Integrating with respect to natural coordinates

I think you'll agree that constant limits of integration are generally easier to deal with than variables ones: $f(x, y)$ is easier to integrate over a rectangle than over a triangle, and $f(x, y, z)$ is easier to integrate over a cuboid than over a sphere. But what if there is no escape from computing a double integral over a non-rectangular region, or a triple integral over a non-cuboidal region? Wouldn't it be easier if, instead of integrating with respect to "Cartesian" coordinates x and y (in two dimensions) or x, y and z (in three), we could use "natural" coordinates with the property that they give us constant integration limits? For example, the natural coordinates for a circular disk in two dimensions would be polar coordinates, the natural coordinates for a cylindrical region in three dimensions would be cylindrical polar coordinates, the natural coordinates for a spherical region in three dimensions would be spherical polar coordinates, and so on.

To illustrate, let's use Cartesian coordinates to calculate the volume V of the circular cylinder with height 1 and curved surface $x^2 + y^2 = a^2$. The problem reduces to that of integrating 1 over the circular disk defined by $x^2 + y^2 \le a^2$. To cover this disk with Cartesian coordinates, we allow x to vary between $-\sqrt{a^2 - y^2}$ and $\sqrt{a^2 - y^2}$, for y varying between $-a$ and a. Thus

$$\begin{aligned} V &= \int_{-a}^{a} \int_{-\sqrt{a^2-y^2}}^{\sqrt{a^2-y^2}} \int_{0}^{1} 1\, dz\, dx\, dy = \int_{-a}^{a} \int_{-\sqrt{a^2-y^2}}^{\sqrt{a^2-y^2}} z\big|_0^1\, dx\, dy = \int_{-a}^{a} \int_{-\sqrt{a^2-y^2}}^{\sqrt{a^2-y^2}} 1\, dx\, dy = \int_{-a}^{a} x\Big|_{-\sqrt{a^2-y^2}}^{\sqrt{a^2-y^2}} dy \\ &= 2\int_{-a}^{a} \sqrt{a^2 - y^2}\, dy \;=\; 4\int_{y=0}^{y=a} \sqrt{a^2 - y^2}\, dy = 4\int_{\theta=0}^{\theta=\pi/2} \sqrt{a^2 - y^2}\,\frac{dy}{d\theta}\, d\theta = 4a^2 \int_{0}^{\pi/2} \cos^2(\theta)\, d\theta \\ &= 2a^2 \int_{0}^{\pi/2} \{1 + \cos(2\theta)\}\, d\theta \;=\; 2a^2 \left\{\theta + \tfrac{1}{2}\sin(2\theta)\right\}\Big|_0^{\frac{\pi}{2}} \;=\; \pi a^2 \end{aligned} \tag{1}$$

after use of the substitution $y = a\sin(\theta)$. That was a lot of work to get an answer we already knew, wasn't it? But we don't have to cover the disk with Cartesian coordinates. We can cover it just as easily with polar coordinates r and θ satisfying $x = r\cos(\theta), y = r\sin(\theta)$ by letting r and θ vary from 0 to a and from 0 to 2π, respectively. So, doesn't that mean that we can calculate V (much more easily) as

$$V \;=\; \int_{0}^{2\pi}\int_{0}^{a} 1\, dr\, d\theta\,? \tag{2}$$

The answer is no! For (2) yields $V = 2a\pi$, which you know to be incorrect.

It isn't difficult to see why (2) won't work. Because our cylinder has height 1, finding its volume V is equivalent to finding the area of its base; and ever since Calculus II you have known that you can calculate the area of a circular disk by chopping it into lots of concentric rings of infinitesimal thickness δr and radius r—and hence inner circumference $2\pi r$, outer circumference $2\pi(r+\delta r)$ and approximate area $2\pi r\delta r$—before summing over all such rings and taking the limit as $\delta r \to 0$. We thus obtain

$$V = \int_0^a 2\pi r\, dr = \int_0^a r\, 2\pi\, dr. \tag{3}$$

But ever since Calculus I you have known that $\int_0^{2\pi} 1\, d\theta = 2\pi$. So (3) implies

$$V = \int_0^a r \left\{ \int_0^{2\pi} 1\, d\theta \right\} dr = \int_0^{2\pi}\int_0^a r\, dr\, d\theta. \tag{4}$$

Comparing (4) with (2), we see that changing variables from x and y to r and θ requires us to multiply the integrand 1 by a factor r to preserve the value of V. The same is true for any other integrand when we change from Cartesian to polar coordinates. That is,

$$\iint_{\text{DISK}} f(x,y)\, dx\, dy = \int_0^{2\pi}\int_0^a r\, f(r\cos(\theta), r\sin(\theta))\, dr\, d\theta \tag{5}$$

for any (continuous) function f. Indeed (5) continues to hold if we replace DISK by any other region.

We should not be surprised that changing integration variables changes the integrand, even though the region of integration is unchanged. We have seen it happen countless times before through integrating ordinary functions of a single variable by substitution: to evaluate $\int_{y_1}^{y_2} f(y)\, dy$ by changing the integration variable from y to, say, θ satisfying $y = \zeta(\theta)$, we evaluate *not* an integral of the form $\int_{\theta_1}^{\theta_2} f(\zeta(\theta))\, d\theta$ with $y_1 = \zeta(\theta_1)$ and $y_2 = \zeta(\theta_2)$, but rather an integral of the form $\int_{\theta_1}^{\theta_2} f(\zeta(\theta))\frac{dy}{d\theta}\, d\theta$. That is, changing the integration variable from y to θ requires us to multiply the original integrand by the conversion factor $\frac{dy}{d\theta} = \zeta'(\theta)$. For illustration, see (1) on Page 1, where $\zeta(\theta) = a\sin(\theta) \Rightarrow \zeta'(\theta) = a\cos(\theta)$.*

Nor—given the analogy between integration by substitution and changing variables in a double integral—should we be surprised to learn that for any given change of variables in a double or triple integral, the conversion factor is always *precisely the same* for any (continuous) function f that we care to integrate. We will denote this conversion factor for a double integral by C_2. More precisely, if

$$x = x(u,v), \quad y = y(u,v) \tag{6}$$

then changing variables from x and y to u and v in a double integral yields

$$\iint_R f(x,y)\, dx\, dy = \iint_R f(x(u,v), y(u,v))C_2(u,v)\, du\, dv \tag{7}$$

for any two-dimensional region R. But how do we calculate C_2?

*Now you understand why I went to the trouble of writing out the calculation in full.

If (7) holds for any function f, then it must at least hold when $f = 1$. Thus

$$\iint_R 1\,dx\,dy \;=\; \iint_R C_2(u,v)\,du\,dv \tag{8}$$

for any two-dimensional region R: area is preserved by a change of coordinates. In particular, (8) holds for an infinitesimal region.

So consider the (infinitesimal) region generated by infinitesimal increases δx, δy in x and y, respectively, corresponding to infinitesimal increases δu, δv in u and v according to (6). In x-y coordinates, the area of this region is given by the left-hand side of (8), which—by the definition of double integral—is approximately $\delta x \delta y$. In u-v coordinates, the area of the region is given by the right-hand side of (8), which—again by the definition of double integral—is approximately $C_2(u,v)\delta u \delta v$. Hence (8) implies

$$\delta x \delta y \;\approx\; C_2(u,v)\delta u \delta v \tag{9}$$

for this infinitesimal region. That is, $C_2(u,v)\delta u \delta v$ is the approximate area in the x-y plane of the result under (6) of increasing u by δu and v by δv.

In the u-v plane, the effect of increasing u by δu and v by δv is to generate an infinitesimal rectangle with corners at the points (u,v), $(u+\delta u, v)$, $(u+\delta u, v+\delta v)$ and $(u, v+\delta v)$, moving counterclockwise around the rectangle. If

$$\mathbf{r}(u,v) \;=\; x(u,v)\mathbf{i} + y(u,v)\mathbf{j} \tag{10}$$

denotes the position vector of the point in the x-y plane corresponding to (u,v) in the u-v plane, then (6) maps the four corners of our rectangle to the points with position vectors $\mathbf{r}(u,v) = x(u,v)\mathbf{i} + y(u,v)\mathbf{j}$, $\mathbf{r}(u+\delta u, v) = x(u+\delta u, v)\mathbf{i} + y(u+\delta u, v)\mathbf{j}$, $\mathbf{r}(u+\delta u, v+\delta v) = x(u+\delta u, v+\delta v)\mathbf{i} + y(u+\delta u, v+\delta v)\mathbf{j}$ and $\mathbf{r}(u, v+\delta v) = x(u, v+\delta v)\mathbf{i} + y(u, v+\delta v)\mathbf{j}$. These four points are the vertices of a curvilinear rectangle; and if δu and δv are sufficiently small (which we assume), then the curvilinear rectangle is approximately a parallelogram. The first side of the parallelogram is represented by the vector

$$\begin{aligned}
\mathbf{r}(u+\delta u, v) - \mathbf{r}(u,v) \;&=\; x(u+\delta u, v)\mathbf{i} + y(u+\delta u, v)\mathbf{j} - \{x(u,v)\mathbf{i} + y(u,v)\mathbf{j}\} \\
&=\; \{x(u+\delta u, v) - x(u,v)\}\,\mathbf{i} + \{y(u+\delta u, v) - y(u,v)\}\,\mathbf{j} \\
&=\; \left(\left\{\frac{x(u+\delta u, v) - x(u,v)}{\delta u}\right\}\mathbf{i} + \left\{\frac{y(u+\delta u, v) - y(u,v)}{\delta u}\right\}\mathbf{j}\right)\delta u \\
&\approx\; \left(\frac{\partial x}{\partial u}\mathbf{i} + \frac{\partial y}{\partial u}\mathbf{j}\right)\delta u \;=\; \frac{\partial (x\mathbf{i} + y\mathbf{j})}{\partial u}\,\delta u \;=\; \frac{\partial \mathbf{r}}{\partial u}\,\delta u \;=\; \mathbf{r}_u\,\delta u
\end{aligned}$$

on using the alternative suffix notation for partial derivatives; and the last side of the parallelogram, which is adjacent to the first, is represented (in the direction away from the original point) by the vector

$$\begin{aligned}
\mathbf{r}(u, v+\delta v) - \mathbf{r}(u,v) \;&=\; x(u, v+\delta v)\mathbf{i} + y(u, v+\delta v)\mathbf{j} - \{x(u,v)\mathbf{i} + y(u,v)\mathbf{j}\} \\
&=\; \{x(u, v+\delta v) - x(u,v)\}\,\mathbf{i} + \{y(u, v+\delta v) - y(u,v)\}\,\mathbf{j} \\
&=\; \left(\left\{\frac{x(u, v+\delta v) - x(u,v)}{\delta v}\right\}\mathbf{i} + \left\{\frac{y(u, v+\delta v) - y(u,v)}{\delta v}\right\}\mathbf{j}\right)\delta v \\
&\approx\; \left(\frac{\partial x}{\partial v}\mathbf{i} + \frac{\partial y}{\partial v}\mathbf{j}\right)\delta v \;=\; \frac{\partial (x\mathbf{i} + y\mathbf{j})}{\partial v}\,\delta v \;=\; \frac{\partial \mathbf{r}}{\partial v}\,\delta v \;=\; \mathbf{r}_v\,\delta v.
\end{aligned}$$

So the area of the parallelogram is approximately $|\mathbf{r}_u\,\delta u\times\mathbf{r}_v\,\delta v| = |\mathbf{r}_u\times\mathbf{r}_v|\delta u\delta v$.[†] This is the approximate area in the x-y plane of the result under (6) of increasing u by δu and v by δv. But we have already established above that precisely the same quantity is approximately equal to $C_2(u,v)\delta u\delta v$. In the limit as $\delta u\to 0$, $\delta v\to 0$ (and, correspondingly, $\delta x\to 0$, $\delta y\to 0$), this approximation becomes exact. So we find that

$$C_2(u,v) \;=\; |\mathbf{r}_u\times\mathbf{r}_v| \;=\; |J(u,v)\mathbf{k}| \;=\; |J(u,v)|\,, \tag{11}$$

where the determinant

$$J(u,v) \;=\; \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix} \tag{12}$$

is called the Jacobian determinant of the transformation (6).

To illustrate: when changing to polar coordinates $u=r$ and $v=\theta$ we have

$$\mathbf{r} \;=\; x\mathbf{i}+y\mathbf{j} \;=\; r\cos(\theta)\mathbf{i}+r\sin(\theta)\mathbf{j} \tag{13}$$

so that

$$\mathbf{r}_r \;=\; \cos(\theta)\mathbf{i}+\sin(\theta)\mathbf{j}, \qquad \mathbf{r}_\theta \;=\; -r\sin(\theta)\mathbf{i}+r\cos(\theta)\mathbf{j} \tag{14}$$

and $\mathbf{r}_r\times\mathbf{r}_\theta = r\{\cos^2(\theta)+\sin^2(\theta)\}\mathbf{k} = r\mathbf{k}$. Hence $C_2(r,\theta)=|r\mathbf{k}|=r$, agreeing with (5).

Note that $\widehat{\mathbf{r}_u}$ and $\widehat{\mathbf{r}_v}$ are the directions in which u and v, respectively, increase. Common notations for these unit vectors are $\mathbf{e}_u$ and $\mathbf{e}_v$ (without hats). That is, we define

$$\mathbf{e}_u \;=\; \widehat{\mathbf{r}_u}, \qquad \mathbf{e}_v \;=\; \widehat{\mathbf{r}_v}. \tag{15}$$

For example, from (14) above, in polar coordinates the directions in which r and θ increase are, respectively,

$$\mathbf{e}_r \;=\; \widehat{\mathbf{r}_r} \;=\; \cos(\theta)\mathbf{i}+\sin(\theta)\mathbf{j} \tag{16}$$

(directly away from the origin) and

$$\mathbf{e}_\theta \;=\; \widehat{\mathbf{r}_\theta} \;=\; -\sin(\theta)\mathbf{i}+\cos(\theta)\mathbf{j} \tag{17}$$

(tangentially to any circle centered at the origin of coordinates, perpendicularly to $\mathbf{e}_r$).

There is a corresponding conversion factor for any triple integral, which we denote by C_3. More precisely, if

$$x \;=\; x(u,v,w), \quad y \;=\; y(u,v,w), \quad z \;=\; z(u,v,w) \tag{18}$$

then replacing x, y and z by u, v and w in a triple integral yields

$$\iiint\limits_R f(x,y,z)\,dx\,dy\,dz \;=\; \iiint\limits_R f(x(u,v,w),y(u,v,w),z(u,v,w))C_3(u,v,w)\,du\,dv\,dw \tag{19}$$

[†]Because if the base and an adjacent side of a parallelogram are represented by $\mathbf{b}$ and $\mathbf{a}$, respectively, and the angle between them is θ, then the area of the parallogram is BASE $\times$ HEIGHT $= ba\sin(\theta) = |ba\sin(\theta)\hat{\mathbf{n}}| = |\mathbf{a}\times\mathbf{b}|$, where $\mathbf{n}$ is normal to the parallelogram.

for any three-dimensional region R.

We proceed from here by analogy with the two-dimensional case. If (19) holds for any function f, then it must at least hold when $f = 1$. Thus

$$\iiint\limits_R 1\,dx\,dy\,dz \;=\; \iiint\limits_R C_3(u,v,w)\,du\,dv\,dw \tag{20}$$

for any three-dimensional region R: volume is preserved by a change of coordinates. In particular, (20) must hold for an infinitesimal region. So consider the region generated when x, y and z increase by δx, δy and δz, respectively, with corresponding increases of δu, δv and δw in u, v and w. In x-y-z coordinates, the volume of this region is given by the left-hand side of (19), which—by the definition of triple integral—is approximately $\delta x \delta y \delta z$. In u-v-w coordinates, the volume of the region is given by the right-hand side of (19), which—again by the definition of triple integral—is approximately $C_3(u,v,w)\delta u \delta v \delta w$. Hence (19) implies

$$\delta x \delta y \delta z \approx C_3(u,v,w)\delta u \delta v \delta w \tag{21}$$

for this infinitesimal region. That is, $C_3(u,v,w)\delta u \delta v \delta w$ is the approximate volume in x-y-z space of the result under (18) of increasing u by δu, v by δv and w by δw.

In u-v-w space, the effect of increasing u by δu, v by δv and w by δw is to generate an infinitesimal cuboid with adjacent corners at the points (u,v,w), $(u+\delta u, v, w)$, $(u, v+\delta v, w)$ and $(u, v, w+\delta w)$. If

$$\boldsymbol{\rho}(u,v,w) \;=\; x(u,v,w)\mathbf{i} + y(u,v,w)\mathbf{j} + z(u,v,w)\mathbf{k} \tag{22}$$

denotes the position vector of the point in x-y-z space corresponding to (u,v,w) in u-v-w space, then (18) maps the four adjacent corners of our cuboid to the points with position vectors $\boldsymbol{\rho}(u,v,w) = x(u,v,w)\mathbf{i} + y(u,v,w)\mathbf{j} + z(u,v,w)\mathbf{k}$, $\boldsymbol{\rho}(u+\delta u,v,w) = x(u+\delta u,v,w)\mathbf{i} + y(u+\delta u,v,w)\mathbf{j} + z(u+\delta u,v,w)\mathbf{k}$, $\boldsymbol{\rho}(u,v+\delta v,w) = x(u,v+\delta v,w)\mathbf{i} + y(u,v+\delta v,w)\mathbf{j} + z(u,v+\delta v,w)\mathbf{k}$ and $\boldsymbol{\rho}(u,w,w+\delta w) = x(u,w,w+\delta w)\mathbf{i} + y(u,w,w+\delta w)\mathbf{j} + z(u,w,w+\delta w)\mathbf{k}$. If δu, δv and δw are sufficiently small (which we assume), then these four points are four adjacent vertices of a parallelipiped. The first and second of the corresponding three adjacent sides of the parallelipiped are represented by the vectors $\boldsymbol{\rho}_u\,\delta u$ and $\boldsymbol{\rho}_v\,\delta v$, by an argument that is virtually identical to that used above; and an analogous argument shows that the third of the three adjacent sides is represented by the vector $\boldsymbol{\rho}_w\,\delta w$. So the parallelipiped's volume is $|(\boldsymbol{\rho}_u\,\delta u \times \boldsymbol{\rho}_v\,\delta v)\cdot\boldsymbol{\rho}_w\,\delta w| = |(\boldsymbol{\rho}_u \times \boldsymbol{\rho}_v)\cdot\boldsymbol{\rho}_w|\,\delta u \delta v \delta w$.[‡] This is the approximate volume in x-y-z space of the result under (18) of increasing u by δu, v by δv and w by δw. But we have already established above that precisely the same quantity is approximately equal to $C_3(u,v,w)\delta u \delta v \delta w$. In the limit as $\delta u \to 0$, $\delta v \to 0$ and $\delta w \to 0$ (and, correspondingly, $\delta x \to 0$, $\delta y \to 0$ and $\delta z \to 0$), this approximation becomes exact. Thus we see that

$$C_3(u,v,w) \;=\; |(\boldsymbol{\rho}_u \times \boldsymbol{\rho}_v)\cdot\boldsymbol{\rho}_w| \;=\; |J(u,v,w)|\,, \tag{23}$$

[‡]Because if adjacent base sides and an adjacent third side of a parallelipiped are represented by **a**, **b** and **c**, respectively, and the angle between **c** and the base's normal is θ, then the parallelipiped's volume is BASE AREA $\times$ HEIGHT $= |\mathbf{a}\times\mathbf{b}||\mathbf{c}|\cos(\theta) = |(\mathbf{a}\times\mathbf{b})\cdot\mathbf{c}|$.

with new Jacobian determinant

$$J(u,v,w) \;=\; \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \\ \frac{\partial x}{\partial w} & \frac{\partial y}{\partial w} & \frac{\partial z}{\partial w} \end{vmatrix}. \tag{24}$$

In place of (15) we have

$$\mathbf{e}_u \;=\; \widehat{\boldsymbol{\rho}_u}, \qquad \mathbf{e}_v \;=\; \widehat{\boldsymbol{\rho}_v}, \qquad \mathbf{e}_w \;=\; \widehat{\boldsymbol{\rho}_w}. \tag{25}$$

To illustrate: for a change to spherical polar coordinates[§] $u=\rho$, $v=\theta$, $w=\phi$ we have

$$\boldsymbol{\rho} \;=\; x\mathbf{i}+y\mathbf{j}+z\mathbf{k} \;=\; \rho\sin(\phi)\cos(\theta)\mathbf{i}+\rho\sin(\phi)\sin(\theta)\mathbf{j}+\rho\cos(\phi)\mathbf{k} \tag{26}$$

so that

$$\begin{aligned} \boldsymbol{\rho}_\rho &= \sin(\phi)\cos(\theta)\mathbf{i}+\sin(\phi)\sin(\theta)\mathbf{j}+\cos(\phi)\mathbf{k} \\ \boldsymbol{\rho}_\theta &= -\rho\sin(\phi)\sin(\theta)\mathbf{i}+\rho\sin(\phi)\cos(\theta)\mathbf{j} \\ \boldsymbol{\rho}_\phi &= \rho\cos(\phi)\cos(\theta)\mathbf{i}+\rho\cos(\phi)\sin(\theta)\mathbf{j}-\rho\sin(\phi)\mathbf{k} \end{aligned} \tag{27}$$

and

$$\begin{aligned} \boldsymbol{\rho}_\rho\times\boldsymbol{\rho}_\theta &= -\rho\sin(\phi)\cos(\phi)\cos(\theta)\mathbf{i}-\rho\sin(\phi)\cos(\phi)\sin(\theta)\mathbf{j} \\ &\qquad +\rho\sin^2(\phi)\{\cos^2(\theta)+\sin^2(\theta)\}\mathbf{k} \\ &= \rho\sin(\phi)\{-\cos(\phi)\cos(\theta)\mathbf{i}-\cos(\phi)\sin(\theta)\mathbf{j}+\sin(\phi)\mathbf{k}\}, \end{aligned} \tag{28}$$

implying that $(\boldsymbol{\rho}_\rho\times\boldsymbol{\rho}_\theta)\cdot\boldsymbol{\rho}_\phi=-\rho^2\sin(\phi)$. Thus

$$C_3(\rho,\theta,\phi) \;=\; |-\rho^2\sin(\phi)| \;=\; \rho^2\sin(\phi). \tag{29}$$

Or, for a change to cylindrical polar coordinates $u=r$, $v=\theta$ and $w=z$ we have $x=r\cos(\theta)$, $y=r\sin(\theta)$ and $z=z$, so that (24) yields

$$C_3(r,\theta,z) \;=\; |J(r,\theta,z)| \;=\; \left|\begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} & \frac{\partial z}{\partial r} \\ \frac{\partial x}{\partial \theta} & \frac{\partial y}{\partial \theta} & \frac{\partial z}{\partial \theta} \\ \frac{\partial x}{\partial z} & \frac{\partial y}{\partial z} & \frac{\partial z}{\partial z} \end{vmatrix}\right| \;=\; \left|\begin{vmatrix} \cos(\theta) & \sin(\theta) & 0 \\ -r\sin(\theta) & r\cos(\theta) & 0 \\ 0 & 0 & 1 \end{vmatrix}\right| \;=\; r. \tag{30}$$

It is intuitively appealing to regard the left-hand sides of (7) and (19) as being transformed into their respective right-hand sides, not by changing the integrand, but rather by keeping the integrand fixed and changing either the area element from $dA=dx\,dy$ to $dA=|J|du\,dv$ in (7) or the volume element from $dV=dx\,dy\,dz$ to $dV=|J|du\,dv\,dw$ in (19). Then, for example, the volume elements for cylindrical or spherical polar coordinates are $dV=r\,dr\,d\theta\,dz$ or $dV=\rho^2\sin(\phi)\,d\rho\,d\theta\,d\phi$, respectively. However, in the end, we always have to calculate a Riemann integral—of a different function—by Method II or III in two dimensions or by one of Methods (i)-(vi) in three dimensions. So our interpretation in terms of integrand conversion factors is arguably more fundamental (even if possibly less appealing).

[§]Defined as in Stewart's text.

Finally, a coordinate system for which $\mathbf{e}_u$, $\mathbf{e}_v$ and $\mathbf{e}_w$ are mutually orthogonal, so that

$$\mathbf{e}_u \cdot \mathbf{e}_v \;=\; \mathbf{e}_u \cdot \mathbf{e}_w \;=\; \mathbf{e}_v \cdot \mathbf{e}_w \;=\; 0, \tag{31}$$

is said to be an orthogonal coordinate system, and it is widely agreed that the most natural kind of orthogonal coordinate system is a right-handed one, that is, one for which—given (31)—we have

$$\mathbf{e}_u \times \mathbf{e}_v \;=\; \mathbf{e}_w, \qquad \mathbf{e}_v \times \mathbf{e}_w \;=\; \mathbf{e}_u, \qquad \mathbf{e}_w \times \mathbf{e}_u \;=\; \mathbf{e}_v. \tag{32}$$

For example, Cartesian coordinates are right-handed, because $\mathbf{e}_x = \mathbf{i}$, $\mathbf{e}_y = \mathbf{j}$, $\mathbf{e}_z = \mathbf{k}$ and we have known for some time that $\mathbf{i} \times \mathbf{j} = \mathbf{k}$, $\mathbf{j} \times \mathbf{k} = \mathbf{i}$ and $\mathbf{k} \times \mathbf{i} = \mathbf{j}$. Cylindrical polars are likewise right-handed, because

$$\boldsymbol{\rho} \;=\; x\mathbf{i} + y\mathbf{j} + z\mathbf{k} \;=\; r\cos(\theta)\mathbf{i} + r\sin(\theta)\mathbf{j} + z\mathbf{k} \tag{33}$$

implies

$$\begin{aligned} \mathbf{e}_r \;&=\; \widehat{\boldsymbol{\rho}_r} \;=\; \cos(\theta)\mathbf{i} + \sin(\theta)\mathbf{j} + 0\mathbf{k} \\ \mathbf{e}_\theta \;&=\; \widehat{\boldsymbol{\rho}_\theta} \;=\; -\sin(\theta)\mathbf{i} + \cos(\theta)\mathbf{j} + 0\mathbf{k} \\ \mathbf{e}_z \;&=\; \widehat{\boldsymbol{\rho}_z} \;=\; 0\mathbf{i} + 0\mathbf{j} + \mathbf{k} \end{aligned} \tag{34}$$

so that (32) holds with $u = r$, $v = \theta$ and $w = z$. Spherical polars are not right-handed if you define them as in Stewart's text, with longitude preceding co-latitude. However, they *are* right-handed if you instead take $u = \rho$ (radial distance), $v = \phi$ (co-latitude) and $w = \theta$ (azimuth or longitude, measured from the "prime" meridional plane)—that is, if you instead define them naturally, as indeed of course you should.