Option Pricing with Monte Carlo Simulations

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1. Motivation

- Options are a financial product that derive their value from another asset
- Given contracts can be esoteric, our research focuses on two forms of contracts, European Call options and Asian Call Options.
- While Monte Carlo Methods can be applied to pricing any option contracts, it is often inefficient to do so compared to other methods
- Our research investigates when/how to apply Monte Carlo pricing (Closed vs Open form solutions)
- Lastly, we carried out a simulation of model stability under increasing levels of Volatility for four Models pricing Arithmetic Mean Asian Call Options to test their robustness

2. Pricing European Options and Asian Options

- European options delivery is only dependent on the final price and has a closed formed solution
- The most the most utilized form of options pricing is the Black Scholes model
- The Black-Scholes equation is a partial differential equation (PDE) which captures the price of an option

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0 \text{ [1]}$$

- For a non-dividend paying European Call the equation can be derived as follows:

$$C(S_t, t) = N(d_1)S_t - N(d_2)Ke^{-r(T-t)} \text{ [2]}$$

$$d_1 = \frac{1}{\sigma\sqrt{T-t}}[ln(\frac{S_t}{K}) + (r + \frac{\sigma^2}{2})(T-t)]$$

$$d_2 = d_1 - \sigma\sqrt{T-t}$$

- Two methods of Pricing European Options are the Binomial Pricing Model and the Finite Difference Methods [3][4]
- The first builds on the Black-Scholes model via a Lattice or Tree based approach while the later uses PDE's
- An important takeaway is that the Binomial and PDE methods can actually be far more efficient compared to Monte Carlo methods for estimating European call option prices given they have a closed formed solution [5]
- Asian Options in contrast are dependent on the average underlying value of the asset over its lifetime rather than at expiration

- Arithmetic Asian Call Options and Geometric Asian Call Options can be expressed as follows:

Arithmetic Asian Option:

$$\Phi(S) = (\frac{1}{T}\int_0^T S(T)dt - K)^+ \text{ [6]}$$

Geometric Asian Option:

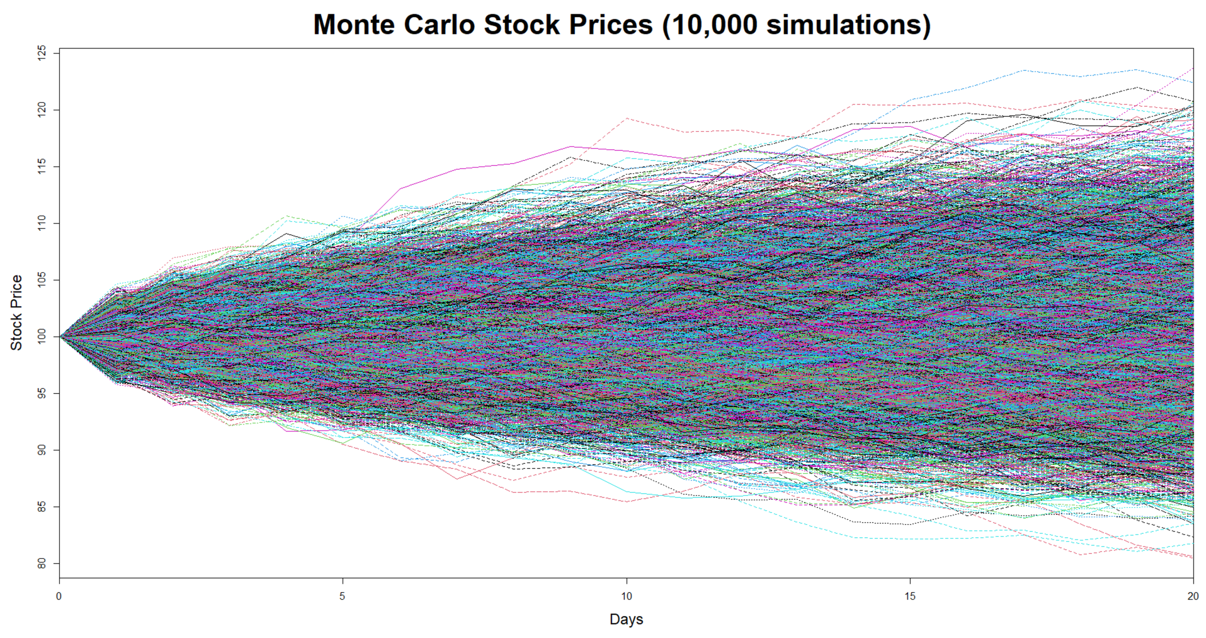
$$\Phi(S) = (e^{\frac{1}{T}\int_0^T logS(t)dt} - K)^+ \text{ [6]}$$

- Considering Arithmetic Asian Option's path dependent structure, no closed formed solution exists and the previous pricing methods grow exponentially in terms of complexity
- Thus, it is fruitful to explore the application of Monte Carlo methods for pricing the Arthemetic Mean Asian Options

3. Options Pricing Algorithm from our PAO package [7]

- When initializing the Monte Carlo algorithm for Options Pricing, we must provide 6 variables (7 for the Asian Options):

S = Asset Value at time 0
K = Strike Price of the Contract
V = Volatility (Annualized volatility of return of the asset in percentage)
tt = The time till expiration (fraction of the number of trading days in a year, 252)
r = risk-free rate
numsim = Number of Monte Carlo Simulations to be run
m = Number of mark to markets (Only for Asian Options)



Algorithm Implementation (In R Code)

1. z <- matrix(rnorm(m *numsim), numsim, m)
2. zcum <- t(apply(z,1,cumsum))
3. h <- tt/m (Only for Asian Options)
4. S <- matrix(1, nrow = numsim, ncol = m)
5. for (i in 1:m){S[,i]<-s*exp((r-0.5*v^2)*h*i+v*$\sqrt{(h)}$*zcum[, i])}
6. At this step, the code (Payoff Eval.) depends on the Option

4. Variance Reduction

- While increasing the number of simulations improves estimates, there are diminishing returns
- ECO(S=100,K=100,vol=0.25,tt=252/252,r=0.02)

numsim	Price	Standard Error
10^3	10.513173	1.057449
10^4	10.9807347	0.3507756
10^5	10.8914305	0.1086549

- We considered two methods for variance reduction, antithetical variates and control variates
- Antithetical variates will reduce variance because the variates X and -X will be negatively correlated

$$\hat{\theta}_a = \frac{1}{n}\sum_{i=1}^{n}\frac{g(X_i)+g(-X_i)}{2}, \text{ with } i.i.d. X_i \sim N(0,1) \text{ [8]}$$

- The control variates require a suitable variate choice to achieve good variance reduction
- The most obvious choice for a control variate is the European Call Option price, but there is a better option

5. Geometric Asian Options

- Surprisingly, while the Arithmetic Asian Option has no close formed solution, the Geometric Asian Option does
- $\Phi(S) = (e^{\frac{1}{T}\int_0^T logS(t)dt} - K)^+$ can be transformed by applying a log transformation with the result:

$$\Phi(S) = \frac{(r-\frac{\sigma^2}{2})T}{2} + \frac{\sigma\sqrt{\frac{T}{m}}\sum_{i=1}^{m} iX_i}{m+1} \text{ [9]}$$

- By the additive mean and variance property of independent normal random varaibles we obtain:

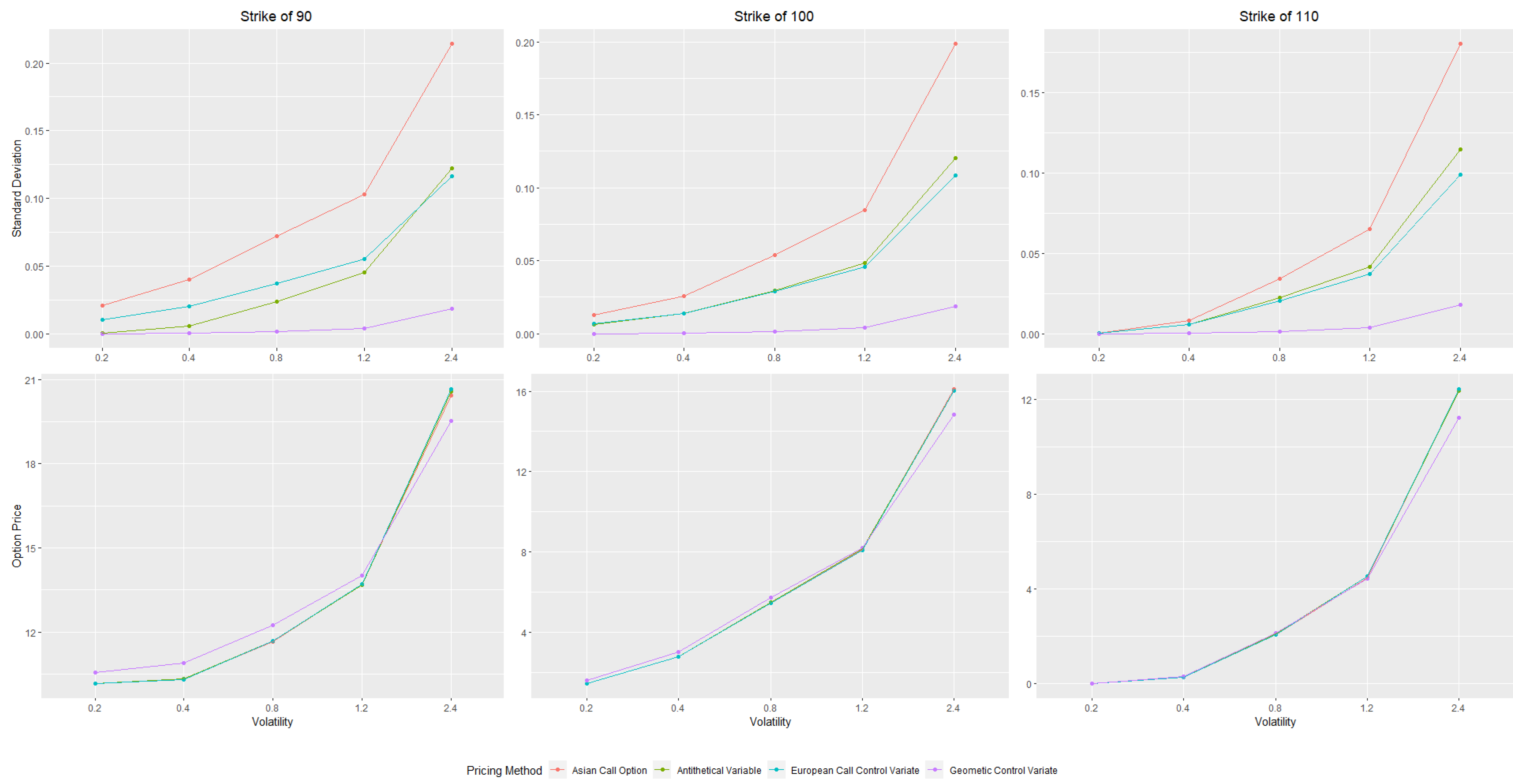
$$\sigma_z = \sigma\sqrt{\frac{2m+1}{6(m+1)}} \text{ and } \rho = \frac{(r-\frac{\sigma^2}{2})+\sigma_z^2}{2} \text{ [9]}$$

- The result produces a new variance and risk-free interest rate which can be plugged into the Black-Scholes formula to produce an exact price for the geometric Asian option
- Under this framework, the Geometric Asian Option can serve as a control variate for the Arithmetic Asian Option

6. Experiment

- Traditionally, volatility has been the bane of the black-scholes model with tail risk often resulting in a complete break down of the pricing model
- We set up a test to investigate how our methods would perform in estimating the price of out the money call options, at the money call options and in the money call options for a period of 20 trading days or one month of time

7. Simulation Results



8. Conclusion

- The Crude Monte Carlo method preformed worst while Geometric Mean Options remained relatively robust even in the case of extremely high levels of volatility.
- Geometric method produces higher option prices under low volatility and lower under high volatility compared to the other three methods
- Out the money options have very precise estimates intially but approach similar levels of Standard Error once volatility increases
- At higher levels of volatitly, the European Control Variate starts to outpreform the estimates of antithetical variates

References

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MONTE CARLO METHODS

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