Partial derivatives

From long ago in Calculus I you are quite familiar with differentiating, say,

$$z \;=\; u(x) \;=\; \sin(xb^2) \tag{1}$$

to obtain

$$\frac{dz}{dx} \;=\; u'(x) \;=\; b^2\cos(xb^2). \tag{2}$$

You think of this as finding the derivative of an ordinary function of a single variable because you say that b is a constant. But what is a "constant"? It is something you are holding fixed for now, but might want to change another time. So you could just as easily think of u as a function of two variables, $u = u(x, b)$, and it wouldn't alter how u varied with x in the least: your calculation of the derivative would be just as valid as it was in Calculus I. You would want, however, to use a different notation, to emphasize that you are holding b fixed and allowing x to vary. So in place of (2) you would write

$$\frac{\partial z}{\partial x} \;=\; u_x(x, b) \;=\; b^2\cos(xb^2). \tag{3}$$

You are equally familiar with differentiating

$$z \;=\; v(y) \;=\; \sin(ay^2) \tag{4}$$

to obtain

$$\frac{dz}{dy} \;=\; v'(y) \;=\; 2ay\cos(ay^2). \tag{5}$$

Again, you think of this as finding the derivative of an ordinary function of a single variable on the grounds that a is constant. But again, a constant is only something you are holding fixed for now, but might allow to change another time. So you could just as easily think of v as a function of two variables, $v = v(a, y)$, and it wouldn't alter how v varied with y in the least: your calculation of the derivative would be just as valid. Again, however, you would want to use a different notation, to emphasize that you are holding a fixed and allowing y to vary. So in place of (5) you would write

$$\frac{\partial z}{\partial y} \;=\; v_y(a, y) \;=\; 2ay\cos(ay^2). \tag{6}$$

We have defined two functions of two variables, namely, u and v, by

$$u(x, b) \;=\; \sin(xb^2), \qquad v(a, y) \;=\; \sin(ay^2). \tag{7}$$

But as soon as we write these equations side by side, we realize that both functions are the same, because the definition of a function is independent of the particular symbols we choose to represent variables. So there is no difference in principle between writing (7) and writing

$$u(\clubsuit, \bigstar) \;=\; \sin(\clubsuit\bigstar^2), \qquad v(\clubsuit, \bigstar) \;=\; \sin(\clubsuit\bigstar^2), \tag{8}$$

from which it is instantly obvious that $u \equiv v$. But ♣ and ★ are a bit clumsy; let's use x and y instead. Then we can dispense with separate u and v, because all we have is a single function of two variables f defined by

$$f(x, y) = \sin(xy^2) \tag{9}$$

with surface graph $z = f(x, y)$ for which

$$\frac{\partial z}{\partial x} = f_x(x, y) = y^2 \cos(xy^2) \tag{10}$$

$$\frac{\partial z}{\partial y} = f_y(x, y) = 2xy \cos(xy^2) \tag{11}$$

in place of (3) and (6). In essence, y being a variable does not prevent us from holding it constant to differentiate with respect to x, and x being a variable does not prevent us from holding it constant to differentiate with respect to y.

We refer to $\frac{\partial z}{\partial x}$ or $\frac{\partial z}{\partial y}$ as the partial derivative of z with respect to x or y, respectively. A partial derivative is simply an ordinary derivative with all other independent variables held fixed: geometrically, at any point (x, y, z) on the surface $z = f(x, y)$, $\frac{\partial z}{\partial x} = f_x(x, y)$ gives you the slope of the surface in the direction of increasing x, or parallel to the x-axis; whereas $\frac{\partial z}{\partial y} = f_y(x, y)$ gives you the slope of the surface in the direction of increasing y, or parallel to the y-axis. For example, suppose you are climbing the hill defined by $z = \sin(xy^2)$ at the point where $x = \frac{2\pi}{3}$, $y = \frac{1}{2}$ (and hence $z = \sin(\pi/6) = \frac{1}{2}$), when you start to feel very tired. Should you continue climbing parallel to the x-axis or the y-axis at that point? From (10)-(11) we have

$$\left.\frac{\partial z}{\partial x}\right|_{\substack{x=2\pi/3\\ y=1/2}} = f_x(2\pi/3, 1/2) = \left(\tfrac{1}{2}\right)^2 \cos(\pi/6) = \tfrac{1}{8}\sqrt{3} \tag{12}$$

$$\left.\frac{\partial z}{\partial y}\right|_{\substack{x=2\pi/3\\ y=1/2}} = f_y(2\pi/3, 1/2) = 2\,\tfrac{2\pi}{3}\tfrac{1}{2} \cos(\pi/6) = \tfrac{\pi}{\sqrt{3}}. \tag{13}$$

So if you're tired, it is better to go east: going north is $\frac{8\pi}{3} \approx 8.4$ times as steep! Similarly,

$$\frac{\partial z}{\partial x} = y^2 \cos(xy^2), \qquad \frac{\partial z}{\partial y} = 2xy \cos(xy^2) \tag{14}$$

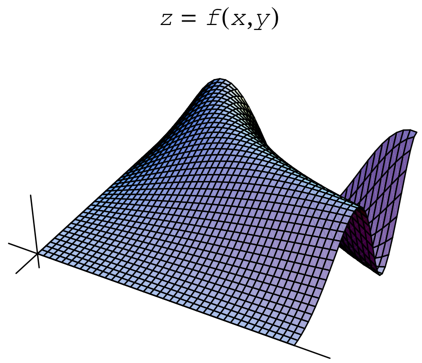
implies that the ground is rising where $0 < xy^2 < \frac{\pi}{2}$ and $y > 0$, regardless of whether you are walking east or north, because $0 < xy^2 < \frac{\pi}{2}$ makes $\cos(xy^2)$ positive and ensures that both $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ are positive. Similarly, the ground is falling where $\frac{\pi}{2} < xy^2 < \frac{3\pi}{2}$ and $y > 0$, regardless of whether you are walking east or north, because $\frac{\pi}{2} < xy^2 < \frac{3\pi}{2}$ makes $\cos(xy^2)$ negative. The surface graph overleaf confirms that the ground indeed rises in both directions up to the ridge along

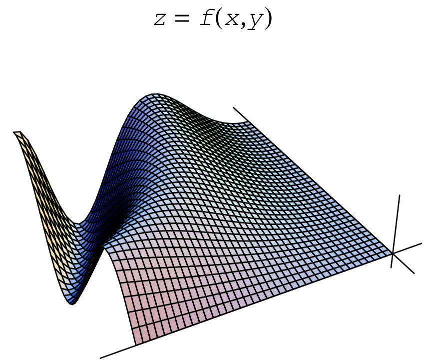
$$y = \sqrt{\frac{\pi}{2x}} \tag{15}$$

and then falls in both directions to the trough at

$$y = \sqrt{\frac{3\pi}{2x}}. \tag{16}$$

Two different views are shown, with the x-axis pointing to the right in the diagram on the left (exactly as in Lecture 1).



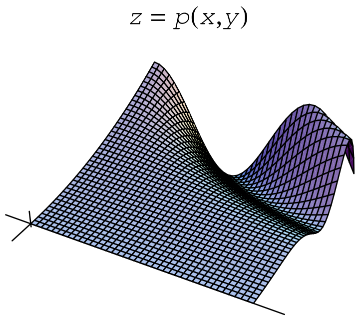


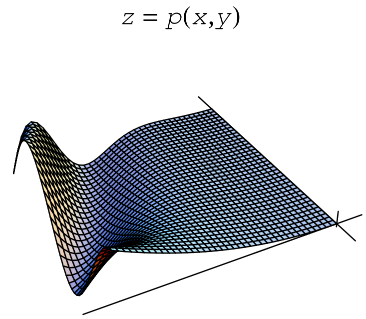
Just as an ordinary derivative gives you a brand new function of single variable, so also does each partial derivative give you a brand new function of two variables (with essentially the same domain). We call the two new functions p and q. That is, we define

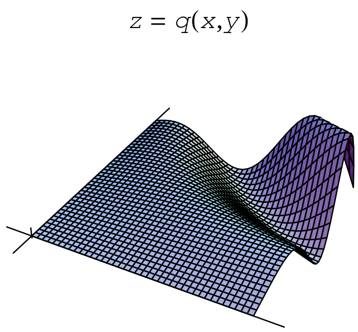
$$p(x,y) = \frac{\partial f}{\partial x} = y^2 \cos(xy^2) \tag{17}$$

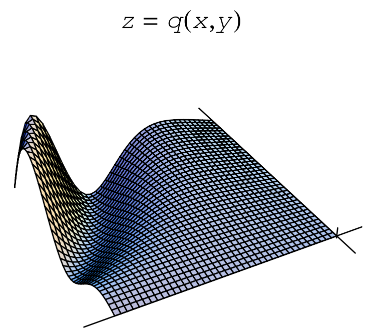
$$q(x,y) = \frac{\partial f}{\partial y} = 2xy \cos(xy^2). \tag{18}$$

The figure below shows the surface graphs of p and q above for $0 \leq x \leq 2, 0 \leq y \leq 2$.









But if being a function of two variables means that f has two derivatives, doesn't being a function of two variables mean that p and q have two derivatives also? Yes, of course. Holding y fixed while we differentiate with respect to x, we obtain

$$\begin{aligned} \frac{\partial p}{\partial x} &= \frac{\partial}{\partial x}\big(y^2 \cos(xy^2)\big) \;=\; y^2 \frac{\partial}{\partial x} \cos(xy^2) \\ &= y^2 \left\{-\sin(xy^2) \times y^2\right\} \;=\; -y^4 \sin(xy^2). \end{aligned} \tag{19}$$

This is yet another function of two variables, just like f, p and q, although this time we won't bother to reserve it a brand new letter for itself. Instead we'll note that because $p = \frac{\partial f}{\partial x}$ is the partial derivative of f with respect to x and $\frac{\partial p}{\partial x}$ is the partial derivative of p with respect to x, we can call it the second partial derivative of f with respect to x and rewrite (19) as

$$\frac{\partial^2 f}{\partial x^2} \;=\; \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) \;=\; -y^4 \sin(xy^2). \tag{20}$$

Similarly, because $q = \frac{\partial f}{\partial y}$ is the partial derivative of f with respect to y, we can differentiate it in turn with respect to y—while holding x constant—to obtain the second partial derivative of f with respect to y. That is, we obtain

$$\begin{aligned}
\frac{\partial^2 f}{\partial y^2} &= \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial q}{\partial y} = \frac{\partial}{\partial y}\left\{2xy\,\cos(xy^2)\right\} \\
&= \frac{\partial}{\partial y}\{2xy\}\,\cos(xy^2) + 2xy\,\frac{\partial}{\partial y}\left\{\cos(xy^2)\right\} \\
&= 2x\,\cos(xy^2) + 2xy\,\left\{-\sin(xy^2)\cdot 2xy\right\} \\
&= 2x\,\left\{\cos(xy^2) - 2xy^2\,\sin(xy^2)\right\}
\end{aligned} \tag{21}$$

The properties of these second partial derivatives are identical to those of the corresponding ordinary derivatives, as long as you remember in which direction you are heading. For example, $\frac{\partial^2 f}{\partial x^2} < 0$ means that f is concave down *in the direction of increasing* x. Similarly, $\frac{\partial^2 f}{\partial y^2} < 0$ means that f is concave down *in the direction of increasing* y.

However, although the properties of these second partial derivatives are identical to those of the corresponding ordinary derivatives, these are not the only second partial derivatives: with two independent variables, there are also the mixed partial derivatives

$$\begin{aligned}
\frac{\partial^2 f}{\partial y \partial x} &= \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = p_y = \frac{\partial}{\partial y}\left\{y^2\,\cos(xy^2)\right\} \\
&= \frac{\partial}{\partial y}\left\{y^2\right\}\,\cos(xy^2) + y^2\,\frac{\partial}{\partial y}\left\{\cos(xy^2)\right\} \\
&= 2y\,\cos(xy^2) + y^2\,\left\{-\sin(xy^2)\cdot 2xy\right\} \\
&= 2y\,\left\{\cos(xy^2) - xy^2\,\sin(xy^2)\right\}
\end{aligned} \tag{22}$$

and

$$\begin{aligned}
\frac{\partial^2 f}{\partial x \partial y} &= \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = q_x = \frac{\partial}{\partial x}\left\{2xy\,\cos(xy^2)\right\} \\
&= \frac{\partial}{\partial x}\{2xy\}\,\cos(xy^2) + 2xy\,\frac{\partial}{\partial x}\left\{\cos(xy^2)\right\} \\
&= 2y\,\cos(xy^2) + 2xy\,\left\{-\sin(xy^2)\cdot y^2\right\} \\
&= 2y\,\left\{\cos(xy^2) - xy^2\,\sin(xy^2)\right\}.
\end{aligned} \tag{23}$$

Note that they are both the same—so there's really only one of them. This turns out to be a general property of mixed partial derivatives, at least for sufficiently well behaved (= sufficiently smooth) functions, which are largely the only ones we consider in this course.

The existence of the mixed partial derivative is obviously a major difference between univariate and bivariate functions. We'll discuss other differences later; for now, the above will suffice to get us started on some problems.