Deltares

Memo

To

To whom it may concern

Date	**Our reference**		**Number of pages**
2025-04-05	SVN: 67762		21
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Subject

PID controller mass-spring-damper system

Contents

enabling delta life

Used references

- Berdahl and III (2007),
- Callafon (2014),
- Rowell (2004),
- Yon-Ping (n.d., eq. 23): PD-Controller, desired set-point will not be reached.
- Rao (2009, pg. 13, eq. 2-39): Integral of *Dirac delta functions* (unit impulse and unit step response)
- Åström and Murray (2016, pg. 47 eq. 2.19): Unit step response solution.
- Seborg et al. (2011, §8.6.1)

To Do

1   Transfer functions.

1   PID controller in SOBEK

The discretised PID-controller in SOBEK2 read:

$$f^n = f^{n-1} + K_p e^n + K_i \sum_{j=0}^{n} e^j + K_d \left(e^n - e^{n-1} \right), \tag{1}$$

No literature references are found for this PID-controller (i.e. Equation (1)).

The proposed discretised PID-controller read:

$$f^n = f^{n-1} + K_p \left(e^n - e^{n-1} \right) + K_i \, \Delta t_n \, e^n + K_d \frac{e^n - 2e^{n-1} + e^{n-2}}{\Delta t} \tag{2}$$

This PID-controller (Equation (2)) is based on the linearisation of the standard PID-controller.

2   Aim of this document

The aim of this document is to find out what each term of the PID-controller does do to a general system. To reach that goal we look to the Mass-Spring-Damper system without loss of generality.

3   Conclusion

Mathematical physics

- Proportional gain factor: influence on equilibrium state. But it does not reach the required set-point.
- Integral gain factor: influence on the equilibrium state. You need this term to reach the equilibrium state.
- Derivative gain factor: Influence transition time to the equilibrium state, but it does not influence the equilibrium state.

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Numerical experiments

Some numerical experiments are performed. Varying time step and varying the time integration method of the PID-controller. The time integration of the Mass-Spring-Damper-system is implicit.

• Explicit implementation (black box approach of the PID-controller) does have a severe draw back on the computation time because the Δt should be decreased w.r.t. the other methods.
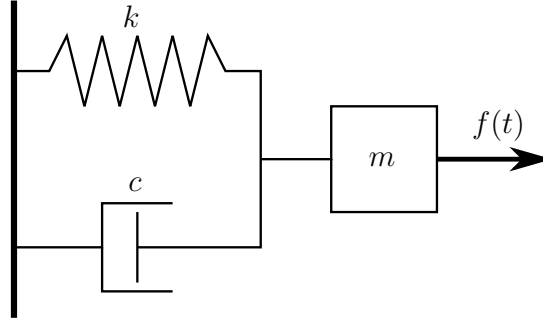
4   Mass-Spring-Damper system

Figure 1: Drawing of Mass-Spring-Damper system.

Equation of Mass-Spring-Damper system ($m > 0$, $c > 0$ and $k > 0$):

$$m\ddot{x} + c\dot{x} + kx = f(t) \quad \Leftrightarrow \quad \ddot{x} + \frac{c}{m}\dot{x} + \frac{k}{m}x = \frac{1}{m}f(t) \tag{3}$$

The natural (free) angular velocity ω_n is ($c = 0$ and $f(t) = 0$):

$$\omega_n = \sqrt{\frac{k}{m}}, \tag{4}$$

The decay time constant τ is:

$$\tau = \frac{2m}{c} \tag{5}$$

and the damping ratio ζ is:

$$\zeta = \frac{c}{2\sqrt{mk}} \tag{6}$$

The damping ratio is related to ω_n.

$mr^2 + cr + k = 0 \Rightarrow r_{1,2} = \frac{-c \pm \sqrt{c^2 - 4mk}}{2m}$

Decay of solution is: $\exp\left(\frac{-c}{2m}\right)$, so the decay time constant is: $\tau = \frac{2m}{c}$

Damping rate is: $\exp\left(-\zeta\omega_n t\right) = \exp\left(\frac{-ct}{2m}\right) \Rightarrow \zeta\omega_n = \frac{c}{2m} \Rightarrow \zeta = \frac{c}{2\omega_n m} \Rightarrow \zeta = \frac{c}{2\sqrt{km}}$.

The equation can also be written as:

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 = 0 \tag{7}$$

$r^2 + 2\zeta\omega_n r + \omega_n^2 = 0 \Rightarrow r_{1,2} = \zeta\omega_n \pm \frac{1}{2}\sqrt{4\zeta^2\omega_n^2 - 4\omega_n^2} \Rightarrow r_{1,2} = \zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$.

Define $\omega_d = \omega_n\sqrt{1 - \zeta^2}$ as the frequency for the under damped oscillations.

$\zeta = 0$ not damped (oscillations)
$\zeta < 1$ under damped (oscillations)
$\zeta = 1$ critical damped
$\zeta > 1$ over damped

4.1 Energy of Mass-Spring-Damper system

Write the Mass-Spring-Damper system as a set of first order of PDE's, without external forces as:

$$\dot{x} = v \tag{8}$$
$$m\dot{v} = -cv - kx, \tag{9}$$

The total energy of the system read:

$$E_{tot} = E_{kin} + E_{pot} = \frac{1}{2}mv^2 + \frac{1}{2}kx^2 \tag{10}$$

Energy conserving:

$$\frac{d}{dt}E_{tot} = \frac{d}{dt}\left(\frac{1}{2}mv^2 + \frac{1}{2}kx^2\right) \tag{11}$$
$$= mv\dot{v} + kx\dot{x} \tag{12}$$

Substituting gives:

$$\frac{d}{dt}E_{tot} = mv\frac{-cv - kx}{m} + kxv = -cv^2 - vkx + kxv = -cv^2 \tag{13}$$

So if $c = 0$ the system is energy conserving, other wise the system is dissipative. If $\frac{d}{dt}E_{tot} = 0$ the Mass-Spring-Damper system is called a Hamiltonian system, if $\frac{d}{dt}E_{tot} \leq 0$ the Mass-Spring-Damper system is called a Lyapunov function.

5 Feedback loop, PID-controller

Implement the feed back loop (external force is dependent on solution, Berdahl and III (2007)) and an external force as step function (i.e. new setpoint)

$$f(t) = -K_p x - K_d \dot{x} + u(t) \tag{14}$$

with

K_p gain factor proportional to x.
K_d gain factor proportional to the time derivative of x (i.e. $\dot{x}$, velocity) and
$u(t)$ the unit step function is defined as Rao (2009, pg. 13, eq. 2-28):

$$u(t) = \begin{cases} 0 & t < 0 \\ 1 & t \geq 0 \end{cases} \tag{15}$$

After substitution of Equation (14) in Equation (3) (consider only $t \geq 0$; assume everything is in rest for $t < 0$) we get:

$$m\ddot{x} + c\dot{x} + kx = -K_p x - K_d \dot{x} + u(t) \tag{16}$$
$$m\ddot{x} + (c + K_d)\dot{x} + (k + K_p)x = u(t) \tag{17}$$

To evaluate the stepfunction $u(t)$ of Equation (15), we first discuss the solution for the *Dirac delta function* (or *unit impulse function*) and than the *unit step function*.

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5.1 Unit impulse function

To evaluate the *Dirac delta function* (or *unit impulse function*):

$$\delta(t - a) = \begin{cases} \infty & t = a \\ 0 & t \neq a \end{cases} \tag{18}$$

which satisfies

$$\int_{-\infty}^{\infty} \delta(t - a)dt = 1 \tag{19}$$

$$\int_{-\infty}^{\infty} f(t)\delta(t - a)dt = f(a) \tag{20}$$

Suppose $g(t)$ is the solution of the system:

$$m\ddot{g} + c\dot{g} + kg = \delta(t) \tag{21}$$

Integrate this equation from 0 to T $(T > 0)$, just the length of the pulse. Resulting in a valid value at the right hand side

$$\int_0^T (m\ddot{g} + c\dot{g} + kg)\, dt = \int_0^T \delta(t)\, dt \tag{22}$$

$$\int_0^T m\ddot{g}\, dt + \int_0^T c\dot{g}\, dt + \int_0^T kg\, dt = \int_0^T \delta(t)\, dt \tag{23}$$

Taking the limit as $T \downarrow 0$, we obtain (term by term):

$$\lim_{T \downarrow 0} \int_0^T m\ddot{g}(t)\, dt = \lim_{T \downarrow 0} m\dot{g}(t)|_0^T = \lim_{T \downarrow 0} m\left[\dot{g}(T) - \dot{g}(0)\right] = m\dot{g}(0^+) \qquad \dot{g}(t) \text{ is discontinue} \tag{24}$$

$$\lim_{T \downarrow 0} \int_0^T c\dot{g}(t)\, dt = \lim_{T \downarrow 0} cg(t)|_0^T = \lim_{T \downarrow 0} c\left[g(T) - g(0)\right] = 0 \qquad g(t) \text{ is continue} \tag{25}$$

$$\lim_{T \downarrow 0} \int_0^T kg(t)\, dt = \lim_{T \downarrow 0} ktg(0)|_0^T = \lim_{T \downarrow 0} k\left[Tg(0) - 0g(0)\right] = 0 \qquad \text{because } g(0) = 0 \tag{26}$$

$$\int_0^T \delta(t)\, dt = 1 \tag{27}$$

and thus

$$m\dot{g}(0^+) = 1 \quad \Rightarrow \quad \dot{g}(0^+) = \frac{1}{m} \tag{28}$$

Now that we know that the response of a second-order resting system is to change the velocity (while leaving position unchanged), we can use this fact to obtain the impulse response $g(t)$. In particular, assuming an underdamped system, we know that the general form of the free response is given as

$$g(t) = e^{-\zeta\omega_n t}\left(A\cos\omega_d t + B\sin\omega_d t\right) \tag{29}$$

Therefore, with $g(0) = 0$ and $\dot{g}(0^+) = 1/m$ the response of the system to a unit impulse at $t = 0$ is given as

$$g(t) = \begin{cases} 0 & t \leq 0 \\ \frac{1}{m\omega_d}e^{-\zeta\omega_n t}\sin\omega_d t & t > 0 \end{cases} \tag{30}$$

5.2 Unit step function

The unit step function is defined as

$$u(t-a) = \begin{cases} 0 & t < a \\ 1 & t \geq a \end{cases} \tag{31}$$

The relation between the $\delta(t)$ and $u(t)$ is as follows:

$$u(t-a) = \int_{-\infty}^{t} \delta(\tau - a)\, d\tau \tag{32}$$

$$\frac{d\, u(t-a)}{dt} = \delta(\tau - a) \tag{33}$$

The function $s(t)$ is called the *step response* and satisfies:

$$m\ddot{s} + c\dot{s} + ks = u(t) \tag{34}$$

The function $s(t)$ will be obtained as follows, starting from the unit impulse solution $g(t)$:

$$m\ddot{g} + c\dot{g} + kg = \delta(t) \tag{35}$$

$$m\ddot{g} + c\dot{g} + kg = \frac{d\, u(t)}{dt} \tag{36}$$

Integrating this equation from $-\infty$ to t:

$$\int_{-\infty}^{t} \left(m\frac{d^2 g}{d\tau^2} + c\frac{d\, g}{d\tau} + kg \right) d\tau = \int_{-\infty}^{t} \frac{d\, u(\tau)}{d\tau}\, d\tau = u(t) \tag{37}$$

Using the fundamental theorem of calculus we get

$$\left(m\frac{d^2}{dt^2} + c\frac{d}{dt} + k \right) \int_{-\infty}^{t} g(\tau)\, d\tau = u(t) \tag{38}$$

It is seen that

$$s(t) = \int_{-\infty}^{t} g(\tau)\, d\tau \tag{39}$$

For $t \leq 0$ we have $s(t) = 0$ and for $t > 0$ we have (integrating Equation (30))

$$s(t) = \int_{0}^{t} \left(\frac{1}{m\omega_d} e^{-\zeta\omega_n \tau} \sin \omega_d \tau \right) d\tau \tag{40}$$

After some calculation, using $\sin x = (e^{ix} - e^{-ix})/2i$ and $(e^{ix} + e^{-ix})/2 = \cos x$ (Rao, 2009), we get

$$s(t) = \begin{cases} 0 & t \leq 0 \\ \frac{1}{m\omega_n^2} \left(1 - e^{-\zeta\omega_n t} \left(\cos \omega_d t + \frac{\zeta\omega_n}{\omega_d} \sin \omega_d t \right) \right) & t > 0 \end{cases} \tag{41}$$

or

$$s(t) = \begin{cases} 0 & t \leq 0 \\ \frac{1}{m\omega_n^2} \left(1 - e^{-\zeta\omega_n t} \left(\cos \omega_d t + \frac{\zeta}{\sqrt{1-\zeta^2}} \sin \omega_d t \right) \right) & t > 0 \end{cases} \tag{42}$$

This solution brings the system to a new equilibrium state (using Equation (4)):

$$\lim_{t \to \infty} s(t) = \frac{1}{m\omega_n^2} = \frac{1}{k} \tag{43}$$

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6 PID controller (positional)

Assume that the equilibrium value x_∞ or setpoint (new equilibrium) is a desired value, the error (deviation) to that value is defined as: $e(t) = x_\infty - x(t)$. This error (deviation) will be investigated.

The PID-controller can been seen as an external force on the Mass-Spring-Damper-system and read:

$$f(t) = K_p e(t) + K_i \int_0^t e(\tau)\, d\tau + K_d \frac{de(t)}{dt}, \tag{44}$$

where

K_p gain factor proportional to $e(t)$,
K_i gain factor proportional to the time integral of $e(t)$ and
K_d gain factor proportional to the time derivative of $e(t)$.

6.1 Proportional term

Consider just the proportional term, so $K_i = K_d = 0$.

The proportial term $K_p e(t)$ does influence the equilibrium state of the Mass-Spring-Damper system, but will not reached the desired equilibrium state x_∞.

$$m\ddot{x} + c\dot{x} + kx = K_p(x_\infty - x) \tag{45}$$
$$m\ddot{x} + c\dot{x} + (k + K_p)x = K_p x_\infty \tag{46}$$

So the particular solution will **not** reach the equilibrium state ($\lim_{t\to\infty}$, and $\ddot{x} = \dot{x} = 0$):

$$\lim_{t\to\infty} x(t) = \frac{K_p}{(k + K_p)} x_\infty \quad \text{and} \quad k + K_p > 0 \tag{47}$$

also the free angular frequency is influenced by the proportional gain K_p

$$\omega_n = \sqrt{\frac{k + K_p}{m}} \quad \text{and} \quad k + K_p > 0 \tag{48}$$

6.2 Derivative term

Consider just the derivative term, so $K_p = K_i = 0$.

The derivative term $K_d \dot{e}(t)$ does **not** influence the equilibrium state of the Mass-Spring-Damper system:

$$m\ddot{x} + c\dot{x} + kx = K_d \frac{d}{dt}(x_\infty - x) \tag{49}$$
$$m\ddot{x} + c\dot{x} + kx = K_d(\dot{x}_\infty - \dot{x}) \tag{50}$$

With $\dot{x}_\infty = 0$ for the equilibrium state, so

$$m\ddot{x} + c\dot{x} + kx = -K_d\dot{x} \tag{51}$$
$$m\ddot{x} + (c + K_d)\dot{x} + kx = 0 \tag{52}$$

So, just the damping factor (resistance, friction) is adjusted by the derivative gain K_d and the equilibrium state is the same as for the homogeneous solution ($\ddot{x} = \dot{x} = 0$):

$$\lim_{t\to\infty} kx(t) = 0 \quad \Rightarrow \quad \lim_{t\to\infty} x(t) = 0. \tag{53}$$

Remark: $c + K_d > 0$ other wise the system is unstable, see section 4.1

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6.3 Integral term

Consider just the integral term, so $K_p = K_d = 0$.

The integral term $K_i \int_0^t e(\tau) \, d\tau$ does influence the equilibrium state of the Mass-Spring-Damper system. Switching a Mass-Spring-Damper system to another equilibrium state is done by prescribing a value other than zero as external force. For example a constant value, the integral of a *Dirac delta function* or another bounded integral. Prescribing a constant value of 1 will give as equilibrium state ($\ddot{x} = \dot{x} = 0$):

$$\lim_{t \to \infty} (m\ddot{x} + c\dot{x} + kx) = 1 \tag{54}$$

$$kx = 1 \quad \Rightarrow \quad x = \frac{1}{k} \tag{55}$$

The integral of the *Dirac delta function* is equal to 1,

$$\int_{-\infty}^{\infty} \delta(t - a) dt = 1 \tag{56}$$

and will give therefore the same solution as above (Equation (55)).

Prescribing the external force as

$$f(t) = \int_0^t (x_\infty - x(\tau)) \, d\tau \tag{57}$$

will force the solution $x(t)$ to x_∞, because

$$\lim_{t \to \infty} (x_\infty - x(t)) = 0 \tag{58}$$

and the integral is bounded ($< \infty$).

7 PID controller (velocity)

Assume that a equilibrium value x_∞ or setpoint (a new equlibrium), is a desired value, the error (deviation) to that value is defined as: $e(t) = x_\infty - x(t)$, and need to be investigated.

The PID-controller can been seen as a external force on the Mass-Spring-Damper-system and read:

$$f_{\mathsf{PID}}(t) = K_p e(t) + K_i \int_0^t e(\tau)\, d\tau + K_d \frac{de(t)}{dt}, \quad \text{equal to Equation (44)} \tag{59}$$

The time derivative (velocity) of the PID-controller (Equation (59)) read:

$$\frac{\partial f_{\mathsf{PID}}}{\partial t} = K_p \frac{de(t)}{dt} + K_i e(t) + K_d \frac{d^2 e(t)}{dt^2} \tag{60}$$

Which comes from the linearisation (i.e. $\delta y = \mathcal{K}\,\delta x$) of the PID-controller:

$$f_{\mathsf{PID}}(t + \delta t) = f_{\mathsf{PID}}(t) + \mathcal{K}\,\delta t \tag{61}$$

$$f_{\mathsf{PID}}(t + \delta t) - f_{\mathsf{PID}}(t) = \mathcal{K}\,\delta t \tag{62}$$

Divide Equation (62) by δt and take $\lim_{\delta t \to 0}$, we get:

$$\lim_{\delta t \to 0} \frac{f_{\mathsf{PID}}(t + \delta t) - f_{\mathsf{PID}}(t)}{\delta t} = \mathcal{K} \tag{63}$$

$$\frac{\partial f_{\mathsf{PID}}}{\partial t} = \mathcal{K} \equiv \text{RHS Equation (60)} \tag{64}$$

In discrete form Equation (60) read, with $\Delta t_n = \Delta t$ is constant, $e^n = x_\infty - x^n$:

$$\frac{f^n - f^{n-1}}{\Delta t_n} = K_p \frac{e^n - e^{n-1}}{\Delta t_n} + K_i\, e^n + K_d \frac{e^n - 2e^{n-1} + e^{n-2}}{\Delta t^2} \tag{65}$$

$$f^n = f^{n-1} + \Delta t_n \left\{ K_p \frac{e^n - e^{n-1}}{\Delta t_n} + K_i\, e^n + K_d \frac{e^n - 2e^{n-1} + e^{n-2}}{\Delta t^2} \right\} \tag{66}$$

$$f^n = f^{n-1} + K_p \left(e^n - e^{n-1} \right) + K_i\, \Delta t_n\, e^n + K_d \frac{e^n - 2e^{n-1} + e^{n-2}}{\Delta t} \tag{67}$$

8 Determine coefficients from experiments

Taken from Callafon (2014)

Estimation of model parameters
With the times t_0, t_n and the values y_0, y_n and y_∞ from step response:
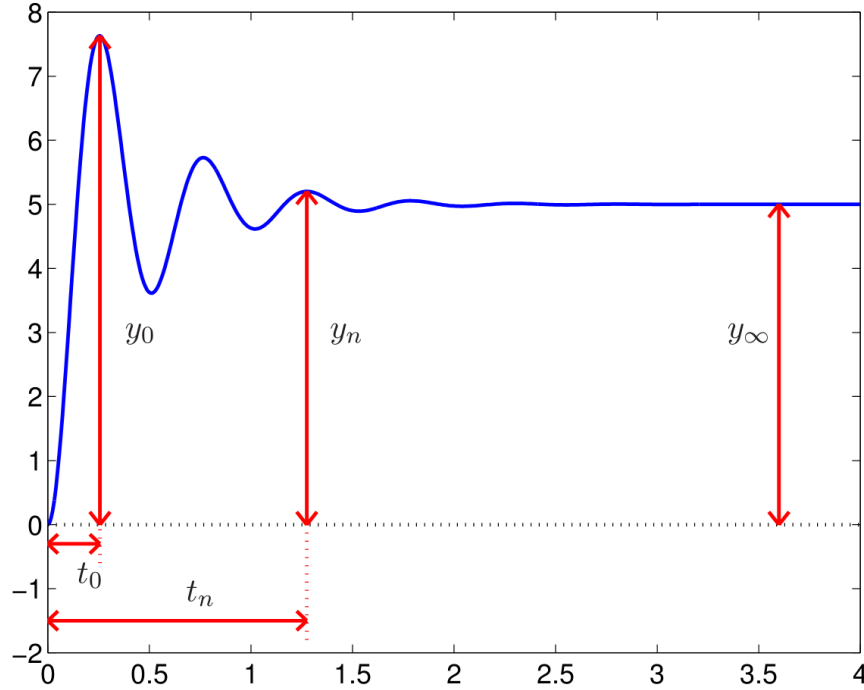

Figure 2: *Collect data from measurement.*

Allows us to estimate:

$$\widehat{\omega}_d = 2\pi \frac{n}{t_n - t_0} \qquad \text{damped resonance frequency} \tag{68}$$

$$\widehat{\beta\omega}_n = \frac{1}{t_n - t_0} \ln\left(\frac{y_0 - y_\infty}{y_0 - y_\infty}\right) \qquad \text{exponential decay term} \tag{69}$$

where n as subscript of t is the *number of oscillations* between t_n and t_0. With the estimations of the demped frequency $\widehat{\omega}_d$ and $\widehat{\beta\omega}_n$ we can now compute the natural frequence ω_n and the damping ratio β:

$$\widehat{\omega}_n = \sqrt{\widehat{\omega}_d^2 + \widehat{\beta\omega}_n^2} \qquad \text{undamped resonance frequency} \tag{70}$$

$$\beta = \frac{\widehat{\beta\omega}_n}{\widehat{\omega}_n} \qquad \text{damping ratio} \tag{71}$$

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9 Numerical discretisation

9.1 Mass-Spring-Damper system as system of first order PDE's

The equation of a Mass-Spring-Damper system

$$m\ddot{x} + c\dot{x} + kx = f(t) \tag{72}$$

can be written as a system of first order partial differential equations, which read:

$$\dot{x} = v \tag{73}$$

$$m\dot{v} = -cv - kx + f(t) \tag{74}$$

with initial conditions: $\dot{x} = v = 0$ and $x = 1$. In matrix notation this equation read:

$$\begin{pmatrix} 1 & 0 \\ 0 & m \end{pmatrix} \begin{pmatrix} \dot{x} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -k & -c \end{pmatrix} \begin{pmatrix} x \\ v \end{pmatrix} + \begin{pmatrix} 0 \\ f(t) \end{pmatrix} \tag{75}$$

This system of equations in discrete form, with $\Delta t_{n+1} = t^{n+1} - t^n$, read:

$$\frac{x^{n+1} - x^n}{\Delta t_{n+1}} = v^n \tag{76}$$

$$m\frac{v^{n+1} - v^n}{\Delta t_{n+1}} = -cv^{n+1} - kx^{n+1} + f(x^{n+1}, x^n, \ldots) \tag{77}$$

Rearranging gives:

$$x^{n+1} = x^n + \Delta t_{n+1} v^n \tag{78}$$

$$\left(1 + \frac{\Delta t_{n+1}}{m}c\right) v^{n+1} = v^n - \frac{\Delta t_{n+1}}{m}kx^{n+1} + \frac{\Delta t_{n+1}}{m}f(x^{n+1}, x^n, \ldots) \tag{79}$$

9.2 PID controller (positional)

9.2.1 Implicit Mass-Spring-Damper system with explicit PID-controller
An explicit discrete form of the PID-controller (Equation (44)) read:

$$f^n = K_p e^n + K_i \sum_{j=0}^{n} e^j \Delta t_j + K_d \frac{e^n - e^{n-1}}{\Delta t_n} \tag{80}$$

The system of equations will than read:

$$x^{n+1} = x^n + \Delta t_{n+1} v^n \tag{81}$$

$$\left(1 + \frac{\Delta t_{n+1}}{m}c\right) v^{n+1} = v^n - \frac{\Delta t_{n+1}}{m}kx^{n+1} +$$

$$\frac{\Delta t_{n+1}}{m}\left[K_p e^n + K_i \sum_{j=0}^{n} e^j \Delta t_j + K_d \frac{e^n - e^{n-1}}{\Delta t_n}\right] \tag{82}$$

9.2.2 Implicit Mass-Spring-Damper system with implicit PID-controller
An implicit discrete form of the PID-controller (Equation (44)) read:

$$f^{n+1} = K_p e^{n+1} + K_i \sum_{j=0}^{n+1} e^j \Delta t_j + K_d \frac{e^{n+1} - e^n}{\Delta t_{n+1}} \tag{83}$$

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The system of equations will than read:

$$x^{n+1} = x^n + \Delta t_{n+1} v^n \tag{84}$$

$$\left(1 + \frac{\Delta t_{n+1}}{m} c\right) v^{n+1} = v^n - \frac{\Delta t_{n+1}}{m} k x^{n+1} +$$

$$\frac{\Delta t_{n+1}}{m} \left[K_p e^{n+1} + K_i \sum_{j=0}^{n+1} e^j \Delta t_j + K_d \frac{e^{n+1} - e^n}{\Delta t_{n+1}} \right] \tag{85}$$

9.2.3 Corrected Mass-Spring-Damper system while using an explicit PID controller

Therefore we will adjust the Mass-Spring-Damper system for the explicit values used by the PID-controller by adding those terms who will make the PID-controller implicit.

We will separate Equation (80) from Equation (83). The individual terms belonging to Equation (80) will be placed between square brackets.

$$f^{n+1} = K_p e^{n+1} + K_i \sum_{j=0}^{n+1} e^j \Delta t_j + K_d \frac{e^{n+1} - e^n}{\Delta t_{n+1}} \tag{86}$$

$$= K_p e^{n+1} - K_p e^n + \left[K_p e^n \right] +$$

$$K_i e^{n+1} \Delta t_{n+1} + \left[K_i \sum_{j=0}^{n} e^j \Delta t_j \right]$$

$$+ K_d \frac{e^{n+1} - e^n}{\Delta t_{n+1}} - K_d \frac{e^n - e^{n-1}}{\Delta t_n} + \left[K_d \frac{e^n - e^{n-1}}{\Delta t_n} \right] \tag{87}$$

The part between square brackets, is equal to Equation (80), which is the PID-controller based on explicit time levels:

$$f^n = K_p e^n + K_i \sum_{j=0}^{n} e^j \Delta t_j + K_d \frac{e^n - e^{n-1}}{\Delta t_n} \tag{88}$$

So we get:

$$f^{n+1} = K_p e^{n+1} - K_p e^n + K_i e^{n+1} \Delta t_{n+1} + K_d \frac{e^{n+1} - e^n}{\Delta t_{n+1}} - K_d \frac{e^n - e^{n-1}}{\Delta t_n} + f^n \tag{89}$$

Substitution of Equation (103) in Equation (79) leads to the following system of equations:

$$x^{n+1} = x^n + \Delta t_{n+1} v^n \tag{90}$$

$$\left(1 + \frac{\Delta t_{n+1}}{m} c\right) v^{n+1} = v^n - \frac{\Delta t_{n+1}}{m} k x^{n+1} +$$

$$\Delta t_{n+1} \frac{1}{m} \left[-K_p (x^{n+1} - x^n) + K_i e^{n+1} \Delta t_{n+1} \right.$$

$$\left. + K_d \frac{e^{n+1} - e^n}{\Delta t_{n+1}} - K_d \frac{e^n - e^{n-1}}{\Delta t_n} + f^n \right] \tag{91}$$

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Some rearranging:

$$x^{n+1} = x^n + \Delta t_{n+1} v^n \tag{92}$$

$$\left(1 + \frac{\Delta t_{n+1}}{m} c\right) v^{n+1} = v^n - \frac{\Delta t_{n+1}}{m} \left(k + K_p\right) x^{n+1} + \frac{\Delta t_{n+1}}{m} K_p x^n +$$

$$+ \frac{\Delta t_{n+1}}{m} K_i e^{n+1} \Delta t_{n+1} +$$

$$+ \frac{\Delta t_{n+1}}{m} K_d \frac{e^{n+1} - e^n}{\Delta t_{n+1}} - \frac{\Delta t_{n+1}}{m} K_d \frac{e^n - e^{n-1}}{\Delta t_n} + \frac{\Delta t_{n+1}}{m} f^n \tag{93}$$

f^n is given by Equation (80).

9.3 PID controller (velocity)

9.3.1 Implicit Mass-Spring-Damper system with explicit velocity PID controller

An explicit discrete form of the PID-controller based on the linearisation (Equation (60)) read:

$$f^n = f^{n-1} + K_p \left(e^n - e^{n-1}\right) + K_i \Delta t \, e^n + K_d \frac{e^n - 2e^{n-1} + e^{n-2}}{\Delta t}, \quad \text{equal to Equation (67)} \tag{94}$$

The system of equations will than read:

$$x^{n+1} = x^n + \Delta t_{n+1} v^n \tag{95}$$

$$\left(1 + \frac{\Delta t_{n+1}}{m} c\right) v^{n+1} = v^n - \frac{\Delta t_{n+1}}{m} k x^{n+1} +$$

$$\frac{\Delta t_{n+1}}{m} \left[f^{n-1} + K_p \left(e^n - e^{n-1}\right) + K_i \Delta t \, e^n + K_d \frac{e^n - 2e^{n-1} + e^{n-2}}{\Delta t} \right] \tag{96}$$

9.3.2 Implicit Mass-Spring-Damper system with implicit velocity PID controller

An implicit discrete form of the PID-controller based on the linearisation (Equation (60)) read:

$$f^{n+1} = f^n + K_p \left(e^{n+1} - e^n\right) + K_i \Delta t \, e^{n+1} + K_d \frac{e^{n+1} - 2e^n + e^{n-1}}{\Delta t}, \quad \text{equal to Equation (67)} \tag{97}$$

The system of equations will than read:

$$x^{n+1} = x^n + \Delta t_{n+1} v^n \tag{98}$$

$$\left(1 + \frac{\Delta t_{n+1}}{m} c\right) v^{n+1} = v^n - \frac{\Delta t_{n+1}}{m} k x^{n+1} +$$

$$\frac{\Delta t_{n+1}}{m} \left[f^n + K_p \left(e^{n+1} - e^n\right) + K_i \Delta t \, e^{n+1} + K_d \frac{e^{n+1} - 2e^n + e^{n-1}}{\Delta t} \right] \tag{99}$$

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9.3.3 Corrected Mass-Spring-Damper system while using an explicit velocity PID controller
Therefore we will adjust the Mass-Spring-Damper system for the explicit values used by the
velocity PID-controller by adding those terms who will make the velocity PID-controller implicit.

We will separate Equation (94) from Equation (97). The individual terms belonging to
Equation (94) will be placed between square brackets. Δt is constant.

$$f^{n+1} = f^n + K_p \left(e^{n+1} - e^n\right) + K_i \, \Delta t \, e^{n+1} + K_d \frac{e^{n+1} - 2e^n + e^{n-1}}{\Delta t} \tag{100}$$

$$
\begin{aligned}
&= f^n - f^{n-1} + \left[f^{n-1}\right] + \\
&+ K_p \left(e^{n+1} - 2e^n + e^{n-1}\right) + \left[K_p \left(e^n - e^{n-1}\right)\right] \\
&+ K_i \, e^{n+1}\Delta t_{n+1} - K_i e^n \Delta t_n + \left[K_i \, e^n \Delta t_n\right] \\
&+ \ldots + \left[K_d \frac{e^n - 2e^{n-1} + e^{n-2}}{\Delta t}\right]
\end{aligned}
\tag{101}
$$

The part between square brackets, is equal to Equation (94), which is the PID-controller
based on explicit time levels:

$$f^n = f^{n-1} + K_p \left(e^n - e^{n-1}\right) + K_i \, \Delta t \, e^n + K_d \frac{e^n - 2e^{n-1} + e^{n-2}}{\Delta t} \tag{102}$$

So we get:

$$f^{n+1} = \ldots + f^n \tag{103}$$

Substitution of Equation (103) in Equation (79) leads to the following system of equations:

$$x^{n+1} = x^n + \Delta t_{n+1} v^n \tag{104}$$

$$
\left(1 + \frac{\Delta t_{n+1}}{m}c\right) v^{n+1} = v^n - \frac{\Delta t_{n+1}}{m} k x^{n+1} +
$$

$$\frac{\Delta t_{n+1}}{m} \left(\ldots + f^n\right) \tag{105}$$

Some rearranging:

$$x^{n+1} = x^n + \Delta t_{n+1} v^n \tag{106}$$

$$\left(1 + \frac{\Delta t_{n+1}}{m}c\right) v^{n+1} = v^n + \ldots + \frac{\Delta t_{n+1}}{m} f^n \tag{107}$$

f^n is given by Equation (80).

10 Experiments

Mass-Spring-Damper-system:

$$m\ddot{x} + c\dot{x} + kx = f(t), \tag{108}$$

$$f(t) = K_p e(t) + K_i \int_0^t e(\tau) \, d\tau + K_d \frac{de(t)}{dt} \tag{109}$$

with
Initial values: $x(0) = 1$ and $\dot{x}(0) = 0$.

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Coefficients: $m = 100$, $c = 2$, $k = 1$.
PID gain factors: $K_p = 10.0$; $K_i = 0.05$; $K_d = 10.0$.
Setpoint: 0.5.

$e(t) = x_\infty - x(t)$.
x_∞ = setpoint.

Table 1: *Equilibrium solution.*

K_p	K_i	K_d	Equili
10.0	0.0	0.0	0.45
0.0	0.05	0.0	0.5
0.0	0.0	10.0	0.0

10.1 Solutions determined by Maplesoft
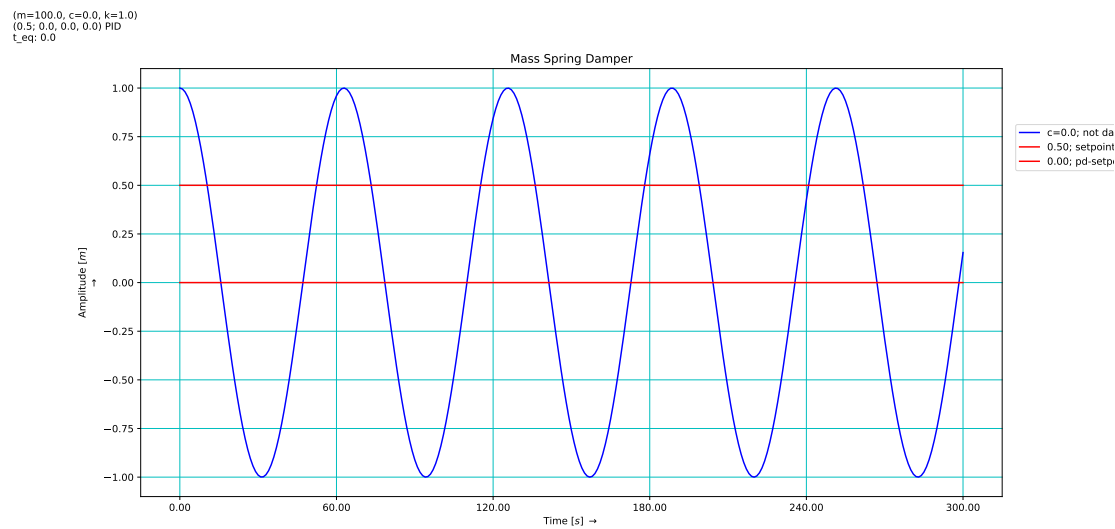
Figure 3: *The natural frequency; $c = K_p = K_i = K_d = 0$.*

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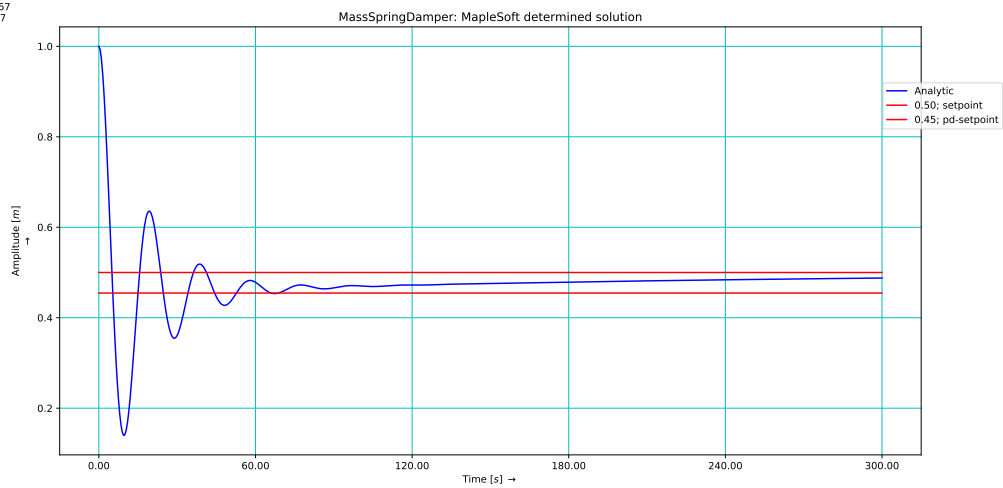**Our reference**
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Figure 4: *PID controller on Mass-Spring-Damper using MapleSoft solution. The integral term is quite small and therefor the influence is noticed after a while.*

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10.2 Numerical experiments

Table 2: *Performed numerical experiments.*

time integration	Δt $1.0\ s$	Δt $0.5\ s$	Δt $0.25\ s$
explicit (§9.2.1)	✓	✓	✓
implicit (§9.3.2)	✓	✓	✓
corrected (§9.2.3)	✓	✓	✓

10.2.1 Time integration method
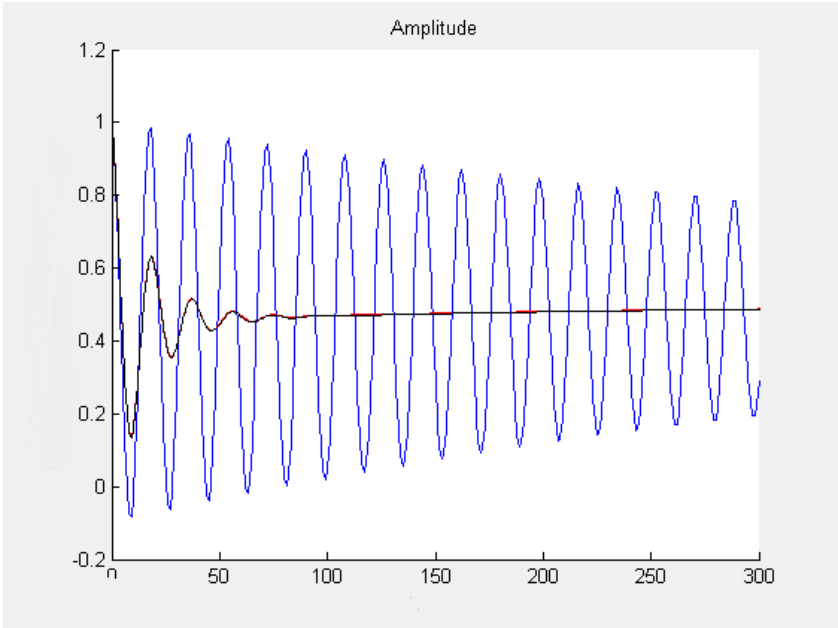
Figure 5: *Numerical experiment ($\Delta t = 1\ s$); explicit (blue, §9.2.1), implicit (red, §9.3.2), corrected (black, §9.2.3).*
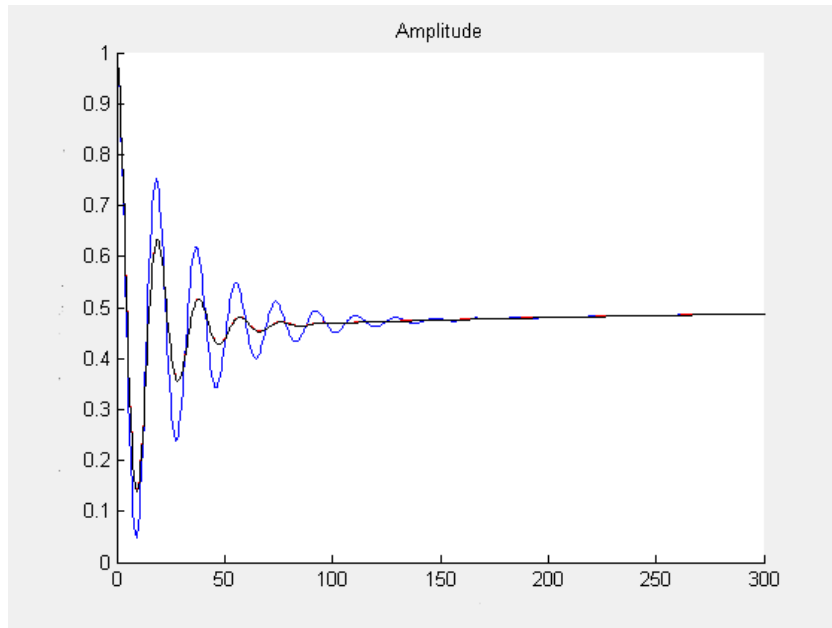
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Figure 6: *Numerical experiment ($\Delta t = 0.5\ s$); explicit (blue, §9.2.1), implicit (red, §9.3.2), corrected (black, §9.2.3).*

Figure 7: *Numerical experiment ($\Delta t = 0.25\ s$); explicit (blue, §9.2.1), implicit (red, §9.3.2), corrected (black, §9.2.3).*
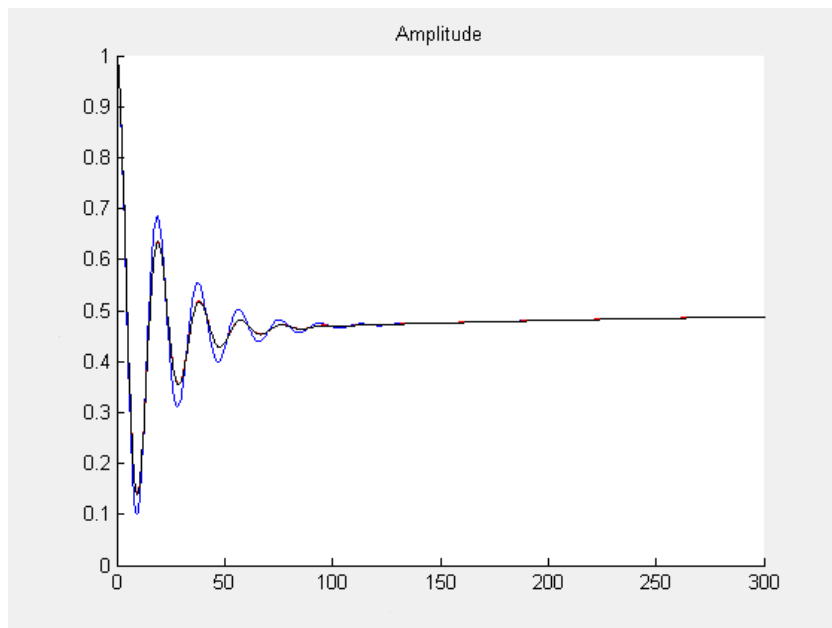
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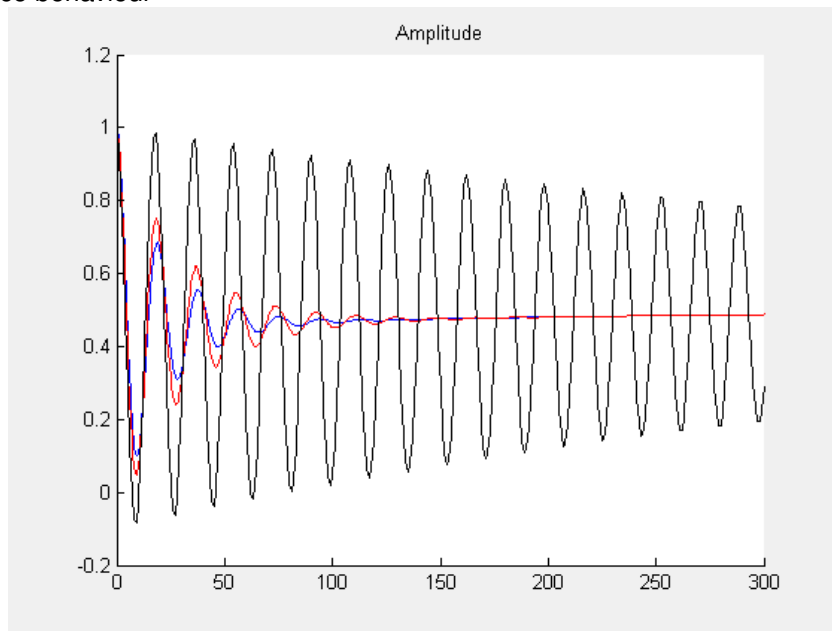19 of 21

10.2.2 Convergence behaviour

Figure 8: *Numerical experiment (explicit §9.2.1); black,* $\Delta t = 1.0\ s$*; red,* $\Delta t = 0.5\ s$*; blue,* $\Delta t = 0.25\ s$*.*
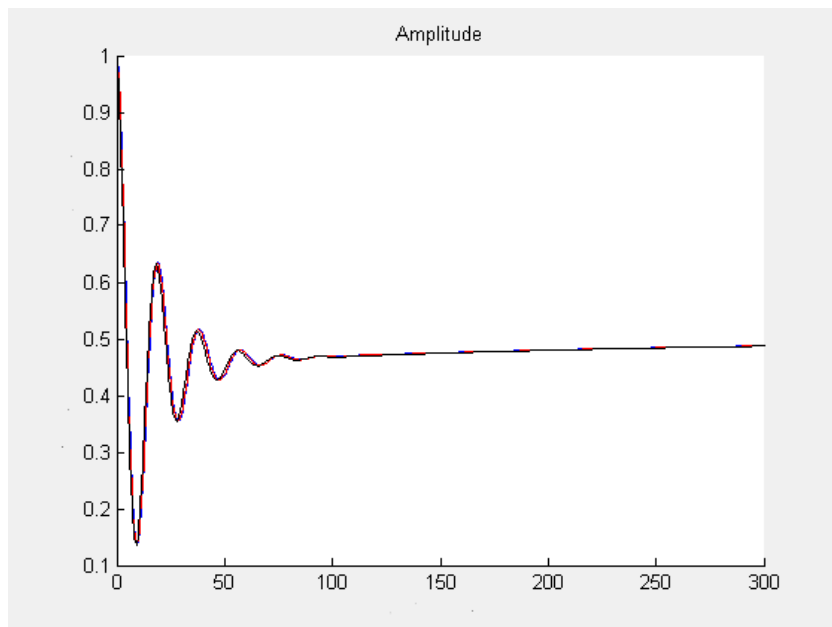

Figure 9: *Numerical experiment (corrected §9.2.3); black,* $\Delta t = 1.0\ s$*; red,* $\Delta t = 0.5\ s$*; blue,* $\Delta t = 0.25\ s$*.*

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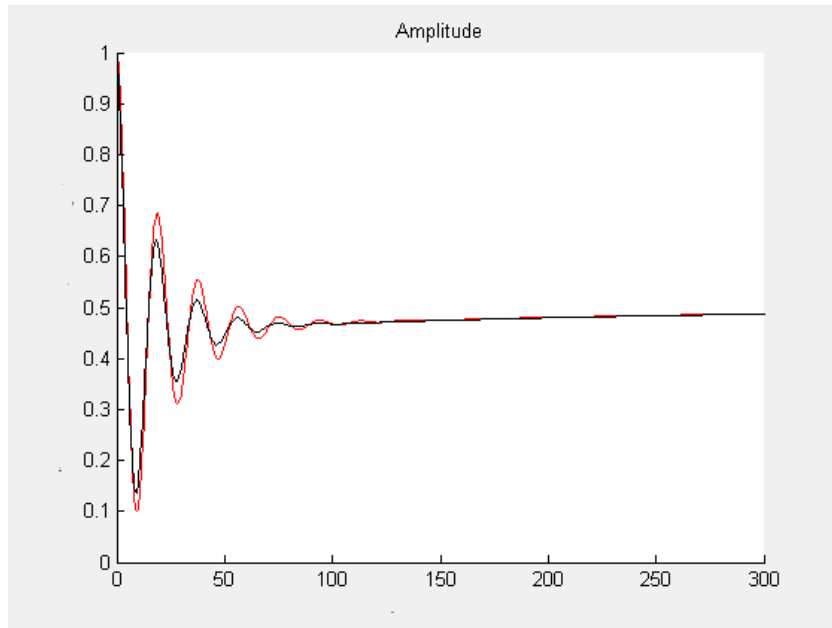**Our reference**
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Figure 10: *Numerical experiment, different Δt and time integration; explicit (red, $\Delta t = 0.25\ s$, §9.2.1), corrected (black, $\Delta t = 1.0\ s$, §9.2.3).*
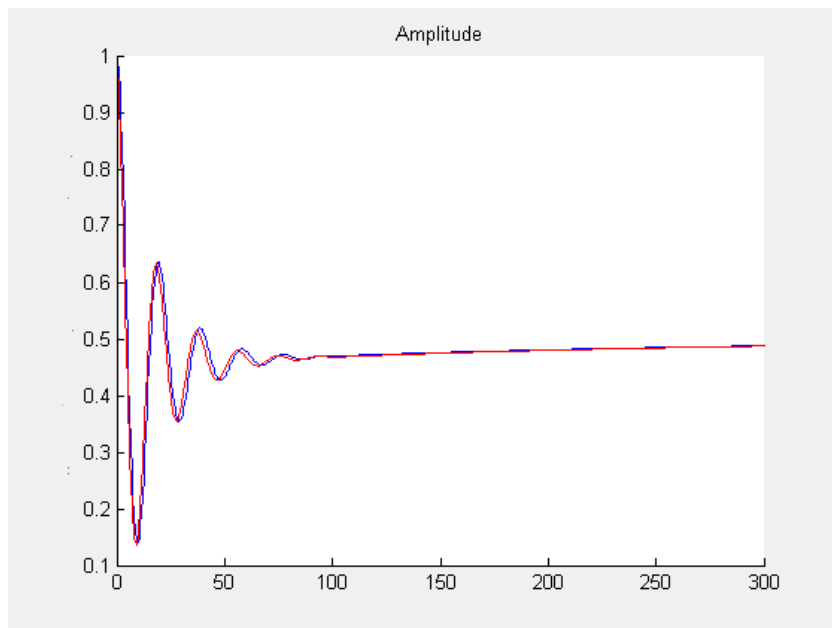

Figure 11: *Maple solution (blue) vs corrected (red, $\Delta t = 1.0\ s$, §9.2.3).*

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