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Frequency Equation of a Curved Beam using the Phase-closure Principle

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Abstract

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A curved beam has the ability to transfer loads through a combined action of bending and stretching. It's mechanical behavior under applied loads produces an efficient component that can be applied in many structures. Therefore, various studies have gained interest in the dynamic behavior of curved beam structures, particularly linear free vibration analysis.

Linear free vibration analysis involves evaluation of natural frequencies and their corresponding mode shapes, which is of great importance to understand the dynamic behavior of curved beams especially at the early stages of design. In order to understand this dynamic behavior in curved beams, several methods have been employed.

However, these methods are very laborious and computationally expensive. As a result, many studies have used the wave propagation method of analysis due to its simplicity in the vibration analysis of complex structures as compared to other methods. The wave perspective of analysis is commonly used with the matrix formulation of reflection and transmission matrices to obtain the numerical solution of matrix eigenvalue problems. The simple approach of using reflection coefficients obtained from the reflection matrix while using the phase-closure principle has received little attention in analysis of curved beams.

In this study, a simplified frequency equation is proposed and used to predict the modal frequencies of a curved beam. The frequency equation is formulated using phase changes of the reflection coefficient obtained from the reflection matrix. Dispersion equations and frequency spectra curves are reviewed. A specific case of set of roots of wave motion is applied with the phase-closure principle using both extensional and inextensional curved beam models with varying support conditions and constant curvature parameters, while neglecting the transverse shear and the rotary inertia effects.

The approach is validated through comparison to numerical results of matrix formulation and previous studies. The results show a good agreement between the predictions by the proposed equation and the numerical results. Therefore the approach can be reliably applied to predict natural frequencies of curved beams with various support conditions.

Keywords: free vibration, curved beams, frequency equation, frequency spectra, phase- closure principle

CONTENTS

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List of Symbols

The following symbols are used in this thesis:

A	Cross-sectional area
a	amplitude of waves
E	Young's Modulus
I	Second moment of inertia
k	Curvature parameter
M	Bending moment
N	Tensile force
R	Radius of curvature
T_0	Characteristic time constant
Γ	Dimensionalized wavenumber
γ	Non-dimensionalized wavenumber
δ	Interface friction angle
θ	Angular co-ordinate
$d\theta$	Arc angle
ρ	Mass density
ω	Non-dimensionalized angular frequency
ω_c	Critical frequency
Ω	Dimensionalized angular frequency

I. INTRODUCTION

1.1 Motivation

Curved beams are common structural elements used in built-up structures which are part of important engineering structures such as aircraft structures, bridges, and modern electric machine parts. A comprehensive study of curved beams includes linear or nonlinear free vibration analysis and planar analysis problems consisting of static and dynamic analysis. Although free vibration analysis of circular beam is more complex than a similar problem in rectangular beam, since its structural configuration depends on two displacement variables, curved beams are more efficient in use than straight beams. This is due to the ability of curved beams to transfer loads through a combined action of bending and stretching.

Various methods are applied to study the dynamics of curved beam structures like Rayleigh-Ritz method, Dynamic Stiffness method, Transfer Matrix method, Finite Element method and the Wave propagation approach. While the Finite Element method is widely used in studies, it is computationally expensive if numerous finite elements are required to be generated. The wave approach is a concise and systematic approach to the analysis of structures since it easily allows efficient variation of the geometry and size of complex structures. Therefore, as compared to other methods, the wave approach efficiently determines natural frequencies and mode shapes of structures.

The wave approach is widely used in deriving reflection and transmission matrices of waves at discontinuities in Euler-Bernoulli straight beams (Mace, 1984). The same approach was applied to derive reflection and transmission matrices of waves was applied for free vibration and forced vibration in Timoshenko beams (Mei, 2005). Studies were later conducted on rods, where equations were developed to analyze the vibration characteristic of rings (Graff, 1970). In curved beams, Lee et al. (2007) obtained natural frequencies using the reflection and transmission matrices for a U-shaped structure. The gradual development of the wave approach has resulted in approaches like the phase-closure principle also known as wave-train closure principle.

The phase-closure principle has been applied in previous studies along with the transfer matrix method to obtain the natural frequencies of structures including curved structures (Kang et al., 2003; Tan and Kang, 1999). However, little attention is given to the use phases of reflection and transmission coefficients while using the phase-closure principle, and yet the use of these coefficients often simplifies vibration analysis of structures.

Therefore, this research involves the use of phases of the reflection coefficients with the phase-closure principle to obtain a frequency equation reliable to predict natural frequencies of curved beams with various support conditions.

1.2 Research Objective

This research applies a frequency equation using the phase-closure principle to predict the modal frequencies of curved beams with varying boundary conditions. It is anticipated that this study will provide a simple method of understanding the dynamic behavior of curved beams so as to improve initial stages of design of curved beam structures.

The objectives of this research are;

(a) Review governing equations and wave solutions for the frequency spectrum.

(b) Derive expressions for the reflection matrices for waves in a curved beam.
(c) Formulate a simplified frequency equation using the phase-closure principle.
(d) Validate the findings of the study using previous research results.

1.3 Research Scope

The research is limited to the following;

Linear free vibration analysis. The curved beam is analysed to evaluate the natural frequencies and associated mode shapes.

A thin curved beam with various support conditions, neglecting the effects of shear deformation, rotary inertia and damping. The support conditions will consist the hinged-hinged, clamped-clamped, free-free support conditions.

The inextensible (without the extension of the neutral axis) and extensible models (with the extension of the neutral axis) of the curved beam undergoing in-plane motion.

Only one case of wave motion is used in vibration analysis (Case III solution) which consists of two propagating and four near fields. The flexural propagating wave is considered and the near fields are neglected while using the phase-closure principle.

1.4 Thesis Outline

The thesis consists of five chapters. A brief overview of each chapter is as follows:

Chapter 1 presents the introduction under which the background, objectives and scope of the research are discussed.

Chapter 2 presents previous research on vibration analysis of curved beams.

Chapter 3 presents the methodology used to study the dynamic behavior of curved beams.

Chapter 4 presents the results and discussion of the research.

Finally, chapter 5 presents the conclusion and recommendations for further studies.

II. LITERATURE REVIEW

2.0 Overview

This chapter presents the literature review on the vibration analysis of curved beams. The literature covers the beam theory and the type of dynamic problem used in the curved beam. The various methods employed for vibration analysis are also discussed, especially the wave propagation method as applied in curved beams.

2.1 Theories in beam analysis

2.1.1 Euler-Bernoulli theory

The Euler-Bernoulli theory is essential in predicting frequencies of flexural vibration of the lower modes of slender beams with sufficient accuracy.

Researchers have used the classical theory of beams to study the vibration motion of structures (Jeong and Ih, 2009; Krishnan and Suresh, 1998; Mead, 1994). In early research, utilized the Euler-Bernoulli beam was used to study the near field along with propagating waves which were frequently studied in the vibrations of stretched strings and torsion bars (Mace, 1984). The near field was also studied in the analysis of long periodic Euler-Bernoulli beam structures (Yong and Lin, 1989).

2.1.2 Timoshenko theory

The Timoshenko theory provides a more accurate approximation of the behavior of the beam when the normality assumption of the Euler-Bernoulli beam theory is not considered (Hu, 2009). However, the Euler-Bernoulli theory provides a simpler approach to modal vibration analysis when effects of rotary inertia and shear deformation are neglected as compared to the Timoshenko theory. For this reason, the Euler-Bernoulli theory is readily used for vibration analysis for slender long beams.

2.2 Dynamic stability analysis

The dynamic problems in free vibrations of curved beams involve in-plane motions which consists of flexural-extensional modes, out-of-plane vibrations involving bending-twisting dynamics and coupled motions consisting of extension, bending, shear and twist.

Various authors have focused on the in-plane vibration of curved structures with various boundary conditions (Yang et al., 2017; Walsh and White, 2000; Yang and Sin, 1995; Mau and Williams, 1988; Petyt and Fleischer, 1971; Sabir and Ashwell, 1971; Seidel and Erdelyi, 1964).

Xiuchang et al. (2013) presented a concise approach to obtain exact solutions for in-plane harmonic motions of a curved beam of uniform bisymmetric cross-section and constant radius of curvature. The dynamic equations of the in-plane harmonic motion were based on the flünge theory. Lee et al. (2007) also applied the flünge theory and in-plane motion of six uncoupled wave components to analyze curved beam structures. Tufekci and Arpaci (1998) also presented an exact solution of circular elastic curved beams of uniform cross-sections under in-plane vibration considering effects of axial extension, transverse shear deformation and rotatory inertia. Issa et al. (1987) obtained natural frequencies of continuous curved beams undergoing in-plane motions. The numerical investigation of the study was based on a dynamic stiffness matrix method for circular curved structures of constant section with shear deformation, extension of neutral axis and rotary inertia effects. Mallik (1976) applied the wave approach to study the in-plane vibrations of inextensional nature of a thin circular ring with radial supports in order to obtain natural frequencies and normal modes. Shoei-sheng Chen (1973) considered a detailed analysis of free vibrations of continuous curved beams vibrating in their own plane in order to determine the frequencies of the curved beams. Davis et al. (1972) incorporated shear deformation and rotary inertial effects of curved beam finite elements with constant curvature in the in-plane vibrations. They were able to obtain frequencies from a more accurate finite element analytical procedure. Tung-Ming Wang (1972) considered minute in-plane motions of circular curved frames with fixed boundary conditions to determine natural frequencies of the multi-span frame structures. Rao and Sundararajan (1969) developed an equation of motion for free, in-plane vibrations of a circular ring that allows for shear deformation and rotatory inertia. As a result, natural frequencies and mode shapes were obtained from an equation of motion. Nelson (1962) performed an analytical study using the Raleigh-Ritz technique along with Lagrangian multipliers for in-plane motions of a simply supported circular ring. Exact and accurate frequencies and mode shapes of asymmetric and symmetric nature were obtained while considering inextensional and extensional vibration.

Other studies consisted of planar vibrations which include both in-plane and out-plane vibrations. Mei and Mace (2005) assumed an existence of in-plane bending waves, axial waves and out-of-plane bending waves when studying reflection and transmission of beams jointed at a right angle. By considering curved beams of constant curvature, Bickford and Strom (1975) performed an analysis to develop in-plane and out-of-plane frequencies and mode shapes of the curved elements. In the analysis of a helical spring, Wittrick (1966) considered existence of coupling between extension of the spring and rotation about its axis. In addition, the author developed expressions for predominant energies of in-plane and out-of-plane deformations of the ring respectively.

While most authors mainly consider in-plane and out-of-plane analysis of structures especially curved structures, other studies emphasized coupling motions in their analysis (Dawe, 1978; Krishnan and Suresh, 1998).

2.3 Vibration analysis using various methods

Various methods are used to study the dynamics of structures. Modal approaches for example the Rayleigh-Ritz method are used to determine the solution of vibration analysis. The Rayleigh-Ritz method depends on the variation principle, which uses an expansion technique from the Ritz method. This method is convenient when a small number of lowest natural frequencies of a vibration system is required, including the fundamental natural frequency. A number of researches have used the Rayleigh-Ritz method to study the vibration nature of circular structures (Nelson, 1962; Den Hartog, 1888). However, this method depends on a large number of displacement functions and requires a trial function (shape function) which satisfies all geometric boundary conditions.

The Galerkin method is a classical analytical method whose trial solution is based on reducing the weighted residual functions. The Galerkin method is easily applicable to conservative and non-conservative continuous systems. Just like the Rayleigh-Ritz method, the trial function in this method should satisfy all boundary conditions of analytical problems. In continuous systems with spatially varying parameters, the Galerkin method may be slow and tedious, and thus effective in obtaining results for the first few shape functions with their corresponding natural frequencies. The Galerkin method was applied to determine natural frequencies and initial buckling stress problems in curved beams (Chen and Shen, 1998; Chidamparam and Leissa, 1995).

Another method is the Dynamic stiffness method. The Dynamic Stiffness method is an analytical method which uses an exact solution of partial differential equations to attain the frequency-dependent stiffness matrix. The Dynamic Stiffness matrix defines the dynamic properties of a system which relate frequency and the stiffness. The method was used to obtain the closed solution of analysis for continuous circular beams (Issa et al., 1987).

The Transfer Matrix technique is also used to analyze the vibration nature of structures. The Transfer Matrix technique describes the dynamic and geometric relationships of an element between two stations. This method consists of a deformation and force vector conveyed from one point to another. The Transfer Matrix approach was easily applied in the analysis of finitely long periodic structures with established boundary conditions or in piecewise periodic structures (Yong and Lin, 1988). This method was utilized to analyze the vibration nature of beam structures (Tufekci and Arpaci, 1998; Tan and Kang, 1999; Bickford and Storm, 1975).

For over the past decade, the Finite Element Method (FEM) has been utilized for modeling a diversity of complex structures in order to obtain the natural frequency and modes shapes and vibration response of these structures. Many researches have utilized FEM to analyze the behavior of complex structures mainly curved structures (Krishnman and Suresh, 1998; Yang and Sin, 1995; Heppler, 1992; Prathap, 1985; Dawe, 1978; Davis et al., 1972; Petyt and Fleischer, 1971; Sabir and Ashwell, 1971; Yong and Lin, 1989).

However, when the number of structural spans is large or high order frequencies are required, most of the mentioned methods become extremely burdensome (Shoei-sheng Chen, 1973). Oftentimes the FEM is computationally expensive when numerous finite elements are required to be generated while performing optimization studies.

2.4 Vibration analysis using wave perspective

Analysis of vibration in curved structures involves studying the physical behavior, for example vibration transmission and energy flow. A couple of studies have focused on energy flow in these structures (Xiuchang et al., 2013; Jeong and Ih, 2009; Mace et al., 2001; Wittrick, 1966). However, most studies have focused on solution methods in the analysis of curved beam structures. Linear vibration analysis especially free vibration analysis, performs the solution analysis which evaluates the natural frequencies and their corresponding mode shapes. Engineers can model or design curved structures which can safely withstand free and forced vibration at the early stages of design.

The wave approach is a concise and systematic approach of analysis of structures since it easily allows efficient variation of the geometry and size of complex structures, thus efficiently determines natural frequencies and mode shapes.

A number of researchers have derived reflection and transmission matrices of waves at discontinuities in Euler-Bernoulli beams (Lee et al. 2007; Mei and Mace, 2005; Mace, 1984). The wave approach has been applied to analyze complex structures with periodic sections and helical springs (Yong and Lin, 1989; Wittrick, 1966). Analysis of curved structures mainly rings using the wave approach are believed to have begun with studies by Den Hartog (1888). Thereafter many studies were conducted to develop equations that analyze the vibration characteristics of rings (Graff, 1970; Mallik and Mead, 1977). In 1975, Graff presented equations for in-plane motion of curved rods excluding the shear and rotary-inertia effects. These equations when derived from the Love's theory and gradually other different theories including Flügge's theory and the correction for rotary inertia and shear deformation were used to analyze curved beams (Lee et al., 2007; Walsh and White, 2000).

The matrix formulation of obtaining natural frequencies using the wave approach has been adopted by many researchers (Lee et al., 2007; Mei and Mace, 2005; Mead, 1994; Yong and Lin, 1989; Mace, 1984). This formulation combines with the transfer matrix method and the reflection, transmission and propagation matrices from and between points of discontinuities to analyze free vibration of structures. Tan and Kang (1999) also applied the same procedure to analyze the free vibration of a rotating Timoshenko shaft system.

A number of studies have involved the use of the transfer matrix method while using the phase-closure principle to obtain the natural frequencies of structures as well as curved structures (Kang et al., 2003; Tan and Kang, 1999). The phase-closure principle states that if the phase shift between incident and reflected waves is an integer multiple of 2π, then the waves propagate at a natural frequency and their motions constitute a vibration mode (Cremer, 2005; Mead, 1994). However, a different approach of wave propagation analysis using the wave train principle is used to obtain natural frequencies of structures. The approach involves formulating an equation of the wave number and frequency relationship of a system which intends to apply the wave closure principle (Mead, 1994). This approach requires a wave propagating along a waveguide, incident upon a discontinuity, which produces reflected and transmitted waves whose amplitudes and phases are obtained by reflection and transmission coefficients (Mace, 1984).

III. METHODOLOGY

3.0 Overview

This chapter presents the methodology on the vibration analysis of curved beams. The wave reflection matrices of varying support conditions are derived from which phases of the reflection coefficients are obtained. The frequency equation and characteristic equation are both derived using the phase-closure principle.

3.1 Governing equations of motion

This section presents governing equations of motion of circular curved beams with a constant radius, neglecting transverse shear deformation and rotary inertia effects. Following the study by Kang et al. 2003, the thin circular curved beam is considered, as shown in Fig.3.1 where N is the tensile force, V is the shear force and M is the bending moment. The radial and tangential displacements of the centroid axis are given as W and U respectively.

$$\frac{EI}{R^3}\frac{\partial^3}{\partial\theta^3}\left(U-\frac{\partial W}{\partial\theta}\right)-\frac{EA}{R}\left(W+\frac{\partial U}{\partial\theta}\right)=\rho RA\frac{\partial^2 W}{\partial T^2} \tag{3.1a}$$

$$\frac{EI}{R^3}\frac{\partial^2}{\partial\theta^2}\left(U-\frac{\partial W}{\partial\theta}\right)+\frac{EA}{R}\frac{\partial}{\partial\theta}\left(W+\frac{\partial U}{\partial\theta}\right)=\rho RA\frac{\partial^2 U}{\partial T^2} \tag{3.1b}$$

where E is the Young's modulus, I is the second moment of inertia of the cross-section area, θ the angular co-ordinate, R is the radius of curvature for a given range of angle, the mass density, A the cross-sectional area, T the time variable.

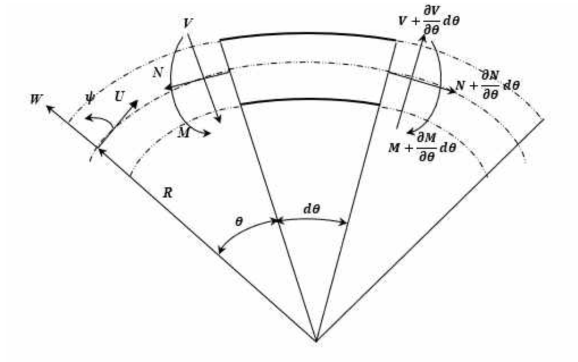


Fig. 3.1. Differential element of a curved beam and sign conventions

The tensile force, N and the bending moment M can be expressed with the displacement and curvature parameter as

$$N = \frac{EA}{R}\left(W + \frac{\partial U}{\partial \theta}\right) \tag{3.2a}$$

$$M = -\frac{EAk^2}{R^2}\frac{\partial}{\partial \theta}\left(U - \frac{\partial W}{\partial \theta}\right) \tag{3.2b}$$

The dimensionless variables and parameters for the curved beam are given as (Kang et al. 2003)

$$u = \frac{U}{R},\ \ w = \frac{W}{R},\ \ t = \frac{T}{T_0},\ \ T_0^2 = R^2\frac{\rho A}{EI},\ \ k^2 = \frac{I}{AR^2},\ \ k = \frac{r}{R} \tag{3.3}$$

where r is the radius of cross-section, T_0 is the a characteristic time constant and $k^2$is the radius of gyration of the cross-section.

$$k^2\frac{\partial^3}{\partial \theta^3}\left(u - \frac{\partial w}{\partial \theta}\right) - \left(w + \frac{\partial u}{\partial \theta}\right) = k^2\frac{\partial^2 w}{\partial t^2} \tag{3.4a}$$

$$k^2 \frac{\partial^2}{\partial\theta^2}\left(u - \frac{\partial w}{\partial\theta}\right) + \frac{\partial}{\partial\theta}\left(w + \frac{\partial u}{\partial\theta}\right) = k^2 \frac{\partial^2 u}{\partial t^2} \tag{3.4b}$$

The curvature parameter *k,* is defined as the ratio of the in-plane thickness and circular radius. The curvature parameter is an important parameter required in describing the frequency spectrum of a curved beam.

3.2 Harmonic wave solutions

Assuming the time-harmonic motion to asses the wave propagation in a curved beam using the following wave solutions,

$$w(\theta,t) = C_w e^{i(\gamma\theta - \omega t)} \tag{3.5a}$$

$$u(\theta,t) = C_u e^{i(\gamma\theta - \omega t)}, \qquad 0 \le \theta \le 2\pi \tag{3.5b}$$

The non-dimensional wavenumber and frequency are expressed as:

$$\gamma = R\Gamma \qquad \omega = \Omega T_0 \tag{3.6}$$

The amplitude ratio of wave amplitudes C_u and C_w is

$$\alpha = \frac{C_u}{C_w} = \frac{i\gamma(1+\gamma^2 k^2)}{\gamma^2(1+k^2) - k^2\omega^2} \tag{3.7}$$

Substituting the harmonic solutions Eq. (3.5a) and (3.5b) into Eq. (3.1a) and (3.1b) respectively gives

$$\begin{vmatrix} 1 + k^2(\gamma^4 - \omega^2) & -i\gamma(1+\gamma^2 k^2) \\ i\gamma(1+\gamma^2 k^2) & -\gamma^2 + k^2(\omega^2 - \gamma^2) \end{vmatrix} \begin{Bmatrix} C_w \\ C_u \end{Bmatrix} = 0 \tag{3.8}$$

The non-trivial solution of the determinant of the matrix in Eq. (3.8) can be obtained as the dispersion equation of wavenumber γ expressed as

$$\gamma_n^6 + (-2 - k^2\omega^2)\gamma_n^4 + \{1 - (1 + k^2)\omega^2\}\gamma_n^2 + (k^2\omega^2 - 1)\omega^2 = 0 \qquad (3.9)$$

Six roots are obtained from the dispersion equation from which four different cases of sets of roots, indicating four distinct wave motions of the extensional and inextensional curved beam (Kang et al. 2003).

Case I

$$\omega(\theta,t) = (C_{w1}^{+} e^{-i\gamma_1\theta} + C_{w2}^{+} e^{-i\gamma_2\theta} + C_{w3}^{+} e^{-i\gamma_3\theta} + C_{w1}^{-} e^{i\gamma_1\theta} + C_{w2}^{-} e^{i\gamma_2\theta} + C_{w3}^{-} e^{i\gamma_3\theta})e^{-i\omega t}$$

$$u(\theta,t) = (C_{u1}^{+} e^{-i\gamma_1\theta} + C_{u2}^{+} e^{-i\gamma_2\theta} + C_{u3}^{+} e^{-i\gamma_3\theta} + C_{u1}^{-} e^{i\gamma_1\theta} + C_{u2}^{-} e^{i\gamma_2\theta} + C_{u3}^{-} e^{i\gamma_3\theta})e^{-i\omega t}$$

(3.10a,b)

Case II

$$\omega(\theta,t) = (C_{w1}^{+} e^{-i\gamma_1\theta} + C_{w2}^{+} e^{-i\gamma_2\theta} + C_{w3}^{+} e^{-i\gamma_3\theta} + C_{w1}^{-} e^{i\gamma_1\theta} + C_{w2}^{-} e^{i\gamma_2\theta} + C_{w3}^{-} e^{i\gamma_3\theta})e^{-i\omega t}$$

$$u(\theta,t) = (C_{u1}^{+} e^{-i\gamma_1\theta} + C_{u2}^{+} e^{-i\gamma_2\theta} + C_{u3}^{+} e^{-i\gamma_3\theta} + C_{u1}^{-} e^{i\gamma_1\theta} + C_{u2}^{-} e^{i\gamma_2\theta} + C_{u3}^{-} e^{i\gamma_3\theta})e^{-i\omega t}$$

(3.11a,b)

Case III

$$\omega(\theta,t) = (C_{w1}^{+} e^{-i\gamma_1\theta} + C_{w2}^{+} e^{i\gamma_2\theta} + C_{w3}^{+} e^{i\gamma_3\theta} + C_{w1}^{-} e^{i\gamma_1\theta} + C_{w2}^{-} e^{i\gamma_2\theta} + C_{w3}^{-} e^{-i\gamma_3\theta})e^{-i\omega t}$$

$$u(\theta,t) = (C_{u1}^{+} e^{-i\gamma_1\theta} + C_{u2}^{+} e^{-i\gamma_2\theta} + C_{u3}^{+} e^{i\gamma_3\theta} + C_{u1}^{-} e^{i\gamma_1\theta} + C_{u2}^{-} e^{i\gamma_2\theta} + C_{u3}^{-} e^{-i\gamma_3\theta})e^{-i\omega t}$$

(3.12a,b)

Case IV

$$\omega(\theta,t) = (C_{w1}^{+} e^{-i\gamma_1\theta} + C_{w2}^{+} e^{i\gamma_2\theta} + C_{w3}^{+} e^{-i\gamma_3\theta} + C_{w1}^{-} e^{i\gamma_1\theta} + C_{w2}^{-} e^{-i\gamma_2\theta} + C_{w3}^{-} e^{i\gamma_3\theta})e^{-i\omega t}$$

$$u(\theta,t) = (C_{u1}^{+} e^{-i\gamma_1\theta} + C_{u2}^{+} e^{i\gamma_2\theta} + C_{u3}^{+} e^{-i\gamma_3\theta} + C_{u1}^{-} e^{i\gamma_1\theta} + C_{u2}^{-} e^{-i\gamma_2\theta} + C_{u3}^{-} e^{i\gamma_3\theta})e^{-i\omega t}$$

(3.13a,b)

where the coefficients C^{+} and C^{-} represent positive-travelling and negative-travelling waves along the curved beam respectively.

The curvature parameter is related to the cut-off frequency in the wave motion. As shown in Fig. 3.2, the non-zero cut-off frequency, ω_c defines the wave-mode transition in the frequency spectrum obtained by attempting the long wave length limits of Eq. (3.14) as follows:

$$\lim_{\gamma \to 0}(3.8) = (k^2\omega^2 - 1)\omega^2 = 0 \qquad (3.14)$$

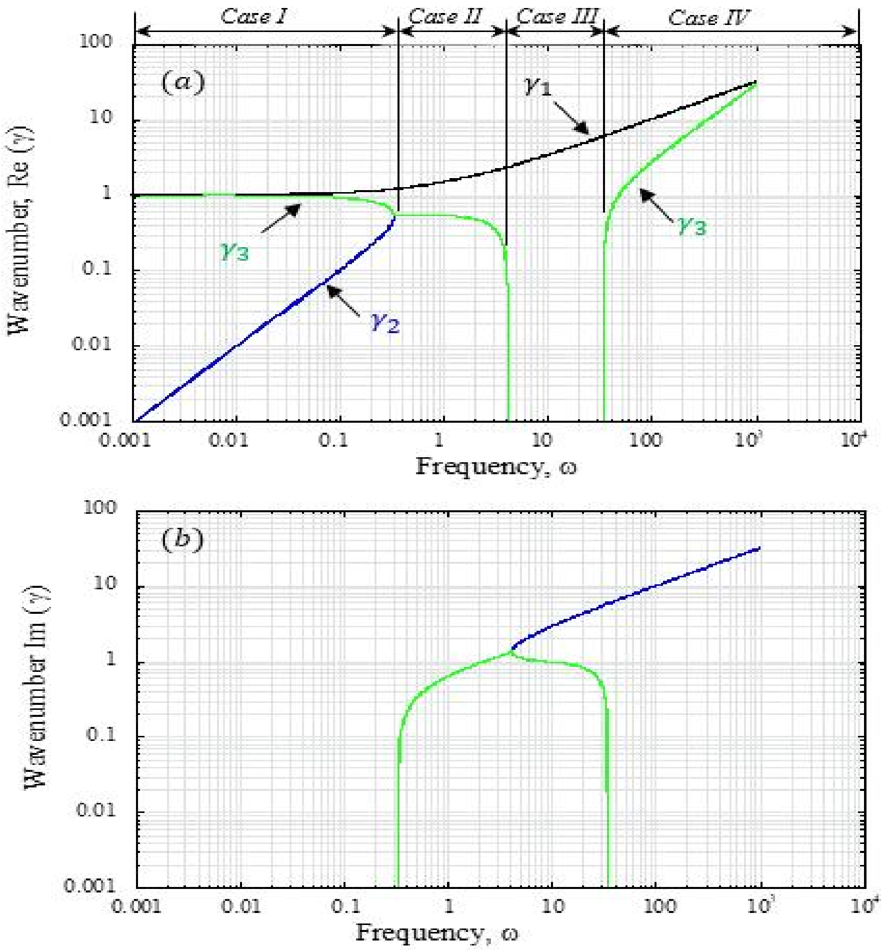
Hence, $\omega = 0, 1/k$ are the limits. The first root substituted in Eq. (3.9) reduces to,

$$\gamma_n^2(\gamma^4 - 2\gamma_n^2 + 1) = 0 \qquad (3.15)$$

from which roots $\gamma_n = 0, 1$ are obtained. The second root of Eq. (3.15) is a special root at zero frequency, which exists when the wavelength of the flexural wave is equal to the circumference, $2\pi R$. The latter root of Eq. (3.14) is the non-zero cut-off frequency ω_c which exists in cylindrical shell dynamics when the wavelength of extensional waves in a straight rod is equal to the circumference, $2\pi R$ (Walsh and White, 1999). The respective values of the curvature parameter k, for the inextensional case and the extensional case are given as k = 0 and k = 0.0289 which corresponds to h/R = 0.1 (Rectangular beam) (Lee et al. 2007).

Figure 3.2. also shows the extensional and flexural mode that exist over the whole frequency spectrum from a low to high range of the non-dimensional frequency. In Case I, the wave modes are all real without any attenuation in the curved beam. In Case II, the wave motion involves two propagating and four spatially decaying waves, which indicate positive-going and negative-going waves. The same wave motion is involved in Case III however, the motion consists of four near field wave components. In Case IV, the wave motion involves four propagating and two near field wave components. The wave components represented by $e^{-i\gamma_3\theta}$ and $e^{i\gamma_3\theta}$ portray the extensional mode. These wave components (γ_3 is real) are important in understanding the nature of wave motion and obtaining general wave motion above the cut-off frequency ω_c. However, in the range of $\omega < \omega_c$ exists spatially decay (γ_3 is complex) or attenuate (γ_3 is imaginary), where the curvature effects are important to the stiffness of a curved beam. There is a minimal contribution of the attenuating wave components to the overall wave motion and can be neglected at hinged boundary ends or at a large span distance between the point of wave generation and the boundary since near fields decay exponentially. In an inextensional curved beam, the flexural mode is highly dispersive at the low and high non-dimensional frequencies represented by the $e^{-i\gamma_1\theta}$ and $e^{i\gamma_1\theta}$. The inextensional curved beam also consists of two others pairs which are near fields when $\omega > 4.1996$. While as for the extensional curved beam model, both the extensional and bending waves dominate when $\omega > \omega_c$. Above the cut-off frequency, the extensional and the flexural waves behave similarly to the quasi-longitudinal waves in a straight rod and purely flexural waves in a straight Euler-Bernoulli beam respectively (Walsh and White, 1999).

Due to the nature of the wave components in the Case III, the phase-closure principle is easily applied with Case III of wave motion since analysis is simplified when one pair of waves is used to calculate the natural frequencies of structure (Mead, 1994). The range of frequency for Case III wave motion is $4.164 < \omega < \omega_c = 34.641$.



Case I
Case II
Case III
Case IV
(a)
Wavenumber, Re (γ)
Frequency, ω
(b)
Wavenumber Im (γ)
Frequency, ω

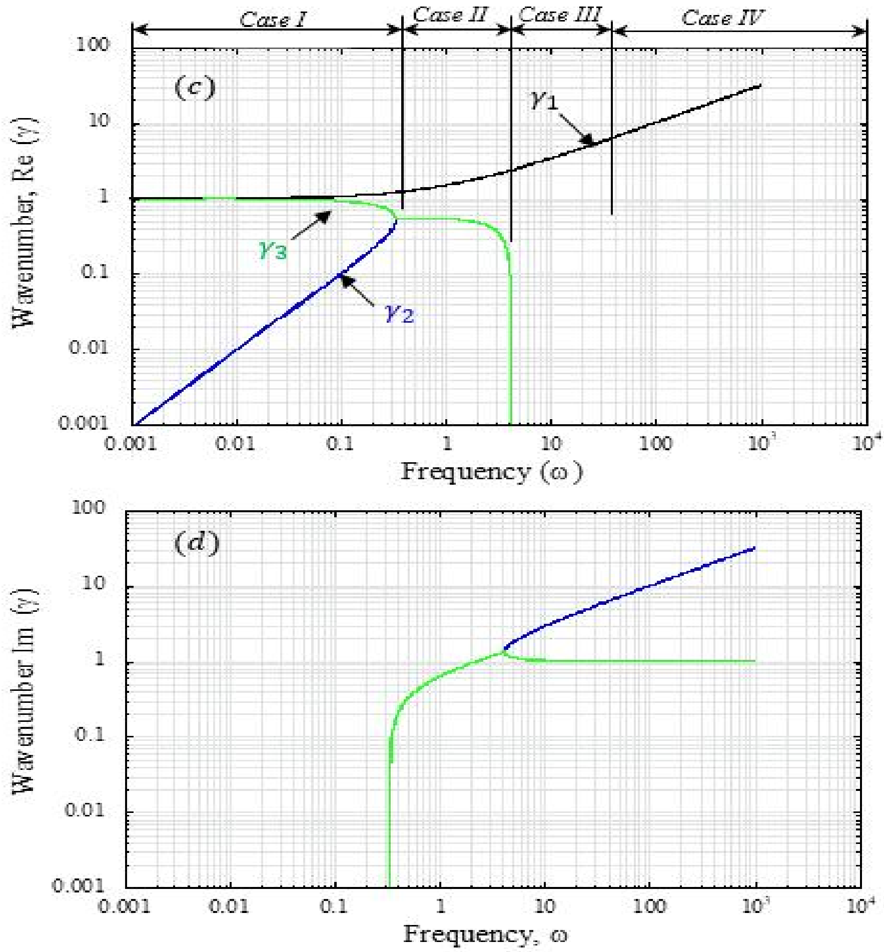


Fig. 3.2. Frequency spectra of a curved beam for extensional case (k = 0.0289) of which (a) real and (b) imaginary branches; for inextensional case (k= 0) of which (c) real and (d) imaginary branches.

3.3 Wave reflection matrices in curved beams

3.3.1 Wave reflection of waves in curved beams

Waves incident upon a discontinuity such as a boundary, are reflected and transmitted depending on the properties of the discontinuity. In this section, reflection of waves incident upon a boundary are considered. Reflection matrices of different boundary conditions are obtained and the phases of reflection matrices for each support condition are applied to the phase-closure principle, to formulate a frequency equation.

3.3.2 Wave reflection at boundaries

The wave components of equations (3.9) to (3.12) and the matrices of the curved beam are represented in order of 3x1 and 3x3 respectively. Using the amplitude ratio given in Eq. (3.6), the wave number and the radial and tangential displacements of Case III are given in Eq. (3.16a) and (3.16b) respectively.

$$C_{w1}^{+} : \gamma = \gamma_1 > 0 \Rightarrow \alpha_1^{+} = \frac{C_{u1}^{+}}{C_{w1}^{+}} = \frac{i\gamma_1(1+\gamma_1^2 k^2)}{\gamma_1^2(1+k^2) - k^2\omega^2} \tag{3.16a}$$

$$C_{w1}^{-} : \gamma = -\gamma_1 \Rightarrow \alpha_1^{-} = \frac{C_{u1}^{-}}{C_{w1}^{-}} = -\alpha_1^{+} \tag{3.16b}$$

(a) Hinged boundary condition

A general boundary condition such as the hinged boundary condition as shown in Fig. 3.3 is considered. The incident waves C^{+} give rise to reflected waves C^{-}, related by

$$C^{-}=r_{III}C^{+} \tag{3.17}$$

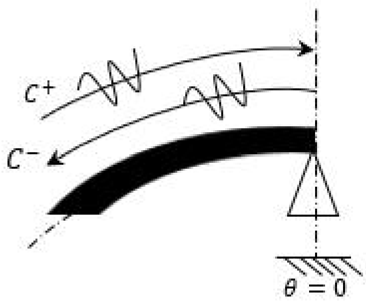


Fig. 3.3. Wave reflection at a hinged boundary

By considering equilibrium and conditions at the boundary, the reflection matrix r_{III} is determined by

(a) Hinged

$$\text{At } \theta=0,\ w^{-}=0,\ u^{-}=0,\ M=\frac{\delta u^{-}}{\delta\theta}-\frac{\delta^2 w^{-}}{\delta\theta^2}=0 \tag{3.18}$$

$$[r_{III}]=-\begin{bmatrix} 1 & 1 & 1 \\ -\alpha_1^{+} & -\alpha_2^{+} & -\alpha_3^{+} \\ (\gamma_1-i\alpha_1^{+})\gamma_1 & (\gamma_2+i\alpha_2^{+})\gamma_2 & (\gamma_3+i\alpha_3^{+})\gamma_3 \end{bmatrix}^{-1}\begin{bmatrix} 1 & 1 & 1 \\ \alpha_1^{+} & \alpha_2^{+} & \alpha_3^{+} \\ (\gamma_1-i\alpha_1^{+})\gamma_1 & (\gamma_2+i\alpha_2^{+})\gamma_2 & (\gamma_3+i\alpha_3^{+})\gamma_3 \end{bmatrix} \tag{3.19}$$

(b) Clamped

$$\text{At } \theta=0,\ w^{-}=0,\ u^{-}=0,\ \psi^{-}=\frac{\delta w^{-}}{\delta\theta}-u^{-}=0 \tag{3.20}$$

$$[r_{III}]=-\begin{bmatrix} 1 & 1 & 1 \\ -\alpha_1^{+} & -\alpha_2^{+} & -\alpha_3^{+} \\ (i\gamma_1+\alpha_1^{+}) & -(i\gamma_2-\alpha_2^{+}) & -(i\gamma_3-\alpha_3^{+}) \end{bmatrix}^{-1}\begin{bmatrix} 1 & 1 & 1 \\ \alpha_1^{+} & \alpha_2^{+} & \alpha_3^{+} \\ -(i\gamma_1+\alpha_1^{+}) & (i\gamma_2-\alpha_2^{+}) & (i\gamma_3-\alpha_3^{+}) \end{bmatrix} \tag{3.21}$$

(c) Free

$$\theta=0,\quad N=\frac{\delta u^-}{\delta\theta}+w^-=0,\quad Q=\frac{\delta^2u^-}{\delta\theta^2}-\frac{\delta^3w^-}{\delta\theta^3}=0,\quad M=\frac{\delta u^-}{\delta\theta}-\frac{\delta^2w^-}{\delta\theta^2}=0 \tag{3.22}$$

$$[r_{III}]=-\begin{bmatrix} -i\alpha_1^+\gamma_1 & i\alpha_2^+\gamma_2 & i\alpha_3^+\gamma_3 \\ (\alpha_1^++i\gamma_1)\gamma_1^2 & (\alpha_2^+-i\gamma_2)\gamma_2^2 & (\alpha_3^+-i\gamma_3)\gamma_3^2 \\ (\gamma_1-i\alpha_1^+)\gamma_1 & (\gamma_2+i\alpha_2^+)\gamma_2 & (\gamma_3+i\alpha_3^+)\gamma_3 \end{bmatrix}^{-1} \begin{bmatrix} -i\alpha_1^+\gamma_1 & i\alpha_2^+\gamma_2 & i\alpha_3^+\gamma_3 \\ (\alpha_1^++i\gamma_1)\gamma_1^2 & -(\alpha_2^+-i\gamma_2)\gamma_2^2 & -(\alpha_3^+-i\gamma_3)\gamma_3^2 \\ (\gamma_1-i\alpha_1^+)\gamma_1 & (\gamma_2+i\alpha_2^+)\gamma_2 & (\gamma_3+i\alpha_3^+)\gamma_3 \end{bmatrix} \tag{3.23}$$

3.3.3 Reflection coefficients

The reflection coefficient r_{11} represents an equal characteristic nature between the propagating and the reflecting wave. The reflection coefficient can be numerically obtained from the reflection matrix and also formulated as follows:

(a) Hinged boundary condition

Substituting Eq. (3.18) into Eq. (3.12a, b)

$$\hat{w}\big|_{\theta=0}=C_{w1}^++C_{w2}^++C_{w3}^++C_{w1}^-+C_{w2}^-+C_{w3}^-=0 \tag{3.24a}$$

$$\hat{u}\big|_{\theta=0}=C_{w1}^+\alpha_1+C_{w2}^+\alpha_2+C_{w3}^+\alpha_3-C_{w1}^-\alpha_1-C_{w2}^-\alpha_2-C_{w3}^-\alpha_3=0 \tag{3.24b}$$

$$\left.\frac{\partial\hat{u}}{\partial\theta}-\frac{\partial^2\hat{w}}{\partial\theta^2}\right|_{\theta=0}=\beta_1C_{w1}^++\beta_2C_{w2}^++\beta_3C_{w3}^++\beta_1C_{w1}^-+\beta_2C_{w2}^-+\beta_3C_{w3}^-=0 \tag{3.24c}$$

where $\beta_i=\gamma_i^2-i\gamma_i\alpha_i$

Eq. (3.24) is simplified as follows by assuming that all incident evanescent wave motions associated with C_{w2}^+, C_{w3}^+, $C_{w2}^+\alpha_2$, $C_{w3}^+\alpha_3$ are negligible in the wave displacements Eq. (3.12a, b)

$$\hat{w}\big|_{\theta=0}=C_{w1}^++C_{w1}^-+C_{w2}^-+C_{w3}^-=0 \tag{3.25a}$$

$$\hat{u}\Big|_{\theta=0} = C_{w1}^{+}\alpha_1 - C_{w1}^{-}\alpha_1 - C_{w2}^{-}\alpha_2 - C_{w3}^{-}\alpha_3 = 0 \tag{3.25b}$$

$$\frac{\partial \hat{u}}{\partial \theta} - \frac{\partial^2 \hat{w}}{\partial \theta^2}\Bigg|_{\theta=0} = \beta_1 C_{w1}^{+} + \beta_1 C_{w1}^{-} + \beta_2 C_{w2}^{-} + \beta_3 C_{w3}^{-} = 0 \tag{3.25c}$$

Adding $\alpha_2 \times$Eq. (3.25a) to Eq. (3.25b) results in

$$\begin{aligned}&\alpha_2 \times \left(C_{w1}^{+} + C_{w1}^{-} + C_{w2}^{-} + C_{w3}^{-}\right) + \\ &C_{w1}^{+}\alpha_1 - C_{w1}^{-}\alpha_1 - C_{w2}^{-}\alpha_2 - C_{w3}^{-}\alpha_3 = 0\end{aligned} \tag{3.26a}$$

$$\Rightarrow (\alpha_1 + \alpha_2) C_{w1}^{+} - (\alpha_1 - \alpha_2) C_{w1}^{-} + (\alpha_2 - \alpha_3) C_{w3}^{-} = 0 \tag{3.26b}$$

Adding $-\beta_2 \times$Eq. (3.25a) to Eq. (3.25c) produces

$$\begin{aligned}&-\beta_2 \times \left(C_{w1}^{+} + C_{w1}^{-} + C_{w2}^{-} + C_{w3}^{-}\right) + \\ &\beta_1 C_{w1}^{+} + \beta_1 C_{w1}^{-} + \beta_2 C_{w2}^{-} + \beta_3 C_{w3}^{-} = 0\end{aligned} \tag{3.27a}$$

$$\Rightarrow (\beta_1 - \beta_2) C_{w1}^{+} + (\beta_1 - \beta_2) C_{w1}^{-} - (\beta_2 - \beta_3) C_{w3}^{-} = 0 \tag{3.27b}$$

Adding $(\beta_2 - \beta_3) \times$Eq. (3.26b) to $(\alpha_2 - \alpha_3) \times$Eq. (3.27b) yields

$$\begin{aligned}&(\beta_2 - \beta_3)(\alpha_1 + \alpha_2) C_{w1}^{+} - (\beta_2 - \beta_3)(\alpha_1 - \alpha_2) C_{w1}^{-} \\ &+ (\alpha_2 - \alpha_3)(\beta_1 - \beta_2) C_{w1}^{+} + (\alpha_2 - \alpha_3)(\beta_1 - \beta_2) C_{w1}^{-} = 0\end{aligned} \tag{3.28a}$$

$$\begin{aligned}&\Rightarrow (\beta_2 - \beta_3)(\alpha_1 + \alpha_2) C_{w1}^{+} - (\beta_2 - \beta_3)(\alpha_1 - \alpha_2) C_{w1}^{-} \\ &- (\alpha_2 - \alpha_3)(\beta_1 + \beta_2) C_{w1}^{+} + (\alpha_2 - \alpha_3)(\beta_1 - \beta_2) C_{w1}^{-} = 0\end{aligned} \tag{3.28b}$$

Using Eq. (3.28b), the reflection coefficient can be expressed as follows:

$$C_{w1}^{-} = \underbrace{\frac{\{(\beta_2 - \beta_3)(\alpha_1 + \alpha_2) + (\alpha_2 - \alpha_3)(\beta_1 - \beta_2)\}}{\{(\beta_2 - \beta_3)(\alpha_1 - \alpha_2) - (\alpha_2 - \alpha_3)(\beta_1 - \beta_2)\}}}_{r_{11}} C_{w1}^{+} \tag{3.29}$$

(b) Clamped boundary condition

Substituting Eq. (3.20) into Eq. (3.12a, b)

$$\hat{w}\Big|_{\theta=0} = C_{w1}^{+} + C_{w2}^{+} + C_{w3}^{+} + C_{w1}^{-} + C_{w2}^{-} + C_{w3}^{-} = 0 \tag{3.30a}$$

$$\hat{u}\Big|_{\theta=0} = C_{w1}^{+}\alpha_1 + C_{w2}^{+}\alpha_2 + C_{w3}^{+}\alpha_3 - C_{w1}^{-}\alpha_1 - C_{w2}^{-}\alpha_2 - C_{w3}^{-}\alpha_3 = 0 \tag{3.30b}$$

$$\frac{\partial \hat{w}}{\partial \theta} - \hat{u}\Bigg|_{\theta=0} = -\beta_1 C_{w1}^{+} - \beta_2 C_{w2}^{+} - \beta_3 C_{w3}^{+} + \beta_1 C_{w1}^{-} + \beta_2 C_{w2}^{-} + \beta_3 C_{w3}^{-} = 0 \tag{3.30c}$$

where $\beta_i = i\gamma_i + \alpha_i$

Eq. (3.30) is simplified as follows, neglecting all incident evanescent wave motions

$$\hat{w}\big|_{\theta=0} = C_{w1}^{+} + C_{w1}^{-} + C_{w2}^{-} + C_{w3}^{-} = 0 \tag{3.31a}$$

$$\hat{u}\big|_{\theta=0} = C_{w1}^{+}\alpha_1 - C_{w1}^{-}\alpha_1 - C_{w2}^{-}\alpha_2 - C_{w3}^{-}\alpha_3 = 0 \tag{3.31b}$$

$$\left.\frac{\partial \hat{w}}{\partial \theta} - \hat{u}\right|_{\theta=0} = -\beta_1 C_{w1}^{+} + \beta_1 C_{w1}^{-} + \beta_2 C_{w2}^{-} + \beta_3 C_{w3}^{-} = 0 \tag{3.31c}$$

Adding $\alpha_2 \times$Eq. (3.31a) to Eq. (3.31b) results in

$$\begin{aligned}&\alpha_2 \times \left(C_{w1}^{+} + C_{w1}^{-} + C_{w2}^{-} + C_{w3}^{-}\right) + \\ &C_{w1}^{+}\alpha_1 - C_{w1}^{-}\alpha_1 - C_{w2}^{-}\alpha_2 - C_{w3}^{-}\alpha_3 = 0\end{aligned} \tag{3.32a}$$

$$\Rightarrow (\alpha_1 + \alpha_2) C_{w1}^{+} - (\alpha_1 - \alpha_2) C_{w1}^{-} + (\alpha_2 - \alpha_3) C_{w3}^{-} = 0 \tag{3.32b}$$

Adding $-\beta_2 \times$Eq. (3.31a) to Eq. (3.31c) produces

$$\begin{aligned}&-\beta_2 \times \left(C_{w1}^{+} + C_{w1}^{-} + C_{w2}^{-} + C_{w3}^{-}\right) + \\ &\beta_1 C_{w1}^{+} + \beta_1 C_{w1}^{-} + \beta_2 C_{w2}^{-} + \beta_3 C_{w3}^{-} = 0\end{aligned} \tag{3.33a}$$

$$\Rightarrow -(\beta_1 + \beta_2) C_{w1}^{+} + (\beta_1 - \beta_2) C_{w1}^{-} - (\beta_2 - \beta_3) C_{w3}^{-} = 0 \tag{3.33b}$$

Adding $(\beta_2 - \beta_3)\times$Eq. (3.32) to $(\alpha_2 - \alpha_3)\times$Eq. (3.33) yields

$$\begin{aligned}&(\beta_2 - \beta_3) \times \left\{(\alpha_1 + \alpha_2) C_{w1}^{+} - (\alpha_1 - \alpha_2) C_{w1}^{-} + (\alpha_2 - \alpha_3) C_{w3}^{-}\right\} \\ &+ (\alpha_2 - \alpha_3)\left[(\beta_1 - \beta_2) C_{w1}^{+} + (\beta_1 - \beta_2) C_{w1}^{-} - (\beta_2 - \beta_3) C_{w3}^{-}\right] = 0\end{aligned} \tag{3.34a}$$

$$\begin{aligned}&\Rightarrow (\beta_2 - \beta_3)(\alpha_1 + \alpha_2) C_{w1}^{+} - (\beta_2 - \beta_3)(\alpha_1 - \alpha_2) C_{w1}^{-} \\ &-(\alpha_2 - \alpha_3)(\beta_1 + \beta_2) C_{w1}^{+} + (\alpha_2 - \alpha_3)(\beta_1 - \beta_2) C_{w1}^{-} = 0\end{aligned} \tag{3.34b}$$

Using Eq. (3.34b), the reflection coefficient can be expressed as follows:

$$C_{w1}^{-} = \underbrace{\frac{\{(\beta_2 - \beta_3)(\alpha_1 + \alpha_2) - (\alpha_2 - \alpha_3)(\beta_1 + \beta_2)\}}{\{(\beta_2 - \beta_3)(\alpha_1 - \alpha_2) - (\alpha_2 - \alpha_3)(\beta_1 - \beta_2)\}}}_{r_{11}} C_{w1}^{+} \tag{3.35}$$

(c) Free boundary condition

Substituting Eq. (3.22) into Eq. (3.12a, b)

$$\left.\frac{\partial \hat{u}}{\partial \theta}+\hat{w}\right|_{\theta=0}=A_1 C_{w1}^{+}+A_2 C_{w2}^{+}+A_3 C_{w3}^{+}+A_1 C_{w1}^{-}+A_2 C_{w2}^{-}+A_3 C_{w3}^{-}=0 \tag{3.36a}$$

$$\left.\frac{\partial^2 \hat{u}}{\partial \theta^2}-\frac{\partial^3 \hat{w}}{\partial \theta^3}\right|_{\theta=0}=-B_1 C_{w1}^{+}-B_2 C_{w2}^{+}-B_3 C_{w3}^{+}+B_1 C_{w1}^{-}+B_2 C_{w2}^{-}+B_3 C_{w3}^{-}=0 \tag{3.36b}$$

$$\left.\frac{\partial \hat{u}}{\partial \theta}-\frac{\partial^2 \hat{w}}{\partial \theta^2}\right|_{\theta=0}=C_1 C_{w1}^{+}+C_2 C_{w2}^{+}+C_3 C_{w3}^{+}+C_1 C_{w1}^{-}+C_2 C_{w2}^{-}+C_3 C_{w3}^{-}=0 \tag{3.36c}$$

where $A_i=i\gamma_i\alpha_i-1$, $B_i=\gamma_i^2(\alpha_i+i\gamma_i)$ and $C_i=\gamma_i^2-i\gamma_i\alpha_i$

Eq. (3.36) is simplified as follows while neglecting incident near fields

$$\left.\frac{\partial \hat{u}}{\partial \theta}+\hat{w}\right|_{\theta=0}=A_1 C_{w1}^{+}+A_1 C_{w1}^{-}+A_2 C_{w2}^{-}+A_3 C_{w3}^{-}=0 \tag{3.37a}$$

$$\left.\frac{\partial^2 \hat{u}}{\partial \theta^2}-\frac{\partial^3 \hat{w}}{\partial \theta^3}\right|_{\theta=0}=-B_1 C_{w1}^{+}+B_1 C_{w1}^{-}+B_2 C_{w2}^{-}+B_3 C_{w3}^{-}=0 \tag{3.37b}$$

$$\left.\frac{\partial \hat{u}}{\partial \theta}-\frac{\partial^2 \hat{w}}{\partial \theta^2}\right|_{\theta=0}=C_1 C_{w1}^{+}+C_1 C_{w1}^{-}+C_2 C_{w2}^{-}+C_3 C_{w3}^{-}=0 \tag{3.37c}$$

Subtracting $B_2\times$Eq. (3.37a) to $A_2\times$Eq. (3.37b) results in

$$\begin{aligned}&B_2\left(A_1 C_{w1}^{+}+A_1 C_{w1}^{-}+A_2 C_{w2}^{-}+A_3 C_{w3}^{-}\right)+\\&A_2\left(B_1 C_{w1}^{+}-B_1 C_{w1}^{-}-B_2 C_{w2}^{-}-B_3 C_{w3}^{-}\right)=0\end{aligned} \tag{3.38a}$$

$$\left(A_1 B_2+A_2 B_1\right)C_{w1}^{+}+\left(A_1 B_2-A_2 B_1\right)C_{w1}^{-}-\left(A_2 B_3-A_3 B_2\right)C_{w3}^{-}=0 \tag{3.38b}$$

Subtracting $C_2\times$Eq. (3.37a) to $A_2\times$Eq. (3.37c) produces

$$\begin{aligned}&C_2\left(A_1 C_{w1}^{+}+A_1 C_{w1}^{-}+A_2 C_{w2}^{-}+A_3 C_{w3}^{-}\right)-\\&A_2\left(C_1 C_{w1}^{+}+C_1 C_{w1}^{-}+C_2 C_{w2}^{-}+C_3 C_{w3}^{-}\right)=0\end{aligned} \tag{3.39a}$$

$$\left(A_1 C_2-A_2 C_1\right)C_{w1}^{+}+\left(A_1 C_2-A_2 C_1\right)C_{w1}^{-}-\left(A_2 C_3-A_3 C_2\right)C_{w3}^{-}=0 \tag{3.39b}$$

Subtracting $(A_2C_3 - A_3C_2)\times$Eq. (3.38b) to $(A_2B_3 - A_3B_2)\times$Eq. (3.39a) yields

$$(A_1B_2 + A_2B_1)(A_2C_3 - A_3C_2)C_{w1}^{+} + (A_1B_2 - A_2B_1)(A_2C_3 - A_3C_2)C_{w1}^{-}$$
$$-(A_1C_2 - A_2C_1)(A_2B_3 - A_3B_2)C_{w1}^{+} - (A_1C_2 - A_2C_1)(A_2B_3 - A_3B_2)C_{w1}^{-} = 0 \tag{3.40}$$

Using Eq. (3.40), the reflection coefficient can be expressed as follows:

$$C_{w1}^{-} = \underbrace{\frac{\{(A_1B_2 + A_2B_1)(A_2C_3 - A_3C_2) - (A_2B_3 - A_3B_2)(A_1C_2 - A_2C_1)\}}{\{(A_2B_3 - A_3B_2)(A_1C_2 - A_2C_1) - (A_1B_2 - A_2B_1)(A_2C_3 - A_3C_2)\}}}_{r_{11}} C_{w1}^{+} \tag{3.41}$$

3.3.4 Vibration analysis using phase-closure principle

The reflection matrix of waves incident upon boundary conditions is used to obtain the reflection coefficient and also combined with the transfer matrix method to analyze the vibration of the curved beam.

Reflection coefficient approach

Figure 3.4 shows the schematic diagram of a curved beam with support conditions from which the phase-closure formulae is applied with the reflection coefficient. The equations are derived as follows,

$$\int d\Gamma - \phi_l - \phi_r = 2n\pi \tag{3.42}$$

where $\int d\Gamma$, ϕ_l and ϕ_r are the phase shifts of the traveling wave in the curved beam, right and left end of the beam respectively, and n is an arbitrary integer.

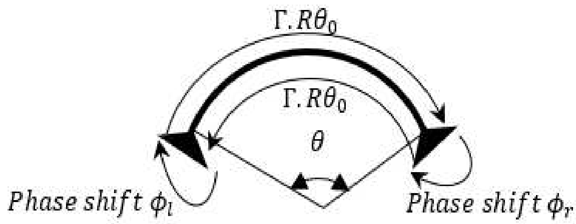


Fig. 3.4. Curved beam with hinged boundary to illustrate the principle of phase closure

$$2\Gamma R\theta_0 - \phi_l - \phi_r = 2n\pi \tag{3.43}$$

Substituting Eq. (3.6) into Eq. (3.43) to obtain the frequency equation as

$$2\gamma\theta_0 - \phi_l - \phi_r = 2n\pi \tag{3.44}$$

The phase sum (ϕ_{sum}) given as is provided as,

$$\phi_{sum} = 2\gamma\theta_0 - (\phi_r + \phi_l + 2n\pi) \tag{3.45}$$

where $\phi_r + \phi_l = 2 \times phase(r_{11})$

Equation (3.45) is used to obtain the natural frequencies for inextensible and extensible curved beams and are compared with the exact values given by Kang et al. (2003). The span angles of 90^0, 180^0 and 360^0 are used since the values of the non-dimensionalized frequencies are dependent on the span angle, and the values for these angles for both the extensional and inextensional beam agree closely.

Determinant matrix approach

The phase-closure principle is applied while using the reflection matrix incident with the transfer matrix method. Figure 3.5 shows a general curved beam structure with boundaries L and R. The incident and reflected waves are denoted at boundaries R and L as w_R^+, w_R^-, w_L^+, and w_L^- respectively.

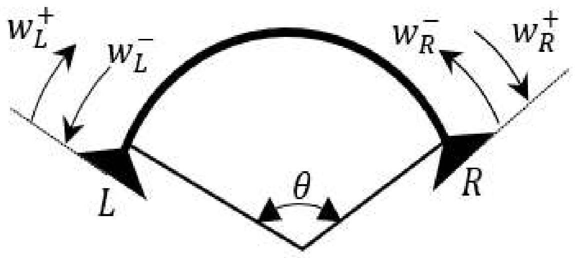


Fig. 3.5. Reflection of waves through a curved beam with constant curvature

The relationships of the waves between the boundaries are given as

$$w_R^- = R_R w_R^+ \tag{3.46a}$$

$$w_L^- = T w_R^- \tag{3.46b}$$

$$w_L^+ = R_L w_L^- \tag{3.46c}$$

$$w_R^+ = T w_L^+ \tag{3.46d}$$

solving the Eqs. (3.46), the characteristic equation is obtained as follows,

$$(R_L T R_R T - 1) w_L^+ = 0 \tag{3.47}$$

The reflection and transmission matrices defined as $R_L = R_R = r_{III}$ and $T = t_{III}$ respectively,
where T is the transmission matrix expressed as

$$T = \begin{bmatrix} e^{-i\gamma_1\theta} & 0 & 0 \\ 0 & e^{i\gamma_2\theta} & 0 \\ 0 & 0 & e^{i\gamma_3\theta} \end{bmatrix} \text{(Kang et al., 2003)} \tag{3.48}$$

For non-trivial solution, the natural frequencies are obtained from the characteristic equation expressed as determinant

$$C(\omega) = Det\left[r_{III} t_{III} r_{III} - I_{3\times 3}\right] = 0 \tag{3.49}$$

IV. RESULTS AND DISCUSSIONS

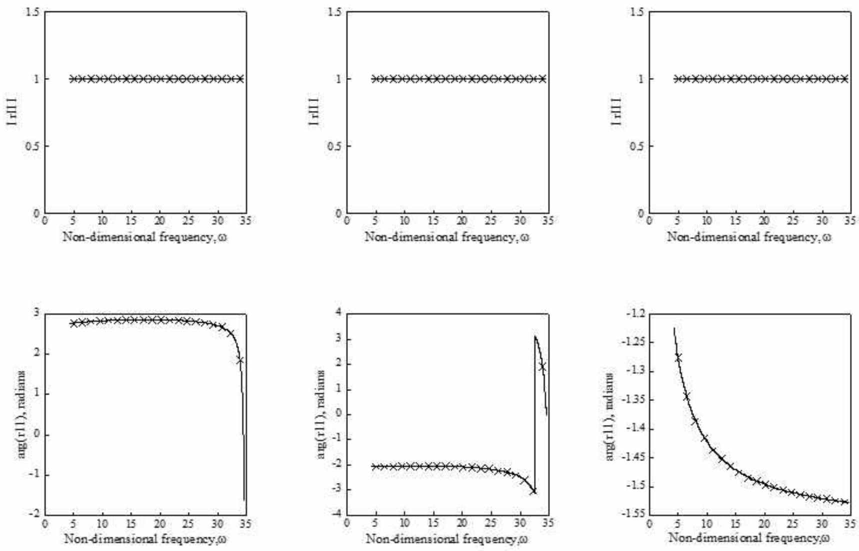
4.0 Overview

This chapter presents the results from the analysis and discussions of the results that provide an understanding of the dynamic behaviour of curved beams. Results of important parameters from the reflection coefficient and the natural frequencies from the frequency equation are presented. These results are validated with numerical results of matrix formulation and study by Kang et al. (2003).

4.1. Results from the reflection coefficient approach

4.1.1 Amplitude and Phase of reflection coefficient

The reflection coefficient obtained as described in chapter III was computed to provide its amplitude and phase as shown in Fig. 4.1.



(a) (b) (c)

Figure 4.1. Amplitudes and Phases for (a) Hinged (b) Clamped (c) Free support conditions

The reflection coefficient for the propagation mode is constant throughout the non-dimensional frequencies. The amplitude of the reflected wave and the incident wave combine to form a wave of lower amplitude (1). The phase of the reflection coefficient gradually decrease for the support conditions but shows a sharp increase for the clamped support condition. The phase between π and $-\pi$ represents low frequencies. Therefore the phase of r_{11} portrays low frequency.

4.1.2 The natural frequencies of the beam

The plots of ϕ_{sum} versus ω are shown in Figs 4.2. to 4.4. Both theories show a decreasing phase shift sum as the non-dimensional frequencies increase. At a specific value of non-dimensional frequency, the phase shift sum increases with increase in non-dimensional frequency.

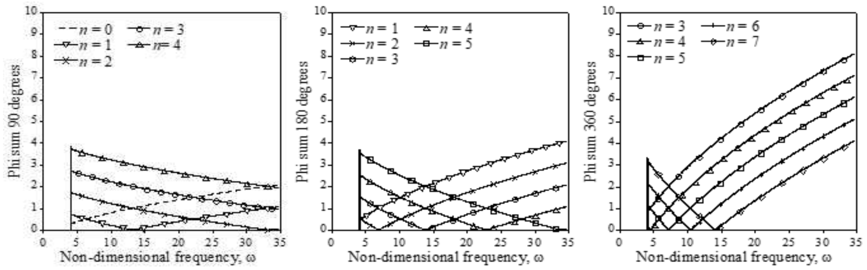


Fig. 4.2a. Natural frequency at a hinged boundary, Inextensional

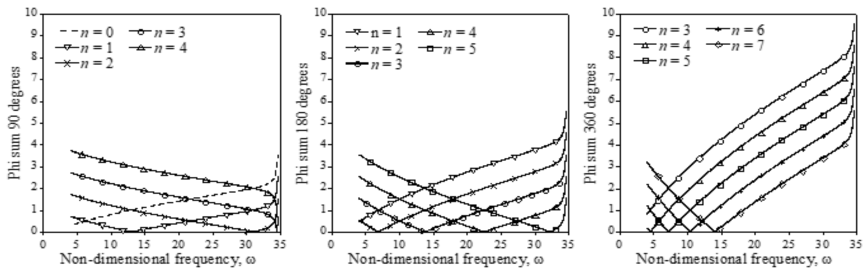


Fig. 4.2b. Natural frequency at a hinged boundary, Extensional

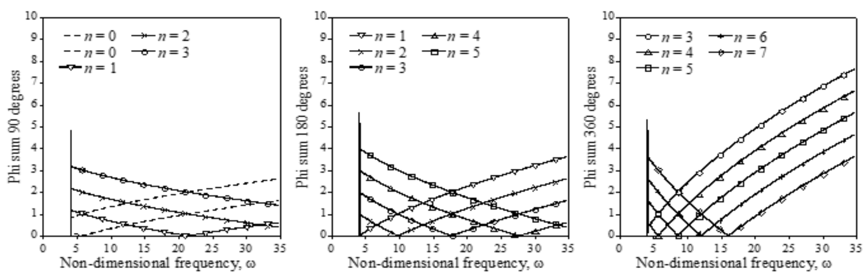


Fig. 4.3a. Natural frequency at a clamped boundary, Inextensional

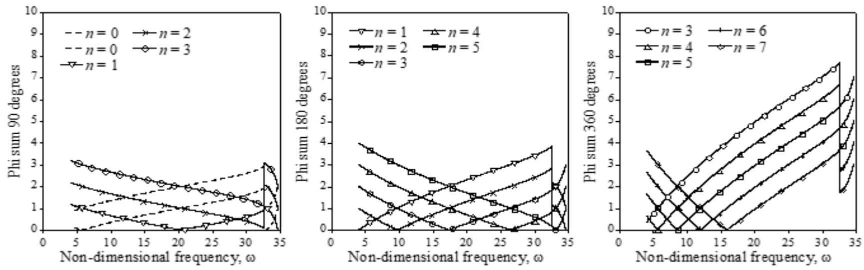


Fig. 4.3b. Natural frequency at a clamped boundary, Extensional

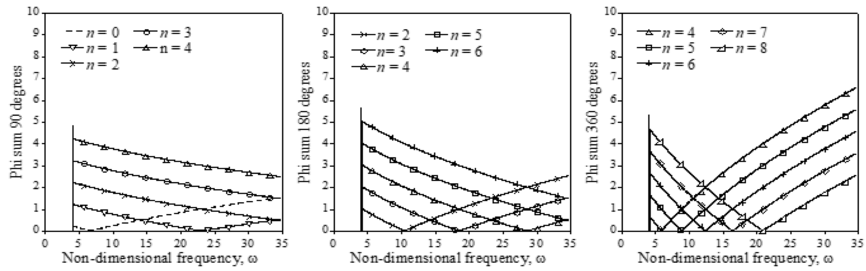


Fig. 4.4a. Natural frequency at a free boundary, Inextensional

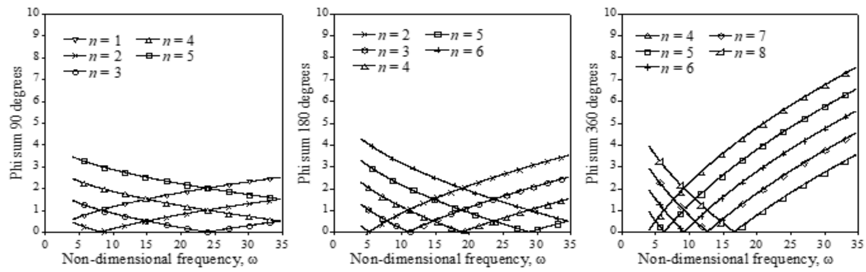


Fig. 4.4b. Natural frequency at a free boundary, Extensional

The specific values at which the curves makes a transition in shape are the natural frequencies. At each span angle, standing waves are obtained at reflection which results in resonance by the phase shift, thus natural frequencies.

4.2. Validation of results with determinant approach

Figures 4.5. to 4.7 show the plot of $C(\omega)$ with non-dimensional frequencies. Frequencies at which real and imaginary values of $C(\omega)$, are the natural frequencies as indicated in the plots.

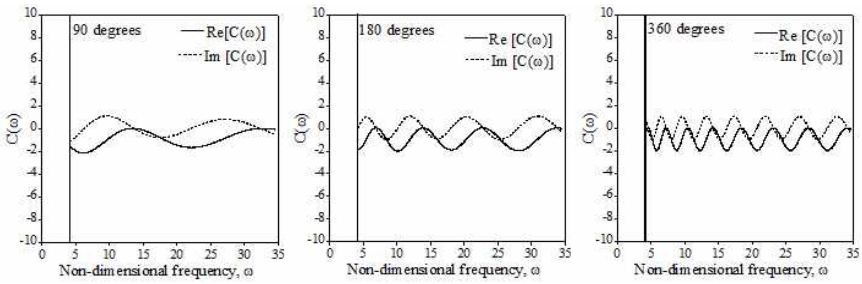


Fig. 4.5a. Determinant $C(\omega)$ for hinged ends, Inextensional

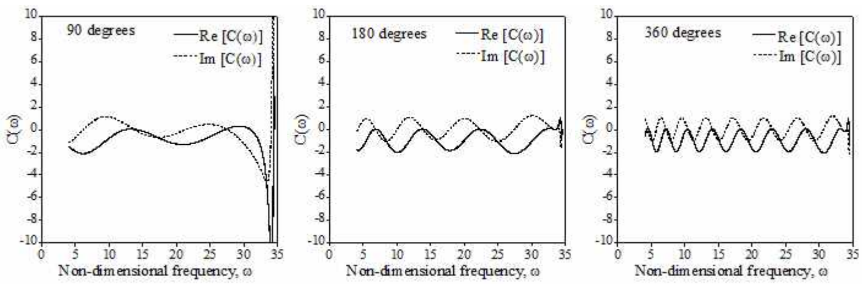


Fig. 4.5b. Determinant $C(\omega)$ for hinged ends, Extensional

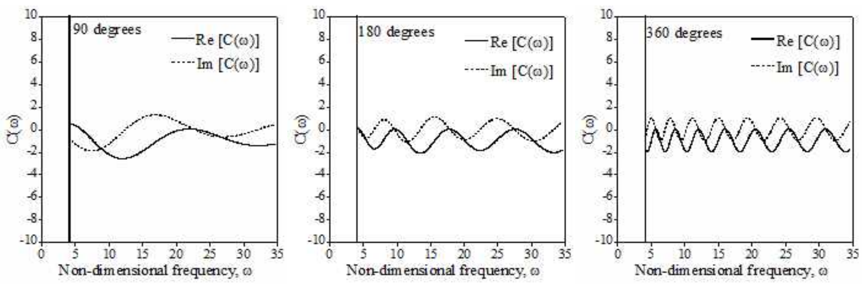


Fig. 4.6a. Determinant $C(\omega)$ for clamped ends, Inextensional

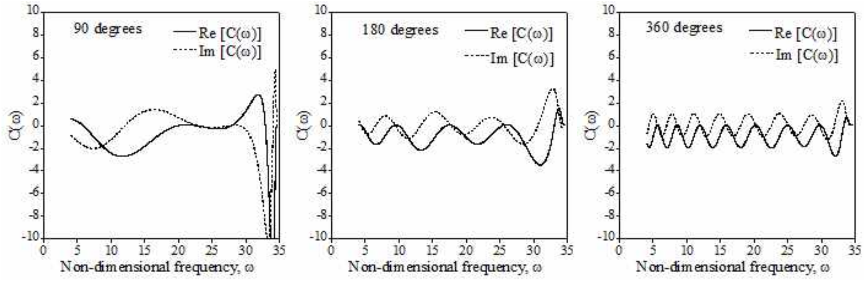


Fig. 4.6b. Determinant $C(\omega)$ for clamped ends, Extensional

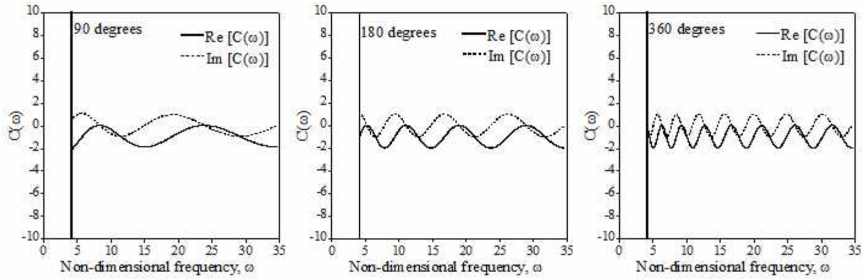


Fig. 4.7a. Determinant $C(\omega)$ for free ends, Inextensional

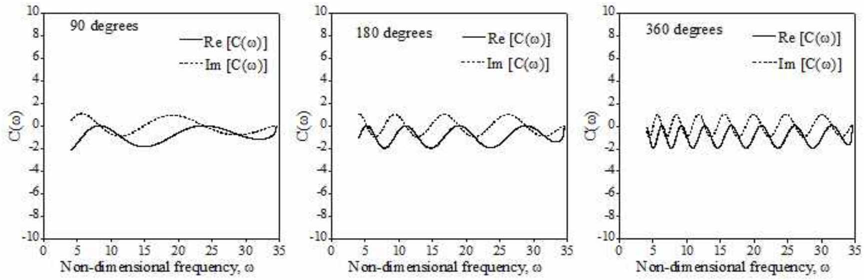


Fig. 4.7b. Determinant $C(\omega)$ for free ends, Extensional

Tables 4.1 and 4.2 show the values of the non-dimensional frequencies for both the inextensional and extensional case of the curved beam.

Table 4.1 Non-dimensional natural frequencies of a curved beam (extensional case $k^2 = 1/1200$)

Span angle	BC	Mode	Extensional			
			Kang et al. (2003)	Matrix Determinant	Phase closure	Difference %
90	H-H	1	13.6873	13.7075	13.1072	4.3794
		2	27.4194	27.4915	-	-
		3	-	-	30.9686	-
	C-C	1	22.4430	22.4430	20.3754	9.2128
		2	28.1125	28.1125	-	-
		3	-	33.0074	32.6828	0.9834
	F-F	1	8.3820	8.3820	8.2681	1.3590
		2	23.8894	23.8894	23.8967	0.0305
180	H-H	1	6.8785	6.8866	6.9627	1.1050
		2	13.8894	13.9040	13.8167	0.6279
		3	22.3344	22.3697	22.5814	0.9464
	C-C	1	9.4983	9.4983	9.7659	2.8177
		2	17.7040	17.7040	17.4076	1.6743
		3	25.6417	25.6417	26.505	3.3667
	F-F	1	5.3029	5.3029	5.3053	0.0461
		2	11.1000	11.1000	11.1000	0.0003
		3	18.9885	18.9885	18.9884	0.0003
360	H-H	4	4.5843	4.5868	4.5877	0.0196
	C-C	4	5.7302	5.7302	5.7352	0.0866
	F-F	4	3.9646	3.9646	4.1638	5.0256

BC = Boundary Condition, **H-H** = Hinged-Hinged, **C-C** = Clamped-Clamped and **F-F** = Free-Free

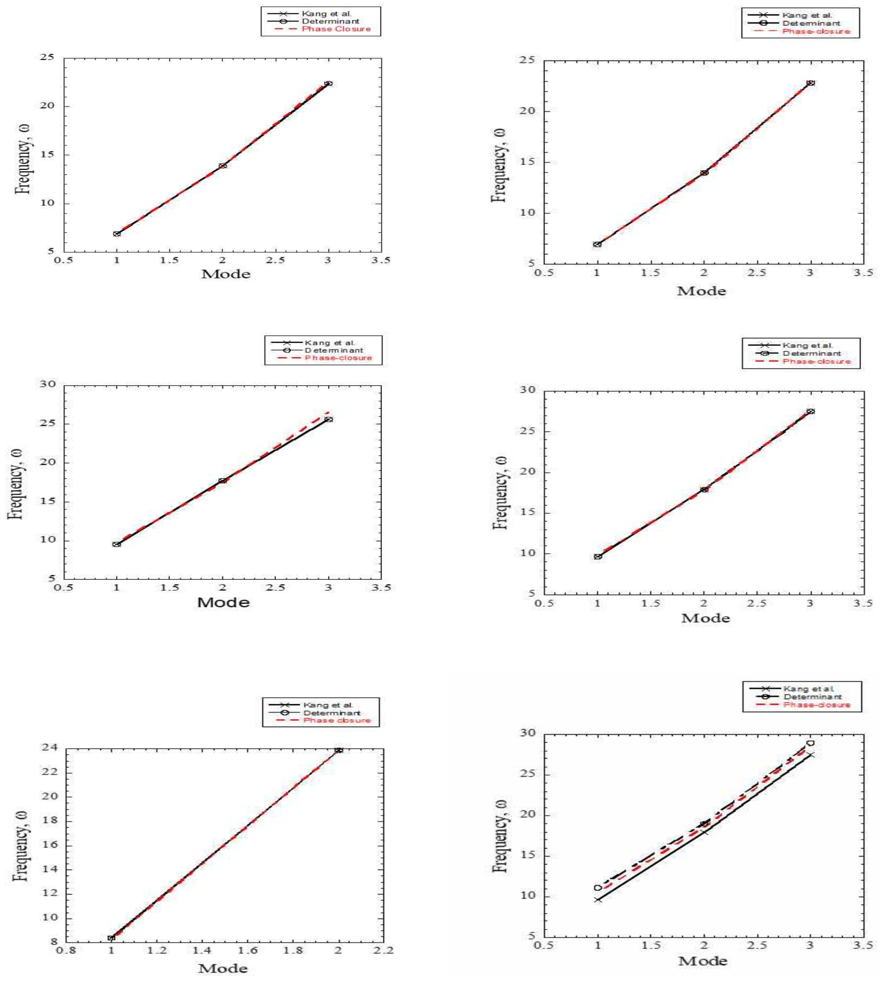
Table 4.2 Non-dimensional natural frequencies of a curved beam (inextensional case $k = 0$)

Span angle	BC	Mode	Inextensional			
			Kang et al. (2003)	Matrix Determinant	Phase closure	Difference %
90	H-H	1	13.7637	13.7637	13.2728	3.5666
		2	32.4036	32.4036	33.2117	2.4939
	C-C	1	22.4430	22.4430	20.3754	9.2128
	F-F	1	22.6251	23.9258	23.1245	3.3491
180	H-H	1	6.9233	6.9233	6.9919	0.9909
		2	13.9777	13.9777	13.9175	0.4305
		3	22.8196	22.8196	22.886	0.2911
	C-C	1	9.6519	9.6519	9.8744	2.3053
		2	17.9218	17.9218	17.7582	0.9128
		3	27.5239	27.5239	27.6939	0.6177
	F-F	1	9.6519	11.1149	10.5063	5.4755
		2	17.9218	19.0168	18.5623	2.3900
		3	27.5239	28.9593	28.5954	1.2566
360	H-H	4	4.5967	4.5967	4.5976	0.0202
	C-C	4	5.7649	5.7649	5.7594	0.0958
	F-F	4	5.7649	6.3676	5.9568	6.4514

BC = Boundary Condition, **H-H** = Hinged-Hinged, **C-C** = Clamped-Clamped and **F-F** = Free-Free

The non-dimensional natural frequencies obtained were in line with the range of non-dimensional frequencies of the Case III motion considered in the analysis of the curved beam. For both theories, the values of span angles greater than 180^0 (2×90^0) are shown to closely agree (Chidamparam and Leissa, 1993). The natural frequencies obtained are compared with those obtained by Kang et al. (2003). The relative error between results of the frequency equation and matrix determinant is significantly noticed in the first mode of all boundary conditions. For the extensional case, the error tends to fluctuate with increase in the mode number as compared to the inextensional case. The error in the inextensional case decreases with increase in the mode number. This error is related to the results obtained by Mead (1994) while applying the phase-closure principle to a single span uniform straight beam when using the propagating wave. Mead (1994) showed that the evanescent wave generated at a boundary slightly had an influence to the motion at the other boundary, especially for third to higher modes of simple beams.

Figure 4.8 shows the relative error (difference %) when the non-dimensional natural frequencies for the 180 degrees span angle are plotted against the mode number for the Case III wave motion. Since the natural frequencies lie within the range for wave motion Case III, the difference in values is shown for the three wave mode numbers.



(a) (b)

Fig. 4.8. (a) Relative error for the extensional case of H-H, C-C, F-F boundary conditions respectively; (b) Relative error for the inextensional case of H-H, C-C, F-F boundary conditions respectively.

V. CONCLUSION AND RECOMMENDATION

5.0 Overview

This chapter presents the conclusion and recommendations for future research on the vibration analysis of curved beams. Conclusions are drawn from the results obtained from the analysis of curved beams and, how the formulated frequency equation can be used in future studies and applied in the field of structural health monitoring.

5.1 Conclusion

This research presented a simplified frequency equation based on wave perspective to study the free vibration of an extensional and inextensional curved beam with various boundary conditions. Reflection matrices of wave motions were derived at boundary conditions. The reflection coefficients were obtained from the reflection matrices whose phases were expressed in the phase-closure principle to form a frequency equation.

The natural frequencies obtained from the equation are compared to numerical values from the characteristic equation of matrix formulation and previous study by Kang et al. (2003). The relative error (difference) between the values of the frequency equation and the matrix determinant decreases with increasing wave mode especially for the inextensional case. Therefore, the formulated frequency equation is simple and easy and more straight forward in obtaining approximate natural frequencies of a curved beam.

5.2 Recommendations for future research

The frequency equation can be applied in predicting high modal frequencies of a curved beam. These high-frequency dynamic responses are essential in sensor technologies which are highly applied in structural health monitoring. Therefore the frequency equation can be used to analyse damage or cracks on curved beams in structural health monitoring.

Attenuating waves can be included in the formulated frequency equation in order to obtain exact solutions of natural frequencies from the vibration analysis.

The developed wave approach can be further used to analyze and design complex built-up curved beam structures with various span length. The wave approach can be used to analyze the dynamic properties of typical non-circular arches (sinusoidal, parabolic and elliptical arches) or curved beams with a varying cross-section.

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V.Appendix A. Derivation of reflection matrix

Hinged Support

$$w^{-}=0 \tag{A.1}$$

$$\Rightarrow\left(C_{w1}^{+}+C_{w2}^{+}+C_{w3}^{+}+C_{w1}^{-}+C_{w2}^{-}+C_{w3}^{-}\right)e^{-i\omega t}=0 \tag{A.2}$$

$$\Rightarrow [1\ 1\ 1]\left\{C_{w1}^{+}\ C_{w2}^{+}\ C_{w3}^{+}\right\}^{T}+[1\ 1\ 1]\left\{C_{w1}^{-}\ C_{w2}^{-}\ C_{w3}^{-}\right\}^{T}=0 \tag{A.3}$$

Let $\{A_w\}=[1\ 1\ 1]$, $\{B_w\}=[1\ 1\ 1]$ (A.4)

$$\Rightarrow \{A_w\}\left\{C_{w1}^{+}\ C_{w2}^{+}\ C_{w3}^{+}\right\}^{T}+\{B_w\}\left\{C_{w1}^{-}\ C_{w2}^{-}\ C_{w3}^{-}\right\}^{T}=0 \tag{A.5}$$

where $\begin{Bmatrix} C_{w1}^{-} \\ C_{w2}^{-} \\ C_{w3}^{-} \end{Bmatrix}=[r_{III}]\begin{Bmatrix} C_{w1}^{+} \\ C_{w2}^{+} \\ C_{w3}^{+} \end{Bmatrix}$

Therefore,

$$\Rightarrow \{A_w\}\left\{C_{w1}^{+}\ C_{w2}^{+}\ C_{w3}^{+}\right\}^{T}+\{B_w\}[r_{III}]\left\{C_{w1}^{-}\ C_{w2}^{-}\ C_{w3}^{-}\right\}^{T}=0 \tag{A.6}$$

$$[\{A_w\}+[r_{III}]\{B_w\}]=0 \tag{A.7}$$

$$u^{-}=0 \tag{A.8}$$

$$\Rightarrow\left(C_{w1}^{+}\alpha_1^{+}+C_{w2}^{+}\alpha_2^{+}+C_{w3}^{+}\alpha_3^{+}-C_{w1}^{-}\alpha_1^{+}-C_{w2}^{-}\alpha_2^{+}-C_{w3}^{-}\alpha_3^{+}\right)e^{-iwt} \tag{A.9}$$

$$\Rightarrow\left[\alpha_1^{+}\ \alpha_2^{+}\ \alpha_3^{+}\right]\left\{C_{w1}^{+}\ C_{w2}^{+}\ C_{w3}^{+}\right\}^{T}+\left[-\alpha_1^{+}\ -\alpha_2^{+}\ -\alpha_3^{+}\right]\left\{C_{w1}^{-}\ C_{w2}^{-}\ C_{w3}^{-}\right\}^{T}=0 \tag{A.10}$$

Let $\{A_u\}=\left[\alpha_1^{+}\ \alpha_2^{+}\ \alpha_3^{+}\right]$, $\{B_u\}=\left[-\alpha_1^{+}\ -\alpha_2^{+}\ -\alpha_3^{+}\right]$ (A.11)

$$\Rightarrow \{A_u\}\left\{C_{u1}^{+}\ C_{u2}^{+}\ C_{u3}^{+}\right\}^{T}+\{B_u\}[r_{III}]\left\{C_{u1}^{-}\ C_{u2}^{-}\ C_{u3}^{-}\right\}^{T}=0 \tag{A.12}$$

$$[\{A_u\}+[r_{III}]\{B_u\}]=0 \tag{A.13}$$

$$M=\frac{\delta u^{-}}{\delta\theta}-\frac{\delta^2 w^{-}}{\delta\theta^2}=0 \tag{A.14}$$

$$\frac{\delta u^{-}}{\delta\theta}=\left(-i\gamma_1 C_{w1}^{+}\alpha_1^{+}+i\gamma_2 C_{w2}^{+}\alpha_2^{+}+i\gamma_3 C_{w3}^{+}\alpha_3^{+}-i\gamma_1 C_{w1}^{-}\alpha_1^{+}+i\gamma_2 C_{w2}^{-}\alpha_2^{+}+i\gamma_3 C_{w3}^{-}\alpha_3^{+}\right)e^{-iwt} \tag{A.15}$$

$$\frac{\delta^2 w^{-}}{\delta\theta^2}=\left(-\gamma_1^2 C_{w1}^{+}-\gamma_2^2 C_{w2}^{+}-\gamma_3^2 C_{w3}^{+}-\gamma_1^2 C_{w1}^{-}-\gamma_2^2 C_{w2}^{-}-\gamma_3^2 C_{w3}^{-}\right)e^{-i\omega t}=0 \tag{A.16}$$

$$\begin{aligned}\Rightarrow & -i\gamma_1 C_{w1}^{+}\alpha_1^{+}+i\gamma_2 C_{w2}^{+}\alpha_2^{+}+i\gamma_3 C_{w3}^{+}\alpha_3^{+}-i\gamma_1 C_{w1}^{-}\alpha_1^{+}+i\gamma_2 C_{w2}^{-}\alpha_2^{+} \\ & +i\gamma_3 C_{w3}^{-}\alpha_3^{+}+\gamma_1^2 C_{w1}^{+}+\gamma_2^2 C_{w2}^{+}+\gamma_3^2 C_{w3}^{+}+\gamma_1^2 C_{w1}^{-}+\gamma_2^2 C_{w2}^{-}+\gamma_3^2 C_{w3}^{-}=0\end{aligned} \tag{A.17}$$

$$\Rightarrow\left(\gamma_1^2-i\gamma_1\alpha_1^+\right)C_{w1}^+ +\left(\gamma_2^2+i\gamma_2\alpha_2^+\right)C_{w2}^+ +\left(\gamma_3^2+i\gamma_3\alpha_3^+\right)C_{w3}^+ +\left(\gamma_1^2-i\gamma_1\alpha_1^+\right)C_{w1}^- + \\ \left(\gamma_2^2+i\gamma_2\alpha_2^+\right)C_{w2}^- +\left(\gamma_3^2+i\gamma_3\alpha_3^+\right)C_{w3}^- =0 \tag{A.18}$$

$$\Rightarrow\left[\left(\gamma_1-i\alpha_1^+\right)\gamma_1 \;\; \left(\gamma_2+i\alpha_2^+\right)\gamma_2 \;\; \left(\gamma_3+i\alpha_3^+\right)\gamma_3\right]\left\{C_{w1}^+ \;\; C_{w2}^+ \;\; C_{w3}^+\right\}^T + \\ \left[\left(\gamma_1-i\alpha_1^+\right)\gamma_1 \;\; \left(\gamma_2+i\alpha_2^+\right)\gamma_2 \;\; \left(\gamma_3+i\alpha_3^+\right)\gamma_3\right]\left\{C_{w1}^- \;\; C_{w2}^- \;\; C_{w3}^-\right\}^T=0 \tag{A.19}$$

$$\text{Let } \{A_M\}=\left[\left(\gamma_1-i\alpha_1^+\right)\gamma_1 \;\; \left(\gamma_2+i\alpha_2^+\right)\gamma_2 \;\; \left(\gamma_3+i\alpha_3^+\right)\gamma_3\right], \\ \{B_M\}=\left[\left(\gamma_1-i\alpha_1^+\right)\gamma_1 \;\; \left(\gamma_2+i\alpha_2^+\right)\gamma_2 \;\; \left(\gamma_3+i\alpha_3^+\right)\gamma_3\right] \tag{A.20}$$

$$\Rightarrow\{A_M\}\left\{C_{w1}^+ \;\; C_{w2}^+ \;\; C_{w3}^+\right\}^T+\{B_M\}\left\{C_{w1}^- \;\; C_{w2}^- \;\; C_{w3}^-\right\}^T=0 \tag{A.21}$$

$$\left[\{A_m\}+[r_{III}]\{B_m\}\right]=0 \tag{A.22}$$

Using the layouts in Eqs. (A.7), (A.13) and (A.22) to obtain the reflection matrix as follows,

$$\begin{bmatrix} A_w \\ A_u \\ A_m \end{bmatrix}+\begin{bmatrix} B_w \\ B_u \\ B_m \end{bmatrix}[r_{III}]=0 \tag{A.23}$$

$$[\beta]=\begin{bmatrix} B_w \\ B_u \\ B_m \end{bmatrix}, \quad [\alpha]=\begin{bmatrix} A_w \\ A_u \\ A_m \end{bmatrix} \tag{A.24}$$

$$[r_{III}]=[\beta]^{-1}[\alpha] \tag{A.25}$$

Clamped support

$$w^-=0 \tag{A.26}$$

$$\Rightarrow\left(C_{w1}^+ + C_{w2}^+ + C_{w3}^+ + C_{w1}^- + C_{w2}^- + C_{w3}^-\right)e^{-i\omega t}=0 \tag{A.27}$$

$$\Rightarrow [1\;\;1\;\;1]\left\{C_{w1}^+ \; C_{w2}^+ \; C_{w3}^+\right\}^T+[1\;\;1\;\;1]\left\{C_{w1}^- \; C_{w2}^- \; C_{w3}^-\right\}^T=0 \tag{A.28}$$

$$\text{Let } \{A_w\}=[1\;\;1\;\;1], \quad \{B_w\}=[1\;\;1\;\;1] \tag{A.29}$$

$$\Rightarrow \{A_w\}\left\{C_{w1}^+ \; C_{w2}^+ \; C_{w3}^+\right\}^T+\{B_w\}\left\{C_{w1}^- \; C_{w2}^- \; C_{w3}^-\right\}^T=0 \tag{A.30}$$

$$\text{where } \begin{Bmatrix} C_{w1}^- \\ C_{w2}^- \\ C_{w3}^- \end{Bmatrix}=[r_{III}]\begin{Bmatrix} C_{w1}^+ \\ C_{w2}^+ \\ C_{w3}^+ \end{Bmatrix}$$

Therefore,

$$\Rightarrow \{A_w\}\left\{C_{w1}^+ \; C_{w2}^+ \; C_{w3}^+\right\}^T+\{B_w\}[r_{III}]\left\{C_{w1}^- \; C_{w2}^- \; C_{w3}^-\right\}^T=0 \tag{A.31}$$

$$\left[\{A_w\}+[r_{III}]\{B_w\}\right]=0 \tag{A.32}$$

$$u^{-}=0 \tag{A.33}$$

$$\Rightarrow\left(C_{w1}^{+}\alpha_1^{+}+C_{w2}^{+}\alpha_2^{+}+C_{w3}^{+}\alpha_3^{+}-C_{w1}^{-}\alpha_1^{+}-C_{w2}^{-}\alpha_2^{+}-C_{w3}^{-}\alpha_3^{+}\right)e^{-iwt} \tag{A.34}$$

$$\Rightarrow\left[\alpha_1^{+}\ \alpha_2^{+}\ \alpha_3^{+}\right]\left\{C_{w1}^{+}\ C_{w2}^{+}\ C_{w3}^{+}\right\}^T+\left[-\alpha_1^{+}\ -\alpha_2^{+}\ -\alpha_3^{+}\right]\left\{C_{w1}^{-}\ C_{w2}^{-}\ C_{w3}^{-}\right\}^T=0 \tag{A.35}$$

Let $\{A_u\}=\left[\alpha_1^{+}\ \alpha_2^{+}\ \alpha_3^{+}\right],\quad \{B_u\}=\left[-\alpha_1^{+}\ -\alpha_2^{+}\ -\alpha_3^{+}\right]$ (A.36)

$$\Rightarrow\{A_u\}\left\{C_{u1}^{+}\ C_{u2}^{+}\ C_{u3}^{+}\right\}^T+\{B_u\}\left[r_{III}\right]\left\{C_{u1}^{-}\ C_{u2}^{-}\ C_{u3}^{-}\right\}^T=0 \tag{A.37}$$

$$\left[\{A_u\}+\left[r_{III}\right]\{B_u\}\right]=0 \tag{A.38}$$

$$\psi^{-}=\frac{\delta w^{-}}{\delta\theta}-u^{-}=0 \tag{A.39}$$

$$\frac{\delta w^{-}}{\delta\theta}=\left(-i\gamma_1 C_{w1}^{+}+i\gamma_2 C_{w2}^{+}+i\gamma_3 C_{w3}^{+}+i\gamma_1 C_{w1}^{-}-i\gamma_2 C_{w2}^{-}-i\gamma_3 C_{w3}^{-}\right)e^{-i\omega t} \tag{A.40}$$

$$\psi^{-}=\begin{pmatrix}-i\gamma_1 C_{w1}^{+}+i\gamma_2 C_{w2}^{+}+i\gamma_3 C_{w3}^{+}+i\gamma_1 C_{w1}^{-}-i\gamma_2 C_{w2}^{-}-i\gamma_3 C_{w3}^{-}\\ -C_{w1}^{+}\alpha_1^{+}-C_{w2}^{+}\alpha_2^{+}-C_{w3}^{+}\alpha_3^{+}+C_{w1}^{-}\alpha_1^{+}\\ +C_{w2}^{-}\alpha_2^{+}+C_{w3}^{-}\alpha_3^{+}\end{pmatrix}e^{-i\omega t}=0 \tag{A.41}$$

$$\Rightarrow\begin{pmatrix}-i\gamma_1 C_{w1}^{+}-C_{w1}^{+}\alpha_1^{+}+i\gamma_2 C_{w2}^{+}-C_{w2}^{+}\alpha_2^{+}+i\gamma_3 C_{w3}^{+}-C_{w3}^{+}\alpha_3^{+}\\ +i\gamma_1 C_{w1}^{-}+C_{w1}^{-}\alpha_1^{+}-i\gamma_2 C_{w2}^{-}+C_{w2}^{-}\alpha_2^{+}-i\gamma_3 C_{w3}^{-}+C_{w3}^{-}\alpha_3^{+}\end{pmatrix}e^{-i\omega t}=0 \tag{A.42}$$

$$\begin{aligned}&\Rightarrow\left[-\left(i\gamma_1+\alpha_1^{+}\right)\ \left(i\gamma_2-\alpha_2^{+}\right)\ \left(i\gamma_3-\alpha_3^{+}\right)\right]\left\{C_{w1}^{+}\ C_{w2}^{+}\ C_{w3}^{+}\right\}^T+\\&\left[\left(i\gamma_1+\alpha_1^{+}\right)\ -\left(i\gamma_2-\alpha_2^{+}\right)\ -\left(i\gamma_3-\alpha_3^{+}\right)\right]\left\{C_{w1}^{-}\ C_{w2}^{-}\ C_{w3}^{-}\right\}^T\end{aligned} \tag{A.43}$$

Let $\{A_\psi\}=\left[-\left(i\gamma_1+\alpha_1^{+}\right)\ \left(i\gamma_2-\alpha_2^{+}\right)\ \left(i\gamma_3-\alpha_3^{+}\right)\right],$ (A.44)
$\{B_\psi\}=\left[\left(i\gamma_1+\alpha_1^{+}\right)\ -\left(i\gamma_2-\alpha_2^{+}\right)\ -\left(i\gamma_3-\alpha_3^{+}\right)\right]$

$$\Rightarrow\{A_\psi\}\left\{C_{w1}^{+}\ C_{w2}^{+}\ C_{w3}^{+}\right\}^T+\{B_\psi\}\left[r_{III}\right]\left\{C_{w1}^{-}\ C_{w2}^{-}\ C_{w3}^{-}\right\}^T \tag{A.45}$$

$$\left[\{A_\psi\}+\left[r_{III}\right]\{B_\psi\}\right]=0 \tag{A.46}$$

Using the layouts in Eqs. (A.32), (A.38) and (A.46) to obtain the reflection matrix as follows,

$$\begin{bmatrix}A_w\\A_u\\A_\psi\end{bmatrix}+\begin{bmatrix}B_w\\B_u\\B_\psi\end{bmatrix}\left[r_{III}\right]=0 \tag{A.47}$$

$$[\beta]=\begin{bmatrix}B_w\\B_u\\B_\psi\end{bmatrix},\quad [\alpha]=\begin{bmatrix}A_w\\A_u\\A_\psi\end{bmatrix} \tag{A.48}$$

$$\left[r_{III}\right]=[\beta]^{-1}[\alpha] \tag{A.49}$$

Free support

$$N=\frac{\delta u^{-}}{\delta\theta}+w^{-}=0 \tag{A.50}$$

$$\frac{\delta u^{-}}{\delta\theta}=\begin{pmatrix}-i\gamma_1 C_{w1}^{+}\alpha_1^{+}+i\gamma_2 C_{w2}^{+}\alpha_2^{+}+i\gamma_3 C_{w3}^{+}\alpha_3^{+}- \\ i\gamma_1 C_{w1}^{-}\alpha_1^{+}+i\gamma_2 C_{w2}^{-}\alpha_2^{+}+i\gamma_3 C_{w3}^{-}\alpha_3^{+}\end{pmatrix}e^{-i\omega t} \tag{A.51}$$

$$N=\begin{pmatrix}-i\gamma_1 C_{w1}^{+}\alpha_1^{+}+i\gamma_2 C_{w2}^{+}\alpha_2^{+}+i\gamma_3 C_{w3}^{+}\alpha_3^{+}-i\gamma_1 C_{w1}^{-}\alpha_1^{+}+ \\ i\gamma_2 C_{w2}^{-}\alpha_2^{+}+i\gamma_3 C_{w3}^{-}\alpha_3^{+}+C_{w1}^{+}+C_{w2}^{+}+C_{w3}^{+}+C_{w1}^{-}+C_{w2}^{-}+C_{w3}^{-}\end{pmatrix}e^{-i\omega t}=0 \tag{A.52}$$

$$\Rightarrow\begin{pmatrix}-i\gamma_1 C_{w1}^{+}\alpha_1^{+}+C_{w1}^{+}+i\gamma_2 C_{w2}^{+}\alpha_2^{+}+C_{w2}^{+}+i\gamma_3 C_{w3}^{+}\alpha_3^{+}+C_{w3}^{+} \\ -i\gamma_1 C_{w1}^{-}\alpha_1^{+}+C_{w1}^{-}+i\gamma_2 C_{w2}^{-}\alpha_2^{+}+C_{w2}^{-}+i\gamma_3 C_{w3}^{-}\alpha_3^{+}+C_{w3}^{-}\end{pmatrix}e^{-i\omega t}=0 \tag{A.53}$$

$$\Rightarrow\left[-\left(i\gamma_1\alpha_1^{+}-1\right)\ \left(i\gamma_2\alpha_2^{+}+1\right)\ \left(i\gamma_3\alpha_3^{+}+1\right)\right]\left\{C_{w1}^{+}\ \ C_{w2}^{+}\ \ C_{w3}^{+}\right\}^{T}+ \\ \left[-\left(i\gamma_1\alpha_1^{+}-1\right)\ \left(i\gamma_2\alpha_2^{+}+1\right)\ \left(i\gamma_3\alpha_3^{+}+1\right)\right]\left\{C_{w1}^{-}\ \ C_{w2}^{-}\ \ C_{w3}^{-}\right\}^{T} \tag{A.54}$$

Let $\{A_N\}=\left[-\left(i\gamma_1\alpha_1^{+}-1\right)\ \left(i\gamma_2\alpha_2^{+}+1\right)\ \left(i\gamma_3\alpha_3^{+}+1\right)\right]$,
$\{B_N\}=\left[-\left(i\gamma_1\alpha_1^{+}-1\right)\ \left(i\gamma_2\alpha_2^{+}+1\right)\ \left(i\gamma_3\alpha_3^{+}+1\right)\right]$ (A.55)

$$\Rightarrow\{A_N\}\left\{C_{w1}^{+}\ \ C_{w2}^{+}\ \ C_{w3}^{+}\right\}^{T}+\{B_N\}\left[r_{III}\right]\left\{C_{w1}^{-}\ \ C_{w2}^{-}\ \ C_{w3}^{-}\right\}^{T} \tag{A.56}$$

$$\left[\{A_N\}+\left[r_{III}\right]\{B_N\}\right]=0 \tag{A.57}$$

$$Q=\frac{\delta^2 u^{-}}{\delta\theta^2}-\frac{\delta^3 w^{-}}{\delta\theta^3}=0 \tag{A.58}$$

$$\frac{\delta^2 u^{-}}{\delta\theta^2}=\begin{pmatrix}-\gamma_1^2 C_{w1}^{+}\alpha_1^{+}-\gamma_2^2 C_{w2}^{+}\alpha_2^{+}-\gamma_3^2 C_{w3}^{+}\alpha_3^{+}+\gamma_1^2 C_{w1}^{-}\alpha_1^{+} \\ +\gamma_2^2 C_{w2}^{-}\alpha_2^{+}+\gamma_3^2 C_{w3}^{-}\alpha_3^{+}\end{pmatrix}e^{-iwt} \tag{A.59}$$

$$\frac{\delta^2 w^{-}}{\delta\theta^2}=\left(i\gamma_1^3 C_{w1}^{+}-i\gamma_2^3 C_{w2}^{+}-i\gamma_3^3 C_{w3}^{+}-i\gamma_1^3 C_{w1}^{-}+i\gamma_2^3 C_{w2}^{-}+i\gamma_3^3 C_{w3}^{-}\right)e^{-i\omega t}=0 \tag{A.60}$$

$$\Rightarrow-\gamma_1^2 C_{w1}^{+}\alpha_1^{+}-\gamma_2^2 C_{w2}^{+}\alpha_2^{+}-\gamma_3^2 C_{w3}^{+}\alpha_3^{+}+\gamma_1^2 C_{w1}^{-}\alpha_1^{+}+\gamma_2^2 C_{w2}^{-}\alpha_2^{+} \\ +\gamma_3^2 C_{w3}^{-}\alpha_3^{+}-i\gamma_1^3 C_{w1}^{+}+i\gamma_2^3 C_{w2}^{+}+i\gamma_3^3 C_{w3}^{+}+i\gamma_1^3 C_{w1}^{-}-i\gamma_2^3 C_{w2}^{-}-i\gamma_3^3 C_{w3}^{-}=0 \tag{A.61}$$

$$\Rightarrow-\gamma_1^2 C_{w1}^{+}\alpha_1^{+}-i\gamma_1^3 C_{w1}^{+}-\gamma_2^2 C_{w2}^{+}\alpha_2^{+}+i\gamma_2^3 C_{w2}^{+}-\gamma_3^2 C_{w3}^{+}\alpha_3^{+}+i\gamma_3^3 C_{w3}^{+} \\ +\gamma_1^2 C_{w1}^{-}\alpha_1^{+}+i\gamma_1^3 C_{w1}^{-}+\gamma_2^2 C_{w2}^{-}\alpha_2^{+}-i\gamma_2^3 C_{w2}^{-}+\gamma_3^2 C_{w3}^{-}\alpha_3^{+}-i\gamma_3^3 C_{w3}^{-}=0 \tag{A.62}$$

$$\Rightarrow\left[-\left(\alpha_1^{+}+i\gamma_1\right)\gamma_1^2\ \ -\left(\alpha_2^{+}-i\gamma_2\right)\gamma_2^2\ \ -\left(\alpha_3^{+}-i\gamma_3\right)\gamma_3^2\right]\left\{C_{w1}^{+}\ \ C_{w2}^{+}\ \ C_{w3}^{+}\right\}^{T}+ \\ \left[\left(\alpha_1^{+}+i\gamma_1\right)\gamma_1^2\ \left(\alpha_2^{+}-i\gamma_2\right)\gamma_2^2\ \left(\alpha_3^{+}-i\gamma_3\right)\gamma_3^2\right]\left\{C_{w1}^{-}\ \ C_{w2}^{-}\ \ C_{w3}^{-}\right\}^{T}=0 \tag{A.63}$$

$\{A_Q\}=\left[-\left(\alpha_1^{+}+i\gamma_1\right)\gamma_1^2\ \ -\left(\alpha_2^{+}-i\gamma_2\right)\gamma_2^2\ \ -\left(\alpha_3^{+}-i\gamma_3\right)\gamma_3^2\right]$,
$\{B_Q\}=\left[\left(\alpha_1^{+}+i\gamma_1\right)\gamma_1^2\ \left(\alpha_2^{+}-i\gamma_2\right)\gamma_2^2\ \left(\alpha_3^{+}-i\gamma_3\right)\gamma_3^2\right]$ (A.64)

$$\Rightarrow\{A_Q\}\left\{C_{w1}^{+}\quad C_{w2}^{+}\quad C_{w3}^{+}\right\}^T+\{B_Q\}\left\{C_{w1}^{-}\quad C_{w2}^{-}\quad C_{w3}^{-}\right\}^T=0 \tag{A.65}$$

$$[\{A_Q\}+[r_{III}]\{B_Q\}]=0 \tag{A.66}$$

$$M=\frac{\delta u^-}{\delta\theta}-\frac{\delta^2 w^-}{\delta\theta^2}=0 \tag{A.67}$$

$$\frac{\delta u^-}{\delta\theta}=\left(-i\gamma_1 C_{w1}^{+}\alpha_1^{+}+i\gamma_2 C_{w2}^{+}\alpha_2^{+}+i\gamma_3 C_{w3}^{+}\alpha_3^{+}-i\gamma_1 C_{w1}^{-}\alpha_1^{+}+i\gamma_2 C_{w2}^{-}\alpha_2^{+}+i\gamma_3 C_{w3}^{-}\alpha_3^{+}\right)e^{-iwt} \tag{A.68}$$

$$\frac{\delta^2 w^-}{\delta\theta^2}=\left(-\gamma_1^2 C_{w1}^{+}-\gamma_2^2 C_{w2}^{+}-\gamma_3^2 C_{w3}^{+}-\gamma_1^2 C_{w1}^{-}-\gamma_2^2 C_{w2}^{-}-\gamma_3^2 C_{w3}^{-}\right)e^{-i\omega t}=0 \tag{A.69}$$

$$\begin{aligned}\Rightarrow &-i\gamma_1 C_{w1}^{+}\alpha_1^{+}+i\gamma_2 C_{w2}^{+}\alpha_2^{+}+i\gamma_3 C_{w3}^{+}\alpha_3^{+}-i\gamma_1 C_{w1}^{-}\alpha_1^{+}+i\gamma_2 C_{w2}^{-}\alpha_2^{+}\\ &+i\gamma_3 C_{w3}^{-}\alpha_3^{+}+\gamma_1^2 C_{w1}^{+}+\gamma_2^2 C_{w2}^{+}+\gamma_3^2 C_{w3}^{+}+\gamma_1^2 C_{w1}^{-}+\gamma_2^2 C_{w2}^{-}+\gamma_3^2 C_{w3}^{-}=0\end{aligned} \tag{A.70}$$

$$\begin{aligned}\Rightarrow &\left(\gamma_1^2-i\gamma_1\alpha_1^{+}\right)C_{w1}^{+}+\left(\gamma_2^2+i\gamma_2\alpha_2^{+}\right)C_{w2}^{+}+\left(\gamma_3^2+i\gamma_3\alpha_3^{+}\right)C_{w3}^{+}+\left(\gamma_1^2-i\gamma_1\alpha_1^{+}\right)C_{w1}^{-}+\\ &\left(\gamma_2^2+i\gamma_2\alpha_2^{+}\right)C_{w2}^{-}+\left(\gamma_3^2+i\gamma_3\alpha_3^{+}\right)C_{w3}^{-}=0\end{aligned} \tag{A.71}$$

$$\begin{aligned}\Rightarrow &\left[(\gamma_1-i\alpha_1^{+})\gamma_1\quad(\gamma_2+i\alpha_2^{+})\gamma_2\quad(\gamma_3+i\alpha_3^{+})\gamma_3\right]\left\{C_{w1}^{+}\quad C_{w2}^{+}\quad C_{w3}^{+}\right\}^T+\\ &\left[(\gamma_1-i\alpha_1^{+})\gamma_1\quad(\gamma_2+i\alpha_2^{+})\gamma_2\quad(\gamma_3+i\alpha_3^{+})\gamma_3\right]\left\{C_{w1}^{-}\quad C_{w2}^{-}\quad C_{w3}^{-}\right\}^T=0\end{aligned} \tag{A.72}$$

Let
$$\begin{aligned}\{A_M\}&=\left[(\gamma_1-i\alpha_1^{+})\gamma_1\quad(\gamma_2+i\alpha_2^{+})\gamma_2\quad(\gamma_3+i\alpha_3^{+})\gamma_3\right],\\ \{B_M\}&=\left[(\gamma_1-i\alpha_1^{+})\gamma_1\quad(\gamma_2+i\alpha_2^{+})\gamma_2\quad(\gamma_3+i\alpha_3^{+})\gamma_3\right]\end{aligned} \tag{A.73}$$

$$\Rightarrow\{A_M\}\left\{C_{w1}^{+}\quad C_{w2}^{+}\quad C_{w3}^{+}\right\}^T+\{B_M\}\left\{C_{w1}^{-}\quad C_{w2}^{-}\quad C_{w3}^{-}\right\}^T=0 \tag{A.74}$$

$$[\{A_m\}+[r_{III}]\{B_m\}]=0 \tag{A.75}$$

Using the layouts in Eqs. (A.57), (A.66) and (A.75) to obtain the reflection matrix as follows,

$$\begin{bmatrix}A_w\\A_u\\A_m\end{bmatrix}+\begin{bmatrix}B_w\\B_u\\B_m\end{bmatrix}[r_{III}]=0 \tag{A.76}$$

$$[\beta]=\begin{bmatrix}B_w\\B_u\\B_m\end{bmatrix},\quad [\alpha]=\begin{bmatrix}A_w\\A_u\\A_m\end{bmatrix} \tag{A.77}$$

$$[r_{III}]=[\beta]^{-1}[\alpha] \tag{A.78}$$

V.Appendix B. Derivation of characteristic equation

Reflection matrix at the right boundary where

$$w_R^- = R_R w_R^+ \tag{B.1}$$

Transmission from the left boundary

$$w_L^- = T w_R^- \tag{B.2}$$

Reflection at the left boundary

$$w_L^+ = R_L w_L^- \tag{B.3}$$

Transmission from the right boundary

$$w_R^+ = T w_L^+ \tag{B.4}$$

Substituting Eq. (B.4) in Eq.(B.1) gives

$$w_R^- = R_R T w_L^+ \tag{B.5}$$

Substituting Eq. (B.5) in Eq.(B.2) provides

$$w_L^- = T R_R T w_L^+ \tag{B.6}$$

Substituting Eq. (B.6) in Eq. (B.3) gives

$$w_L^+ = R_L T R_R T w_L^+ \tag{B.7}$$

From Eq. (B.7)

$$(R_L T R_R T - 1) w_L^+ = 0 \tag{B.8}$$

국문초록

위상폐합원리를 이용한 곡선보의 진동수 방정식

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곡선보는 굽힘과 스트레칭에 의해 동시에 발생하는 하중을 전달할 수 있다. 가해지는 하중조건에서 곡선보의 역학적 거동은 많은 구조물들에 효율적으로 적용될 수 있는 장점을 가지고 있다. 따라서 선형 자유 진동 해석에 관련한 곡선보의 동적거동에 대한 여러 선행연구들이 주목을 받아왔다.

선형 자유 진동 해석은 특히 초기 설계 단계에서 곡선보의 동적거동을 이해하는 데 매우 중요한 고유진동수와 그에 상응하는 모드 형상의 평가가 포함된다. 곡선보에서 이러한 동적거동을 이해하기 위해 여러 가지 기법들이 사용되어 왔다. 그러나 기존 기법들은 상대적으로 복잡하고 계산 비용이 많이 소요되는 단점이 있다. 결과적으로, 다른 기법들에 비해 구조물의 진동 해석이 단순해지는 점 때문에 많은 연구들이 파동역학에 기반을 둔 기법들을 제시하였다. 일반적으로 파동역학 관점에서의 해석은 고유행렬의 수치해를 구하기 위해 반사행렬과 투과행렬을 함께 활용한다. 반면, 반사행렬에서 얻은 반사계수를 활용한 위상폐합원리에 기반을 둔 접근법은 거의 주목을 받지 못했다.

이 연구에서는 곡선보의 고유진동수를 예측하기 위해 단순화된 주파수 방정식을 제안한다. 주파수 방정식은 반사 행렬에서 얻은 반사 계수의 위상 변화를 사용하여 정식화된다. 곡선보의 분산방정식과 주파수 스펙트럼이 검토된다. 여러 지점 조건과 일정한 곡률을 가진 곡선보에 대해 분산방정식의 특정해의 영역에서 위상폐합원리를 적용한다. 축변형이 가능한 경우와 축변

형을 무시한 경우 모두 고려하고 전단변형 및 회전 관성 효과는 고려하지 않는다. 제안된 방법은 행렬 정식화 및 기존 연구 수치 해석 결과와 비교하여 검증한다. 비교 결과 제안된 방정식에 의한 예측과 기존의 수치 해석 결과가 잘 일치함을 확인할 수 있다. 따라서 제안된 방법은 다양한 지지 조건을 가진 곡선보의 고유진동수를 쉽고 간편하게 예측할 수 있다.

주요어: 자유 진동, 곡선보, 진동수 방정식, 주파수 스펙트럼, 위상폐합원리

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