Area and volume as multiple integrals

Suppose that you wish to find the volume V of the three-dimensional region bounded above by the graph of a nonnegative, continuous function f and below by its domain D—which, you will recall, is some two-dimensional subset of the x-y plane (or $z = 0$, if you prefer). If f is a constant function, then all we have to do is multiply that constant height f by the "base" area A of the function's domain, and immediately we have the volume: $V = fA$.

But what if f is not a constant function? Then we can't just multiply height by area to get volume, because the height isn't constant. However, we can approximate the volume as follows. First we cover D with a very large number mn of rectangular subregions D_{ij} of infinitesimal area $\Delta x \Delta y$, each of which is so small that f cannot vary across it by much. Thus the volume lying between D_{ij} and the infinitesimal piece of graph lying directly above it must be well approximated by $\Delta x \Delta y$ times any value of $f(x, y)$ in the subregion D_{ij}. Next we sum these infinitesimal volumes over all D_{ij} by allowing i to range between 1 to m and j to range between 1 to n. Finally, we take the limit as both $m \to \infty$ and $n \to \infty$. In this way we discover that the volume bounded above by the graph of f and below by its domain D is just a double integral

$$V \;=\; \iint\limits_D f(x,y)\,dA \;=\; \lim_{\substack{m\to\infty\\ n\to\infty}} \sum_{i=1}^{m}\sum_{j=1}^{n} \left\{ \begin{matrix} \text{A value of } f(x,y) \text{ in} \\ \text{the rectangle } D_{ij} \end{matrix} \right\} \cdot \Delta x \Delta y \tag{1}$$

that, at least in principle, is readily calculated via iterated integration, using Method II or Method III, whichever is more convenient.

On the one hand, this result fits a pattern. Ever since Calculus I you have known that the area between the x-axis (or $y = 0$) and the graph $y = f(x)$ of the nonnegative univariate function f defined on the domain $[a, b]$ is $\int_a^b f(x)\,dx$. Now you know that the volume between the x-y plane (or $z = 0$) and the graph $z = f(x, y)$ of the nonnegative bivariate function f defined on the domain D is $\iint_D f(x, y)\,dA$.

On the other hand, there is a sense in which both of these results are unnatural: area is intrinsically two-dimensional, whereas volume is intrinsically three-dimensional. It is much more natural to regard both area and volume as special cases of the result that the total amount of stuff in a region is the integral over that region of the amount of stuff per unit region, regardless of whether that region is two- or three-dimensional. But area per unit area equals 1, and volume per unit volume also equals 1. Thus the area between the x axis, where $y = 0$, and the graph $y = f(x)$ of the nonnegative univariate function f defined on the domain $[a, b]$ is (by Method II)

$$\iint\limits_{\substack{a\le x\le b\\ 0\le y\le f(x)}} 1\,dA \;=\; \int_a^b \int_0^{f(x)} 1\,dy\,dx \;=\; \int_a^b y\Big|_0^{f(x)}\,dx \;=\; \int_a^b f(x)\,dx, \tag{2}$$

agreeing with the result you already know. Similarly, the volume between the x-y plane, where $z = 0$, and the graph $z = f(x, y)$ of the nonnegative bivariate function f defined on the domain D is the "triple integral"

$$\iiint\limits_{\substack{(x,y)\in D\\0\leq z\leq f(x,y)}} 1\,dV \;=\; \iint\limits_D \int\limits_0^{f(x,y)} 1\,dz\,dx\,dy \;=\; \iint\limits_D z\big|_0^{f(x,y)}\,dx\,dy \;=\; \iint\limits_D f(x,y)\,dx\,dy, \tag{3}$$

again in agreement with the result you already know. But note what we have gained: the left-hand side of (2) is now intrinsically two-dimensional, as area is supposed to be. Likewise, the left-hand side of (3) is now intrinsically three-dimensional, as volume is supposed to be.

The formal definition of a triple integral parallels that of a double integral: infinitesimal rectangles D_{ij} of area $\Delta x\Delta y$ are replaced by infinitesimal cuboids E_{ijk} of volume $\Delta x\Delta y\Delta z$ where k is a third index that sums in the z direction, and in place of summations over i or j between 1 and m or n, respectively, followed by limits as $m, n \to \infty$, we have summations over i, j or k between 1 and m, n or l, respectively, followed by limits as $m, n, l \to \infty$ to yield

$$\iiint\limits_E f(x,y,z)\,dV \;\triangleq\; \lim_{\substack{m\to\infty\\n\to\infty\\l\to\infty}} \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{l} \left\{\begin{matrix}\text{A value of } f(x,y,z) \text{ in}\\ \text{the cuboid } E_{ijk}\end{matrix}\right\}\cdot\Delta x\Delta y\Delta z \tag{4}$$

where E denotes a "volumetric region"—which would be just a volume in everyday speech, because in practice the word volume is used to denote either a three dimensional region or a measure of its magnitude, with the intended meaning virtually always obvious from context. But mathematicians tend to dislike such ambiguity and so prefer to distinguish, using E for the region and V for its magnitude, that is, its volume. Two special cases are as follows. First, if $f(x, y, z) = 1$, then (4) yields the volume of E:

$$V \;=\; \iiint\limits_E 1\,dV. \tag{5}$$

Second, if $f(x, y, z)$ denotes volumetric density, that is, mass per unit volume, then (4) yields the mass of E:

$$M \;=\; \iiint\limits_E f(x,y,z)\,dV. \tag{6}$$

Yet, again as in the case of a double integral, in practice definition (6) is never used to calculate a triple integral. Instead we use iterated ("partial") integration. But now, instead of just an inner and an outer integration, we have an innermost integration, an outermost integration, and a piggy-in-the-middle integration as well. The three integrations can be done in any order, so there are six different methods by which the triple integral can be calculated, corresponding to six different methods by which the region E can be "diced" into infinitesimal cuboids as indicated in the tables on p. 3—and needless to say, all six methods must yield the same answer! In practice, it is important to keep in mind that the integration limits for the innermost variable can now depend on either of the other two variables, and that the integration limits for the piggy-in-the-middle variable can depend on the outermost variable (whereas the integration limits for the outermost variable are always constant).

Six three-step methods for "dicing" volumetric region E into tiny cuboids (method # & step # indicated by Roman & Arabic numerals, respectively):

		How to slice	Result
(i)	1	Perpendicular to the x-axis	Lots of thin $x =$ constant plates all parallel to the y-z plane and ranging from $x = x_S$ to $x = x_R$
	2	Perpendicular to the y-axis	Lots of thin $y =$ constant strips all parallel to the z-axis and ranging from $y = y_F(x)$ to $y = y_B(x)$
	3	Perpendicular to the z-axis	Lots of infinitesimal cuboids ranging from $z = z_L(x,y)$ to $z = z_U(x,y)$
(ii)	1	Perpendicular to the y-axis	Lots of thin $y =$ constant plates all parallel to the x-z plane and ranging from $y = y_F$ to $y = y_B$
	2	Perpendicular to the x-axis	Lots of thin $x =$ constant strips all parallel to the z-axis and ranging from $x = x_S(y)$ to $x = x_R(y)$
	3	Perpendicular to the z-axis	Lots of infinitesimal cuboids ranging from $z = z_L(x,y)$ to $z = z_U(x,y)$
(iii)	1	Perpendicular to the z-axis	Lots of thin $z =$ constant plates all parallel to the x-y plane and ranging from $z = z_L$ to $z = z_U$
	2	Perpendicular to the x-axis	Lots of thin $x =$ constant strips all parallel to the y-axis and ranging from $x = x_S(z)$ to $x = x_R(z)$
	3	Perpendicular to the y-axis	Lots of infinitesimal cuboids ranging from $y = y_F(x,z)$ to $y = y_B(x,z)$
(iv)	1	Perpendicular to the x-axis	Lots of thin $x =$ constant plates all parallel to the y-z plane and ranging from $x = x_S$ to $x = x_R$
	2	Perpendicular to the z-axis	Lots of thin $z =$ constant strips all parallel to the y-axis and ranging from $z = z_L(x)$ to $z = z_U(x)$
	3	Perpendicular to the y-axis	Lots of infinitesimal cuboids ranging from $y = y_F(x,z)$ to $y = y_B(x,z)$
(v)	1	Perpendicular to the y-axis	Lots of thin $y =$ constant plates all parallel to the x-z plane and ranging from $y = y_F$ to $y = y_B$
	2	Perpendicular to the z-axis	Lots of thin $z =$ constant strips all parallel to the x-axis and ranging from $z = z_L(y)$ to $z = z_U(y)$
	3	Perpendicular to the x-axis	Lots of infinitesimal cuboids ranging from $x = x_S(y,z)$ to $x = x_R(y,z)$
(vi)	1	Perpendicular to the z-axis	Lots of thin $z =$ constant plates all parallel to the x-y plane and ranging from $z = z_L$ to $z = z_U$
	2	Perpendicular to the y-axis	Lots of thin $y =$ constant strips all parallel to the x-axis and ranging from $y = y_F(z)$ to $y = y_B(z)$
	3	Perpendicular to the x-axis	Lots of infinitesimal cuboids ranging from $x = x_S(y,z)$ to $x = x_R(y,z)$

Six corresponding ways to calculate a triple integral (note the sense in which last becomes first and first becomes last):

		How to integrate	Resulting form of $\iiint\limits_E f(x,y,z)\,dV$
(i)	1	From $z = z_L(x,y)$ to $z = z_U(x,y)$	$\int\limits_{x_S}^{x_R}\int\limits_{y_F(x)}^{y_B(x)}\int\limits_{z_L(x,y)}^{z_U(x,y)} f(x,y,z)\,dz\,dy\,dx$
	2	From $y = y_F(x)$ to $y = y_B(x)$	
	3	From $x = x_S$ to $x = x_R$	
(ii)	1	From $z = z_L(x,y)$ to $z = z_U(x,y)$	$\int\limits_{y_F}^{y_B}\int\limits_{x_S(y)}^{x_R(y)}\int\limits_{z_L(x,y)}^{z_U(x,y)} f(x,y,z)\,dz\,dx\,dy$
	2	From $x = x_S(y)$ to $x = x_R(y)$	
	3	From $y = y_F$ to $y = y_B$	
(iii)	1	From $y = y_F(x,z)$ to $y = y_B(x,z)$	$\int\limits_{z_L}^{z_U}\int\limits_{x_S(z)}^{x_R(z)}\int\limits_{y_F(x,z)}^{y_B(x,z)} f(x,y,z)\,dy\,dx\,dz$
	2	From $x = x_S(z)$ to $x = x_R(z)$	
	3	From $z = z_L$ to $z = z_U$	
(iv)	1	From $y = y_F(x,z)$ to $y = y_B(x,z)$	$\int\limits_{x_S}^{x_R}\int\limits_{z_L(x)}^{z_U(x)}\int\limits_{y_F(x,z)}^{y_B(x,z)} f(x,y,z)\,dy\,dz\,dx$
	2	From $z = z_L(x)$ to $z = z_U(x)$	
	3	From $x = x_S$ to $x = x_R$	
(v)	1	From $x = x_S(y,z)$ to $x = x_R(y,z)$	$\int\limits_{y_F}^{y_B}\int\limits_{z_L(y)}^{z_U(y)}\int\limits_{x_S(y,z)}^{x_R(y,z)} f(x,y,z)\,dx\,dz\,dy$
	2	From $z = z_L(y)$ to $z = z_U(y)$	
	3	From $y = y_F$ to $y = y_B$	
(vi)	1	From $x = x_S(y,z)$ to $x = x_R(y,z)$	$\int\limits_{z_L}^{z_U}\int\limits_{y_F(z)}^{y_B(z)}\int\limits_{x_S(y,z)}^{x_R(y,z)} f(x,y,z)\,dx\,dy\,dz$
	2	From $y = y_F(z)$ to $y = y_B(z)$	
	3	From $z = z_L$ to $z = z_U$	

Note that the region E is regarded as being viewed along the y-axis in the positive direction, which points into the page, while the x-axis points to the right and the z-axis points upward. Thus z increases from a lower (L) to an upper (U) limit, y increases from a lower limit at the front (F) to an upper limit at the back (B) and x increases from a left-hand or sinistral (S) limit to a right-hand (R) limit.

To illustrate, we will use Method (v) to calculate the volume of a tetrahedron with vertices at $(a, 0, 0)$, $(0, b, 0)$ and $(0, 0, c)$, whose upper face is the plane with equation

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} \;=\; 1. \tag{7}$$

Following method (v), we dice this tetrahedron by first of all slicing it into lots of thin triangular plates perpendicular to the y-axis, then slicing perpendicular to the z-axis to turn each plate into horizontal strips that are parallel to the x-axis, and finally slicing perpendicular to the x-axis to end up with tiny cuboids.

To cover the volumetric region E we must, in effect, reassemble the tetrahedron by reversing the above steps. We first revert from cuboids to strips by integrating along the generic strip—whose y and z coordinates are fixed—from the end at $x = 0$ to the end that intersects the surface defined by (7) or, which is exactly the same thing, from $x = 0$ to $x = a\left(1 - \frac{y}{b} - \frac{z}{c}\right)$. In terms of the table on the previous page, we have $x_S(y, z) = 0$ and $x_R(y, z) = a\left(1 - \frac{y}{b} - \frac{z}{c}\right)$. Next we revert from strips to plates by integrating over all such strips—whose y coordinates are fixed—from the lowest such strip at $z = 0$* to the highest such strip at $z = c\left(1 - \frac{y}{b}\right)$. In terms of the table on the previous page, we have $z_L(y) = 0$ and $z_U(y) = c\left(1 - \frac{y}{b}\right)$. Finally, we revert from plates to the tetrahedron itself by integrating over all such triangular plates, from the plate at $y = 0$† to the plate at $y = b$. In terms of the same table, we have $y_F = 0$ and $y_B = b$. Thus the volume of the tetrahedron is

$$\begin{aligned}
V \;&=\; \int_0^b \int_0^{c\left(1-\frac{y}{b}\right)} \int_0^{a\left(1-\frac{y}{b}-\frac{z}{c}\right)} 1\,dx\,dz\,dy \;=\; \int_0^b \int_0^{c\left(1-\frac{y}{b}\right)} x\Big|_0^{a\left(1-\frac{y}{b}-\frac{z}{c}\right)} dz\,dy \\
&=\; \int_0^b \int_0^{c\left(1-\frac{y}{b}\right)} a\left(1-\frac{y}{b}-\frac{z}{c}\right) dz\,dy \;=\; \int_0^b a\left\{\left(1-\frac{y}{b}\right)z - \frac{1}{2c}z^2\right\}\Big|_0^{c\left(1-\frac{y}{b}\right)} dy \\
&=\; a\int_0^b \left\{\left(1-\frac{y}{b}\right)c\left(1-\frac{y}{b}\right) - \frac{1}{2c}c^2\left(1-\frac{y}{b}\right)^2\right\} dy \\
&=\; \frac{1}{2}ac\int_0^b \left(1-\frac{y}{b}\right)^2 dy \\
&=\; \frac{1}{2}ac\left\{-\frac{1}{3}b\left(1-\frac{y}{b}\right)^3\right\}\Big|_0^b \;=\; \tfrac{1}{2}ac\left\{0 - \left(-\tfrac{1}{3}b\right)\right\} \;=\; \tfrac{1}{6}abc \\
&=\; \tfrac{1}{3}\times\tfrac{1}{2}ab\times c \;=\; \tfrac{1}{3}\times \text{AREA OF BASE}\times\text{HEIGHT},
\end{aligned}$$

agreeing with a familiar result that you have probably known for a very long time.

*Which just happens to have length $a\left(1 - \frac{y}{b}\right)$, while the highest strip has length 0.
†Which just happens to have area $\frac{1}{2}ac$, while the other "endplate" at $y = b$ has area 0.