A sort of potted geometry of three-dimensional space

With a view to the big picture, this is a quick look back at some—but only some—of what is really important for Calculus III. Specifically, it is a kind of catalog of the four possible kinds of object that you can have when objects are distinguished in terms of their dimensionality.

Points

Of the four basic objects, the simplest is a point. It has dimension zero, or no degrees of freedom.* A generic point has Cartesian coordinates (x, y, z) and hence position vector $\boldsymbol{\rho} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$. A specific point typically has Cartesian coordinates (x_l, y_l, z_l) and hence position vector $\boldsymbol{\rho} = \boldsymbol{\rho}_l = x_l\mathbf{i} + y_l\mathbf{j} + z_l\mathbf{k}$ where the subscript l is used to distinguish different fixed points. (If there is only one point, then we will likely use $l = 0$; if there are two points, then we may use $l = 0$ and $l = 1$ or $l = 1$ and $l = 2$, as the whim takes us; and so on.)

The equation of a point

The equation of a point is very simple. It is just

$$\boldsymbol{\rho} \;=\; \boldsymbol{\rho}_0$$

in vector form or $(x, y, z) = (x_0, y_0, z_0)$ or $x = x_0$, $y = y_0$, $z = z_0$ in terms of components.†

Curves

A curve—or "space" curve—has dimension one, or one degree of freedom.

The vector equation of a curve

As discussed in Lectures 8 and 14, the vector equation of a curve has the form

$$\boldsymbol{\rho} \;=\; \boldsymbol{\rho}(t) \;=\; x(t)\mathbf{i} \;+\; y(t)\mathbf{j} \;+\; z(t)\mathbf{k}, \qquad t_{\text{min}} \le t \le t_{\text{max}}$$

where t is the parameter with the one degree of freedom to roam between $t = t_{\text{min}}$ (typically 0) and $t = t_{\text{max}}$ (not atypically 1). An equivalent component or "parametric" form is

$$x \;=\; x(t), \quad y \;=\; y(t), \quad z \;=\; z(t), \qquad t_{\text{min}} \le t \le t_{\text{max}}.$$

The scalar equations for a curve

Two surfaces, say those with equations $\phi_1(x, y, z) = d_l$ for $l = 1, 2$, typically intersect in a curve, in which case, that curve is equivalently defined by the pair of scalar equations:

$$\phi_1(x, y, z) \;=\; d_1, \qquad \phi_2(x, y, z) \;=\; d_2.$$

*To be sure, there are infinitely many of them, but that freedom lies entirely in the grand collection or ensemble of them, as opposed to in a point per se.

†By analogy with the scalar equations for a curve or surface (see below), you could also in principle implicitly define a point as the intersection of three surfaces that just happen to meet in that point, in which case, the point would have three equations of the form $\phi_l(x, y, z) = d_l$ for $l = 1, 2, 3$—but rarely if ever would you want to do this in practice.

Direction of a curve whose vector equation is known

At the point $\boldsymbol{\rho}_0 = \boldsymbol{\rho}(t_0) = x(t_0)\mathbf{i} \ + \ y(t_0)\mathbf{j} \ + \ z(t_0)\mathbf{k}$, the curve has direction $\mathbf{T} = \hat{\mathbf{w}}$ where

$$\mathbf{w} \ = \ \dot{\boldsymbol{\rho}}(t_0) \ = \ \dot{x}(t_0)\mathbf{i} \ + \ \dot{y}(t_0)\mathbf{j} \ + \ \dot{z}(t_0)\mathbf{k}$$

and an overdot denotes differentiation; see Lecture 14.

Direction of a curve whose scalar equations are known

Because a curve of intersection between two surfaces must lie in both surfaces, at the point with position vector $\boldsymbol{\rho}_0 = x_0\mathbf{i} \ + \ y_0\mathbf{j} \ + \ z_0\mathbf{k}$ its direction must be parallel to the tangent plane of each surface, and hence must be normal to the normal to each tangent plane.‡ So the curve has direction $\hat{\mathbf{w}}$ where

$$\mathbf{w} \ = \ \pm \nabla\phi_1 \times \nabla\phi_2$$

(by Lecture 8).

The special case of a line

An important special case of a curve is a line. The line through $\boldsymbol{\rho}_0 = x_0\mathbf{i} \ + \ y_0\mathbf{j} \ + \ z_0\mathbf{k}$ has vector equation

$$\boldsymbol{\rho} \ = \ \boldsymbol{\rho}_0 + t\mathbf{w}, \qquad -\infty < t < \infty$$

(Lecture 3). Correspondingly, the line segment between any two points, say those with position vectors $\boldsymbol{\rho}_1 = \boldsymbol{\rho}_0 + t_1\mathbf{w}$ and $\boldsymbol{\rho}_2 = \boldsymbol{\rho}_0 + t_2\mathbf{w}$, has vector equation

$$\boldsymbol{\rho} \ = \ \boldsymbol{\rho}_0 + t\mathbf{w}, \qquad t_1 \le t \le t_2.$$

An equivalent component or "parametric" form is

$$x \ = \ x_0 + w_1 t, \qquad y \ = \ y_0 + w_2 t, \qquad z \ = \ z_0 + w_3 t, \qquad t_1 \le t \le t_2$$

where $\mathbf{w} = w_1\mathbf{i} + w_2\mathbf{j} = w_3\mathbf{k}$.

The line may also be regarded as the intersection of two planes, which are surfaces having equations of the form $\phi_l(x, y, z) = d_l$ for $l = 1, 2$, where $\phi_l(x, y, z) = a_l x + b_l y + c_l z$. Then the line has scalar equations of the form

$$a_1 x + b_1 y + c_1 z \ = \ d_1, \qquad a_2 x + b_2 y + c_2 z \ = \ d_2$$

and hence direction $\hat{\mathbf{w}}$ for $\mathbf{w} = \pm \nabla\phi_1 \times \nabla\phi_2 = \pm \{a_1\mathbf{i} + b_1\mathbf{j} + c_1\mathbf{k}\} \times \{a_2\mathbf{i} + b_2\mathbf{j} + c_2\mathbf{k}\}$.

Surfaces

A surface has dimension two, or two degrees of freedom.

‡Which, don't forget, is also what we mean by the normal to the surface at that point.

The vector equation of a surface

The vector equation of a surface has the form

$$\boldsymbol{\rho} \;=\; \boldsymbol{\rho}(u,v) \;=\; x(u,v)\mathbf{i} \;+\; y(u,v)\mathbf{j} \;+\; z(u,v)\mathbf{k}, \qquad u_{\min} \le u \le u_{\max}, \quad v_{\min} \le v \le v_{\max}$$

where u and v are the parameters with the two degrees of freedom, allowing them to roam where $u_{\min} \le u \le u_{\max}$, $v_{\min} \le v \le v_{\max}$ (Lecture 17). Possibly either $u_{\min}$ and $u_{\max}$ depend on v or $v_{\min}$ and $v_{\max}$ depend on u (but not both at the same time), although not atypically $u_{\min}$, $u_{\max}$, $v_{\min}$ and $v_{\max}$ are all constant.

The scalar equation for a surface

The scalar equation of a surface has the form

$$\phi(x,y,z) \;=\; d.$$

Direction of normal to a surface whose vector equation is known

At the point parameterized by u and v, the surface has normal $\boldsymbol{\rho}_u \times \boldsymbol{\rho}_v$, hence unit normal

$$\mathbf{n} \;=\; \pm\,\widehat{\boldsymbol{\rho}_u \times \boldsymbol{\rho}_v} \;=\; \pm\,\frac{\boldsymbol{\rho}_u \times \boldsymbol{\rho}_v}{|\boldsymbol{\rho}_u \times \boldsymbol{\rho}_v|}$$

(Lecture 17).

Direction of normal to a surface whose whose scalar equation is known

The surface has normal $\nabla\phi$ (Lecture 8) and hence unit normal

$$\mathbf{n} \;=\; \pm\,\widehat{\nabla\phi} \;=\; \pm\,\frac{\nabla\phi}{|\nabla\phi|}.$$

The special case of a triangular region

An important special case of a planar region is a triangular with corners having position vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$, for which

$$\boldsymbol{\rho} \;=\; \boldsymbol{\rho}(u,v) \;=\; \mathbf{a} \;+\; u(\mathbf{b}-\mathbf{a}) \;+\; v(\mathbf{c}-\mathbf{b}), \qquad 0 \le v \le u, \quad 0 \le u \le 1$$

by Lecture 18. Here $v_{\max} = u$ depends on u (whereas $v_{\min} = 0$, $u_{\min} = 0$ and $u_{\max} = 1$ are constant). Correspondingly,

$$\boldsymbol{\rho}(u,v) \;=\; \mathbf{a} \;+\; u(\mathbf{b}-\mathbf{a}) \;+\; v(\mathbf{c}-\mathbf{b}), \qquad 0 \le v \le 1, \quad 0 \le u \le 1$$

yields the parallelogram whose corners have position vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ and $\mathbf{a}-\mathbf{b}+\mathbf{c}$, and

$$\boldsymbol{\rho}(u,v) \;=\; \mathbf{a} \;+\; u(\mathbf{b}-\mathbf{a}) \;+\; v(\mathbf{c}-\mathbf{b}), \qquad -\infty < v < \infty, \quad -\infty < u < \infty$$

yields the entire plane through the same four points. This surface has normal

$$\mathbf{m} \triangleq \boldsymbol{\rho}_u \times \boldsymbol{\rho}_v = (\mathbf{b}-\mathbf{a}) \times (\mathbf{c}-\mathbf{b}) = \mathbf{a}\times\mathbf{b}+\mathbf{b}\times\mathbf{c}+\mathbf{c}\times\mathbf{a}$$

and hence (by Lecture 4) scalar equation

$$\mathbf{m}\cdot\boldsymbol{\rho} = \mathbf{m}\cdot\mathbf{a} = \mathbf{m}\cdot\mathbf{b} = \mathbf{m}\cdot\mathbf{c}$$

or

$$m_1 x + m_2 y + m_3 z = \mathbf{m}\cdot\mathbf{a} = \mathbf{m}\cdot\mathbf{b} = \mathbf{m}\cdot\mathbf{c}$$

where, of course, $\boldsymbol{\rho} = \boldsymbol{\rho} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ and $\mathbf{m} = m_1\mathbf{i} + m_2\mathbf{j} = m_3\mathbf{k}$. (See "The vector and scalar equations of a plane" under Supplementary Materials.)

Volumetric regions

A volumetric region E[§]—such as the interior of a sphere, ellipsoid or cylinder—has dimension three, or three degrees of freedom. Such a region has vector equation

$$\boldsymbol{\rho} = \boldsymbol{\rho}(u,v,w) = x(u,v,w)\mathbf{i} + y(u,v,w)\mathbf{j} + z(u,v,w)\mathbf{k},$$
$$u_{\min} \le u \le u_{\max},\ v_{\min} \le v \le v_{\max},\ w_{\min} \le w \le w_{\max}$$

where u, v and w are the parameters with the three degrees of freedom, allowing them to roam where $u_{\min} \le u \le u_{\max}$, $v_{\min} \le v \le v_{\max}$ and $w_{\min} \le w \le w_{\max}$ (Lectures 12, 13 and 20). Possibly one—but at most one—of the three pairs of coordinate limits (namely, $u_{\min}$ and $u_{\max}$, $v_{\min}$ and $v_{\max}$, $w_{\min}$ and $w_{\max}$) depends on the parameters associated with the other two; and possibly one—but at most one—of the two pairs that remain depends on the parameter associated with the third; but at least one of the three pairs is a pair of constants (and in natural coordinates all six limits may be constant).

Finally, don't forget what the above implies for volume-integral calculation. For the sake of argument, let us suppose that $w_{\min}$ and $w_{\max}$ depend upon u and v, and that $v_{\min}$ and $v_{\max}$ depend upon u, so that $u_{\min}$ and $u_{\max}$ are—of necessity—constant. Then any integral over E must be calculated with w as the innermost variable of integration and u as the outermost. That is, if f is the (trivariate) function to be integrated, then we must calculate

$$\iiint\limits_E f\,dx\,dy\,dz = \int_{u_{\min}}^{u_{\max}} \int_{v_{\min}}^{v_{\max}} \int_{w_{\min}}^{w_{\max}} |J|\,dw\,dv\,du$$

where

$$J(u,v,w) = (\boldsymbol{\rho}_u \times \boldsymbol{\rho}_v)\cdot\boldsymbol{\rho}_w = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \\ \frac{\partial x}{\partial w} & \frac{\partial y}{\partial w} & \frac{\partial z}{\partial w} \end{vmatrix}$$

is the Jacobian determinant.

[§]In everyday speech, a volumetric region is called a volume, and we are arguably being a bit pedantic here; however, in mathematics it is sometimes necessary to be more careful about distinguishing between the number that measures the size of a three-dimensional region and the region itself.