Fundamental Ideas in Vector Calculus
(and how to keep them separate)

First, a rundown of what kinds of questions will be on your final.

The new stuff

This will account for approximately 20%–30% of the points.

- one (perhaps multi-part) question on the divergence theorem (§16.9);
- one (perhaps multi-part) question on Stokes' theorem (§16.8);

Test 4 stuff

This will account for 20%–45% of the points.

- one question like number 3;
- ≥ 1 line integral (with one requiring Green's theorem);
- ≥ 1 surface integral/flux problem.

Note that these can be combined with the new stuff in obvious ways. For example, you may have to find a line integral using the old methods for part (a) of a question, and then find the same line integral in part (b) using Stokes' theorem.

The old stuff

This will account for the remaining points, and not all of the listed concepts will be tested and/or tested equally.

- using double and triple integrals (which may or may not include polar, cylindrical, or spherical coordinates) to find areas/volumes and volumes/hypervolumes, respectively (e.g. questions 2 and 4 on exam 3);
- directional derivatives/gradients (e.g. questions 3b, 3c, and 3d on exam 2);
- optimization of 2-variable functions (e.g. question 4 on exam 2);
- tangents/normals/curvature/etc. of vector-valued functions (e.g. question 3 on exam 1).

Miscellany

Here's some random info about the final not covered above.

- there **won't be** any matching questions (e.g. question 4 on exam 1);
- there **may or may not be** true/false questions; if there are, it will **only** be about the Chapter 16 stuff;
- you **won't** have to know trig identities;
- there **will not** be a formula sheet.

Now, let's answer some questions some of you may be struggling with!

Question: What is a vector field?

Short Answer: It's a function!

Longer Answer: It's a function that takes in a 2- or 3-dimensional vector and outputs a vector of the same length. Because it's a function, you're able to do all the normal stuff with it: You can do arithmetic, you can compose things by plugging vector functions into it, and you can do calculus things.

Question: What is a line integral?

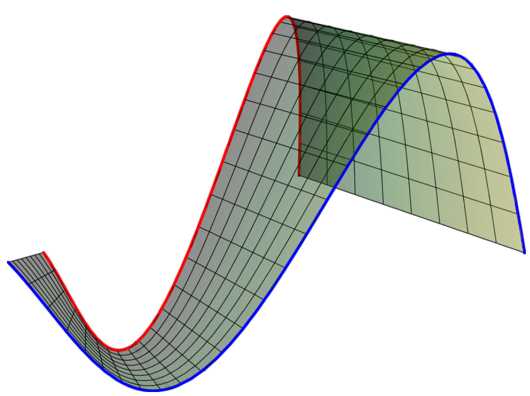
Answer: It's the "space curve" equivalent of a "regular integral" from Calculus I and II.

In those courses, your world is the xy-plane: You imagine the graph of a function $y = f(x)$ "living over" the x-axis and the "regular integral" represents the area bounded between the graph and the axis.

In this course, your world is $\mathbb{R}^3$: You can imagine making the "old x-axis" curvy and putting it in space, calling this "curvy axis" C, then considering the part of the graph of a two-variable function $z = f(x, y)$ which lives over C. The area of that chunk of graph is **precisely** what we mean when we write $\oint_C f(x, y)\, ds$!

Question: Right, okay...but what is that *vertical curtain* thing you talked about so much?

Answer: It's just a useful visualization of what the above-mentioned chunk of graph looks like!



In this picture, the "curvy axis" C is the red curve and the green "curtain" is the part of the graph of the two-var function $z = f(x, y)$ that lives above C (with the blue "top" just there for visual precision).

With this visual, $\oint_C f(x, y)\, ds$ represents the area of that green curtain.

Question: Okay, so how do we compute line integrals?

Answer: This answer kinda sorta depends on what the ingredients of the line integral look like.

If you have a function $z = f(x, y)$, then your steps to compute $\oint_C f(x, y)\, ds$ are as follows:

- Find a parametrization of the form $x = x(t)$, $y = y(t)$, $a \le t \le b$ for C. For example: If C is the upper half of the circle $x^2 + y^2 = 1$, then you can use $x = \cos t$, $y = \sin t$, $0 \le t \le \pi$.
- Plug $x = x(t)$ and $y = y(t)$ into $f(x, y)$ to get $f(x(t), y(t))$.
- Replace ds with $\sqrt{[x'(t)]^2 + [y'(t)]^2}\, dt$.
- Evaluate $\int_a^b f(x(t), y(t))\sqrt{[x'(t)]^2 + [y'(t)]^2}\, dt$. Inside, this is multiplication of two scalars.

Question: How do we compute the line integral of a vector field?

Answer: The process is very similar to the above.

If you have a vector field $\mathbf{F}$, then the steps for computing $\oint_C \mathbf{F} \cdot d\mathbf{r}$ are as follows:

- Find a vector function $\mathbf{r}(t)$, $a \leq t \leq b$, which parametrizes C. For example: If C is the upper half of the circle $x^2 + y^2 = 1$ like before, then you can use $\mathbf{r}(t) = \langle \cos t, \sin t \rangle$, $0 \leq t \leq \pi$.
- Plug $\mathbf{r}(t)$ into $\mathbf{F}$ to get $\mathbf{F}(\mathbf{r}(t))$.

  If $\mathbf{r}$ and $\mathbf{F}$ are two-dimensional, this means you identify $\mathbf{r}(t)$ as $\mathbf{r}(t) = \langle x(t), y(t) \rangle$ and replace all x's and y's inside $\mathbf{F}$ with the $x(t)$ and $y(t)$ from $\mathbf{r}$; if they're three-dimensional, then $\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$ and you replace the x's, y's, and z's inside $\mathbf{F}$ with $x(t)$, $y(t)$, and $z(t)$.
- Replace $d\mathbf{r}$ with $\mathbf{r}'(t)dt$.
- Evaluate $\int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt$. Inside, this is dot product of two vectors (and thus a scalar).

Question: What does $\oint_C \mathbf{F} \cdot d\mathbf{r}$ represent *physically*?

Answer: That line integral represents the amount of work done by $\mathbf{F}$ on an object moving along C.

Question: What is a conservative vector field?

Answer: A vector field $\mathbf{F}$ (of any dimension) is *conservative* if there exists a function f for which $\mathbf{F} = \nabla f$.

Such vector fields are important, particularly in physics, because they tend to *conserve* particular quantities in their respective systems.

Question: How can I tell if $\mathbf{F}$ is conservative?

Answer: There are lots of ways, but here are two good criteria (one for each of dimensions 2 and 3).

Assuming P and Q have continuous first-order partial derivatives and that the 2D vector field $\mathbf{F} = P\mathbf{i} + Q\mathbf{j}$ is defined on an open, simply-connected domain in $\mathbb{R}^2$, then $\mathbf{F}$ is conservative if and only if

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}.$$

Assuming P, Q, and R have continuous partial derivatives and that the 3D vector field $\mathbf{F} = P\mathbf{i}+Q\mathbf{j}+R\mathbf{k}$ is defined on an open, simply-connected region in $\mathbb{R}^3$, then $\mathbf{F}$ is conservative if and only if $\operatorname{curl} \mathbf{F} = \mathbf{0}$. This is true if and only if

$$\frac{\partial R}{\partial y} = \frac{\partial Q}{\partial z} \qquad \frac{\partial P}{\partial z} = \frac{\partial R}{\partial x} \qquad \frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$$

all hold.

On your test, it's a 99.9% certainty that the vector fields you're given will be defined on all of $\mathbb{R}^2$ or $\mathbb{R}^3$, in which case the conclusions of these criteria may be used without checking the hypotheses.