Telescoping Series Common Subproblem Generalized Solution

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STATEMENT OF GOAL:

Suppose that:

$$\begin{aligned} S &= \sum_{i=L}^{i=H} \left(f(i)-g(i)\right) \\ (1): \quad &= \sum_{i=L}^{i=H} \left(f(i)\right) - \sum_{i=L}^{i=H} \left(g(i)\right) \end{aligned}$$

GOAL: Find a simple, closed (if possible) form for the telescoping series S .

PROOF OUTLINE:

We make the following assumption about f and g since this will allow us to re-write (1) later on as 4 separate summations, 2 of which will cancel out:

$$(2): \quad \exists (n,m) \in \mathbb{N}_0^2 \left(\forall i \in \mathbb{N}_0 \left(\left((L \le i+n \le H) \wedge (L \le i+m \le H) \right) \Rightarrow \left(f(i+n) = g(i+m) \right) \right) \right)$$

Read as: there are 2 natural numbers n and m that we can find where the value of f will equal the value of g , along with other values of f and g that will be equal at some point. As long as the variable i is within the bounds of the summation given in (1) , then we can safely use this equivalence property to cancel these terms later.

Once (2) has been proven, this allows us to re-write the bounds of (1) using these n and m :

$$(3): \quad S = \sum_{i=L-n}^{i=H-n} \left(f(i+n)\right) - \sum_{i=L-m}^{i=H-m} \left(g(i+m)\right)$$

The following 2 assumptions will also be made in order to complete the proof:

$$\begin{aligned} (4): \quad n &\le m \\ (5): \quad L-n &\le H-m \end{aligned}$$

Assumption (5) must (somehow) be made since it is a necessary condition for S to be a telescoping series with canceling terms at all, while assumption (4) is made so that the summations of S can be broken apart properly. If assumption (4) cannot be made, then a simple series of substitutions can be used to satisfy it:

$$\begin{aligned} S_{Alt} &= -S \\ f_{Alt}(x) &= g(x) \\ g_{Alt}(x) &= f(x) \\ n_{Alt} &= m \\ m_{Alt} &= n \end{aligned}$$

The problem will then be solvable using the new variables in conjunction with the end result of this proof.

However, given that the assumptions are fulfilled to begin with, we can then safely subdivide the summations of equation (3) into the following summations:

$$\begin{aligned} S &= \sum_{i=H-m+1}^{i=H-n}(f(i+n))+\sum_{i=L-n}^{i=H-m}(f(i+n))-\sum_{i=L-n}^{i=H-m}(g(i+m))-\sum_{i=L-m}^{i=L-n-1}(g(i+m)) \\ (6): \quad &= \sum_{i=H-m+1}^{i=H-n}(f(i+n))+\sum_{i=L-n}^{i=H-m}(f(i+n)-g(i+m))-\sum_{i=L-m}^{i=L-n-1}(g(i+m)) \end{aligned}$$

But, by (2) , we know that, within the bounds of the 2nd summation:

$$\begin{aligned} f(i+n) &= g(i+m) \\ f(i+n)-g(i+m) &= 0 \end{aligned}$$

Thus:

$$\begin{aligned} S &= \sum_{i=H-m+1}^{i=H-n}(f(i+n))+\sum_{i=L-n}^{i=H-m}(0)-\sum_{i=L-m}^{i=L-n-1}(g(i+m)) \\ (7): \quad &= \sum_{i=H-m+1}^{i=H-n}(f(i+n))-\sum_{i=L-m}^{i=L-n-1}(g(i+m)) \end{aligned}$$

Which should be much simpler to evaluate than (1) as $H-L$ approaches infinity since n and m are both constants.

EXAMPLE/SANITY CHECK (Taken from [1]):

Let:

$$S = \sum_{i=1}^{i=2015} \left(\frac{1}{i^2+3 \cdot i+2} \right)$$

Let us re-write the inner term using partial fractions. In the following, we must find an A and a B such that the following is true for all values of i :

$$\begin{aligned} \frac{1}{i^2+3 \cdot i+2} &= \frac{A}{i+1}+\frac{B}{i+2} \\ 1 &= A \cdot (i+2)+B \cdot (i+1) \end{aligned}$$

Setting $i=-1$, we have:

$$\begin{aligned} 1 &= A \cdot (-1+2)+B \cdot (-1+1) \\ 1 &= A \end{aligned}$$

Setting $i=-2$, we have:

$$\begin{aligned} 1 &= A \cdot (-2+2)+B \cdot (-2+1) \\ 1 &= -B \\ -1 &= B \end{aligned}$$

Thus:

$$\frac{1}{i^2+3 \cdot i+2} = \frac{1}{i+1}+\frac{-1}{i+2}$$

And so:

$$S = \sum_{i=1}^{i=2015} \left(\frac{1}{i+1}+\frac{-1}{i+2} \right)$$

Identifying some key values of the original proof:

$$\begin{aligned} H &= 2015 \\ L &= 1 \\ f(x) &= \frac{1}{i+1} \\ g(x) &= \frac{1}{i+2} \end{aligned}$$

By observing that $f(x+1)=g(x+0)$ for all values of x between H and L inclusive, we can observe that $n=1$ and $m=0$ satisfy the following relationship for all relevant values of i :

$$f(i+n)=g(i+m)$$

Unfortunately, this wouldn't satisfy assumption (4) of the original proof. However, we can perform the recommended substitutions upon the original problem in order to fulfill it:

$$\begin{aligned}
S_{Alt} &= -S \\
f_{Alt}(x) &= g(x) \\
g_{Alt}(x) &= f(x) \\
n_{Alt} &= m \\
m_{Alt} &= n
\end{aligned}$$

Thus:

$$\begin{aligned}
S_{Alt} &= \sum_{i=L}^{i=H} \left(f_{Alt}(i)-g_{Alt}(i)\right) \\
&= \sum_{i=1}^{i=2015} \left(\frac{1}{i+2}-\frac{1}{i+1}\right)
\end{aligned}$$

We can also show that (2) is satisfied since, for all relevant values of i :

$$f_{Alt}(i+n_{Alt})=g_{Alt}(i+m_{Alt})$$

We can show that assumptions (4) and (5) are now satisfied since:

$$\begin{aligned}
n_{Alt} &\leq m_{Alt} \\
0 &\leq 1
\end{aligned}$$

And:

$$\begin{aligned}
L-n_{Alt} &\leq H-m_{Alt} \\
1-0 &\leq 2015-1
\end{aligned}$$

Which allows us to use (7) to show that:

$$
\begin{aligned}
S_{Alt} &= \sum_{i=H-m_{Alt}+1}^{i=H-n_{Alt}} \left(f_{Alt}(i+n_{Alt})\right) - \sum_{i=L-m_{Alt}}^{i=L-n_{Alt}-1} \left(g_{Alt}(i+m_{Alt})\right) \\
&= \sum_{i=2015-1+1}^{i=2015} \left(\frac{1}{i+2+0}\right) - \sum_{i=1-1}^{i=1-0-1} \left(\frac{1}{i+1+1}\right) \\
&= \frac{1}{2015+2} - \frac{1}{0+2} \\
&= \frac{1}{2017} - \frac{1}{2} \\
S &= \frac{1}{2} - \frac{1}{2017}
\end{aligned}
$$

Which is consistent with the answer given in the referenced example.

CAVEATS AND DISCLAIMER:

This work ignores some of the finer details regarding the proper use of quantified/first-order logic to address the given problem, though still attempts to be as accurate and accessible as possible to an audience that is familiar with the material. I, Stephen Fedele, claim no responsibility, liability, etc. in the case where damages may occur from the use of the material provided herein.

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