Surface integrals

We have already seen that work has a natural mathematical representation as a line integral, a one-dimensional integral along a curve. Another quantity of physical interest, the flux of a vector field, say $\mathbf{q}$, has a natural mathematical representation as a two-dimensional integral over a surface. Here we will study this representation.

Before we can study such a surface integral, however, we must first have a representation for the surface itself. We know the surface if we know the position vector of every point on it; and because the surface is two-dimensional, any point is uniquely determined by a pair of coordinates, say u and v. The corresponding representation for the surface is an equation or equations of the form

$$\mathbf{r} \;=\; \mathbf{r}(u,v), \quad u_{min} \le u \le u_{max}, \;\; v_{min} \le v \le v_{max} \tag{1}$$

with u and v appropriately restricted. We always use natural coordinates, e.g., Cartesian coordinates for part of the surface of a cuboid, cylindrical polars for part of the surface of a cylinder, spherical polars for part of the surface of a sphere, and so on.

Consider, for example, a vertical cylinder of height h and radius a with its floor at $z = 0$ and its axis aligned along the the z-axis. The natural coordinates are cylindrical polars R, θ and z. Why have I listed three coordinates, if you need only two to represent a surface? The answer is that the two you need are not always the same ones. Any point on the floor of the cylinder has a position vector of the form $R\cos(\theta)\mathbf{i} + R\sin(\theta)\mathbf{j} + 0\,\mathbf{k}$ for some value of R between 0 and a and some value of θ between 0 and 2π. Thus the floor of the cylinder has equation

$$\mathbf{r} \;=\; R\cos(\theta)\mathbf{i} + R\sin(\theta)\mathbf{j}, \quad 0 \le R \le a, \;\; 0 \le \theta \le 2\pi. \tag{2}$$

Similarly, the roof has equation

$$\mathbf{r} \;=\; R\cos(\theta)\mathbf{i} + R\sin(\theta)\mathbf{j} + h\,\mathbf{k}, \quad 0 \le R \le a, \;\; 0 \le \theta \le 2\pi. \tag{3}$$

Both are special cases of (1) with $u = R$ and $v = \theta$. On the curved surface of the cylinder, however, points have position vectors of the form $a\cos(\theta)\mathbf{i} + a\sin(\theta)\mathbf{j} + z\,\mathbf{k}$ for some value of θ between 0 and 2π and some value of z between 0 and h. Thus the vertical wall has equation

$$\mathbf{r} \;=\; a\cos(\theta)\mathbf{i} + a\sin(\theta)\mathbf{j} + z\,\mathbf{k}, \quad 0 \le \theta \le 2\pi, \;\; 0 \le z \le h. \tag{4}$$

This is still a special case of (1), but now with $u = \theta$ and $v = z$ instead of $u = R$ and $v = \theta$. Note that the coordinates remain in their natural order.

As a second example, consider a hemisphere of radius a whose pole is positioned along the positive z-axis at $\mathbf{r} = a\mathbf{k}$. The natural coordinates are now spherical polars ρ (distance from origin), ϕ (colatitude) and θ (longitude), for which, as you know, the generic position vector is given by

$$\mathbf{r} \;=\; x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k} \;=\; \rho\sin(\phi)\cos(\theta)\mathbf{i} + \rho\sin(\phi)\sin(\theta)\mathbf{j} + \rho\cos(\phi)\mathbf{k}; \tag{5}$$

but $\rho = a$ on the hemisphere, so the only coordinates we need are ϕ and θ. In other words, the hemisphere has equation

$$\begin{aligned}\mathbf{r} \;=\; & a\sin(\phi)\cos(\theta)\mathbf{i} + a\sin(\phi)\sin(\theta)\mathbf{j} + a\cos(\phi)\mathbf{k}, \\ & 0 \le \phi \le \tfrac{1}{2}\pi, \;\; 0 \le \theta \le 2\pi.\end{aligned} \tag{6}$$

These two examples illustrate that ordinary (two-sided) surfaces are of two kinds. A two-sided surface is closed if it has a well defined inside and outside *and* you can't move from one to the other without making a hole in the surface to pass through; and otherwise the surface is open. For example, our cylinder is closed but our hemisphere is open: although it has a well defined inside and outside, you don't have to make a hole in it to move from the inside to the outside (or vice versa). But any closed surface can be pieced together from two or more open surfaces, each with its own equation, as our cylinder illustrates; and sometimes the only difference between closed and open lies in the range of coordinates. For example,

$$\begin{aligned} \mathbf{r} \;&=\; a\sin(\phi)\cos(\theta)\mathbf{i} + a\sin(\phi)\sin(\theta)\mathbf{j} + a\cos(\phi)\mathbf{k}, \\ &\qquad 0 \le \phi \le \pi,\;\; 0 \le \theta \le 2\pi. \end{aligned} \tag{7}$$

is the equation of a sphere, which is closed, but the only difference between (6) and (7) is the range of the colatitude.

As you know, at any given point on a surface, there are two possible unit normal vectors: if the surface has equation $\psi(\mathbf{r}) = 0$, then those two directions are $\pm\widehat{\nabla\psi}$. Which one should be regarded as "the" normal—the positive normal direction? For a closed surface, the standard convention is that the normal points out; for example, the normal to a sphere with center at the origin points away from the origin. Then the outer surface is the positive surface and the inner surface is the negative. For an open surface, in principle, the direction of the normal must be defined, but in practice again there are standard conventions. In particular, if the open surface is part of a closed surface, then its normal is chosen to be consistent with the convention for closed surfaces. Thus, for example, the roof of our cylinder has positive normal $\mathbf{n} = \mathbf{k}$, whereas the floor has positive normal $\mathbf{n} = -\mathbf{k}$: one is up and the other is down, but both are out.

At last we are ready to talk about fluxes. We proceed by analogy with the steps we took to introduce the line integral, starting with the simplest possible case. So suppose that $q_\perp$, a constant, is the amount of some physical quantity that crosses a surface per unit area per unit time *in a direction perpendicular to the surface*: this, by definition, is the flux per unit area. For example, if volume of water is the physical quantity, then the flux per unit area is simply the water's velocity.

If we know the flux per unit area, then—provided that it is constant—all we have to do to get the flux is multiply by the total area. For example, if $q_\perp$ is constant over a sphere of radius a, then the total flux, say Q, is given by $Q = 4\pi a^2 q_\perp$, because $4\pi a^2$ is the sphere's total surface area. Similarly, if $q_\perp$ were constant over the roof of our cylinder, then the total upward flux would be $Q = \pi a^2 q_\perp$.

Now, if $\mathbf{q}$ represents the magnitude and direction of the flow per unit area of some physical quantity, then it won't be true in general that $\mathbf{q}$ points directly away from—that is, perpendicular to—whatever surface we happen to have in mind. Rather, at any point on the surface, $\mathbf{q}$ can be resolved into three mutually perpendicular components, two of which are parallel to the tangent plane, and therefore do not represent flow across the surface; on the contrary, they represent flow along it. But a flow must cross a surface to yield a flux. So we are only interested in the third component of $\mathbf{q}$; and because a flux across a surface is always counted as positive in the direction of the surface's positive side, we have

$$q_\perp \;=\; \mathbf{q}\cdot\mathbf{n} \tag{8}$$

(where $\mathbf{n}$ is the positive *unit* normal, i.e., $|\mathbf{n}| = 1$.)

In general, however, flux per unit area is not constant, but rather varies over a surface according to some equation of the form $q_\perp = q_\perp(u, v)$. So consider the infinitesimal element of surface area δS generated when u increases by δu and v increases by δv. By the argument of Lecture 13, (1) implies that

$$\delta S \;\approx\; |\mathbf{r}_u\,\delta u \times \mathbf{r}_v\,\delta v| = |\mathbf{r}_u \times \mathbf{r}_v|\,\delta u\delta v. \tag{9}$$

Over this element of surface area, $q_\perp$ will differ so little from $q_\perp(u, v)$ that we can approximate $q_\perp$ by $q_\perp(u, v)$. Hence we can approximate the flux across the surface element, say δQ, by

$$\delta Q \;\approx\; q_\perp\,\delta S \;\approx\; q_\perp\,|\mathbf{r}_u\,\delta u \times \mathbf{r}_v\,\delta v| \;=\; q_\perp\,|\mathbf{r}_u \times \mathbf{r}_v|\,\delta u\delta v. \tag{10}$$

Furthermore, we can subdivide any surface into zillions and zillions of little surface elements, each with the property that the flux per unit area $q_\perp$ hardly varies across it. Hence we can approximate the total flux Q across the surface by summing over all the little pieces:

$$Q \;\approx\; \sum \delta Q \;\approx\; \sum q_\perp\,\delta S \;\approx\; \sum q_\perp\;|\mathbf{r}_u \times \mathbf{r}_v|\,\delta u\delta v.$$

In the limit as $\delta S \to 0$ (implying both $\delta u \to 0$ and $\delta v \to 0$), the above relationship becomes exact in the form of a double integral:

$$Q \;=\; \iint\limits_S q_\perp\,dS \;=\; \iint\limits_S q_\perp\;|\mathbf{r}_u \times \mathbf{r}_v|\;du\,dv \;=\; \iint\limits_S \mathbf{q}\cdot\mathbf{n}\;|\mathbf{r}_u \times \mathbf{r}_v|\;du\,dv \tag{11}$$

on using (8), where S denotes the surface with surface area element

$$dS \;=\; |\mathbf{r}_u \times \mathbf{r}_v|\;du\,dv. \tag{12}$$

But (1) implies

$$\mathbf{n} \;=\; \pm\frac{\mathbf{r}_u \times \mathbf{r}_v}{|\mathbf{r}_u \times \mathbf{r}_v|} \tag{13}$$

by an argument used in Lecture 13, the sign being chosen to ensure that $\mathbf{n}$ points toward the positive side of the surface. Moreover, it is traditional to write

$$\mathbf{dS} \;=\; \mathbf{n}\,dS \;=\; \mathbf{n}\,|\mathbf{r}_u \times \mathbf{r}_v|\,du\,dv \;=\; \pm(\mathbf{r}_u \times \mathbf{r}_v)\,du\,dv \tag{14}$$

(and interpret $\mathbf{dS}$ as an element of vector area). So (11) yields

$$Q \;=\; \iint\limits_S \mathbf{q}\cdot\mathbf{dS} \;=\; \iint\limits_S \mathbf{q}\cdot\mathbf{n}\,dS \;=\; \pm\iint\limits_S \mathbf{q}\cdot(\mathbf{r}_u \times \mathbf{r}_v)\,du\,dv. \tag{15}$$

This is our desired representation of flux in terms of a surface integral.

Often both $\mathbf{n}$ and dS are known without invoking (13)-(14). We illustrate by calculating the flux through our cylinder of the vector field

$$\mathbf{q} \;=\; ax\,\mathbf{i} + yz\mathbf{j} + x^2\mathbf{k}. \tag{16}$$

To decompose the closed surface into a union of open ones, we write

$$S = S_1 \cup S_2 \cup S_3, \tag{17}$$

where S_1 is the floor, S_2 is the vertical wall and S_3 is the roof. Then

$$Q = \iint\limits_S \mathbf{q}\cdot\mathbf{dS} = \iint\limits_{S_1} \mathbf{q}\cdot\mathbf{n}\,dS + \iint\limits_{S_2} \mathbf{q}\cdot\mathbf{n}\,dS + \iint\limits_{S_3} \mathbf{q}\cdot\mathbf{n}\,dS, \tag{18}$$

and we calculate each of the three contributions in turn.

First, on the floor of the cylinder, where $z = 0$, we have[1]

$$\mathbf{q} = ax\,\mathbf{i} + x^2\,\mathbf{k}, \; dS = R\,dR\,d\theta, \; \mathbf{n} = -\mathbf{k}. \tag{19}$$

Thus $\mathbf{q}\cdot\mathbf{n} = -x^2 = -R^2\cos^2(\theta)$, implying

$$\begin{aligned} \iint\limits_{S_1} \mathbf{q}\cdot\mathbf{n}\,dS &= \int\limits_{\theta=0}^{\theta=2\pi}\int\limits_{R=0}^{R=a} \{-R^2\cos^2(\theta)\}R\,dR\,d\theta \\ &= -\int\limits_0^{2\pi}\cos^2(\theta)\,d\theta \int\limits_0^a R^3\,dR = -\tfrac{1}{4}\pi a^4. \end{aligned} \tag{20}$$

Second, on the vertical wall where $R = a$, $x = a\cos(\theta)$ and $y = a\sin(\theta)$ so that $\mathbf{r} = a\cos(\theta)\,\mathbf{i} + a\sin(\theta)\,\mathbf{j} + z\mathbf{k}$, we have[2]

$$\mathbf{q} = a^2\cos(\theta)\,\mathbf{i} + a\sin(\theta)z\,\mathbf{j} + a^2\cos^2(\theta)^2\,\mathbf{k} \tag{21}$$

$$dS = a\,d\theta\,dz, \; \mathbf{n} = \cos(\theta)\,\mathbf{i} + \sin(\theta)\,\mathbf{j}. \tag{22}$$

Thus $\mathbf{q}\cdot\mathbf{n} = a^2\cos^2(\theta) + a\sin^2(\theta)z$, implying

$$\begin{aligned} \iint\limits_{S_2} \mathbf{q}\cdot\mathbf{n}\,dS &= \int\limits_{\theta=0}^{\theta=2\pi}\int\limits_{z=0}^{z=h} \{a^2\cos^2(\theta) + a\sin^2(\theta)z\}a\,d\theta\,dz \\ &= a^3\int\limits_0^{2\pi}\cos^2(\theta)\,d\theta\int\limits_0^h dz + a^2\int\limits_0^{2\pi}\sin^2(\theta)\,d\theta\int\limits_0^h z\,dz \\ &= \pi a^3 h + \tfrac{1}{2}\pi a^2 h^2. \end{aligned} \tag{23}$$

Finally, on the roof where $z = h$, $\mathbf{q} = aR\cos(\theta)\,\mathbf{i} + R\sin(\theta)h\,\mathbf{j} + R^2\cos^2(\theta)\,\mathbf{k}$, $dS = R\,dR\,d\theta$, $\mathbf{n} = \mathbf{k}$ and $\mathbf{q}\cdot\mathbf{n} = R^2\cos^2(\theta)$, implying

$$\iint\limits_{S_3} \mathbf{q}\cdot\mathbf{n}\,dS = \int\limits_{\theta=0}^{\theta=2\pi}\int\limits_{R=0}^{R=a} \{R^2\cos^2(\theta)\}R\,dR\,d\theta = \tfrac{1}{4}\pi a^4. \tag{24}$$

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[1] Note that $\mathbf{r}_R \times \mathbf{r}_\theta = \{\cos(\theta)\mathbf{i} + \sin(\theta)\mathbf{j}\} \times R\{-\sin(\theta)\mathbf{i} + \cos(\theta)\mathbf{j}\} = R\mathbf{k}$ points to the negative side of this surface: the negative sign would have to be taken if (13) were used.

[2] We also have $\mathbf{r}_\theta = -a\sin(\theta)\,\mathbf{i} + a\cos(\theta)\,\mathbf{j} + 0\,\mathbf{k}$, $\mathbf{r}_z = 0\,\mathbf{i} + 0\,\mathbf{j} + \mathbf{k} = \mathbf{k}$, implying $\mathbf{r}_\theta \times \mathbf{r}_z = a\cos(\theta)\,\mathbf{i} + a\sin(\theta)\,\mathbf{j}$ and hence $\mathbf{n} = \cos(\theta)\,\mathbf{i} + \sin(\theta)\,\mathbf{j}$ with $dS = |\mathbf{r}_\theta \times \mathbf{r}_z|d\theta\,dz = a\,d\theta\,dz$. However, it is more efficient to obtain (22) from simple geometric considerations.

Comparing with (20), we see that the flux through the roof exactly cancels the flux through the floor: what goes in through the floor comes out through the roof, because the vertical component of $\mathbf{q}$ is independent of z (and the roof is the same size as the floor). Thus, from (18) and (23):

$$Q \;=\; \iint_S \mathbf{q}\cdot\mathbf{dS} \;=\; \tfrac{1}{2}\pi a^2 h(h+2a). \tag{25}$$

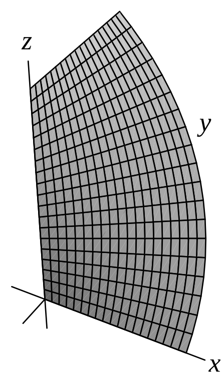

Figure 1: An open helicoidal surface

Sometimes, however, there is no alternative to using (12)-(14). To illustrate, we find the area A of the open helicoidal surface S defined by

$$\mathbf{r} \;=\; u\cos(v)\,\mathbf{i} + u\sin(v)\,\mathbf{j} + v\,\mathbf{k}, \quad 0 \le u \le 1,\; 0 \le v \le \tfrac{1}{2}\pi \tag{26}$$

and depicted in Figure 1. Differentiating (26), we obtain

$$\begin{aligned} \mathbf{r}_u &= \cos(v)\,\mathbf{i} + \sin(v)\,\mathbf{j}, \\ \mathbf{r}_v &= -u\sin(v)\,\mathbf{i} + u\cos(v)\,\mathbf{j} + \mathbf{k} \end{aligned} \tag{27}$$

and hence

$$\mathbf{r}_u \times \mathbf{r}_v \;=\; \sin(v)\,\mathbf{i} - \cos(v)\,\mathbf{j} + u\mathbf{k}, \tag{28}$$

so that[3]

$$\left|\mathbf{r}_u \times \mathbf{r}_v\right| \;=\; \sqrt{\{\sin(v)\}^2 + \{-\cos(v)\}^2 + u^2} \;=\; \sqrt{1+u^2}. \tag{29}$$

[3]In principle, because this surface is open, it is our prerogative to decide which side of it is positive; in practice, whenever an open surface has a top and a bottom, we virtually always choose the top side to be positive. Thus the direction of the positive normal is

$$\mathbf{n} = \widehat{\mathbf{r}_u \times \mathbf{r}_v} = \tfrac{1}{\sqrt{1+u^2}}\{-u\sin(v)\,\mathbf{i} + u\cos(v)\,\mathbf{j} + \mathbf{k}\};$$

however, it is largely irrelevant in calculating surface area, because $\mathbf{n}\cdot\mathbf{n} = 1$, regardless.

Now we can calculate A, which is just the flux of the unit normal over the surface:

$$A \;=\; \iint\limits_S dS \;=\; \iint\limits_S 1\,dS \;=\; \iint\limits_S \mathbf{n}\cdot\mathbf{n}\,dS \;=\; \iint\limits_S \mathbf{n}\cdot\mathbf{dS}. \tag{30}$$

On using (12), we obtain

$$\begin{aligned} A \;&=\; \iint\limits_S dS \;=\; \iint\limits_S \left|\mathbf{r}_u \times \mathbf{r}_v\right|\, du\,dv \\ &=\; \int\limits_{v=0}^{v=\pi/2}\int\limits_{u=0}^{u=1} \sqrt{1+u^2}\,du\,dv \;=\; \int\limits_0^{\pi/2} dv \int\limits_0^1 \sqrt{1+u^2}\,du \\ &=\; \tfrac{1}{2}\pi\left(\tfrac{1}{2}\{\ln(u+\sqrt{1+u^2}) + u\sqrt{1+u^2}\}\right)\Big|_0^1 \\ &=\; \tfrac{1}{4}\{\sqrt{2}+\ln(1+\sqrt{2})\}\pi. \end{aligned} \tag{31}$$