

# rQRd algorithm, a robust and fast denoising algorithm

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**Abstract**—This paper addresses the denoising of harmonic signals encountered in many analytical techniques such as NMR, FTIR, FTICR etc. The denoising algorithm presented, based on a parametric approach for harmonic analysis, makes use of a *matrix approximation* through *random projection* instead of classical SVD. It exhibits better performances, both in denoising, robustness and speed over its deterministic counterpart.

**Index Terms**—low rank approximation, random projection, denoising, harmonic signals, harmonic analysis

## I. INTRODUCTION

In harmonic analysis, parametric methods [1][2] are well suited for denoising since they remove efficiently undesirable information in data which does not correspond to an underlying model while conserving a high resolution. Nevertheless this high resolution advantage is counterbalanced by a lack of robustness : the signal integrity is very dependent on apriori knowledge about the signal. Here, this approach is revisited through recent improvements in the field of matrix approximation.

In SVD (singular value decomposition) based parametric methods, a matrix approximation is implicitly performed by a truncated decomposition which is equivalent to a strong model constraint. This constraint can be relaxed by the use of alternative matrix approximation techniques. *Random projection* was recently proved to perform this task efficiently in spite of being a non deterministic method [3]. Our algorithm integrates this powerful new concept inside the specific framework of the parametric methods based on AR(AutoRegressive) processes which is widely used for denoising in many areas such as speech recognition, seismology, spectral analysis, signal restoration, etc [4].

The paper is organized as follows: first we introduce the notions of AR process and *matrix approximation*, then we present the rQRd (Random QR denoising) algorithm principles and its geometrical interpretation, finally simulations on synthetic data are provided for illustrating the robustness, the speed and the denoising performances of rQRd.

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## A. Model-based Autoregressive process

If a signal sampled at regular time intervals (time-series) can be described with an analytical model as the finite sum of damped sinusoids, it is shown that its time-series can be considered as an autoregressive process. This allows to transform an analytical viewpoint to a pure linear algebra formulation [5]. This model based approach appears to be very effective for noise removal if the number of sinusoids is well known [6].

Let  $x$  be such a time-series of length  $L$  containing  $P$  frequencies,

$$x = (x_1, x_2, \dots, x_L) \quad (1)$$

from  $x$  we can build  $H$ , a Hankel matrix of order  $M$ . The matrix is  $(M \times N)$  with  $M + N = L + 1$ .

If  $x$  is noiseless,  $H$  is of rank  $P$  and has  $P$  linearly independent lines. On the contrary if  $x$  contains noise the linear independence of the lines does not hold anymore and the rank of  $H$  is equal to the order  $M$ . From the matrix  $H$ , it is possible to build a *low rank* matrix of rank  $k$  :  $\tilde{H}_k$ . This rank reduction has the effect of removing partially the noise while keeping most of the signal.

For achieving such a low rank reduction a classical method is to take the Singular Value Decomposition (SVD) of  $H$ . The highest  $P$  singular values are associated to the signal, the remaining ones to the noise. Keeping only the first  $P$  singular values allows a rank reduction of the matrix  $H$  from  $M$  to  $P$ . But this method is prone to introducing spurious artefacts if  $P$  is not exactly the number of sinusoids in the signal. This SVD truncature operation can be viewed as an approximation of  $H$  and it is not a trivial one. It is a *low rank approximation* of  $H$  under the strong constraint of being the closest as possible to the AR-based model.

Recently the field of *matrix approximation* explored many alternative ways for approaching a matrix at a given rank. The main objective was to speed up the rank reduction for data compression in machine learning [7] or datamining [8] but in the same time it permitted a deep renewal in rank reduction techniques. This field of investigation is still nowadays a very active domain.

The rank reduction however comes with the loss of the Hankel matrix structure. Some authors proposed to carry out concomitantly rank reduction and structure conservation [9][10]. But averaging all the antidiagonal of the Hankel matrix [11] after the rank reduction was proved to be a very satisfactory heuristic solution to this problem even if non

optimal. For this reason we took into account only matrix approximation.

## II. LOW RANK APPROXIMATION

A very efficient technique for matrix rank reduction approximation is *the projection on a subspace*. For understanding the notion of projection, it must be noticed that geometrically a matrix  $H(M \times N)$  can be seen as a collection of  $N$  column-vectors in a Hilbert space of dimension  $M$ . A matrix approximation  $\tilde{H}_k$  of rank  $k$  from this point of view is the projection of the matrix  $H$  on a subspace formed by a set of  $k$  vectors forming an orthonormalized basis of dimension  $k$ . Let  $B_k$  be such a basis, the approximate matrix  $\tilde{H}_k$  is given by

$$\tilde{H}_k = B_k B_k^* H \quad (2)$$

where  $*$  notes the hermitian conjugate. The SVD truncature mentioned earlier corresponds to the special case where  $B_k$  is the matrix  $U_k$  formed by the first  $k$  singular vectors of  $H$ .

$$\tilde{H}_k = U_k U_k^* H \quad (3)$$

where the SVD decomposition of  $H$  is given by

$$H = U \times \Sigma \times V^* \quad (4)$$

$\Sigma$  is a pseudodiagonal matrix containing the singular values ordered from the highest to the lowest.

$$\sigma_1 > \sigma_2 > \sigma_3 \dots$$

According to the Young-Eckardt theorem [12], this projection gives the best approximation of  $H$  of rank  $k$  in term of Euclidian distance.

The choice of  $U_k$  as a basis of projection is not optimal for ensuring that the global shape of the signal is preserved. Indeed the basis  $U_k$  spans the main directions of  $H$  but neglects completely the smaller ones. This drawback can be avoided by using a basis that covers the main directions but also partially the smaller ones. This way of proceeding guarantees the signal integrity.

A straightforward approach for applying this correction consists in linearly combining the column vectors of  $U$  in  $k$  normed vectors for having a suitable basis. But the calculus of the SVD can be very difficult if the matrix is large.

A way of proceeding without SVD calculation is to build such a weighted basis by sorting the column vectors of  $H$  according to their weight, conserving the  $k$  heaviest ones and then orthonormalizing the set of picked vectors [13]. Even if the computational burden is smaller than for SVD, this method needs also some heavy calculation for evaluating the weights in the case of a large matrix. A third option is to use *randomness* i.e forming  $k$  random combinations of all the column-vectors and then to orthonormalize them. This ensures that all the directions will be represented and mainly the "heaviest" ones. The advantage above other methods is that it relies on simple matrix products.

## III. RQRD ALGORITHM

The first attempts for using randomness for algorithms that would compete with deterministic methods in routine calculation became more and more conceivable at the end of the last twenty years [14]. But random techniques were still considered until recently as the last resort when no deterministic method was available.

A new era was initiated with techniques issued from the Johnson-Lindenstrauss lemma [15] where approximation and randomness get connected through the notion of *random projection*. The lemma ensures very generally that, for a set of points in a high-dimensional Euclidean space, there exists a projection into a low-dimensional Euclidean space such that the distance between any two points changes is tightly conserved. The lemma was proved using a random directions for the projections. Contrarily to the common acceptance this demonstrates that *randomness* can lead to better results than *deterministic methods* since it helps to retain salient information much more efficiently than systematic deterministic processes. Since then, this new paradigm has been used for rewriting many classical linear algebra algorithms [16], it has been also heavily used in datamining [17]. Random projection has very strong links with the field of *Compressed Sensing* [18][19].

Recently an algorithm in this vein was proposed for matrix approximation by the use of a *self weighted random basis* instead of a pure gaussian random projection basis [3]. This different basis of projection is very well suited for denoising since it retains more the highest weighted components of the matrix vectors in the Hilbert space (generally associated to the signal) than the lightest ones (associated to the noise). We reused this matrix approximation method in the framework of Autoregressive process and called it rQRd (standing for *random QR denoising*). It was shown that the error on the approximation  $\tilde{H}_k$  of  $H$  by this algorithm is bounded as (in term of spectral norm <sup>1</sup>) [20] :

$$\|H - \tilde{H}_k\| \leq \beta \sigma_{k+1}$$

where

$$\beta = \left[ 1 + 9\sqrt{k+p} \cdot \sqrt{\min(m,n)} \right]$$

with a probability longer than  $1 - 3p^{-p}$ , where  $k$  is the rank (the number of vectors in the basis of projection) and  $p$  is a small number of additional samplings. As the factor  $\beta$  is far superior to the unit the bound is much less tight than for a SVD approximation for which it is equal to the unit. It expresses the relaxation from the strong model SVD model based viewpoint.

Behind rQRd are the notions of dimension reduction, weighted basis, matrix structure conservation and random projection.

### A. The algorithm

The algorithm relies on two main steps : the projection of the Hankel matrix  $H$  on a random basis and the matrix

<sup>1</sup>The spectral norm is defined as the highest singular value of the SVD of the matrix.

reconstruction in Hankel structure from which is retrieved the filtered signal.

the algorithm steps are :

- 1) First, form the Hankel matrix  $H$  ( $M \times N$ ) from time domain data  $x$ .
- 2) Then multiply  $H$  by a gaussian random matrix  $\Omega$  of size  $N \times k$  ( $k$  being the rank parameter) to obtain a random matrix  $Y$ .

$$Y = H \times \Omega \quad (5)$$

- 3) decompose the basis  $Y$  using the QR algorithm for orthonormalization.

$$Y = QR \quad (6)$$

the result is the orthonormalized basis  $Q$

- 4) Find an approximate  $\tilde{H}_k$  of  $H$  by projection of  $H$  on the basis  $Q$ .  $\tilde{H}_k$  is expressed in the canonical basis by the product of  $QQ^*$  with  $H$ .

$$\tilde{H}_k = QQ^*H \quad (7)$$

- 5) Finally, retrieve the filtered signal  $\tilde{x}$  by averaging on each antidiagonal of  $\tilde{H}_k$ .

#### B. Analysis of the random denoising process

We give hereafter insights about the core of the algorithm through the geometrical process underpinning the denoising.

1) *from hypersphere to hyperellipsoid*: Let consider eq (5), the random matrix  $\Omega$  is a collection of  $k$  normalized random column vectors  $\Omega_i$  of length  $N$ . Chosing the normal law distribution for  $\Omega_i$ ,

$$\Omega_i \sim \mathcal{N}(0, 1)$$

implies that  $\Omega$  is a set of isotropically distributed vectors on the unit hypersphere.

Now let see  $H$  as an operator from  $R^N$  into  $R^M$ . And let observe the formation of a random vector  $Y_i$ .

$$Y_i = H\Omega_i \quad (8)$$

Considering  $H$  through its SVD decomposition,  $H$  will transform each column vector  $\Omega_i$  to some vectors  $Y_i$  through three successive operations of rotation ( $V^*$ ), homothety ( $\Sigma$ ), and rotation ( $U$ ).

$$Y_i = U\Sigma V^*\Omega_i \quad (9)$$

It follows that each column vector of  $Y$  can be written as a random linear combination of the singular vectors  $u_i$  with coefficients  $\alpha_{ij}$  weighted by the singular values  $\sigma_i$ .

$$Y_i = \sum_{j=1}^N \sigma_j \alpha_{ij} u_j \quad (10)$$

With  $\alpha_i \sim \mathcal{N}(0, 1)$  since  $V^*$  is a rotation matrix and

$$\alpha_{ij} = (V^*\Omega_i)_j \quad (11)$$

The vectors  $Y_i$  map an hyperellipsoid which axes lengths are determined by the singular values  $\sigma_i$  (Fig. 1).

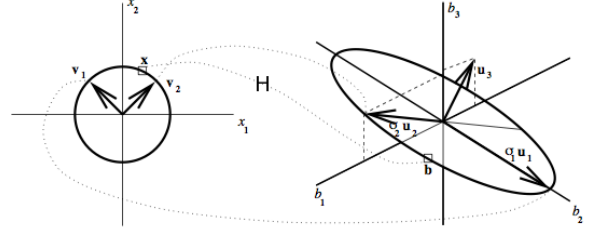


Fig. 1. Transformation of the unit hypersphere hyperellipsoid through the action of the operator  $H$ . The main axes of the hyperellipsoid correspond to the signal, the small axes to the noise. A random collection of vectors  $\Omega$  on the sphere being isotropic (gaussian distribution), its image  $H\Omega$  by the operator  $H$  will be a collection of vectors capturing more the signal directions than the noise directions. Such a vectors set is well suited as a projection basis for denoising.

Each column vector  $Y_i$  can be decomposed to a signal subspace  $Sub_s$  and a noise subspace  $Sub_n$  by separating the singular values associated respectively to the noise and to the signal. Noting that  $\sigma_{j(j \leq P)}$  and  $\sigma_{j(j > P)}$  are the singular values associated respectively to the signal and to the noise,  $Y_i$  can be decomposed to

$$Y_i = \sum_{j=1}^P \alpha_{ij} \sigma_j u_j + \sum_{j=P+1}^M \alpha_{ij} \sigma_j u_j \quad (12)$$

With  $\sigma_{j(j \leq P)} > \sigma_{j(j > P)}$  the vectors  $Y_i$  will be more oriented toward signal subspace than the noise subspace. In other words the matrix  $Y$  will capture more directions parallel to the signal than directions parallel to the noise. The captured directions will be the same for  $Q$  which is a simple orthonormalisation of  $Y$ .

#### IV. MATERIAL AND METHOD

The algorithm was applied on different synthetic time-series.

A complex time-series of length  $L = 1000$  containing fifteen sinusoids with a line width of 1.1 Hz and linearly increasing intensities was generated. A white additive gaussian noise was added to it. This data-set was processed with various values of the rank  $k$  (Fig. 2). The same data-set were processed for a full range of rank values, for three values for the order parameter for the Hankel matrix were chosen to  $L/2$ ,  $L/4$  and  $L/8$  (Fig. 3) This data-set was then used to compare rQRd and the Cadzow algorithm (4).

A complex time-series containing ten sinusoids with a line width of 1.1 Hz and linearly increasing intensities was then generated. The size of this series was varied from 1,000 to 64,000 points. The data-set was processed with the rQRd and Cadzow algorithms, with order  $M = L/4$  and a rank of 100. Processing time and Signal-to-Noise Ratio (SNR) gains are given for each length.

SNR gains were computed as the difference of SNR between before and after denoising; and are expressed as :

$$\text{SNR gain} = 10 \log \frac{\|\tilde{x} - x_o\|^2}{\|x_n - x_o\|^2} \quad (13)$$

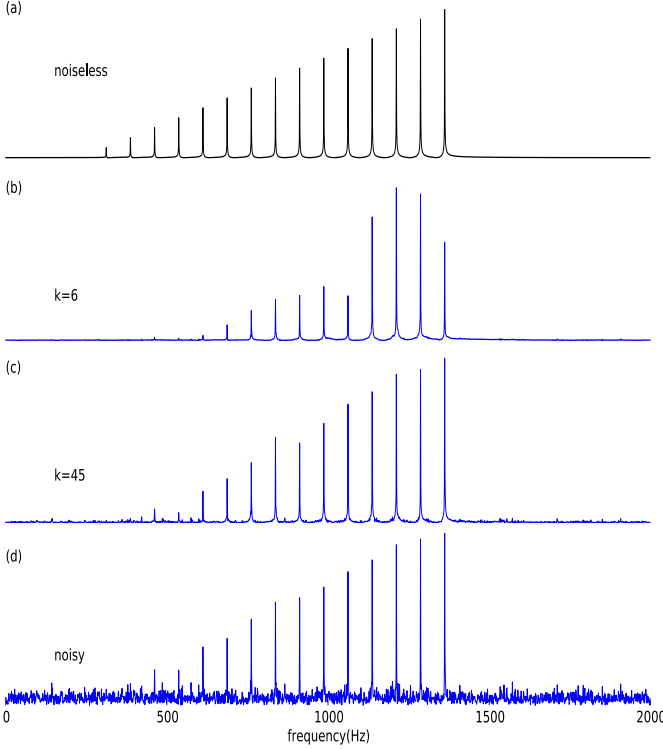


Fig. 2. Example of denoising of a synthetic dataset as a function of the rank. a) the noise-free signal containing fifteen sinusoids. b) the same signal with noise, partially denoised with a rank  $k = 6$  too small for restituting the sinusoids satisfactorily. The basis of projection  $Q$ , in this case does not cover the signal subspace enough. c) same dataset processed with  $k = 45$ , the SNR is optimal. All the sinusoids are retrieved with full amplitude and the noise is minimal. d) the unprocessed noisy signal.

where  $x_o$  is the original clean data-set,  $x_n$  is the noisy data-set, and  $\tilde{x}$  is the processed one. Results are shown in Fig. 5 and Fig. 6.

## V. RESULTS AND DISCUSSION

### A. Influence of the rank

Fig. 2 shows that the denoising afforded by rQRd is very efficient when  $k$  is larger than the number of expected lines. The intensities and the positions of the lines are perfectly conserved.

When a too small value is chosen (here  $k = 6$ ) signal distortions appear. However, even for such a grossly wrong value of the order, it can be seen that the overall shape of the signal is conserved. More detailed analysis of the effect is further analyzed in the following.

### B. Influence of the order on the optimal rank

Fig. 3 shows the SNR gains which can be obtained for varying rank and order values. The dataset used for this simulation is 1000 complex points long. It can be seen that

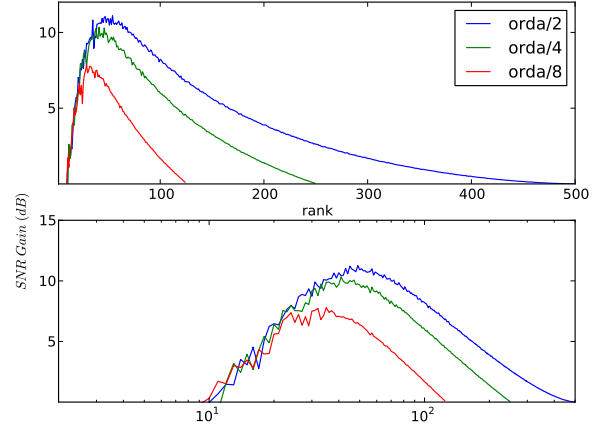


Fig. 3. Result of applying rQRd on the same data as in figure 2. rQRd SNR gain as a function of the rank, achieved for various values of the order parameter. *top* : logarithmic scale for the rank. *bottom* : linear scale for the rank. It appears that a larger order allows a better denoising.

even for such a short time series, gains over 10dB can be observed.

The maximal achievable gain depends on the order and increases with larger order values. Indeed, the order characterizes the correlation length of the autoregressive analysis. A longer correlation length increases the contrast between the signal which is correlated and the noise which does not display any correlation. The logarithmic scale highlights the shift of the position of the optimal rank for increasing order values. A large order lengthen the processing time, hence a trade-off between speed and SNR.

On the figure, the randomized appearance of the curves can be noticed for low ranks. This comes from the random projection step of the algorithm, which is performed on  $k$  vectors, and displays its random nature for small  $k$ .

### C. Robustness

Fig. 4 shows that the Cadzow algorithm has to be matched to the number of sinusoids present in the signal to give its optimum. In contrast, while rQRd is slightly less optimal in terms of SNR, its dependence on the rank is much less acute. For rQRd, the precise knowledge of the number of frequencies in the signal can be replaced by a rough estimation of the number of frequencies while having a better denoising than Cadzow.

This is due to the fact that Cadzow relies on the *best rank approximation* of  $H$ , which implies that for each rank above the number of frequencies, Cadzow will be closer to the noisy dataset than rQRd.

### D. Processing time

Fig. 5 compares the processing speed of rQRd and Cadzow on the same dataset. The speed difference between rQRd and Cadzow can be explained by the fact that the  $Y$  matrix handled

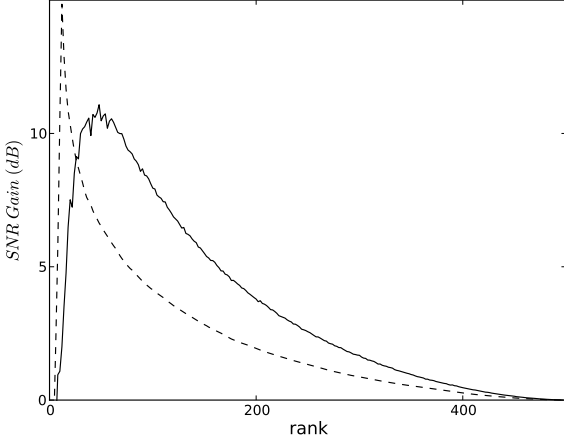


Fig. 4. rQRd (plain line) and Cadzow (dotted line) denoising performances varying with rank for order equal to  $L/2$ . While Cadzow allows a slightly better gain of rQRd at its optimum, rQRd exhibits here a better SNR gain than Cadzow on a large range of rank values.

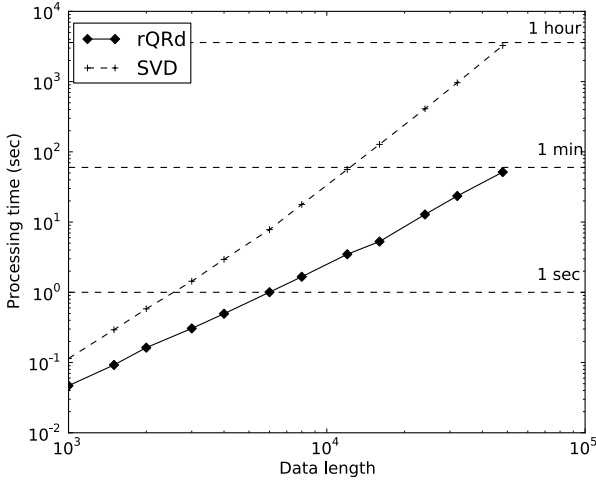


Fig. 5. Processing time rQRd and Cadzow for different data lengths. Asymptotic behavior fitted on the graph are Cadzow  $\sim n^{2.61}$  and rQRd  $\sim n^{1.77}$ . For a data length of 48,000 points, rQRd processing time is 51 sec whereas it is 3264 sec for Cadzow. All computation were performed on a Macintosh *Mac Pro* dual Xeon for a total of 12 cores. The machine was equipped with 32 GB of memory and was running MacOSX 10.6.8. All processing are parallelized on all available cores.

in rQRd is quite smaller than  $H$  which has to be processed in Cadzow.

It is noticeable that for a processing on a rank 100, one order of magnitude in speed-up is achieved for small data length and nearly two order of magnitude for data length of about  $5 \cdot 10^4$  data points.

#### E. SNR gain

The acceleration allowed by rQRd for denoising is accompanied by better performance in terms of SNR gain. Fig. 5 shows that rQRd is always superior to the Cadzow method,

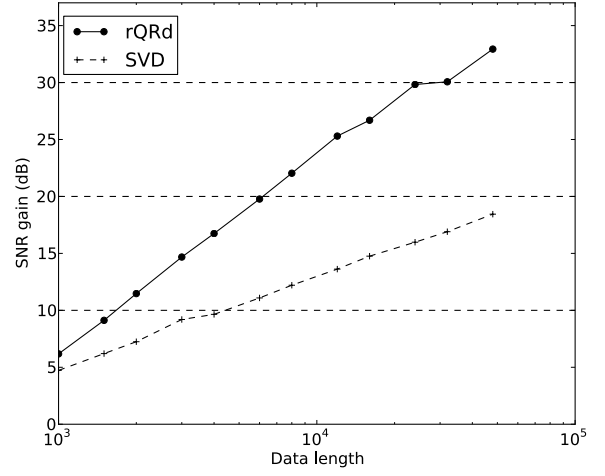


Fig. 6. The SNR gain of rQRd is compared with Cadzow method for different data lengths. For both methods the gain increases with the size of the data. For a length around 48,000 data points, gains are of 32.9 dB for rQRd and 18.4 dB for Cadzow.

and that the difference increases with larger datasets.

#### F. Estimating signal retrieval

In the case where the speed is not stringent, the rank parameter can be optimized so as to retrieve most of the signal with as less noise as possible. This requires an estimation of the signal part retrieved as a function of the rank  $k$ . This can be performed by analyzing the time-series norm retrieval as a function of the rank.

It can be estimated that

$$\frac{\|\tilde{x}_k\|}{\|x\|} \approx \frac{\|\tilde{H}_k\|}{\|H\|} = \left\| \sum_{i=1}^M \sigma_i \zeta_i^k \mathbf{u}_i \right\| / \left\| \sum_{i=1}^M \sigma_i \right\| \quad (14)$$

where  $\zeta_i^k$  is the coverage of the direction  $\mathbf{u}_i$  for a basis  $Q$  of dimension  $k$ .

This general equation can be replaced by the equivalent one of a time-series for which all the  $P$  sinusoids are of equal amplitudes. In this case an analytical expression can be found in function of  $k$  and four parameters  $a$ ,  $b$ ,  $c$  and  $d$  :

$$\frac{\|\tilde{x}_k\|}{\|x\|} \approx c \zeta_p^k(a, b) + d \zeta_m^k \quad (15)$$

with

$$\zeta_p^k(a, b) = 1 - \frac{\sqrt{((a/b)k - a + 1)^2 + 4a} - (a/b)k + a - 1}{2a} \quad (16)$$

where  $a = \text{SNR}^2$ ,  $b = P$ .  $\zeta_m^k$  is estimated to :

$$\zeta_m^k = \frac{k}{M} \quad (17)$$

In these expressions,  $\zeta_p^k$  is the common coverage of all the directions associated to the signal at rank  $k$  while  $\zeta_m^k$  is the common coverage of all the directions associated to the noise at rank  $k$ .

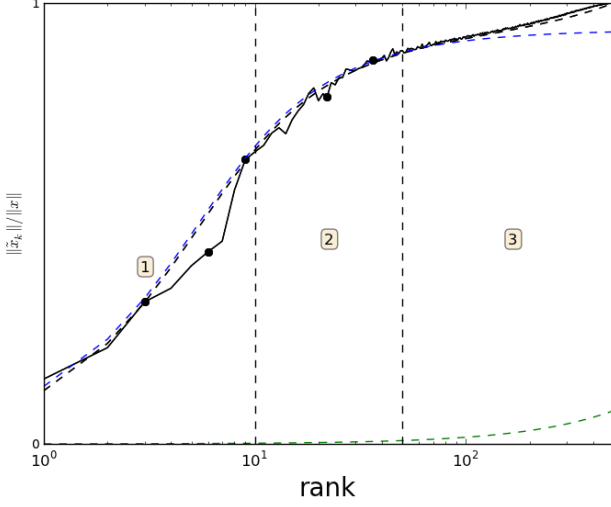


Fig. 7. Evolution of  $\|\tilde{x}_k\|/\|x\|$  as a function of the rank (solid black line). Three regions can be observed: region 1 is signal dominant, region 2 is a transition region while region 3 is noise dominant. Equation 15 is fitted here (dotted black) on five rQRd computations (black bullets). The experimental curve superimposes very well with the fit. The fitted curve can be decomposed in signal component  $\zeta_p^k$  (dotted blue) and noise component  $\zeta_m^k$  (dotted green).

Thanks to eq 15, the parameters  $a$ ,  $b$ ,  $c$ ,  $d$  can be fitted from a series of rQRd computations performed for various  $k$ . From this fit the optimal rank can be estimated and used.

Fig. 7 presents the procedure for a synthetic dataset. The fit is performed here from 5 rQRd computations, realized mostly with small rank values. Thanks to this small values and the rapid processing times, the overhead of this procedure is acceptable.

## VI. CONCLUSION

The hybridization of parametric harmonic analysis with random projection has been proved to be a very efficient way for performing denoising. It does not require any accurate *a priori* knowledge about the number of frequencies in the signal. The random nature of the algorithm affords processing times much shorter than with classical methods since dealing with a smaller computing burden. This method has also the advantage of relying only on few matrices multiplications. This feature allows an easy parallelization of the algorithm.

In the case where speed is not the main constraint, it is possible to fit from a few rapid rQRd computations, a model of the signal norm recovery in function of the rank. This fit allows to find the optimal rank value for which denoising SNR gain is maximum.

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