# Model-based reinforcement learning for infinite-horizon approximate optimal tracking

Rushikesh Kamalapurkar, Lindsey Andrews, Patrick Walters, and Warren E. Dixon

Abstract—This paper provides an approximate online adaptive solution to the infinite-horizon optimal tracking problem for control-affine continuous-time nonlinear systems with unknown drift dynamics. Model-based reinforcement learning is used to relax the persistence of excitation condition. Model-based reinforcement learning is implemented using a concurrent learning-based system identifier to simulate experience by evaluating the Bellman error over unexplored areas of the state space. Tracking of the desired trajectory and convergence of the developed policy to a neighborhood of the optimal policy are established via Lyapunov-based stability analysis. Simulation results demonstrate the effectiveness of the developed technique.

Index Terms—reinforcement learning, optimal control, datadriven control, nonlinear control, system identification

# I. INTRODUCTION

In the past few decades, reinforcement learning (RL)-based techniques have been effectively utilized to obtain online approximate solutions to optimal control problems for systems with finite state-action spaces, and stationary environments (cf. [1], [2]). However, progress for systems with continuous state-action spaces has been slow due to various technical challenges (cf. [3], [4]). Various implementations of RL-based learning strategies to solve deterministic optimal regulation problems can be found in results such as [5]–[16].

Offline and online approaches to solve infinite-horizon tracking problems are proposed in results such as [17]–[22]. Results such as [18], [21]–[23] solve optimal tracking problems for linear and nonlinear systems online, where persistence of excitation (PE) of the error states is used to establish convergence. In general, it is impossible to guarantee PE a priori; hence, a probing signal designed using trial and error is added to the controller to ensure PE. However, the probing signal is not considered in the stability analysis. In this paper, the objective is to employ data-driven model-based RL to design an online approximate optimal tracking controller for continuous-time uncertain nonlinear systems under a relaxed finite excitation condition.

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Submitted to the special issue on New Developments in Neural Network Structures for Signal Processing, Autonomous Decision, and Adaptive Control RL in systems with continuous state and action spaces is realized via value function approximation, where the value function corresponding to the optimal control problem is approximated using a parametric universal approximator. The control policy is generally derived from the approximate value function; hence, obtaining a good approximation of the value function is critical to the stability of the closed-loop system. In trajectory tracking problems, the value function depends explicitly on time. Since universal function approximators can approximate functions with arbitrary accuracy only on compact domains, value functions for infinite-horizon optimal tracking problems can not be approximated with arbitrary accuracy [17], [23].

If the desired trajectory can be expressed as the output of an autonomous dynamical system, then the value function can be expressed as a stationary (time-independent) function of the state and the desired trajectory. Hence, universal function approximators can be employed to approximate the value function with arbitrary accuracy by using the system state, augmented with the desired trajectory, as the training input (cf. [17], [21]–[23]).

The technical challenges associated with the nonautonomous nature of the trajectory tracking problem are addressed in the author's previous work in [23], where it is established that under a matching condition on the desired trajectory, the optimal trajectory tracking problem can be reformulated as a stationary optimal control problem. Since the value function associated with a stationary optimal control problem is time-invariant, it can be approximated using traditional function approximation techniques.

The aforementioned reformulation in [23] requires computation of the steady-state tracking controller, which depends on the system model; hence, the development in [23] requires exact model knowledge. Obtaining an accurate estimate of the desired steady-state controller, and injecting the resulting estimation error in the stability analysis are the major technical challenges in extending the work in [23] to uncertain systems. In this paper and in the preliminary work in [24], a concurrent learning (CL)-based system identifier is used to estimate the desired steady-state controller and model-based RL is used to simulate experience by evaluating the Bellman error (BE) over unexplored areas of the state space [24]-[27]. The error between the actual steady-state controller and its estimate is included in the stability analysis by formulating the Hamilton-Jacobi-Bellman equation in terms of the actual steady-state controller, and the effectiveness of the developed technique is demonstrated via numerical simulations.

The main contributions of this work include: 1) Approx-

imate model inversion using a CL-based system identifier to approximate the desired steady-state controller in the presence of uncertainties in the drift dynamics, 2) Implementation of model-based RL to relax the PE condition to a finite excitation condition, 3) Simulation results that demonstrate approximation of the optimal policy without an added exploration signal.

### II. PROBLEM FORMULATION AND EXACT SOLUTION

Consider a control affine system described by the differential equation  $\dot{x} = f(x) + g(x)u$ , where  $x \in \mathbb{R}^n$  denotes the state,  $u \in \mathbb{R}^m$  denotes the control input, and  $f : \mathbb{R}^n \to \mathbb{R}^n$  and  $g: \mathbb{R}^n \to \mathbb{R}^{n \times m}$  are locally Lipschitz continuous functions that denote the drift dynamics, and the control effectiveness, respectively. The control objective is to optimally track a time-varying desired trajectory  $x_d \in \mathbb{R}^n$ . To facilitate the subsequent control development, an error signal  $e \in \mathbb{R}^n$  is defined as  $e \triangleq x - x_d$ . Since the steady-state control input that is required for the system to track a desired trajectory is, in general, not identically zero, an infinite-horizon total-cost optimal control problem formulated in terms of a quadratic cost function containing e and u always results in an infinite cost. To address this issue, an alternative cost function is formulated in terms of the tracking error and the mismatch between the actual control signal and the desired steady-state control [17], [21]-[23]. The following assumptions facilitate the determination of the desired steady-state control.

**Assumption 1.** [23] The function g is bounded, the matrix g(x) has full column rank for all  $x \in \mathbb{R}^n$ , and the function  $g^+: \mathbb{R}^n \to \mathbb{R}^{m \times n}$  defined as  $g^+ \triangleq \left(g^T g\right)^{-1} g^T$  is bounded and locally Lipschitz.

**Assumption 2.** [23] The desired trajectory is bounded by a known positive constant  $d \in \mathbb{R}$  such that  $||x_d|| \leq d$ , and there exists a locally Lipschitz function  $h_d : \mathbb{R}^n \to \mathbb{R}^n$  such that  $\dot{x}_d = h_d(x_d)$  and

$$g\left(x_{d}\right)g^{+}\left(x_{d}\right)\left(h_{d}\left(x_{d}\right)-f\left(x_{d}\right)\right)=h_{d}\left(x_{d}\right)-f\left(x_{d}\right),$$

for all  $t \in \mathbb{R}_{\geq t_0}$ .

Based on Assumptions 1 and 2, the steady-state control policy  $u_d: \mathbb{R}^n \to \mathbb{R}^m$  required for the system to track the desired trajectory  $x_d$  can be expressed as  $u_d(x_d) = g_d^+(h_d(x_d) - f_d)$ , where  $f_d \triangleq f(x_d)$  and  $g_d^+ \triangleq g^+(x_d)$ . The error between the actual control signal and the desired steady-state control signal is defined as  $\mu \triangleq u - u_d(x_d)$ . Using  $\mu$ , the system dynamics can be expressed in the autonomous form

$$\dot{\zeta} = F(\zeta) + G(\zeta)\mu,\tag{1}$$

where the concatenated state  $\zeta \in \mathbb{R}^{2n}$  is defined as  $\zeta \triangleq \begin{bmatrix} e^T, \ x_d^T \end{bmatrix}^T$ , and the functions  $F: \mathbb{R}^{2n} \to \mathbb{R}^{2n}$  and  $G: \mathbb{R}^{2n} \to \mathbb{R}^{2n}$  are defined as

$$F\left(\zeta\right)\triangleq\begin{bmatrix}f^{T}\left(e+x_{d}\right)-h_{d}^{T}+u_{d}^{T}\left(x_{d}\right)g^{T}\left(e+x_{d}\right)&h_{d}^{T}\end{bmatrix}^{T}$$

 $^1$ For notational brevity, unless otherwise specified, the domain of all the functions is assumed to be  $\mathbb{R}_{\geq 0}$ . Furthermore, time-dependence is suppressed in equations and definitions. For example, the trajectory  $x:\mathbb{R}_{\geq 0}\to\mathbb{R}^n$  is defined by abuse of notation as  $x\in\mathbb{R}^n$  and unless otherwise specified, an equation of the form f+h(y,t)=g(x) is interpreted as f(t)+h(y(t),t)=g(x(t)) for all  $t\in\mathbb{R}_{\geq 0}$ .

and

$$G(\zeta) \triangleq \begin{bmatrix} g^T(e+x_d) & \mathbf{0}_{m\times n} \end{bmatrix}^T$$
.

The control error  $\mu$  is treated hereafter as the design variable. The control objective is to solve the infinite-horizon optimal regulation problem online, i.e., to simultaneously synthesize and utilize a control signal  $\mu$  online to minimize the cost functional

$$J\left(\zeta,\mu\right)\triangleq\int\limits_{t_{0}}^{\infty}r\left(\zeta\left(\tau\right),\mu\left(\tau\right)\right)d\tau,$$

under the dynamic constraint

$$\dot{\zeta} = F(\zeta) + G(\zeta) \mu,$$

while tracking the desired trajectory, where  $r: \mathbb{R}^{2n} \times \mathbb{R}^m \to \mathbb{R}$  is the local cost defined as

$$r(\zeta, \mu) \triangleq Q(e) + \mu^T R \mu,$$

 $R\in\mathbb{R}^{m\times m}$  is a positive definite symmetric matrix of constants, and  $Q:\mathbb{R}^n\to\mathbb{R}$  is a continuous positive definite function.

Assuming that an optimal policy exists, the optimal policy can be characterized in terms of the value function  $V^*$ :  $\mathbb{R}^{2n} \to \mathbb{R}$  defined as

$$V^{*}(\zeta) \triangleq \min_{\mu(\tau) \in U \mid \tau \in \mathbb{R}_{\geq t}} \int_{t}^{\infty} r\left(\phi^{\mu}(\tau, t, \zeta), \mu\left(\tau\right)\right) d\tau,$$

where  $U \in \mathbb{R}^m$  is the action space and the notation  $\phi^{\mu}(t;t_0,\zeta_0)$  denotes the trajectory of  $\dot{\zeta}=F(\zeta)+G(\zeta)\mu$ , under the control signal  $\mu:\mathbb{R}_{\geq 0}\to\mathbb{R}^m$  with the initial condition  $\zeta_0\in\mathbb{R}^{2n}$  and initial time  $t_0\in\mathbb{R}_{\geq 0}$ . Assuming that a minimizing policy exists and that  $V^*$  is continuously differentiable, a closed-form solution for the optimal policy can be obtained as [28]

$$\mu^*(\zeta) = -\frac{1}{2} R^{-1} G^T(\zeta) (\nabla_{\zeta} V^*(\zeta))^T,$$

where  $\nabla_{\zeta}(\cdot) \triangleq \frac{\partial(\cdot)}{\partial \zeta}$ . The optimal policy and the optimal value function satisfy the Hamilton-Jacobi-Bellman (HJB) equation [28]

$$\nabla_{\zeta} V^*(\zeta) (F(\zeta) + G(\zeta) \mu^*(\zeta)) + \overline{Q}(\zeta) + \mu^{*T}(\zeta) R \mu^*(\zeta) = 0, \quad (2)$$

with the initial condition  $V^*\left(0\right)=0$ , where the function  $\overline{Q}:\mathbb{R}^{2n}\to\mathbb{R}$  is defined as

$$\overline{Q}\left(\begin{bmatrix} e^T & x_d^T \end{bmatrix}^T\right) = Q\left(e\right), \ \forall e, \ x_d \in \mathbb{R}^n.$$

Remark 1. Assumptions 1 and 2 can be eliminated if a discounted cost optimal tracking problem is considered instead of the total cost problem considered in this article. The discounted cost tracking problem considers a value function of the form

$$V^{*}(\zeta) \triangleq \min_{u(\tau) \in U \mid \tau \in \mathbb{R}_{\geq t}} \int_{t}^{\infty} e^{\kappa(t-\tau)} r\left(\phi^{u}(\tau, t, \zeta), u(\tau)\right) d\tau,$$

where  $\kappa \in \mathbb{R}_{>0}$  is a constant discount factor, and the control effort u is minimized instead of the control error  $\mu$ .

The control effort required for a system to perfectly track a desired trajectory is generally nonzero even if the initial system state is on the desired trajectory. Hence, in general, the optimal value function for a discounted cost problem does not satisfy  $V^*\left(0\right)=0$ . Online continuous-time RL techniques are generally analyzed using the optimal value function as a candidate Lyapunov function. Since the optimal value function for a discounted cost problem does not evaluate to zero at the origin, it can not be used as a candidate Lyapunov function. Hence, analyzing the stability of a discounted cost optimal controller during the learning phase is complex.

# III. BELLMAN ERROR

Since a closed-form solution of the HJB is generally infeasible to obtain, an approximate solution is sought. In an approximate actor-critic-based solution, the optimal value function  $V^*$  is replaced by a parametric estimate  $\hat{V}\left(\zeta,\hat{W}_c\right)$  and the optimal policy  $\mu^*$  by a parametric estimate  $\hat{\mu}\left(\zeta,\hat{W}_c\right)$ , where  $\hat{W}_c \in \mathbb{R}^L$  and  $\hat{W}_a \in \mathbb{R}^L$  denote vectors of estimates of the ideal parameters. The objective of the critic is to learn the parameters  $\hat{W}_c$ , and the objective of the actor is to learn the parameters  $\hat{W}_a$ . Substituting the estimates  $\hat{V}$  and  $\hat{\mu}$  for  $V^*$  and  $\mu^*$  in the HJB equation, respectively, yields a residual error  $\delta: \mathbb{R}^{2n} \times \mathbb{R}^L \times \mathbb{R}^L \to \mathbb{R}$ , called the BE, is defined as

$$\delta\left(\zeta, \hat{W}_{c}, \hat{W}_{a}\right) = \overline{Q}\left(\zeta\right) + \hat{\mu}^{T}\left(\zeta, \hat{W}_{a}\right) R\hat{\mu}\left(\zeta, \hat{W}_{a}\right) + \nabla_{\zeta}\hat{V}\left(\zeta, \hat{W}_{c}\right) \left(F\left(\zeta\right) + G\left(\zeta\right)\hat{\mu}\left(\zeta, \hat{W}_{a}\right)\right). \tag{3}$$

Specifically, to solve the optimal control problem, the critic aims to find a set of parameters  $\hat{W}_c$  and the actor aims to find a set of parameters  $\hat{W}_a$  such that

$$\delta\left(\zeta, \hat{W}_c, \hat{W}_a\right) = 0,$$

and

$$\hat{u}\left(\zeta,\hat{W}_{a}\right) = -\frac{1}{2}R^{-1}G^{T}(\zeta)\left(\nabla_{\zeta}\hat{V}\left(\zeta,\hat{W}_{a}\right)\right)^{T},$$

for all  $\zeta \in \mathbb{R}^{2n}$ . Since an exact basis for value function approximation is generally not available, an approximate set of parameters that minimizes the BE is sought. In particular, to ensure uniform approximation of the value function and the policy over a compact operating domain  $\mathcal{C} \subset \mathbb{R}^{2n}$ , it is desirable to find parameters that minimize the error  $E_s: \mathbb{R}^L \times \mathbb{R}^L \to \mathbb{R}$  defined as

$$E_{s}\left(\hat{W}_{c}, \hat{W}_{a}\right) \triangleq \sup_{\zeta \in \mathcal{C}} \left| \delta\left(\zeta, \hat{W}_{c}, \hat{W}_{a}\right) \right|.$$

Computation of the error  $E_s$ , and computation of the control signal u require knowledge of the system drift dynamics f. Two prevalent approaches employed to render the control design robust to uncertainties in the system drift dynamics are integral RL (cf. [15] and [29]) and state derivative estimation (cf. [12] and [23]).

Integral RL exploits the fact that for all T>0 and  $t>t_0+T$ , the BE in (3) has an equivalent integral form

$$\delta_{int}\left(t,\hat{W}_{c},\hat{W}_{a}\right) = \hat{V}\left(\phi^{\hat{\mu}}\left(t-T,t_{0},\zeta_{0}\right),\hat{W}_{c}\right)$$

$$-\int_{t-T}^{t} r\left(\phi^{\hat{\mu}}\left(\tau, t_{0}, \zeta_{0}\right), \hat{\mu}\left(\phi^{\hat{\mu}}\left(\tau, t_{0}, \zeta_{0}\right), \hat{W}_{a}\right)\right) d\tau$$
$$-\hat{V}\left(\phi^{\hat{\mu}}\left(t, t_{0}, \zeta_{0}\right), \hat{W}_{c}\right).$$

Since the integral form does not require model knowledge, policies designed based on  $\delta_{int}$  can be implemented without knowledge of f.

State derivative estimation-based techniques exploit the fact that the BE in (3) can be expressed as

$$\delta_{d}\left(\zeta,\dot{\zeta},\hat{W}_{a},\hat{W}_{c}\right) = \nabla_{\zeta}\hat{V}\left(\zeta,\hat{W}_{c}\right)\dot{\zeta} + \overline{Q}\left(\zeta\right) + \hat{\mu}^{T}\left(\zeta,\hat{W}_{a}\right)R\hat{\mu}\left(\zeta,\hat{W}_{a}\right).$$

Hence, an estimate of the BE can be computed without model knowledge if an estimate of the derivative  $\dot{\zeta}$  is available. An adaptive derivative estimator such as [30] could be used to estimate  $\dot{\zeta}$  online.

The integral form of the BE is inherently dependent on the state trajectory, and since adaptive derivative estimators approximate the derivative only along the trajectory, derivative estimation-based techniques are also dependent on the state trajectory. Hence, in techniques such as [12], [15], [23], [29] the BE can only be evaluated along the system trajectory. Thus, the error  $E_s$  is approximated by the instantaneous integral error

$$\hat{E}\left(t\right) \triangleq \int_{t_{0}}^{t} \delta^{2}\left(\phi^{\hat{\mu}}\left(\tau, t_{0}, \zeta_{0}\right), \hat{W}_{c}\left(t\right), \hat{W}_{a}\left(t\right)\right) d\tau.$$

Intuitively, for  $\hat{E}$  to approximate E over an operating domain, the state trajectory  $\phi^{\hat{\mu}}(t,t_0,\zeta_0)$  needs to visit as many points in the operating domain as possible. This intuition is formalized by the fact that techniques such as [12], [15], [23], [29], [31] require PE to achieve convergence. The PE condition is relaxed in [15] to a finite excitation condition by using integral RL along with experience replay, where each evaluation of the BE  $\delta_{int}$  is interpreted as gained experience, and these experiences are stored in a history stack and are repeatedly used in the learning algorithm to improve data efficiency.

In this paper, a different approach is used to improve data efficiency. A dynamic system identifier is developed to generate a parametric estimate  $\hat{F}\left(\zeta,\hat{\theta}\right)$  of the drift dynamics F, where  $\hat{\theta}$  denotes the estimate of the matrix of unknown parameters. Given  $\hat{F},\hat{V}$ , and  $\hat{\mu}$ , an estimate of the BE can be evaluated at any  $\zeta\in\mathbb{R}^{2n}$ . That is, using  $\hat{F}$ , experience can be simulated by extrapolating the BE over unexplored off-trajectory points in the operating domain. Hence, if an identifier can be developed such that  $\hat{F}$  approaches F exponentially fast, learning laws for the optimal policy can utilize simulated experience along with experience gained and stored along the state trajectory.

If parametric approximators are used to approximate F, convergence of  $\hat{F}$  to F is implied by convergence of the parameters to their unknown ideal values. It is well known that adaptive system identifiers require PE to achieve parameter

convergence. To relax the PE condition, a CL-based (cf. [24]–[27]) system identifier that uses recorded data for learning is developed in the following section.

# IV. SYSTEM IDENTIFICATION

On any compact set  $\mathcal{C} \subset \mathbb{R}^n$  the function f can be represented using a neural network (NN) as

$$f(x) = \theta^{T} \sigma_{f} \left( Y^{T} x_{1} \right) + \epsilon_{\theta} \left( x \right),$$

where  $x_1 \triangleq \begin{bmatrix} 1 & x^T \end{bmatrix}^T \in \mathbb{R}^{n+1}$ ,  $\theta \in \mathbb{R}^{p+1 \times n}$  and  $Y \in \mathbb{R}^{n+1 \times p}$  denote the constant unknown output-layer and hidden-layer NN weights,  $\sigma_f : \mathbb{R}^p \to \mathbb{R}^{p+1}$  denotes a bounded NN basis function,  $\epsilon_\theta : \mathbb{R}^n \to \mathbb{R}^n$  denotes the function reconstruction error, and  $p \in \mathbb{N}$  denotes the number of NN neurons. Using the universal function approximation property of single layer NNs, given a constant matrix Y such that the rows of  $\sigma_f(Y^Tx_1)$  form a proper basis, there exist constant ideal weights  $\theta$  and known constants  $\overline{\theta}$ ,  $\overline{\epsilon_\theta}$ , and  $\overline{\epsilon'_\theta} \in \mathbb{R}$  such that  $\|\theta\|_F \leq \overline{\theta} < \infty$ ,  $\sup_{x \in \mathcal{C}} \|\epsilon_\theta(x)\| \leq \overline{\epsilon_\theta}$ , and  $\sup_{x \in \mathcal{C}} \|\nabla_x \epsilon_\theta(x)\| \leq \overline{\epsilon'_\theta}$ , where  $\|\cdot\|_F$  denotes the Frobenius norm [32].

Using an estimate  $\hat{\theta} \in \mathbb{R}^{p+1 \times n}$  of the weight matrix  $\theta$ , the function f can be approximated by the function  $\hat{f}: \mathbb{R}^{2n} \times \mathbb{R}^{p+1 \times n} \to \mathbb{R}^n$  defined as

$$\hat{f}\left(\zeta,\hat{\theta}\right) \triangleq \hat{\theta}^T \sigma_{\theta}\left(\zeta\right),\tag{4}$$

where  $\sigma_{\theta}: \mathbb{R}^{2n} \to \mathbb{R}^{p+1}$  is defined as  $\sigma_{\theta}(\zeta) = \sigma_{f}\left(Y^{T}\begin{bmatrix}1 & e^{T} + x_{d}^{T}\end{bmatrix}^{T}\right)$ . An estimator for online identification of the drift dynamics is developed as

$$\dot{\hat{x}} = \hat{\theta}^T \sigma_\theta \left( \zeta \right) + q\left( x \right) u + k\tilde{x},\tag{5}$$

where  $\tilde{x} \triangleq x - \hat{x}$ , and  $k \in \mathbb{R}$  is a positive constant learning gain.

**Assumption 3.** [26] A history stack containing recorded stateaction pairs  $\{x_j, u_j\}_{j=1}^M$  along with numerically computed state derivatives  $\{\dot{\bar{x}}_j\}_{j=1}^M$  that satisfies

$$\lambda_{\min}\left(\sum_{j=1}^{M} \sigma_{fj} \sigma_{fj}^{T}\right) = \underline{\sigma_{\theta}} > 0, \quad \|\dot{\bar{x}}_{j} - \dot{x}_{j}\| < \overline{d}, \ \forall j,$$

is available a priori, where  $\sigma_{fj} \triangleq \sigma_f \left( Y^T \begin{bmatrix} 1 & x_j^T \end{bmatrix}^T \right)$ ,  $\overline{d} \in \mathbb{R}$  is a known positive constant,  $\dot{x}_j = f(x_j) + g(x_j)u_j$ , and  $\lambda_{\min}\left(\cdot\right)$  denotes the minimum eigenvalue.<sup>2</sup>

The weight estimates  $\hat{\theta}$  are updated using the following CL-based update law:

$$\dot{\hat{\theta}} = \Gamma_{\theta} \sigma_f (Y^T x_1) \tilde{x}^T + k_{\theta} \Gamma_{\theta} \sum_{j=1}^{M} \sigma_{fj} (\dot{\bar{x}}_j - g_j u_j - \hat{\theta}^T \sigma_{fj})^T, \quad (6)$$

 $^2$ A priori availability of the history stack is used for ease of exposition, and is not necessary. Provided the system states are exciting over a finite time interval  $t \in [t_0, t_0 + \overline{t}]$  (versus  $t \in [t_0, \infty)$  as in traditional PE-based approaches) the history stack can also be recorded online. The controller developed in [23] can be used over the time interval  $t \in [t_0, t_0 + \overline{t}]$  while the history stack is being recorded, and the controller developed in this result can be used thereafter. The use of two different controllers results in a switched system with one switching event. Since there is only one switching event, the stability of the switched system follows from the stability of the individual subsystems.

where  $k_{\theta} \in \mathbb{R}$  is a constant positive CL gain, and  $\Gamma_{\theta} \in \mathbb{R}^{p+1\times p+1}$  is a constant, diagonal, and positive definite adaptation gain matrix. Using (4), the BE in (3) can be approximated as

$$\hat{\delta}\left(\zeta,\hat{\theta},\hat{W}_{c},\hat{W}_{a}\right) = \overline{Q}\left(\zeta\right) + \hat{\mu}^{T}\left(\zeta,\hat{W}_{a}\right)R\hat{\mu}\left(\zeta,\hat{W}_{a}\right),$$

$$+\nabla_{\zeta}\hat{V}\left(\zeta,\hat{W}_{a}\right)\left(F_{\theta}\left(\zeta,\hat{\theta}\right) + F_{1}\left(\zeta\right) + G\left(\zeta\right)\hat{\mu}\left(\zeta,\hat{W}_{a}\right)\right) \tag{7}$$

where

$$F_{\theta}\left(\zeta,\hat{\theta}\right) \triangleq \begin{bmatrix} \hat{\theta}^{T}\sigma_{\theta}\left(\zeta\right) - g\left(x\right)g^{+}\left(x_{d}\right)\hat{\theta}^{T}\sigma_{\theta}\left(\begin{bmatrix}\mathbf{0}_{n\times1}\\x_{d}\end{bmatrix}\right) \\ 0 \end{bmatrix},$$

and

$$F_{1}\left(\zeta\right)\triangleq\begin{bmatrix}-h_{d}+g\left(e+x_{d}\right)g^{+}\left(x_{d}\right)h_{d}\\h_{d}\end{bmatrix}.$$

# V. VALUE FUNCTION APPROXIMATION

Since  $V^*$  and  $\mu^*$  are functions of the state  $\zeta$ , the minimization problem stated in Section II is intractable. To obtain a finite-dimensional minimization problem, the optimal value function is represented over any compact operating domain  $\mathcal{C} \subset \mathbb{R}^{2n}$  using a NN as  $V^*\left(\zeta\right) = W^T\sigma\left(\zeta\right) + \epsilon\left(\zeta\right)$ , where  $W \in \mathbb{R}^L$  denotes a vector of unknown NN weights,  $\sigma: \mathbb{R}^{2n} \to \mathbb{R}^L$  denotes a bounded NN basis function,  $\epsilon: \mathbb{R}^{2n} \to \mathbb{R}^L$  denotes the function reconstruction error, and  $L \in \mathbb{N}$  denotes the number of NN neurons. Using the universal function approximation property of single layer NNs, for any compact set  $\mathcal{C} \subset \mathbb{R}^{2n}$ , there exist constant ideal weights W and known positive constants  $\overline{W}, \overline{\epsilon}$ , and  $\overline{\epsilon'} \in \mathbb{R}$  such that  $\|W\| \leq \overline{W} < \infty$ ,  $\sup_{\zeta \in \mathcal{C}} \|\epsilon\left(\zeta\right)\| \leq \overline{\epsilon'}$ , and  $\sup_{\zeta \in \mathcal{C}} \|\nabla_{\zeta}\epsilon\left(\zeta\right)\| \leq \overline{\epsilon'}$  [32].

A NN representation of the optimal policy is obtained as

$$\mu^* \left( \zeta \right) = -\frac{1}{2} R^{-1} G^T \left( \zeta \right) \left( \nabla_{\zeta} \sigma^T \left( \zeta \right) W + \nabla_{\zeta} \epsilon^T \left( \zeta \right) \right). \tag{8}$$

Using estimates  $\hat{W}_c$  and  $\hat{W}_a$  for the ideal weights W, the optimal value function and the optimal policy are approximated as

$$\hat{V}\left(\zeta, \hat{W}_{c}\right) \triangleq \hat{W}_{c}^{T} \sigma\left(\zeta\right),$$

$$\hat{\mu}\left(\zeta, \hat{W}_{a}\right) \triangleq -\frac{1}{2} R^{-1} G^{T}\left(\zeta\right) \nabla_{\zeta} \sigma^{T}\left(\zeta\right) \hat{W}_{a}.$$
(9)

The optimal control problem is thus reformulated as the need to find a set of weights  $\hat{W}_c$  and  $\hat{W}_a$  online, to minimize the error

$$\hat{E}_{\hat{\theta}}\left(\hat{W}_{c}, \hat{W}_{a}\right) \triangleq \sup_{\zeta \in \Upsilon} \left| \hat{\delta}\left(\zeta, \hat{\theta}, \hat{W}_{c}, \hat{W}_{a}\right) \right|,$$

for a given  $\hat{\theta}$ , while simultaneously improving  $\hat{\theta}$  using (6), and ensuring stability of the system using the control law

$$u = \hat{\mu}\left(\zeta, \hat{W}_a\right) + \hat{u}_d\left(\zeta, \hat{\theta}\right),\tag{10}$$

where

$$\hat{u}_d\left(\zeta,\hat{\theta}\right) \triangleq g_d^+ \left(h_d - \hat{\theta}^T \sigma_{\theta d}\right),$$

and  $\sigma_{\theta d} \triangleq \sigma_{\theta} \left( \begin{bmatrix} \mathbf{0}_{1 \times n} & x_d^T \end{bmatrix}^T \right)$ . The error between  $u_d$  and  $\hat{u}_d$  is included in the stability analysis based on the fact that

the error trajectories generated by the system  $\dot{e}=f\left(x\right)+g\left(x\right)u-\dot{x}_{d}$  under the controller in (10) are identical to the error trajectories generated by the system  $\dot{\zeta}=F\left(\zeta\right)+G\left(\zeta\right)\mu$  under the control law

$$\mu = \hat{\mu} \left( \zeta, \hat{W}_a \right) + g_d^+ \tilde{\theta}^T \sigma_{\theta d} + g_d^+ \epsilon_{\theta d}, \tag{11}$$

where  $\epsilon_{\theta d} \triangleq \epsilon_{\theta} (x_d)$ .

# VI. SIMULATION OF EXPERIENCE

Since computation of the supremum in  $\hat{E}_{\hat{\theta}}$  is intractable in general, simulation of experience is implemented by minimizing a squared sum of BEs over finitely many points in the state space. The following assumption facilitates the aforementioned approximation.

**Assumption 4.** [24] There exists a finite set of points  $\{\zeta_i \in \mathcal{C} \mid i=1,\cdots,N\}$  and a constant  $\underline{c} \in \mathbb{R}$  such that

$$0 < \underline{c} \triangleq \frac{1}{N} \left( \inf_{t \in \mathbb{R}_{\geq t_0}} \left( \lambda_{min} \left\{ \sum_{i=1}^{N} \frac{\omega_i \omega_i^T}{\rho_i} \right\} \right) \right),$$

where  $\rho_i \triangleq 1 + \nu \omega_i^T \Gamma \omega_i \in \mathbb{R}$ , and

$$\omega_{i} \triangleq \nabla_{\zeta} \sigma\left(\zeta_{i}\right) \left(F_{\theta}\left(\zeta_{i}, \hat{\theta}\right) + F_{1}\left(\zeta_{i}\right) + G\left(\zeta_{i}\right) \hat{\mu}\left(\zeta_{i}, \hat{W}_{a}\right)\right).$$

Using Assumption 4, simulation of experience is implemented by the weight update laws

$$\dot{\hat{W}}_{c} = -\eta_{c1} \Gamma \frac{\omega}{\rho} \hat{\delta}_{t} - \frac{\eta_{c2}}{N} \Gamma \sum_{i=1}^{N} \frac{\omega_{i}}{\rho_{i}} \hat{\delta}_{ti}, \tag{12}$$

$$\dot{\Gamma} = \left(\beta \Gamma - \eta_{c1} \Gamma \frac{\omega \omega^{T}}{\rho^{2}} \Gamma\right) \mathbf{1}_{\left\{\|\Gamma\| \leq \overline{\Gamma}\right\}}, \|\Gamma(t_{0})\| \leq \overline{\Gamma}, \tag{13}$$

$$\dot{\hat{W}}_{a} = -\eta_{a1} \left(\hat{W}_{a} - \hat{W}_{c}\right) - \eta_{a2} \hat{W}_{a}$$

$$+ \left(\frac{\eta_{c1} G_{\sigma}^{T} \hat{W}_{a} \omega^{T}}{4\rho} + \sum_{i=1}^{N} \frac{\eta_{c2} G_{\sigma i}^{T} \hat{W}_{a} \omega_{i}^{T}}{4N\rho_{i}}\right) \hat{W}_{c}, \tag{14}$$

where

$$\omega\triangleq\nabla_{\zeta}\sigma\left(\zeta\right)\left(F_{\theta}\left(\zeta,\hat{\theta}\right)+F_{1}\left(\zeta\right)+G\left(\zeta\right)\hat{\mu}\left(\zeta,\hat{W}_{a}\right)\right),$$

 $\Gamma \in \mathbb{R}^{L \times L}$  is the least-squares gain matrix,  $\overline{\Gamma} \in \mathbb{R}$  denotes a positive saturation constant,  $\beta \in \mathbb{R}$  denotes a constant forgetting factor,  $\eta_{c1}, \eta_{c2}, \eta_{a1}, \eta_{a2} \in \mathbb{R}$  denote constant positive adaptation gains,  $\mathbf{1}_{\{\cdot\}}$  denotes the indicator function of the set  $\{\cdot\}$ ,  $G_{\sigma} \triangleq \nabla_{\zeta}\sigma(\zeta)\,G(\zeta)\,R^{-1}G^{T}(\zeta)\,\nabla_{\zeta}\sigma^{T}(\zeta)$ , and  $\rho \triangleq 1 + \nu\omega^{T}\Gamma\omega$ , where  $\nu \in \mathbb{R}$  is a positive normalization constant. In (12)-(14) and in the subsequent development, for any function  $\xi(\zeta,\cdot)$ , the notation  $\xi_{i}$ , is defined as  $\xi_{i} \triangleq \xi(\zeta_{i},\cdot)$ , and the instantaneous BEs  $\hat{\delta}_{t}$  and  $\hat{\delta}_{ti}$  are given by

$$\hat{\delta}_t = \hat{\delta} \left( \zeta, \hat{W}_c, \hat{W}_a, \hat{\theta} \right),\,$$

and

$$\hat{\delta}_{ti} = \hat{\delta} \left( \zeta_i, \hat{W}_c, \hat{W}_a, \hat{\theta} \right).$$

# VII. STABILITY ANALYSIS

If the state penalty function  $\overline{Q}$  is positive definite, then the optimal value function  $V^*$  is positive definite, and serves as a Lyapunov function for the concatenated system under the optimal control policy  $\mu^*$ ; hence,  $V^*$  is used (cf. [11], [12], [29]) as a candidate Lyapunov function for the closed-loop system under the policy  $\hat{\mu}$ . The function  $\overline{Q}$ , and hence, the function  $V^*$  are positive semidefinite; hence, the function  $V^*$  is not a valid candidate Lyapunov function. However, the results in [23] can be used to show that a nonautonomous form of the optimal value function denoted by  $V_t^*: \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ , defined as

$$V_{t}^{*}\left(e,t\right)=V^{*}\left(\begin{bmatrix}e\\x_{d}\left(t\right)\end{bmatrix}\right),\;\forall e\in\mathbb{R}^{n},\;t\in\mathbb{R},$$

is positive definite and decrescent. Hence,  $V_t^* (0,t) = 0, \ \forall t \in \mathbb{R}$  and there exist class  $\mathcal{K}$  functions  $\underline{v} : \mathbb{R} \to \mathbb{R}$  and  $\overline{v} : \mathbb{R} \to \mathbb{R}$  such that

$$\underline{v}\left(\|e\|\right) \le V_t^*\left(e,t\right) \le \overline{v}\left(\|e\|\right),\tag{15}$$

for all  $e \in \mathbb{R}^n$  and for all  $t \in \mathbb{R}$ .

To facilitate the stability analysis, a candidate Lyapunov function  $V_0:\mathbb{R}^n\times\mathbb{R}^{p+1\times n}\to\mathbb{R}$  is selected as

$$V_0\left(\tilde{x},\tilde{\theta}\right) = \frac{1}{2}\tilde{x}^T\tilde{x} + \frac{1}{2}\operatorname{tr}\left(\tilde{\theta}^T\Gamma_{\theta}^{-1}\tilde{\theta}\right),\tag{16}$$

where  $\tilde{\theta} \triangleq \theta - \hat{\theta}$  and tr  $(\cdot)$  denotes the trace of a matrix. Using (5)-(6), the following bound on the time derivative of  $V_0$  is established:

$$\dot{V}_{0} \leq -k \|\tilde{x}\|^{2} - k_{\theta} \underline{\sigma_{\theta}} \|\tilde{\theta}\|_{F}^{2} + \overline{\epsilon_{\theta}} \|\tilde{x}\| + k_{\theta} \overline{d_{\theta}} \|\tilde{\theta}\|_{F}, \quad (17)$$

where

$$\overline{d_{\theta}} \triangleq \overline{d} \sum_{j=1}^{M} \|\sigma_{\theta j}\| + \sum_{j=1}^{M} (\|\epsilon_{\theta j}\| \|\sigma_{\theta j}\|).$$

A concatenated state  $Z \in \mathbb{R}^{2n+2L+n(p+1)}$  is defined as

$$Z \triangleq \begin{bmatrix} e^T & \tilde{W}_c^T & \tilde{W}_a^T & \tilde{x}^T & \left(\operatorname{vec}\left(\tilde{\theta}\right)\right)^T \end{bmatrix}^T,$$

and a candidate Lyapunov function is defined as

$$V_L(Z,t) \triangleq V_t^*(e,t) + \frac{1}{2} \tilde{W}_c^T \Gamma^{-1} \tilde{W}_c + \frac{1}{2} \tilde{W}_a^T \tilde{W}_a + V_0(\tilde{\theta}, \tilde{x}),$$
(18)

where  $\text{vec}\left(\cdot\right)$  denotes the vectorization operator and  $V_0$  is defined in (16). The saturated least-squares update law in (13) ensures that there exist positive constants  $\underline{\gamma}, \overline{\gamma} \in \mathbb{R}$  such that

$$\gamma \le \|\Gamma^{-1}(t)\| \le \overline{\gamma}, \ \forall t \in \mathbb{R}.$$
(19)

Using (16), the bounds in (19) and (15), and the fact that  $\operatorname{tr}\left(\tilde{\theta}^T\Gamma_{\theta}^{-1}\tilde{\theta}\right) = \left(\operatorname{vec}\left(\tilde{\theta}\right)\right)^T\left(\Gamma_{\theta}^{-1}\otimes\mathbb{I}_{p+1}\right)\left(\operatorname{vec}\left(\tilde{\theta}\right)\right)$ , the candidate Lyapunov function in (18) can be bounded as

$$\underline{v_l}\left(\|Z\|\right) \le V_L\left(Z,t\right) \le \overline{v_l}\left(\|Z\|\right),\tag{20}$$

for all  $Z \in \mathbb{R}^{2n+2L+n(p+1)}$  and for all  $t \in \mathbb{R}$ , where  $\underline{v_l} : \mathbb{R} \to \mathbb{R}$  and  $\overline{v_l} : \mathbb{R} \to \mathbb{R}$  are class  $\mathcal{K}$  functions.

For notational brevity, the dependence of the functions F, G,  $\sigma$ ,  $\sigma'$ ,  $\epsilon$ ,  $\epsilon'$ ,  $\sigma_{\theta}$ ,  $\epsilon_{\theta}$ , and g on the system states is suppressed hereafter. To facilitate the stability analysis, the approximate

BE in (7) is expressed in terms of the weight estimation errors as

$$\hat{\delta}_t = -\omega^T \tilde{W}_c - W^T \sigma' F_{\tilde{\theta}} + \frac{1}{4} \tilde{W}_a^T G_{\sigma} \tilde{W}_a + \Delta, \tag{21}$$

where  $F_{\tilde{\theta}} \triangleq F_{\theta}\left(\zeta, \tilde{\theta}\right)$  and  $\Delta = O\left(\overline{\epsilon}, \overline{\epsilon'}, \overline{\epsilon_{\theta}}\right)$ . Given any compact set  $\chi \subset \mathbb{R}^{2n+2L+n(p+1)}$  containing an open ball of radius  $\rho \in \mathbb{R}$  centered at the origin, a positive constant  $\iota \in \mathbb{R}$  is defined as

$$\iota \triangleq \frac{3\left(\frac{(\eta_{c1} + \eta_{c2})\overline{W}^{2} \|G_{\sigma}\|}{16\sqrt{\nu\underline{\Gamma}}} + \frac{\|(\overline{W}^{T}G_{\sigma} + \epsilon'G_{r}\sigma'^{T})\|}{4} + \frac{\eta_{a2}\overline{W}}{2}\right)^{2}}{(\eta_{a1} + \eta_{a2})} + \frac{3\left(\left(\|\overline{W}^{T}\sigma'Gg_{d}^{+}\| + \|\epsilon'Gg_{d}^{+}\|\right)\overline{\sigma_{g}} + k_{\theta}\overline{d_{\theta}}\right)^{2}}{4k_{\theta}\underline{\sigma_{\theta}}} + \frac{(\eta_{c1} + \eta_{c2})^{2} \|\overline{\Delta}\|^{2}}{4\nu\underline{\Gamma}\eta_{c2}\underline{c}} + \frac{\overline{\epsilon_{\theta}}^{2}}{2k} + \|\epsilon'Gg_{d}^{+}\epsilon_{\theta d}\| + \|\frac{1}{2}W^{T}\sigma'G_{r}\epsilon'^{T}\| + \|W^{T}\sigma'Gg_{d}^{+}\epsilon_{\theta d}\|, \quad (22)$$

where  $G_r \triangleq GR^{-1}G^T$ , and  $G_{\epsilon} \triangleq \epsilon' G_r (\epsilon')^T$ . Let  $v_l : \mathbb{R} \to \mathbb{R}$  be a class  $\mathcal{K}$  function such that

$$v_{l}(\|Z\|) \leq \frac{\underline{q}(\|e\|)}{2} + \frac{\eta_{c2}\underline{c}}{8} \|\tilde{W}_{c}\|^{2} + \frac{(\eta_{a1} + \eta_{a2})}{6} \|\tilde{W}_{a}\|^{2} + \frac{k}{4} \|\tilde{x}\|^{2} + \frac{k_{\theta}\underline{\sigma}_{\theta}}{6} \|\operatorname{vec}(\tilde{\theta})\|^{2}.$$
(23)

The sufficient gain conditions used in the subsequent Theorem 1 are

$$v_{l}^{-1}(\iota) < \overline{v_{l}}^{-1}\left(\underline{v_{l}}(\rho)\right)$$

$$\eta_{c2}\underline{c} > \frac{3\left(\eta_{c2} + \eta_{c1}\right)^{2} \overline{W}^{2} \|\overline{\sigma'}\|^{2} \overline{\sigma_{g}}^{2}}{4k_{\theta} \underline{\sigma_{\theta}} \nu \underline{\Gamma}}$$

$$(\eta_{a1} + \eta_{a2}) > \frac{3\left(\eta_{c1} + \eta_{c2}\right) \overline{W} \|G_{\sigma}\|}{8\sqrt{\nu}\underline{\Gamma}}$$

$$+ \frac{3}{c\eta_{c2}} \left(\frac{\left(\eta_{c1} + \eta_{c2}\right) \overline{W} \|G_{\sigma}\|}{8\sqrt{\nu}\underline{\Gamma}} + \eta_{a1}\right)^{2}.$$

$$(26)$$

In (22)-(26), for any function  $\varpi: \mathbb{R}^l \to \mathbb{R}, \ l \in \underline{\mathbb{N}}$ , the notation  $\|\varpi\|$ , denotes  $\sup_{y \in \chi \cap \mathbb{R}^l} \|\varpi(y)\|$ , and  $\overline{\sigma_g} \triangleq \|\sigma_\theta\| + \|gg_d^+\| \|\sigma_{\theta d}\|$ .

The sufficient condition in (24) requires the set  $\chi$  to be large enough based on the constant  $\iota$ . Since the NN approximation errors depend on the compact set  $\chi$ , in general, for a fixed number of NN neurons, the constant  $\iota$  increases with the size of the set  $\chi$ . However, for a fixed set  $\chi$ , the constant  $\iota$  can be reduced by reducing function reconstruction errors, i.e., by increasing number of NN neurons, and by increasing the learning gains provided  $\underline{\sigma}_{\theta}$  is large enough. Hence a sufficient number of NN neurons and extrapolation points are required to satisfy the condition in (24).

**Theorem 1.** Provided Assumptions 2-4 hold, and the control gains, number of NN neurons, and BE extrapolation points are selected based on (24)-(26), the controller in (10), along with the weight update laws (12)-(14), and the identifier in

(5) along with the weight update law (6) ensure that the system states remain bounded, the tracking error is ultimately bounded, and that the control policy  $\hat{\mu}$  converges to a neighborhood around the optimal control policy  $\mu^*$ .

Proof: Using (1) and the fact that

$$\dot{V}_{t}^{*}\left(e\left(t\right),t\right)=\dot{V}^{*}\left(\zeta\left(t\right)\right),\,\forall t\in\mathbb{R},$$

the time-derivative of the candidate Lyapunov function in (18) is

$$\dot{V}_{L} = \nabla_{\zeta} V^{*} (F + G\mu^{*}) - \tilde{W}_{c}^{T} \Gamma^{-1} \dot{\hat{W}}_{c} - \frac{1}{2} \tilde{W}_{c}^{T} \Gamma^{-1} \dot{\Gamma} \Gamma^{-1} \tilde{W}_{c} - \tilde{W}_{a}^{T} \dot{\hat{W}}_{a} + \dot{V}_{0} + \nabla_{\zeta} V^{*} G\mu - \nabla_{\zeta} V^{*} G\mu^{*}.$$
(27)

Using (2), (8), (9), and (11) the expression in (27) is bounded as

$$\dot{V}_{L} \leq -\overline{Q}(\zeta) - \tilde{W}_{c}^{T} \Gamma^{-1} \dot{\hat{W}}_{c} - \frac{1}{2} \tilde{W}_{c}^{T} \Gamma^{-1} \dot{\Gamma} \Gamma^{-1} \tilde{W}_{c} - \tilde{W}_{a}^{T} \dot{\hat{W}}_{a} 
+ \dot{V}_{0} + \frac{1}{2} \left( W^{T} G_{\sigma} + \epsilon' G_{r} \sigma'^{T} \right) \tilde{W}_{a} + W^{T} \sigma' G g_{d}^{+} \tilde{\theta}^{T} \sigma_{\theta d} 
+ \epsilon' G g_{d}^{+} \tilde{\theta}^{T} \sigma_{\theta d} + \frac{1}{2} G_{\epsilon} + \frac{1}{2} W^{T} \sigma' G_{r} \epsilon'^{T} + W^{T} \sigma' G g_{d}^{+} \epsilon_{\theta d} 
- (\mu^{*})^{T} R \mu^{*} + \epsilon' G g_{d}^{+} \epsilon_{\theta d}. \tag{28}$$

Using the update laws in (12)-(14), the bound in (17), and (21), the expression in (28) is bounded as

$$\begin{split} \dot{V}_{L} &\leq -\overline{Q}\left(\zeta\right) - \sum_{i=1}^{N} \tilde{W}_{c}^{T} \frac{\eta_{c2}}{N} \frac{\omega_{i} \omega_{i}^{T}}{\rho_{i}} \tilde{W}_{c} - k_{\theta} \underline{\sigma_{\theta}} \left\| \tilde{\theta} \right\|_{F}^{2} \\ &- \left(\eta_{a1} + \eta_{a2}\right) \tilde{W}_{a}^{T} \tilde{W}_{a} - k \left\| \tilde{x} \right\|^{2} - \eta_{c1} \tilde{W}_{c}^{T} \frac{\omega}{\rho} W^{T} \sigma' F_{\tilde{\theta}} \\ &+ \eta_{c1} \tilde{W}_{c}^{T} \frac{\omega}{\rho} \Delta + \eta_{a1} \tilde{W}_{a}^{T} \tilde{W}_{c} + \eta_{a2} \tilde{W}_{a}^{T} W \\ &+ \frac{1}{4} \eta_{c1} \tilde{W}_{c}^{T} \frac{\omega}{\rho} \tilde{W}_{a}^{T} G_{\sigma} \tilde{W}_{a} - \sum_{i=1}^{N} \tilde{W}_{c}^{T} \frac{\eta_{c2}}{N} \frac{\omega_{i}}{\rho_{i}} W^{T} \sigma'_{i} F_{\tilde{\theta}i} \\ &+ \sum_{i=1}^{N} \frac{1}{4} \tilde{W}_{c}^{T} \frac{\eta_{c2}}{N} \frac{\omega_{i}}{\rho_{i}} \tilde{W}_{a}^{T} G_{\sigma i} \tilde{W}_{a} + \tilde{W}_{c}^{T} \frac{\eta_{c2}}{N} \sum_{i=1}^{N} \frac{\omega_{i}}{\rho_{i}} \Delta_{i} \\ &- \tilde{W}_{a}^{T} \left( \frac{\eta_{c1} G_{\sigma}^{T} \hat{W}_{a} \omega^{T}}{4\rho} + \sum_{i=1}^{N} \frac{\eta_{c2} G_{\sigma i}^{T} \hat{W}_{a} \omega_{i}^{T}}{4N\rho_{i}} \right) \hat{W}_{c} \\ &+ \overline{\epsilon_{\theta}} \left\| \tilde{x} \right\| + k_{\theta} \overline{d_{\theta}} \left\| \tilde{\theta} \right\|_{F} + \frac{1}{2} \left( W^{T} G_{\sigma} + \epsilon' G_{r} \sigma'^{T} \right) \tilde{W}_{a} \\ &+ W^{T} \sigma' G g_{d}^{+} \tilde{\theta}^{T} \sigma_{\theta d} + \epsilon' G g_{d}^{+} \tilde{\theta}^{T} \sigma_{\theta d} + \frac{1}{2} G_{\epsilon} \\ &+ \frac{1}{2} W^{T} \sigma' G_{r} \epsilon'^{T} + W^{T} \sigma' G g_{d}^{+} \epsilon_{\theta d} + \epsilon' G g_{d}^{+} \epsilon_{\theta d}. \end{split}$$

Segregation of terms, completion of squares, and the use of Young's inequalities yields

$$\begin{split} \dot{V}_{L} &\leq -\overline{Q}\left(\zeta\right) - \frac{\eta_{c2}\underline{c}}{4} \left\| \tilde{W}_{c} \right\|^{2} - \frac{\left(\eta_{a1} + \eta_{a2}\right)}{3} \left\| \tilde{W}_{a} \right\|^{2} \\ &- \frac{k}{2} \left\| \tilde{x} \right\|^{2} - \frac{k_{\theta}\underline{\sigma_{\theta}}}{3} \left\| \tilde{\theta} \right\|_{F}^{2} \\ &- \left( \frac{\eta_{c2}\underline{c}}{4} - \frac{3\left(\eta_{c2} + \eta_{c1}\right)^{2} \overline{W}^{2} \left\| \underline{\sigma'} \right\|^{2} \overline{\sigma_{g}^{2}}}{16k_{\theta}\underline{\sigma_{\theta}}\nu\underline{\Gamma}} \right) \left\| \tilde{W}_{c} \right\|^{2} \end{split}$$

$$-\left(\frac{(\eta_{a1} + \eta_{a2})}{3} - \frac{(\eta_{c1} + \eta_{c2})\overline{W}\|G_{\sigma}\|}{8\sqrt{\nu}\underline{\Gamma}}\right)\|\tilde{W}_{a}\|^{2}$$

$$+ \frac{1}{\underline{c}\eta_{c2}}\left(\frac{(\eta_{c1} + \eta_{c2})\overline{W}\|G_{\sigma}\|}{8\sqrt{\nu}\underline{\Gamma}} + \eta_{a1}\right)^{2}\|\tilde{W}_{a}\|^{2}$$

$$+ \frac{3\left(\frac{(\eta_{c1} + \eta_{c2})\overline{W}^{2}\|G_{\sigma}\|}{16\sqrt{\nu}\underline{\Gamma}} + \frac{\|(W^{T}G_{\sigma} + \epsilon'G_{r}\sigma'^{T})\|}{4} + \frac{\eta_{a2}\|W\|}{2}\right)^{2}}{(\eta_{a1} + \eta_{a2})}$$

$$+ \frac{3\left(\left(\|W^{T}\sigma'Gg_{d}^{+}\| + \|\epsilon'Gg_{d}^{+}\|\right)\overline{\sigma_{g}} + k_{\theta}\overline{d_{\theta}}\right)^{2}}{4k_{\theta}\underline{\sigma_{\theta}}}$$

$$+ \frac{(\eta_{c1} + \eta_{c2})^{2}\|\overline{\Delta}\|^{2}}{4\nu\underline{\Gamma}\eta_{c2}\underline{c}} + \frac{\overline{\epsilon_{\theta}}^{2}}{2k} + \frac{1}{2}G_{\epsilon}\| + \frac{1}{2}W^{T}\sigma'G_{r}\epsilon'^{T}\|$$

$$+ \frac{\|W^{T}\sigma'Gg_{d}^{+}\epsilon_{\theta d}\| + \|\epsilon'Gg_{d}^{+}\epsilon_{\theta d}\|, (29)$$

for all  $Z \in \chi$ . Provided the sufficient conditions in (25)-(26) are satisfied, the expression in (29) yields

$$\dot{V}_L \le -v_l(\|Z\|), \ \forall \|Z\| \ge v_l^{-1}(\iota), \ \forall Z \in \chi. \tag{30}$$

Using (20), (24), and (30) Theorem 4.18 in [33] can be invoked to conclude that every trajectory Z(t) satisfying  $\|Z(t_0)\| \leq \overline{v_l}^{-1}\left(\underline{v_l}\left(\rho\right)\right)$ , is bounded for all  $t \in \mathbb{R}$  and satisfies  $\lim\sup_{t\to\infty}\|Z(t)\| \leq \underline{v_l}^{-1}\left(\overline{v_l}\left(v_l^{-1}\left(\iota\right)\right)\right)$ .

# VIII. CONCLUSION

A concurrent-learning based implementation of model-based RL is developed to obtain an approximate online solution to infinite horizon optimal tracking problems for nonlinear continuous-time control-affine systems. The desired steady-state controller is used to facilitate the formulation of a feasible optimal control problem, and the system state is augmented with the desired trajectory to facilitate the formulation of a stationary optimal control problem. A CL-based system identifier is developed to remove the dependence of the desired steady-state controller on the system drift dynamics, and to facilitate simulation of experience via BE extrapolation. Simulation results are provided to demonstrate the effectiveness of the developed technique.

Similar to the PE condition in RL-based online optimal control literature, Assumption 4 can not, in general, be guaranteed a priori. However, Assumption 4 can be heuristically met by oversampling, i.e., by selecting  $N\gg L$ . Furthermore, unlike PE, the satisfaction of Assumption 4 can be monitored online; hence, threshold-based algorithms can be employed to preserve rank by selecting new points if the minimum singular value falls below a certain threshold. Provided the minimum singular value does not decrease during a switch, the trajectories of the resulting switched system can be shown to be uniformly bounded using a common Lyapunov function. Formulation of sufficient conditions for Assumption 4 that can be verified a priori is a topic for future research.

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