Line integrals

Suppose that a particle is moving in a straight line under the influence of a constant force of magnitude F: perhaps the particle is a marble constrained to roll in a groove, or perhaps it's a bead constrained to slide on a wire. For the sake of argument, suppose that the straight line is the x-axis, and that the force is parallel to it. Then the work done by the force on the particle in moving it an infinitesimal distance δx in an infinitesimal time δt is $\delta W = F\,\delta x$, and the rate at which the force does work on the particle is

$$\frac{\delta W}{\delta t} \;=\; F\,\frac{\delta x}{\delta t}. \tag{1}$$

Rate of doing work is known as power.

The above relationship is exact, because F is constant. If F depends on time t or displacement x, however, then the relationship is only approximate, because F varies a little as time increases from t to $t + \delta t$ and displacement increases from x to $x + \delta x$:

$$\frac{\delta W}{\delta t} \;\approx\; F\,\frac{\delta x}{\delta t}. \tag{2}$$

But we can obtain an exact relationship by taking a limit as $\delta t \to 0$, which implies that $\delta x \to 0$ as well; then (2) becomes

$$\frac{dW}{dt} \;=\; F\,\frac{dx}{dt}. \tag{3}$$

Now suppose that the force, $\mathbf{F}$, is no longer parallel to the particle's line of motion, but acts instead in the x-y plane at an angle θ to the x-axis. We resolve $\mathbf{F}$ into its two components, $F\cos(\theta)$ in the direction of motion and $F\sin(\theta)$ perpendicular to that direction:

$$\mathbf{F} \;=\; F\cos(\theta)\,\mathbf{i} + F\sin(\theta)\,\mathbf{j}. \tag{4}$$

The second component does no work, because it cannot move the particle in the direction of the y-axis, because the particle is constrained to move only in the direction of the x-axis: only the first component does work. So the power equation becomes

$$\frac{dW}{dt} \;=\; F\cos(\theta)\,\frac{dx}{dt} \;=\; F\cos(\theta)\,v \tag{5}$$

where v denotes speed. Because the particle is constrained to satisfy

$$\frac{dy}{dt} \;=\; 0 \tag{6}$$

(and hence $\hat{\mathbf{v}} = \mathbf{i}$), we can rewrite (5) in vector form as

$$\frac{dW}{dt} = F\cos(\theta)\,\frac{dx}{dt} + F\sin(\theta)\,\frac{dy}{dt} = \big(F\cos(\theta)\,\mathbf{i} + F\sin(\theta)\,\mathbf{j}\big)\cdot\Big(\frac{dx}{dt}\,\mathbf{i} + \frac{dy}{dt}\,\mathbf{j}\Big) = \mathbf{F}\cdot\frac{d\mathbf{r}}{dt} = \mathbf{F}\cdot\mathbf{v} \tag{7}$$

where

$$\mathbf{v} \;=\; \frac{d\mathbf{r}}{dt} \;=\; \frac{dx}{dt}\,\mathbf{i} + \frac{dy}{dt}\,\mathbf{j} \tag{8}$$

is the velocity of the particle.

Now suppose that the particle is constrained to move, not along a line, but instead along an arbitrary two-dimensional curve with equation $\mathbf{r} = \mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}$. It is no longer true that $\hat{\mathbf{v}} = \mathbf{i}$; rather, from Lecture 14, $\hat{\mathbf{v}} = \mathbf{T}$, where $\mathbf{T}$ is the unit tangent vector. In fact, $\mathbf{v} = v\mathbf{T}$, where v denotes speed. But it is still true that only the force component parallel to the direction of motion does any work. Let us resolve the force into tangential and normal components, denoted by F_T and F_N, respectively:

$$\mathbf{F} \;=\; F_T\mathbf{T} + F_N\mathbf{N} \tag{9}$$

where, from Lecture 14, $\mathbf{N}$ is the principal unit normal vector. Because all the work is done by the tangential component F_T, in place of (7) we obtain

$$\frac{dW}{dt} \;=\; F_T v \;=\; F_T\frac{ds}{dt}, \tag{10}$$

where s denotes distance travelled along the curve, again as in Lecture 14. From (9), however, we have

$$\mathbf{F}\cdot\mathbf{v} \;=\; (F_T\mathbf{T}+F_N\mathbf{N})\cdot(v\mathbf{T}) \;=\; v(F_T\mathbf{T}\cdot\mathbf{T}+F_N\mathbf{N}\cdot\mathbf{T}) \;=\; v(F_T\cdot 1+vF_N\cdot 0) \;=\; F_T v \tag{11}$$

because $\mathbf{T}$ and $\mathbf{N}$ are orthogonal unit vectors. From (10)–(11) we now obtain

$$\frac{dW}{dt} \;=\; \mathbf{F}\cdot\mathbf{v} \;=\; \mathbf{F}\cdot\frac{d\mathbf{r}}{dt} \tag{12}$$

as before. In other words, (7) still holds.

Because there is no component of force in the direction of $\mathbf{B}$ (the binormal), the equation $\frac{dW}{dt} = \mathbf{F}\cdot\frac{d\mathbf{r}}{dt}$ readily generalizes to three dimensions; in fact, vector equations always do (and hence are said to have invariant form). That is, $\mathbf{F}\cdot\frac{d\boldsymbol{\rho}}{dt}$ is the rate at which any three-dimensional force $\mathbf{F}$ does work on a particle whose trajectory in three dimensions is $\boldsymbol{\rho} = \boldsymbol{\rho}(t)$. Now we have a dilemma. On the one hand, if we want to talk about results that are true in both two and three dimensions, we should probably denote the generic position vector by $\boldsymbol{\rho}$, because three dimensions are more inclusive than two. On the other hand, much of the world has been using $\mathbf{r}$ for the generic position vector in both two and three dimensions since long before you and even I were born: and for the most part the world does not get confused, because it simply recognizes that the meaning of $\mathbf{r}$ is

$$\mathbf{r} \;=\; \begin{cases} x\mathbf{i} + y\mathbf{j} & \text{in two dimensions} \\ x\mathbf{i} + y\mathbf{j} + z\mathbf{k} \quad (\;=\;\boldsymbol{\rho}) & \text{in three dimensions.} \end{cases}$$

Moreover, if there is any danger at all of confusion in three dimensions, then one can denote $\sqrt{x^2+y^2}$ by R instead of by r (and so use r as an alternative to ρ for the magnitude of $x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$). What shall we do? Make a point of using $\mathbf{r}$ in two dimensions and $\boldsymbol{\rho}$ in three, so that, in effect, what is really only a single result has to be written out twice every time (thus largely destroying the beauty of using vectors to begin with)? Or use $\boldsymbol{\rho}$ for both? Or follow the prevailing convention and use $\mathbf{r}$ for both? . . . At least for now, we will follow convention. Thus, in either two or three dimensions, the total work that the force $\mathbf{F}$ does on our particle between times $t = a$ and $t = b$ can be written as

$$W_{tot} \;=\; \int_a^b \frac{dW}{dt}\, dt \;=\; \int_a^b \mathbf{F} \cdot \frac{d\mathbf{r}}{dt}\, dt, \tag{13}$$

provided that the particle's trajectory is known. You will often see the above equation written in shorthand as

$$W_{tot} \;=\; \int_C dW \;=\; \int_C \mathbf{F} \cdot \mathbf{dr} \tag{14}$$

where C denotes the curve on which the particle moves between its initial position vector $\mathbf{r}(a)$ and its final position vector $\mathbf{r}(b)$. But that doesn't alter the simple fact that (13) and (14) mean exactly the same thing. The integral

$$\int_C \mathbf{F} \cdot \mathbf{dr} \;=\; \int_a^b \mathbf{F} \cdot \frac{d\mathbf{r}}{dt}\, dt \;=\; \int_a^b \mathbf{F} \cdot \mathbf{v}\, dt \tag{15}$$

is called a line integral (even when C is not a straight line, which it usually isn't). Its value is independent of how the curve is parameterized, and in the special case where $\mathbf{r}(b) = \mathbf{r}(a)$ this value is called the *circulation* of $\mathbf{F}$ around the curve. Then, to indicate that the curve is closed, the integral is usually written in the form

$$\oint_C \mathbf{F} \cdot \mathbf{dr}. \tag{16}$$

Note that if the curve is parameterized by s (arc length), then because $\frac{d\mathbf{r}}{ds} = \mathbf{T}$, we have

$$\int_C \mathbf{F} \cdot \mathbf{dr} \;=\; \int_{s_a}^{s_b} \mathbf{F} \cdot \mathbf{T}\, ds \tag{17}$$

by (15), where $s = s_a$ when $t = a$ and $s = s_b$ when $t = b$.

We should also note that $\mathbf{F}$ is often a function of x, y and z in the first instance, that is, there exist (sufficiently smooth) functions F_1, F_2 and F_3 such that

$$\mathbf{F} \;=\; F_1(x, y, z)\mathbf{i} + F_2(x, y, z)\mathbf{j} + F_3(x, y, z)\mathbf{k}, \tag{18}$$

but because x, y and z all depend on t along C, so also $\mathbf{F}$ depends only on t along C. Whenever (18) holds, we say that $\mathbf{F}$ is a vector field.

Because a line integral's value is independent of how the curve C is parameterized, we always try to choose the most natural representation, often breaking the line integral up into segments so that, e.g., we can use a Cartesian coordinate to parameterize C on any segment that is parallel to a coordinate axis and a polar coordinate to parameterize C where it forms an arc of a circle. For illustration, suppose that

$$\mathbf{F} \;=\; -x(1 + \sqrt{y})\mathbf{i} + y(\sqrt{x} - 1)\mathbf{j} \tag{19}$$

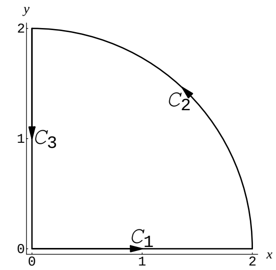

Figure 1: A closed curve $C = C_1 \cup C_2 \cup C_3$

and $C = C_1 \cup C_2 \cup C_3$ consists of the two straight-line segments and quarter-circle depicted in Figure 1. Then $\mathbf{F}$'s circulation is

$$\oint\limits_C \mathbf{F} \cdot \mathbf{dr} \;=\; \int\limits_{C_1} \mathbf{F} \cdot \mathbf{dr} + \int\limits_{C_2} \mathbf{F} \cdot \mathbf{dr} + \int\limits_{C_3} \mathbf{F} \cdot \mathbf{dr}. \tag{20}$$

Along C_1 we have $\mathbf{dr} = dx\,\mathbf{i} + dy\,\mathbf{j} = dx\,\mathbf{i}$ (because $dy = 0$ on C_1), implying $\mathbf{F} \cdot \mathbf{dr} = -x\,dx$ (because $\sqrt{y} = 0$ on C_1). Thus the natural parameter for C_1 is the Cartesian coordinate x:

$$\int\limits_{C_1} \mathbf{F} \cdot \mathbf{dr} \;=\; -\int_{x=0}^{x=2} x\,dx \;=\; -\tfrac{1}{2}x^2\Big|_0^2 \;=\; -\tfrac{1}{2}(2^2 - 0^2) \;=\; -2. \tag{21}$$

Because C_2 is an arc of a circle with radius 2 and center (0,0), its natural parameter is the polar coordinate θ: with $x = 2\cos(\theta)$ and $y = 2\sin(\theta)$ or

$$\mathbf{r} \;=\; 2\cos(\theta)\mathbf{i} + 2\sin(\theta)\mathbf{j}, \tag{22}$$

as θ increases from 0 to $\frac{1}{2}\pi$ we traverse C_2 from $(1, 0)$ to $(0, 1)$ in a counterclockwise direction; see Figure 1 . Note that (22) implies

$$\frac{d\mathbf{r}}{d\theta} \;=\; -2\sin(\theta)\mathbf{i} + 2\cos(\theta)\mathbf{j}, \tag{23}$$

and hence

$$\begin{aligned} \mathbf{F} \cdot \frac{d\mathbf{r}}{d\theta} \;&=\; -2\cos(\theta)\{1 + \sqrt{2\sin(\theta)}\} \cdot \{-2\sin(\theta)\} + \\ &\qquad\qquad 2\sin(\theta)\{\sqrt{2\cos(\theta)} - 1\} \cdot \{2\cos(\theta)\} \\ &=\; 4\sqrt{2}\left(\{\sin(\theta)\}^{3/2}\cos(\theta) + \{\cos(\theta)\}^{3/2}\sin(\theta)\right). \end{aligned} \tag{24}$$

So

$$
\begin{aligned}
\int\limits_{C_2} \mathbf{F}\cdot d\mathbf{r} \;&=\; \int_{\theta=0}^{\theta=\pi/2} \mathbf{F}\cdot\frac{d\mathbf{r}}{d\theta}\,d\theta \\
&=\; 4\sqrt{2}\int_{\theta=0}^{\theta=\pi/2}\left(\{\sin(\theta)\}^{3/2}\cos(\theta)+\{\cos(\theta)\}^{3/2}\sin(\theta)\right)d\theta \\
&=\; 4\sqrt{2}\int_0^{\pi/2}\frac{d}{d\theta}\left(\tfrac{2}{5}\{\sin(\theta)\}^{5/2}-\tfrac{2}{5}\{\cos(\theta)\}^{5/2}\right)d\theta \\
&=\; 4\sqrt{2}\left(\tfrac{2}{5}\{\sin(\theta)\}^{5/2}-\tfrac{2}{5}\{\cos(\theta)\}^{5/2}\right)\Big|_0^{\pi/2} \\
&=\; \tfrac{8}{5}\sqrt{2}\left(\{\sin(\pi/2)\}^{5/2}+\{\cos(0)\}^{5/2}\}\right) \;=\; \tfrac{16}{5}\sqrt{2}.
\end{aligned}
\tag{25}
$$

Finally, along C_3 we have $\mathbf{dr} = dx\,\mathbf{i} + dy\,\mathbf{j} = dy\,\mathbf{j}$ (because $dx = 0$ on C_3), implying $\mathbf{F}\cdot\mathbf{dr} = -y\,dy$ (because $\sqrt{x} = 0$ on C_3). Thus the natural parameter for C_3 is the Cartesian coordinate y:

$$
\int\limits_{C_3} \mathbf{F}\cdot d\mathbf{r} \;=\; -\int_{y=2}^{y=0} y\,dy \;=\; -\tfrac{1}{2}y^2\Big|_2^0 \;=\; -\tfrac{1}{2}(0^2-2^2) \;=\; 2. \tag{26}
$$

Substituting from (21) and (25)-(26) into (20) now yields

$$
\oint\limits_{C} \mathbf{F}\cdot d\mathbf{r} \;=\; \tfrac{16}{5}\sqrt{2}. \tag{27}
$$

Irrotational fields

A vector field $\mathbf{F}$ is called irrotational if its circulation around *any* closed curve is zero. An irrotational field must automatically also have the property that its line integral is the same for any path between two different points. Why? Let C_1 and C_2 be any two paths between the points, and let C_3 denote C_2 traversed in the opposite direction. Then

$$
\int\limits_{C_1} \mathbf{F}\cdot d\mathbf{r} - \int\limits_{C_2} \mathbf{F}\cdot d\mathbf{r} \;=\; \int\limits_{C_1} \mathbf{F}\cdot d\mathbf{r} + \int\limits_{C_3} \mathbf{F}\cdot d\mathbf{r} \;=\; \oint\limits_{C_1\cup C_3} \mathbf{F}\cdot d\mathbf{r} \;=\; 0 \tag{28}
$$

if $\mathbf{F}$ is irrotational, because $C_1 \cup C_3$ is a closed curve. For this reason, irrotational fields are also known as path-independent fields.

All irrotational fields have the form $\mathbf{F} = \nabla\phi(\mathbf{r})$, where $\phi(\mathbf{r})$ is a shorthand for $\phi(x, y, z)$ in three dimensions but for $\phi(x, y)$ in two; the scalar ϕ is known as the potential. To see why all gradients must be irrotational, let an arbitrary curve C be parameterized by t for $a \le t \le b$. Then its endpoints are $\mathbf{r}(a)$ and $\mathbf{r}(b)$, so that

$$
\int\limits_{C} \nabla\phi\cdot d\mathbf{r} \;=\; \int\limits_{t=a}^{t=b} \nabla\phi\cdot\frac{d\mathbf{r}}{dt}\,dt \;=\; \int\limits_{t=a}^{t=b} \frac{d\phi}{dt}\,dt \;=\; \phi(\mathbf{r}(b)) - \phi(\mathbf{r}(a)), \tag{29}
$$

by the chain rule (Lecture 8, p. 3). But if C is closed, then its endpoints coincide, implying $\mathbf{r}(b) = \mathbf{r}(a)$, which reduces (29) to zero. It is more difficult to show that all irrotational fields are gradients—but it happens to be true, although we postpone the proof to Lecture 21.

Meanwhile, a practical consequence of the above result is that line integrals of irrotational fields are easy to calculate, by (29). For example, suppose that

$$\mathbf{F} = \{2xy + 3\}\mathbf{i} + \{x^2 - 3y^2\}\mathbf{j} \tag{30}$$

and that C_1, C_2 and C_3 are still the arcs depicted in Figure 1. Then because $\mathbf{F} = \nabla\phi$ for

$$\phi(x, y) = x^2y - y^3 + 3x \tag{31}$$

as you can readily check, we obtain

$$\begin{aligned}
\int_{C_1} \mathbf{F}\cdot d\mathbf{r} &= \phi(2,0) - \phi(0,0) = 6 - 0 = 6 \\
\int_{C_2} \mathbf{F}\cdot d\mathbf{r} &= \phi(0,2) - \phi(2,0) = -8 - 6 = -14 \\
\int_{C_3} \mathbf{F}\cdot d\mathbf{r} &= \phi(0,0) - \phi(0,2) = 0 - (-8) = 8
\end{aligned} \tag{32}$$

(where the notation is arguably a tad inconsistent but nevertheless self-explanatory). These three line integrals add up to zero, as they must, because the irrotationality of $\mathbf{F}$ implies that (16) equals zero for *any* closed curve.