Level surface area elements for orthogonal coordinates

We already know from Lecture 17 that the standard representation of a surface has the form

$$\mathbf{r} \;=\; \mathbf{r}(u,v), \quad u_{min} \le u \le u_{max},\;\; v_{min} \le v \le v_{max} \tag{1}$$

where u and v are appropriately restricted curvilinear coordinates, and the corresponding surface area element is

$$dS \;=\; \big|\mathbf{r}_u \times \mathbf{r}_v\big| du\, dv. \tag{2}$$

Often—but by no means always*—these two curvilinear coordinates are orthogonal, that is, the direction in which u increases is perpendicular to the direction in which v increases, or

$$\mathbf{r}_u \cdot \mathbf{r}_v \;=\; 0. \tag{3}$$

Then

$$\mathbf{r}_u \times \mathbf{r}_v \;=\; \big|\mathbf{r}_u\big|\,\big|\mathbf{r}_v\big| \sin\left(\tfrac{1}{2}\pi\right) \widehat{\mathbf{r}_u \times \mathbf{r}_v} \;=\; \big|\mathbf{r}_u\big|\,\big|\mathbf{r}_v\big|\widehat{\mathbf{r}_u \times \mathbf{r}_v}, \tag{4}$$

implying $|\mathbf{r}_u \times \mathbf{r}_v| = |\mathbf{r}_u|\,|\mathbf{r}_v||\widehat{\mathbf{r}_u \times \mathbf{r}_v}| = |\mathbf{r}_u|\,|\mathbf{r}_v| \cdot 1 = |\mathbf{r}_u|\,|\mathbf{r}_v|$. Hence if—and only if—the curvilinear coordinates are orthogonal, then by (2) the surface area element is the product of the arc element $|\mathbf{r}_u|\,du$ generated when u increases infinitesimally while v is held fixed and the arc element $|\mathbf{r}_v|\,dv$ generated when v increases infinitesimally while u is held fixed:

$$dS \;=\; \big|\mathbf{r}_u\big|\,\big|\mathbf{r}_v\big| du\, dv \;=\; \big|\mathbf{r}_u\big| du\,\big|\,\big|\mathbf{r}_v\big| dv. \tag{5}$$

Often these arc elements—and hence the corresponding surface area element—can be readily determined by simple geometry, thus obviating the need to calculate $\mathbf{r}_u \times \mathbf{r}_v$.

For example, suppose that the underlying coordinate system is that of spherical polar coordinates ρ (radial extension, or distance from origin), θ (azimuth or longitude) and ϕ (colatitude). No matter where you are in space, you are at distance ρ from the origin on both a "great circle" (meridional circle) of radius ρ and a "small circle" (circle of latitude) of radius $(r =)\ \rho\sin(\phi)$, which is the distance from the axis between north and south poles. If you increase ρ a little bit while holding ϕ and θ constant, then you just move a little bit further directly away from the origin, and so the arc element is just $d\rho$; if you increase ϕ a little bit while holding ρ and θ constant, then you move a little bit further away from the north pole along your great circle in the direction of decreasing latitude, and so the arc element is RADIUS $\cdot\, d\phi = \rho\, d\phi$; and if you increase θ a little bit while holding ρ and ϕ constant, then you move a little bit further east along your small circle in the direction of increasing longitude, and so the arc element is RADIUS $\cdot\, d\theta = \rho\sin(\phi)\, d\theta$. Gathering our results together, the three arc elements are as follows:

$$\begin{aligned}
\big|\boldsymbol{\rho}_\rho\big|\, d\rho \;&=\; d\rho && \text{(increasing } \rho\text{; } \theta \text{ and } \phi \text{ held constant)}\\
\big|\boldsymbol{\rho}_\theta\big|\, d\theta \;&=\; \rho\sin(\phi)\, d\theta && \text{(increasing } \theta\text{; } \rho \text{ and } \phi \text{ held constant)}\\
\big|\boldsymbol{\rho}_\phi\big|\, d\phi \;&=\; \rho\, d\phi && \text{(increasing } \phi\text{; } \rho \text{ and } \theta \text{ held constant)}
\end{aligned} \tag{6}$$

*The most obvious exception is the triangular region of Lecture 18, whose coordinates are not orthogonal unless the triangle itself is right-angled.

If we multiply all three arc elements together, then (6) tells us that the volume element for spherical polars is $dV = d\rho \cdot \rho \sin(\phi)\, d\theta \cdot \rho\, d\phi = \rho^2 \sin(\phi)\, d\rho\, d\theta\, d\phi$, as of course we have now known for several weeks. However, volume elements do not concern us here: we are interested only in surface area elements. And to create a surface, we must fix one coordinate.

There are, of course, precisely three possibilities. First, if we fix ρ, say $\rho = a$ in (6), then we obtain a sphere of radius a, for which

$$dS \;=\; \left|\boldsymbol{\rho}_\theta\right| d\theta \cdot \left|\boldsymbol{\rho}_\phi\right| d\phi \;=\; a \sin(\phi)\, d\theta \cdot a\, d\phi \;=\; a^2 \sin(\phi)\, d\theta\, d\phi. \tag{7a}$$

The corresponding unit outward normal is

$$\mathbf{n} \;=\; \hat{\boldsymbol{\rho}} \;=\; \frac{\boldsymbol{\rho}}{|\boldsymbol{\rho}|} \;=\; \frac{\boldsymbol{\rho}}{a} \;=\; \frac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{a} \;=\; \sin(\phi)\cos(\theta)\mathbf{i} + a\sin(\phi)\sin(\theta)\mathbf{j} + a\cos(\phi)\mathbf{k} \tag{7b}$$

because $\rho = a$ on the sphere, implying $x = \{a\sin(\phi)\}\cos(\theta)$, $y = \{a\sin(\phi)\}\sin(\theta)$ and $z = \{a\cos(\phi)\}$. Thus we have determined $\mathbf{n}\, dS$ without needing to calculate $\boldsymbol{\rho}_\theta \times \boldsymbol{\rho}_\phi$.

If we fix θ, say $\theta = \omega$ in (6), then we obtain a meridional plane, for which

$$dS \;=\; \left|\boldsymbol{\rho}_\rho\right| d\rho \cdot \left|\boldsymbol{\rho}_\phi\right| d\phi \;=\; d\rho \cdot \rho\, d\phi \;=\; \rho\, d\rho\, d\phi. \tag{8a}$$

In this case—unlike the first—the corresponding unit outward normal is a constant vector

$$\mathbf{n} \;=\; -\sin(\omega)\mathbf{i} + \cos(\omega)\mathbf{j} + 0\,\mathbf{k} \tag{8b}$$

because the plane is vertical and makes a fixed angle $\theta = \omega$ with the x-z plane, from which the azimuthal angle is measured; note that $\mathbf{n}$ points east, the direction in which azimuth would be increasing if we hadn't fixed it. Here, in effect, we have standard two-dimensional polar coordinates turned around and shifted up into a vertical plane, with r, θ and $z = 0$ replaced by ρ, ϕ and $\theta = \omega$, respectively. But rarely does a use for (8) arise in practice—whereas (7) constantly earns its keep.

Finally, if we fix ϕ, say $\phi = \gamma$ in (6) with $0 < \gamma < \frac{1}{2}\pi$, then we obtain a cone, for which

$$dS \;=\; \left|\boldsymbol{\rho}_\rho\right| d\rho \cdot \left|\boldsymbol{\rho}_\theta\right| d\theta \;=\; d\rho \cdot \rho \sin(\gamma)\, d\theta \;=\; \sin(\gamma)\rho\, d\rho\, d\theta \tag{9a}$$

with corresponding downward unit normal

$$\begin{aligned} \mathbf{n} \;&=\; \cos(\gamma)\hat{\mathbf{r}} - \sin(\gamma)\mathbf{k} \\ &=\; \cos(\gamma)\cos(\theta)\mathbf{i} + \cos(\gamma)\sin(\theta)\mathbf{j} - \sin(\gamma)\mathbf{k} \end{aligned} \tag{9b}$$

where $\mathbf{r} = r\,\cos(\theta)\mathbf{i} + r\,\sin(\theta)\mathbf{j}$. Here, in effect, we have obtained $\mathbf{n}$ by first resolving it in terms of its horizontal (parallel to $\hat{\mathbf{r}}$) and vertical (parallel to $\mathbf{k}$) components in an arbitrary meridional plane. For that we need only a simple sketch. Thus we have determined $\mathbf{n}\, dS$ without needing to calculate $\boldsymbol{\rho}_\rho \times \boldsymbol{\rho}_\theta$, just as in the other cases.

In cylindrical polar coordinates, the three expressions corresponding to (7)–(9) are

$$dS \;=\; \left|\boldsymbol{\rho}_r\right| dr \cdot \left|\boldsymbol{\rho}_\theta\right| d\theta \;=\; dr\; \cdot r\, d\theta \;=\; r\, dr\, d\theta \tag{10a}$$

with unit normal

$$\mathbf{n} \;=\; \pm\,\mathbf{k} \tag{10b}$$

for a horizontal ($z =$ constant) plane,

$$dS \;=\; \left|\boldsymbol{\rho}_r\right| dr \cdot \left|\boldsymbol{\rho}_z\right| dz \;=\; r\, dr\, dz \tag{11a}$$

with unit normal

$$\mathbf{n} \;=\; \pm\{-\sin(\omega)\mathbf{i} + \cos(\omega)\mathbf{j} + 0\,\mathbf{k}\} \tag{11b}$$

for a meridional ($\theta =$ constant $= \omega$) plane and

$$dS \;=\; \left|\boldsymbol{\rho}_\theta\right| d\theta \cdot \left|\boldsymbol{\rho}_z\right| dz \;=\; a\, d\theta\, dz \tag{12a}$$

with outward unit normal

$$\mathbf{n} \;=\; \hat{\mathbf{r}} \;=\; \cos(\theta)\mathbf{i} + \sin(\theta)\mathbf{j} + 0\,\mathbf{k} \tag{12b}$$

for a vertical ($r =$ constant $= a$) cylinder with axis of symmetry along the z-axis. Rarely does a use for (11)—which corresponds, in effect, to Cartesian coordinates in a meridional plane—arise in practice, but (10) and (12) both earn their keep.