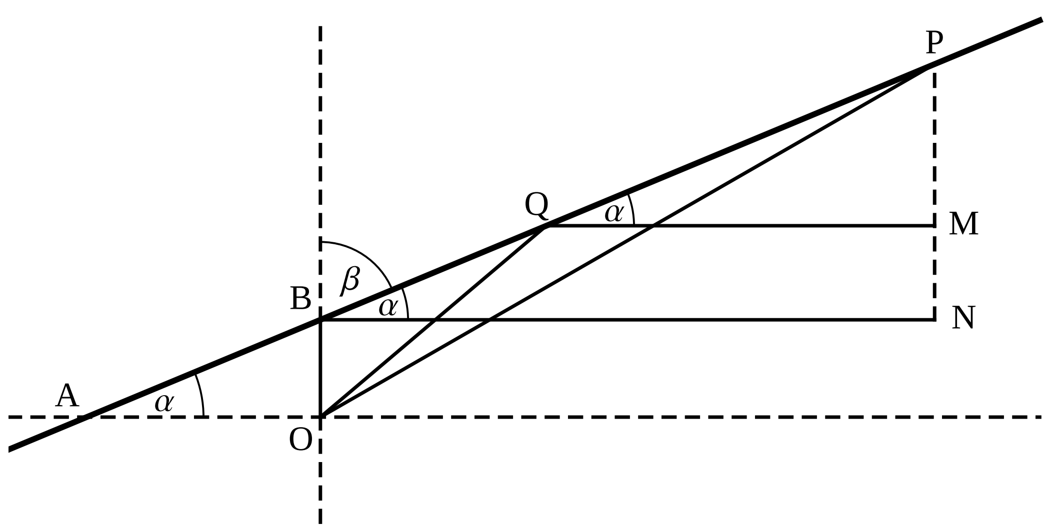
Then $\overrightarrow{OP} = \overrightarrow{OB} + \overrightarrow{BP} =$ yields

$$\begin{aligned} \mathbf{r} &= \mathbf{b} + \mathbf{s} \\ &= b\mathbf{j} + s\{\cos(\alpha)\mathbf{i} + \sin(\alpha)\mathbf{j}\} \\ &= s\cos(\alpha)\mathbf{i} + \{b + s\sin(\alpha)\}\mathbf{j}. \end{aligned} \tag{26}$$

But $\mathbf{r} = x\mathbf{i} + y\mathbf{j}$, and vectors are equal if and only if their components are identical. Hence

$$x \;=\; s\cos(\alpha), \qquad y \;=\; b + s\sin(\alpha) \tag{27}$$

implying $y = b + s\sin(\alpha) = b + \tan(\alpha)\cdot s\cos(\alpha) = b + \tan(\alpha)x = b + mx$, where $m = \tan(\alpha)$ is the slope of the line. By varying the value of s, we can obtain the position vector of any point on the line: $s = 0$ corresponds to B; $s > 0$ corresponds to points above and hence to the right of B; and $s < 0$ corresponds to points below and hence to the left of B. For example, $s = -b\,\text{cosec}(\alpha)$ corresponds to A.



Nevertheless, we are not obliged to go through B on our way to P; moreover, we are not obliged to use a unit vector to represent the direction of the line, and it is often convenient not to do so. For example, we can instead go through A; and if $\overrightarrow{OA} = \mathbf{a}$, then the vector

$$\mathbf{u} \;=\; \overrightarrow{AB} \;=\; \mathbf{b} - \mathbf{a} \tag{28}$$

represents the direction of the line just as surely as $\hat{\mathbf{s}}$ does. Let us define

$$t \;=\; \frac{AP}{AB}, \tag{29}$$

that is, let t be the ratio of the distance between A and P to the distance between A and B. Then, because a displacement from A to P has exactly the same direction $\hat{\mathbf{u}}$ as a displacement from A to B but is t times as far, we can represent the position vector of an arbitrary point P on the line by going through A as follows:

$$\mathbf{r} \;=\overrightarrow{OA} \;+\; \overrightarrow{AP} = \overrightarrow{OA} \;+ AP\hat{\mathbf{u}} \;= \overrightarrow{OA} \;+ tAB\hat{\mathbf{u}} \;= \overrightarrow{OA} \;+ t\,\overrightarrow{AB} = \;\mathbf{a} \;+\; t(\mathbf{b} - \mathbf{a}) \tag{30}$$

on using (29). By varying the value of t, we can obtain the position vector of any point on the line: $t = 0$ corresponds to A; $t = 1$ corresponds to B; $0 < t < 1$ corresponds to points between A and B; $t > 1$ corresponds to points above and hence to the right of B; and $t < 0$ corresponds to points below and hence to the left of A.[4] We say that t parameterizes the line. Moreover, we could just as easily go through any other point on the line, e.g., Q with position vector $\mathbf{q} = q_1\mathbf{i} + q_2\mathbf{j}$. With

$$\tau \;=\; \frac{QP}{AB} \tag{31}$$

we obtain

$$\mathbf{r} \;=\overrightarrow{OQ} \;+\; \overrightarrow{QP} = \overrightarrow{OQ} \;+ QP\hat{\mathbf{u}} \;= \overrightarrow{OQ} \;+ \tau AB\hat{\mathbf{u}} \;= \overrightarrow{OQ} \;+ \tau\,\overrightarrow{AB} = \;\mathbf{q} \;+\; \tau\mathbf{u} \tag{32}$$

in place of (30), so that now τ parameterizes the line.[5]

A few moments' thought now reveals, however, that the above representation immediately generalizes to lines in three-dimensional space. Essentially the only differences are that vectors have three instead of two components; that $\mathbf{r} = x\mathbf{i} + y\mathbf{j}$ is replaced by $\boldsymbol{\rho} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ $(= \mathbf{r} + z\mathbf{k})$; and that the diagram represents the plane through the line and the origin (in which $\overrightarrow{OA}$, $\overrightarrow{OB}$, $\overrightarrow{OQ}$, $\overrightarrow{OP}$ and $\overrightarrow{AP}$ all must lie, and which is not in general the x-y plane). Thus, generalizing from (30) and (32), the vector equation of the line through $\mathbf{a}$ and $\mathbf{b}$ is

$$\boldsymbol{\rho} \;=\; \mathbf{a} \;+\; t(\mathbf{b} - \mathbf{a}) \;=\; (1 - t)\mathbf{a} \;+\; t\mathbf{b}, \quad -\infty < t < \infty \tag{33}$$

where $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$ are the position vectors of *any* two points in three-dimensional space; and the vector equation of the line with direction $\hat{\mathbf{u}}$ through *any* $\mathbf{q}$ is

$$\boldsymbol{\rho} \;=\; \mathbf{q} \;+\; \tau\mathbf{u}, \quad -\infty < \tau < \infty. \tag{34}$$

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[4] And in case you are worried, it goes without saying that $y = b + mx$ still holds, just as surely as before. Why? With $\mathbf{a} = -a\mathbf{i}$ and $\mathbf{b} = b\mathbf{j}$, we have $x\mathbf{i} + y\mathbf{j} = \mathbf{r} = \mathbf{a} + t(\mathbf{b} - \mathbf{a}) = -a\mathbf{i} + t(a\mathbf{i} + b\mathbf{j}) = a(t-1)\mathbf{i} + bt\mathbf{j}$, implying $x = a(t - 1)$ and $y = bt$, so that $y = b(x + a)/a = b + mx$, where $m = b/a$ is the slope of the line.

[5] And in case you are *still* worried, well, don't be, $y = b + mx$ still holds! Why? With $\mathbf{a} = -a\mathbf{i}$ and $\mathbf{b} = b\mathbf{j}$, we now have $x\mathbf{i} + y\mathbf{j} = \mathbf{r} = \{q_1 + \tau a\}\mathbf{i} + \{q_2 + \tau b\}\mathbf{j}$, implying $x = q_1 + \tau a$ and $y = q_2 + \tau b$, so that (on eliminating τ) we obtain $y = b + mx$ as before, because $(q_2 - b)/q_1 = b/a$ from the diagram.