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Optimal Taxation and Public Production I: Production Efficiency

By PETER A. DIAMOND AND JAMES A. MIRRLEES*

Theories of optimal production in a planned economy have usually assumed that the tax system can allow the government to achieve any desired redistribution of property.¹ On the other hand, some recent discussions of public investment criteria have tended to ignore taxation as a complementary method of controlling the economy.² Although lump sum transfers of the kind required for full optimality³ are not feasible today, commodity and income taxes can certainly be used to increase welfare.⁴ We shall therefore examine the maximization of social welfare using

both taxes and public production as control variables. In doing so, we intend to bring together the theories of taxation, public investment, and welfare economics.

There are two main results of the study: the demonstration of the desirability of aggregate production efficiency in a wide variety of circumstances provided that taxes are set at the optimal level; and an examination of that optimal tax structure. It is widely known that aggregate production efficiency is desired as one part of achieving a Pareto optimum. It is also widely known that when the desired Pareto optimum cannot be achieved, aggregate production efficiency may not be desirable. Our conclusion differs from these results in that production efficiency is desirable although a full Pareto optimum is not achieved. In the optimum position, the presence of commodity taxes implies that marginal rates of substitution are not equal to marginal rates of transformation. Furthermore, the absence of lump sum taxes implies that the income distribution is not the best that can be conceived. Yet, the presence of optimal commodity taxes will be shown to imply the desirability of aggregate production efficiency.

This result is similar to that derived by Marcel Boiteux, although he considered an economy where lump sum redistributions of income were possible. Boiteux also examined the optimal tax structure that was necessary for this result. The optimal tax structure for the case of a single consumer (or equivalently with lump sum redistribution) has also been examined by Frank

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¹ For a discussion of this literature, see Abram Bergson.

² For a survey of this literature, see Alan Prest and Ralph Turvey.

³ We wish to distinguish here between lump sum taxes, which may vary from individual to individual while being unaffected by the individual's behavior, and poll taxes which are the same for all individuals, or perhaps for all individuals within several large groups, distinguished perhaps by age, sex, or region.

⁴ For another study of the general equilibrium impact of taxation, which does not explore the optimality question, see Gerard Debreu (1954).

Ramsey and Paul Samuelson.⁵ Our results move beyond theirs in considering the problem of income redistribution together with that of raising revenue. Even in the absence of government revenue requirements, if lump sum redistribution is impossible, the government will want to use its excise tax powers to improve income distribution. It will subsidize and tax different goods so as to alter individual real incomes. The optimal redistribution by this method occurs when there is a balance between the equity improvements and the efficiency losses from further taxation.

The general situation we want to discuss is an economy in which there are many consumers, public and private production, public consumption, and many different kinds of feasible tax instruments. We think that it is easier to understand the problem if we present the analysis first for a single consumer, no public consumption, and only commodity taxation, although this case has little intrinsic interest. The main point of the paper is that the analysis of this special case carries over in the main to the general case.

The first two sections are devoted to this special case. In the first, the situation is portrayed geometrically (for a two-commodity world with no private production); in the second, production efficiency and conditions for the optimal taxes are derived by application of the calculus. The use of the calculus here and elsewhere is not perfectly rigorous for the usual reasons. These issues are taken up in Section IV. In the third section, we extend the analysis of production to an economy with many consumers, elucidating precise conditions under which production efficiency is desirable (and presenting certain exceptions).

Section IV provides a rigorous statement of the theorems. In the fifth section, we discuss briefly certain applications and extensions of the basic efficiency result.

A following paper, referred to here as Diamond-Mirrlees II, will appear in the June 1971 *Review*. In it we will examine the optimality rules for commodity taxes, for other taxes including income taxes, and for public consumption. We will also give a rigorous statement of conditions under which the first-order conditions obtained (heuristically) below are indeed necessary conditions.

I. One-Consumer Economy— Geometric Analysis

We begin by considering an economy with a single, price-taking consumer and two commodities. We assume, for the moment, that all production possibilities are controlled by the government. While there is no scope for redistribution of income in this economy, the government might need to raise revenue to cover losses if there are increasing returns to scale or if there are fixed expenditures (such as defense) and constant returns to scale. Alternatively, the technology might exhibit decreasing returns to scale, facing the government with the problem of disposing of a surplus if all transactions are carried out at market prices. The optimal solution to either raising or disposing of revenue is well known. A poll tax or subsidy, as the case may be, will permit the hiring of the needed resources and permit the economy to achieve a Pareto optimum, which, in a one-consumer economy, is equivalent to the maximization of the consumer's utility. While this is a reasonable possibility in a one-consumer economy, lump sum taxes varying from individual to individual do not seem feasible in a much larger economy. An identical problem of distributing a surplus among many people arises if it is desired to improve income distribution.

⁵ For a detailed history of analysis of this problem, see William Baumol and David Bradford. A summary and discussion of the work of Boiteux has been given by Jacques Drèze.

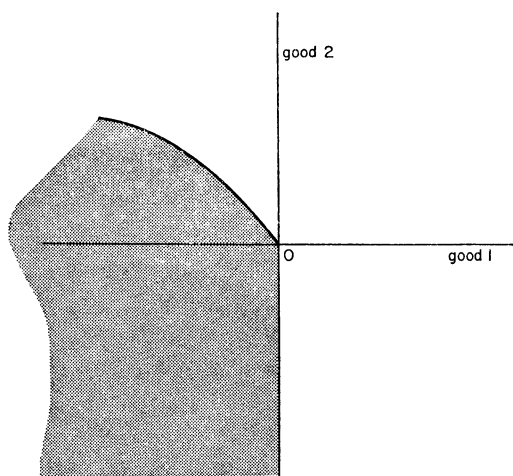


FIGURE 1

Thus we shall consider the use of commodity taxes when lump sum taxes are not permitted to the government, not for the intrinsic interest of this question in a one-consumer economy, but as an introduction to the many-consumer case. Furthermore we shall hold constant the government expenditure pattern which directly affects consumer utility. Thus we can ignore it, since the utility function already reflects its impact. The addition of choice for public consumption will be considered in Diamond-Mirrlees II.

Assuming free disposal, the technological constraint on the planner is that the government supply be on or under the production frontier. Such a constraint is shown by the shaded area in Figure 1. Let us measure on the axes the quantities supplied to the consumer. Thus, the output being produced (good 2) is measured positively, while the input (good 1) is measured negatively. The case drawn is the familiar one of decreasing returns to scale. If the government needed a fixed bundle of resources, for national defense say, then the production possibility frontier (describing the potential transactions with the consumer) would not pass through

the origin. With constant returns to scale this might appear as in Figure 2, where a units of good 1 are needed for defense. (It is perhaps convenient to think of good 1 as labor and good 2 as a consumption good.)

In a totally planned economy, where the planner selects a fixed consumption bundle (including labor to be supplied) for each consumer, the planner would have no further constraint and could choose any point that was technologically feasible. Again, this is not implausible for the planner in a one-consumer economy, but becomes so as the number of households grows. A more realistic assumption, then, is to assume that the planner can only deal with consumers through the market place, hiring labor and selling the consumer good. Assume further that the planner is constrained to charge uniform prices. The planner must now set the price of the consumer good relative to the wage (or inversely the real wage), and is constrained to transactions which the consumer is willing to undertake at some relative price. The locus of consumption bundles which the consumer is willing to achieve by trade from the origin is the offer curve or price-consumption locus. It represents the

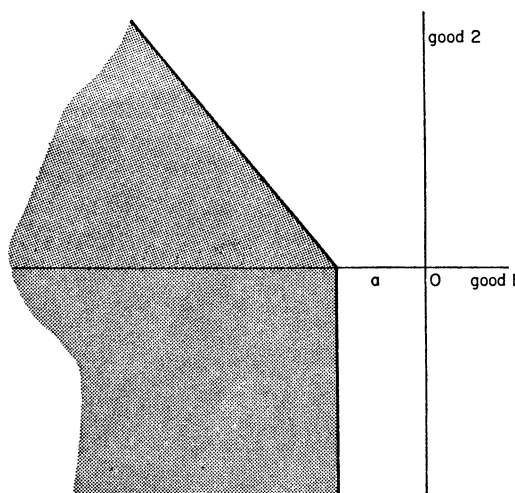


FIGURE 2

bundles of goods that the consumer would purchase at different possible price ratios. Figure 3 contains an example of an offer curve with several hypothetical budget lines and the corresponding indifference curves drawn in. The planner thus has two constraints: he must choose a point which is both technologically feasible and an equilibrium bundle from the point of view of the consumer. Combining these two constraints, the range of consumption bundles which are both feasible and potential consumer equilibria is shown as the heavy line in Figure 4.

We can state these two constraints algebraically. Let us denote by $z = (z_1, \dots, z_n)$ the vector of government supply. The production constraint is then written

$$(1) \quad G(z) \leq 0, \quad \text{or, equivalently,} \\ z_1 \leq g(z_2, z_3, \dots, z_n)$$

The constraint that the government sup-

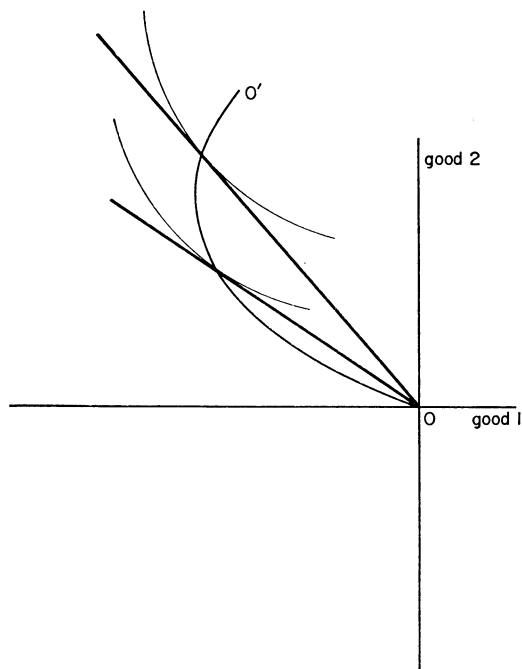


FIGURE 3

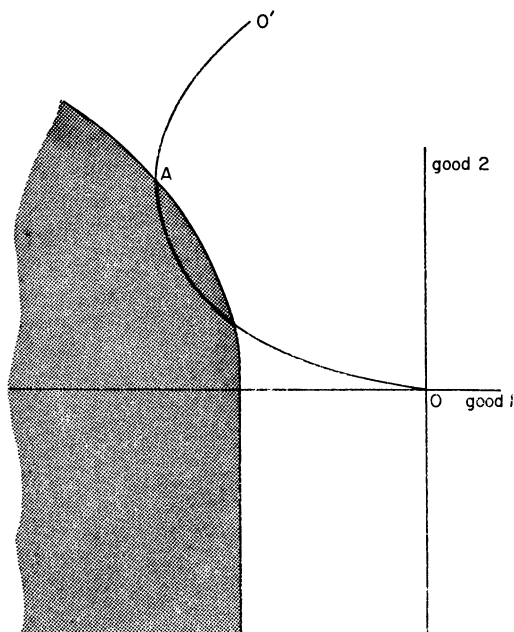


FIGURE 4

ply equal the consumer demand for some price can be written in vector notation

$$(2) \quad x(q) = z,$$

where $x = (x_1, \dots, x_n)$ is the vector of consumer demands and $q = (q_1, \dots, q_n)$ is the vector of prices faced by the consumer.

Now consider the government's objectives. Since the consumer's equilibrium position is determined by the prices he faces, we can, in the usual circumstances, describe the objective function as a function of prices, say $v(q)$. The problem is to choose q so as to

$$(3) \quad \begin{aligned} &\text{Maximize } v(q) \\ &\text{subject to } G(x(q)) \leq 0 \end{aligned}$$

This simply formulated problem is the focus of attention of the paper and can take on a variety of interpretations. The reader may note that the consideration of many consumers does not alter the form of this problem. This is a major advantage

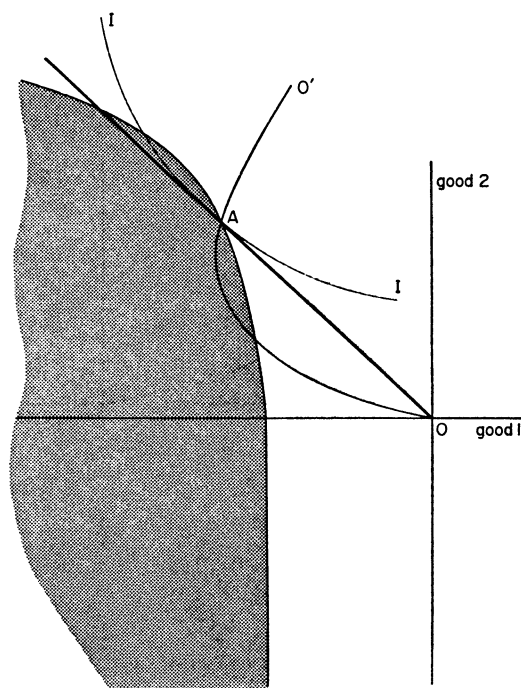


FIGURE 5

of using prices rather than quantities as the focus of the analysis.

Let us consider the case where the planner seeks to maximize the same function of consumption as the consumer's utility function. The welfare function is said to be *individualistic*, or to respect individual preferences, since welfare can be written as a function of individual utility. Returning to Figure 3 we see that the consumer moves to higher indifference curves as he proceeds along the offer curve away from the origin. Thus, in Figure 4 we wish to move as far along OO' as possible, subject to the constraint of the shaded production possibility set. The optimal point is therefore A , where the offer curve and the production frontier intersect.

The prices which will induce the consumer to purchase the optimal consumption bundle are defined by the budget line OA . In Figure 5 we show the optimal point and the implied budget line, and indiffer-

ence curve II . All the points above II and in the shaded production set are Pareto-superior to A and technologically feasible, but not attainable by market transactions without lump sum transfers. For contrast, in Figure 6, we show the Pareto optimal point, B , and the implied budget line, and indifference curve $I'I'$, which will permit decentralization. In the case drawn, the consumer's budget line does not pass through the origin; this represents his payment of a lump sum tax to cover government expenditures in excess of profits from production.

We see that the optimal point is on the production possibility frontier of the economy, not inside it. This important property of the optimum can easily be seen to carry over to the case of many commodities, but still one consumer. With many commodities, the offer curve is a union of loci, each of which is obtained by holding the prices of all but one commodity constant and varying the price of that one commodity. Doing this for each com-

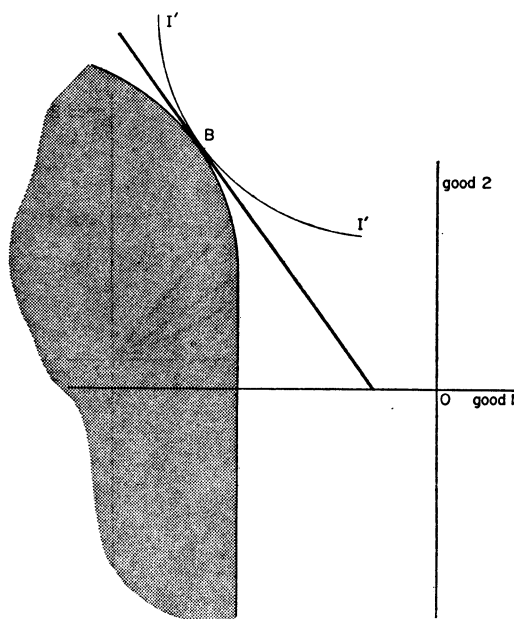


FIGURE 6

modity, and for all possible configurations of prices for the other commodities, generates all the loci. The offer curve is the union of such loci. On each locus, the point which is also on the production frontier is better than the other points on the locus. Thus, any point which is not on the production frontier is dominated by some point which is on the frontier. Therefore, the optimal point is one of the points on the frontier. The implications of this result will be seen more clearly below, when we consider both public and private production. For this result to carry over to the case of many consumers requires one further, mild assumption which will be discussed in the third section. First, we treat the one consumer economy algebraically, with both public and private production, showing by calculus the desirability of aggregate production efficiency, and obtaining the optimal relationship between consumer prices and the slope of the production possibilities. This relationship defines the optimal tax structure.

II. One-Consumer Economy— Algebraic Analysis

We assume constant returns to scale in the private production sector and the presence of competitive conditions there. In equilibrium there are, therefore, no profits. (This is a critical assumption for the efficiency analysis.) We also assume, for the present, that the only taxes used by the government are commodity taxes.⁶ Consumer prices, q , therefore determine the choices available to the consumer, and we may write the welfare function as a function of consumer prices, $v(q)$. Notice that this covers the case where the government's assessment of welfare does not coincide with the consumer's utility, al-

though depending on what he consumes. In the special case where social preferences coincide with those of the single consumer, his utility may be taken to measure welfare. Then we have

$$(4) \quad v(q) = u(x(q))$$

We shall not use this special form for $v(q)$ in the analysis below until we come to evaluate the tax structure explicitly. Until that point, the analysis applies also to welfare functions that are not individualistic. For later use let us express the derivatives of v in this special case. Writing $v_k = \partial v / \partial q_k$, $u_i = \partial u / \partial x_i$, and using (4), we have

$$(5) \quad v_k = \sum u_i \frac{\partial x_i}{\partial q_k} = -\alpha x_k,$$

where α is a positive constant (i.e., independent of k), the marginal utility of income. Equation (5) follows from the budget constraint,

$$(6) \quad \sum q_i x_i = 0,$$

which on differentiation with respect to q_k yields

$$(7) \quad x_k + \sum q_i \frac{\partial x_i}{\partial q_k} = 0$$

Since utility-maximization implies that $u_i = \alpha q_i$, (5) now follows from (7).

Production

Let us denote the vector of prices faced by private producers by $p = (p_1, \dots, p_n)$. Because of taxes, t , these may differ from the prices faced by consumers: $q_i = p_i + t_i$ ($i = 1, \dots, n$). $y = (y_1, \dots, y_n)$ is the vector of commodities privately supplied (inputs will thus appear as negative supplies), and we write the private production constraint,

$$(8) \quad y_1 = f(y_2, \dots, y_n)$$

Notice that we assume *equality* in the

⁶ This assumption is made solely for simplicity. In Diamond-Mirrlees II, the general principles will be seen to carry over with additional taxes, including a progressive income tax.

production constraint, that is, that production is efficient in the private sector. This follows from profit maximization if there are no zero prices. We assume that f is a differentiable function, and that $y_i \neq 0$ ($i=1, \dots, n$). Then, profit maximization means that

$$(9) \quad p_i = -p_1 f_i(y_2, \dots, y_n), \\ (i = 2, \dots, n)$$

where f_i denotes the derivative of f with respect to y_i . Also, by the assumption of constant returns to scale, maximized profits are zero in equilibrium:

$$(10) \quad \sum p_i y_i = 0$$

So that we may conveniently employ calculus, we shall assume that the government production constraint, (1), is satisfied with an equality rather than an inequality:

$$(11) \quad z_1 = g(z_2, \dots, z_n)$$

Thus we do not give the government the option of inefficient government production. Rather, we shift our attention to *aggregate* production efficiency. Efficiency will be present if marginal rates of transformation are the same in publicly and privately controlled production. It will then be seen quite easily that the assumption of efficiency in the public sector is justified.

Walras' Law

We have chosen an objective function and expressed the government's production constraint above. To complete the formulation of the maximization problem, it remains to add the requirement that the economy be in equilibrium. The conditions that all markets clear can be stated in terms of the vectors x , y , and z .

$$(12) \quad x(q) = y + z$$

The reader may be puzzled that at no place in this formulation has a budget

constraint been introduced for the government. (Other readers may be puzzled by our failure to include only $n-1$ markets in our market clearance equations. These are aspects of the same phenomenon.) Walras' Law implies that if all economic agents satisfy their budget constraints and all markets but one are in equilibrium, then the last market is also in equilibrium. It also implies that when all markets clear and all economic agents but one are on their budget constraints, then the last economic agent is on his budget constraint. In setting up our problem, we have assumed that the household and the private firms are on their budget constraints. Thus, if we assume that all markets clear, this will imply that the government is satisfying its budget constraint,⁷ which we can express as

$$(13) \quad \sum (q_i - p_i)x_i + \sum p_i z_i = 0 \\ = \sum t_i x_i + \sum p_i z_i$$

Alternatively, if we consider the government budget balance as one of the constraints, then it is only necessary to impose market clearance in $n-1$ of the markets.

In this model we can make two price normalizations, one for each price structure. Since both consumer demand and firm supply are homogeneous of degree zero in their respective prices, changing either price level without altering relative prices leaves the equilibrium unchanged. As normalizations let us assume,

$$(14) \quad p_1 = 1, \quad q_1 = 1, \quad t_1 = 0$$

It may seem surprising that it does not matter whether the government can tax good one. But the reader should remember the budget balance of the consumer. Since there are no lump sum transfers to the

⁷ In an intertemporal interpretation of this model, the government budget is in balance over the horizon of the model, not year by year.

consumer, net expenditures are zero. Thus, levying a tax at a fixed proportional rate on all consumer transactions results in no revenue. (It should be noticed that a positive tax rate applied to a good supplied by the consumer is in effect a subsidy and results in a loss of revenue to the government.)

Welfare Maximization

We can now state the maximization problem. In the statement we shall use the two sets of prices as control variables. It would be a more natural approach to use the taxes which the government actually controls as decision variables. However, once we have determined the optimal p and q vectors we have determined the optimal taxes. Using taxes as decision variables complicates the mathematical formulation and leads to a control problem since the tax vector may not uniquely determine equilibrium.

Rather than calculate the first-order conditions from the formulation spelled out above, we shall alter the problem to simplify the derivation. We have to choose

$$(15) \quad q_2, \dots, q_n, \quad p_2, \dots, p_n, \quad z_1, \dots, z_n$$

to maximize $v(q)$ subject to

$$x_i(q) - y_i - z_i = 0 \quad (i = 1, 2, \dots, n),$$

where y maximizes $\sum p_i y_i$ subject to

$$y_1 = f(y_2, \dots, y_n),$$

and

$$z_1 = g(z_2, \dots, z_n)$$

Since the choice of producer prices can be used to obtain any desired behavior on the part of private producers, we can use any vector y consistent with the production constraint (8). Producer prices are then determined by equation (9). Using the equations

$$y_2 = x_2 - z_2, \dots, y_n = x_n - z_n,$$

we reduce the constraints in (15) to the

single constraint

$$\begin{aligned} x_1(q) &= y_1 + z_1 \\ &= f(x_2 - z_2, \dots, x_n - z_n) + g(z_2, \dots, z_n) \end{aligned}$$

We have therefore simplified the problem (15) to:

$$(16) \quad \text{Choose } q_2, \dots, q_n, \quad z_2, \dots, z_n$$

to maximize $v(q)$ subject to

$$\begin{aligned} x_1(q) - f(x_2(q) - z_2, \dots, x_n(q) - z_n) \\ - g(z_2, \dots, z_n) = 0 \end{aligned}$$

Forming a Lagrangian expression from (16), with multiplier λ ,

$$\begin{aligned} L &= v(q) - \lambda [x_1(q) \\ &\quad - f(x_2 - z_2, \dots, x_n - z_n) \\ &\quad - g(z_2, \dots, z_n)], \end{aligned} \quad (17)$$

we can differentiate with respect to q_k :

$$\begin{aligned} (18) \quad v_k - \lambda \left(\frac{\partial x_1}{\partial q_k} - \sum_{i=2}^n f_i \frac{\partial x_i}{\partial q_k} \right) &= 0 \\ k &= 2, 3, \dots, n \end{aligned}$$

Making use of the equations (9) for producer prices, this can be written

$$\begin{aligned} (19) \quad v_k - \lambda \sum_{i=1}^n p_i \frac{\partial x_i}{\partial q_k} &= 0 \\ k &= 2, 3, \dots, n \end{aligned}$$

Differentiating L with respect to z_k we have

$$(20) \quad \lambda(f_k - g_k) = 0 \quad k = 2, 3, \dots, n$$

Provided that λ is unequal to zero (i.e., that there is a social cost to a marginal need for additional resources), equation (20) implies equal marginal rates of transformation in public and private production and thus aggregate production efficiency as was argued above. The assumption that $\lambda \neq 0$ needs justification. This is provided by the rigorous arguments of Sections III and IV.

If we had introduced *several* public

production sectors, each described by a constraint like (11), we should have obtained an equation of the form (20) for *each* sector. Thus marginal rates of transformation in all public sectors should be equal, since they are all to be equal to the private marginal rates of transformation. This argument—which we only sketch here, since the conclusion will be proved more directly in the next section—justifies our assumption that there should be production efficiency in the public sector.

Optimal Tax Structure

The relations (19) determine the optimal tax structure, since they show how producer and consumer prices should be related. These equations show that consumer prices should be at a level such that further increases in any price result in an increase in social welfare, v_k , which is the same ratio, λ , to the cost of satisfying the change in demand arising from the price increase. Reintroducing taxes explicitly into the problem we can obtain an alternative interpretation for the first-order conditions.

Since x_i is a function of $p + t$,

$$\frac{\partial x_i}{\partial q_k} = \frac{\partial x_i}{\partial t_k}$$

(p is held constant in this latter derivative.) Consequently, the optimal tax structure, (19), can be rewritten:

$$(21) \quad v_k = \lambda \sum p_i \frac{\partial x_i}{\partial t_k} = \lambda \frac{\partial}{\partial t_k} \sum p_i x_i$$

Since $\sum p_i x_i = \sum q_i x_i - \sum t_i x_i = - \sum t_i x_i$ (by the consumer's budget constraint (6)), we have

$$(22) \quad v_k = - \lambda \frac{\partial}{\partial t_k} (\sum t_i x_i)$$

This last set of equations asserts the

proportionality of the marginal utility of a change in the price of a commodity to the change in tax revenue resulting from a change in the corresponding tax rate, calculated at constant producer prices. Like the first-order conditions for the optimum in standard welfare economics, our first-order conditions are expressions in constant prices. The tax administrator, like the production planner, need not be concerned with the response of prices to government action when looking at the first-order conditions.

If we now make the further assumption that the welfare function is individualistic, we can use equation (5) to replace v_k . The first-order conditions then become

$$(23) \quad x_k = \frac{\lambda}{\alpha} \frac{\partial (\sum t_i x_i)}{\partial t_k}$$

Thus for all commodities the ratio of marginal tax revenue from an increase in the tax on that commodity to the quantity of the commodity is a constant. This form of the first-order conditions has the advantage of showing the information needed to test whether a tax structure is optimal. The amount of information does not seem excessive relative to the data and knowledge which a planner in an advanced country should have.

The statements of the first-order conditions thus far do not directly indicate the size of the tax rates required, nor the impact upon demand that the optimal tax rates would have. In his pioneering study of optimal tax structure, Frank Ramsey manipulated the first-order conditions so as to shed light on the latter question. He employed the concept of demand curves calculated at a constant marginal utility of income. Paul Samuelson reformulated this using the more familiar demand curves calculated at a constant level of utility. We shall return to this question in Diamond-Mirrlees II.

III. Production Efficiency in the Many-Consumer Economy

We have remarked already that many of the results carry over directly to an economy of many consumers, even when lump sum taxation is excluded. We notice at once that the device of expressing welfare as a function of the prices, q , faced by consumers can be used perfectly well. Explicitly, we assume that there are H households, with utility and demand functions u^h and x^h ($h=1, 2, \dots, H$). If, as we may generally suppose, in the absence of externalities from producers to consumers, social welfare can be expressed as a function of the consumption of the various consumers in the economy, $U(x^1, x^2, \dots, x^H)$, it may also be written

$$(24) \quad V(q) = U(x^1(q), x^2(q), \dots, x^H(q)),$$

where we assume that there are no lump sum incomes or transfers that would be influenced by producer prices or government policy. In the case where social welfare depends only on individual utility and there are no externalities, we can write

$$(25) \quad V(q) = W[u^1(x^1(q)), u^2(x^2(q)), \dots, u^H(x^H(q))],$$

where W is presumed to be strictly increasing in each of its arguments.

Using this indirect welfare function, we can carry out the analysis already presented for the one-consumer economy, and conclude in the same way that aggregate production efficiency is desirable. For that argument to be correct, we must confirm that the Lagrange multiplier λ is not zero. Rather than attempt to do this directly, we shall present a different argument for the desirability of production efficiency. A further condition will be required to secure our conclusion. In considering this problem, we shall concentrate on the case where all production is under government con-

trol. The desirability of production efficiency in this case will be seen to imply the same conclusion when there is also a private sector (provided that private producers are price takers, and profits, if any, are transferred to the government). Assume then (as we did in Section I) that all production takes place in the public sector: our problem is to find q that will

$$(26) \quad \text{Maximize } V(q),$$

$$\text{subject to } G(X(q)) \leq 0,$$

where we define $X(q) = \sum_h x^h(q)$ as aggregate demand at prices q . We shall also express the production constraint a little more generally by saying that $X(q)$ is to belong to the production set G , the set of technologically feasible production plans. (Thus the letter G denotes both the production set, and also the function that can be used to describe it; but we shall hardly ever use the *function* G explicitly).

Suppose we establish that, at the optimum for problem (26), production is efficient. Consider an economy with the same technological possibilities, partly under the control of private, competitive producers. The government can induce private firms to produce any efficient net output bundle by suitable choice of producer prices p . In particular, it can obtain the production plan that would be optimal if the government controlled all production. The choice of p does not affect consumer demands or welfare, since pure profit arising from decreasing returns to scale go to the government, and since, any commodity taxes being possible, q can be chosen independently of p . Thus, if the solution to (26) is efficient, the same equilibrium can be achieved when some production is under private control, and is optimal in that case too. Proof that production efficiency is desirable in the "special" case (26) therefore implies that pro-

duction efficiency is desirable in the more general case.

Examples of Inefficiency

Before considering the argument for efficiency, it is useful to consider some limitations on that argument as demonstrated by the following examples of desired inefficiency. It will be recollected that a production plan is efficient if any other feasible production plan provides a smaller net supply of at least one commodity. We shall use a different concept: we say that a production plan is *weakly efficient* if it is on the production frontier. It is possible for a production plan to be weakly efficient without being efficient if the production frontier has vertical or horizontal portions. For matters of economic importance, such as the existence of shadow prices, weak efficiency is all that is required. It is easy to see that if all the prices corresponding to a weakly efficient production plan are positive, the plan is in fact efficient in the usual sense.

Even with this slightly weakened concept of efficiency, it is not necessarily true that, when an optimum exists, optimal production has to be weakly efficient. We present two examples.

Example a is portrayed in Figure 7. It is a one-consumer economy where social preferences, as depicted in the social indifference curve II , do not coincide with individual preferences. It is evident that, in the case shown, the optimal production plan is actually in the interior of the production set.

In the second example, social preferences do respect household preferences, but again optimal production lies in the interior of the production set, and is therefore not weakly efficient: suitable producer prices cannot be found, and the social optimum cannot be obtained when there is private control of production.

Example b. There are two commodities and two households. One has utility function x^2y , the other has utility function xy^2 ; each has the nonnegative quadrant $\{(x, y) | x \geq 0, y \geq 0\}$ as consumption set. The first consumer has three units of the first commodity initially; the second, one unit of the second commodity. The welfare function is

$$-\frac{1}{x_1 y_1} - \frac{1}{x_2 y_2}$$

The second commodity can be transformed into the first according to the production relation $x + 10y \leq 0$, ($x \geq 0$). Let the prices of the commodities be q_1, q_2 . Then the first household's net demands are

— 1 of the first commodity,
 q_1/q_2 of the second commodity.

The second household has net demands

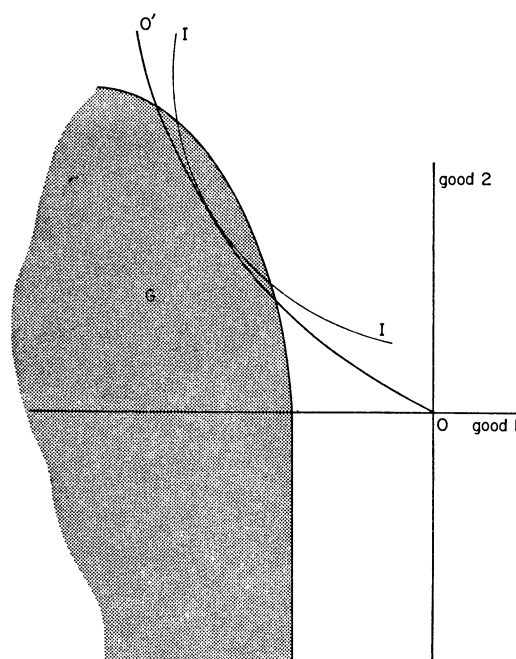


FIGURE 7

$$\frac{1}{3}(q_2/q_1) \quad \text{and} \quad -\frac{1}{3}$$

Thus, the net market demand for the commodities is

$$x = \frac{1}{3}(q_2/q_1) - 1 \quad \text{and} \quad y = (q_1/q_2) - \frac{1}{3}$$

These must satisfy

$$x + 10y \geq 0, \quad x \geq 0$$

Welfare is $-q_2/4q_1 - 27q_1/4q_2$ which is maximized when $q_2/q_1 = 3\sqrt{3}$: the corresponding production vector $\sqrt{3}-1, \frac{1}{3}(\sqrt{3}-1)$ is actually interior to the production set, not on the frontier. This example has the unimportant peculiarity that initial endowments of the consumers are on the frontiers of their consumption sets. More complicated examples avoiding the peculiarity have been constructed.

The Efficiency Argument

Despite these examples, the following argument shows that optimal production will generally be on the production frontier. Suppose that the aggregate demand functions, $X(q)$, are continuous. Then any small change in the prices, q , will not change aggregate production requirements by much. Therefore, if optimal production were in the interior of the production set, small changes in consumer prices would still result in technologically feasible aggregate demands. Thus, if we are at the optimum, small changes in consumer prices cannot increase welfare. If we can argue that, at the optimum, there exists a small price change which would increase $V(q)$, we can conclude that production for the optimum must occur on the production frontier. For any unsatiated single consumer, utility can be increased either by lowering the price of a supplied good or raising the price of a demanded good (as we can see, algebraically, in equation (5)). With a single consumer, we need not argue further, provided the equilibrium involves some trade. When there are many con-

sumers, we can be certain of increasing welfare if we raise some consumer's utility without lowering that of anyone else. If there is a commodity that no consumer purchases, but some consumer supplies (such as certain labour skills); or a good (with positive price) which no consumer supplies, but some consumer purchases (such as electricity), we could alter the price of that commodity in such a way as to bring about an unambiguous increase in welfare. In that case, we conclude that efficient production is required for the maximization of individualistic social welfare. In example *b*, it will be seen that neither of the commodities is supplied, or demanded, by both consumers. The very simplicity of the case appears to be misleading.

A formal presentation of this argument is given in the next section: these technical details can be omitted without loss of continuity. We conclude this section by introducing further taxes into the discussion.

First, consider the case of a poll tax (or subsidy)—that is, a tax is paid by a household on the basis of some unalterable property, such as its sex or age distribution. Such a tax is, of course, a lump sum tax, although its availability is not, in general, sufficient to enable the full optimum to be achieved. To fix ideas, suppose there is a single transfer, τ , to be made to all households. Then welfare can be written $V(q, \tau)$, and we are to

$$(27) \quad \text{Maximize } V(q, \tau)$$

subject to $X(q, \tau)$ being in G

The standard efficiency argument can be used. Let (q^*, τ^*) be the optimum: if any small change in q or τ would increase V , optimal production, $X(q^*, \tau^*)$ must be on the production frontier (assuming that X is a continuous function). Now a poll subsidy must make everyone better off,

unless some are already satiated, and so must a small increase in subsidy. Thus so long as a poll subsidy is possible (and it surely is) and not every household is satiated, optimal production must be on the frontier.

Adding further tax instruments to the government's armoury in no way weakens the efficiency conclusion. We simply note that if there are other tax variables which are independent of producer prices and quantities, denoted collectively by ζ , we can hold them constant at their optimum values ζ^* , and then apply the efficiency argument to the problem (27) or (26), where V and X are evaluated for $\zeta = \zeta^*$.

Our final conclusion is that whatever the class of possible tax systems, if all possible commodity taxes are available to the government, then in general, and certainly if a poll subsidy is possible, optimal production is weakly efficient. We would not expect this conclusion to be valid if there were constraints on the possibilities of commodity taxation, or more generally, on the possible relationship between producer prices and consumer demand. The presence of pure profits is one example of such a relationship. To show what goes wrong, suppose, by way of another example, that *no* commodity taxes are possible, but a poll tax is possible, and that part of production is privately controlled, in such a way that it is uniquely determined by producer prices. Then we have to choose a public production vector z and a poll tax τ to

(28) Maximize $V(p, \tau)$

subject to $X(p, \tau) - y(p) = z$ being in G ,

where $y(p)$ is the private production vector when prices are p . Following the argument used above, we consider τ smaller than τ^* , the optimum level, and note that $V(p^*, \tau) > V(p^*, \tau^*)$. This implies that $X(p^*, \tau) - y(p^*)$ is not in G , and therefore z^* , the optimal z , is efficient in G . But the

argument does not imply that the aggregate optimal production plan, $y(p^*) + z^*$ is efficient. Of course, in an economy where all production is under public control, these problems do not arise. Even when some of the q_k are fixed, the efficiency argument holds, for there can be no necessary relation between q and p .

IV. Theorems on Optimal Production

In this section, we explore the existence of the optimum, and the efficiency of optimal production, rigorously. We rely on Debreu (1959) for the results of general equilibrium theory that are required.

Assumptions

There are H households in the economy, each household choosing a preferred net consumption vector x from his consumption set C subject to the budget constraint $q \cdot x \leq 0$ where q is the vector of prices charged to consumers. (Consumption is measured net of initial endowment for convenience, since the latter is unaltered in the analysis.) As usual the net demand vector x has, in general, both positive and negative components corresponding to purchases and sales by the household.

The assumptions used below will be selected from the following list (the superscript h refers to the index of households; all assumptions, when made, hold for all h):

- (a.1) C^h is closed, convex, bounded below by a vector a^h , and contains a vector with every component negative.
- (a.2) The preference ordering is continuous.
- (a.3) The preference ordering is strongly convex. Formally, if x^2 is preferred or indifferent to x^1 and $0 < t < 1$, then $tx^2 + (1-t)x^1$ is strictly preferred to x^1 .
- (a.4) There is no satiation consumption in C^h .

Assumptions (a.1) and (a.2) guarantee the existence of continuous utility functions, which we shall write u^h (see Debreu Section 4.6). Furthermore, under (a.1)–(a.3), when the demand vector $x^h(q)$ is defined, it is uniquely defined. When C^h is bounded, assumptions (a.1)–(a.3) imply that $x^h(q)$ is defined and continuous at all non-zero nonnegative q . (See Debreu, Section 4.10.)

Let us denote aggregate demand by $X(q) = \sum_h x^h(q)$.

It is assumed that all production is controlled by the government. The assumptions on the production possibility set, G , will be taken from the following set:

- (b.1) Every production plan in which nothing is produced in a positive quantity is possible: i.e., if $z \leq 0$, z is in G .
- (b.2) Complete inactivity is possible: i.e., 0 is in G .
- (b.3) G is closed.
- (b.4) There exists a vector \bar{a} such that $z \leq \bar{a}$ for all nonnegative z in the convex closure of G . (i.e., the closure of the convex hull of G).⁸
- (b.5) G is convex.

The welfare function will be denoted by $U(x^1, \dots, x^H)$. When demands are functions of prices only we can define the indirect welfare function as

$$V(q) = U(x^1(q), \dots, x^H(q))$$

Similarly we can define an individual's indirect utility function by

$$v^h(q) = u^h(x^h(q))$$

We shall say that the welfare function *respects household preferences* when U can be written

$$U(x^1, \dots, x^H) = W(u^1(x^1), \dots, u^H(x^H))$$

⁸ When G is convex, this assumption is similar to the assumption that inputs are required to obtain outputs, but permits the government to own a vector of inputs.

with W increasing in each argument. We shall assume

- (c.1) U is a continuous function of (x^1, \dots, x^H)

We can now state our problem as trying to find q^* to maximize $V(q)$ subject to $X(q)$ being in G . A commodity vector will be called *attainable* if it is feasible and if there exists prices such that aggregate demand equals the vector. The set of all such vectors, the *attainable set*, is the intersection of G with the set of vectors $X(q)$ for all nonnegative q .

Existence of an Optimum

If we assume that the attainable set is nonempty and bounded, we obtain

THEOREM 1. *If assumptions (a.1)–(a.3), (b.3), and (c.1) hold, and if the attainable set is nonempty and bounded, an optimum exists.*

PROOF:

Consider an economy in which the consumption sets are truncated by removing from them all points x with $\|x\| > M$, where all vectors in the attainable set satisfy $\|x\| < M$. For this truncated economy, the demand functions are continuous at all price vectors not equal to zero. Since the attainable set, and demands for any q corresponding to an attainable vector, are the same in the original and truncated economies, an optimum for the truncated economy is an optimum for the original economy. In other words, we may, without loss of generality, assume that demands are continuous at $q \neq 0$. Since the demand functions are homogeneous of degree zero in the prices, we can restrict our attention to q satisfying $q \geq 0$ and $\sum_i q_i = 1$.

We next demonstrate that the set $\{q | X(q) \text{ in } G\}$ is closed. Let q_n be a sequence of price vectors converging to q' , with $X(q_n)$ in G for all n . Let x' be a limit point of $\{X(q_n)\}$. Since G is closed, x' is

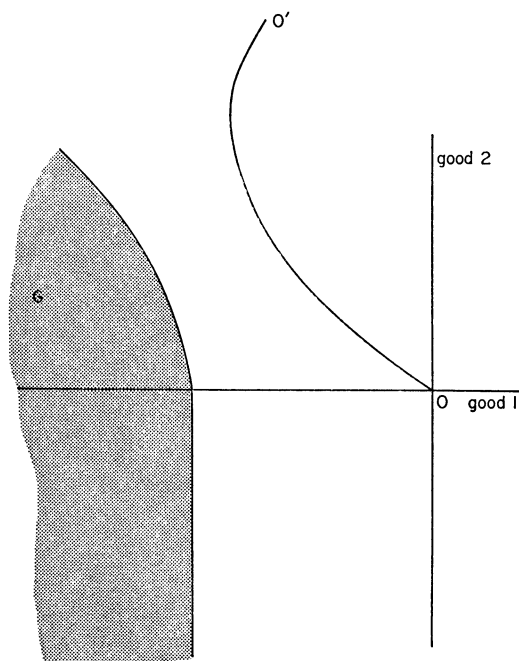


FIGURE 8

in G . At the same time, $x' = X(q')$, by the continuity of X . Thus q' is in $\{q | X(q) \in G\}$, which is therefore closed.

Since the attainable set is nonempty, and prices are in any case bounded, $\{q | X(q) \in G\}$ is closed, bounded, and nonempty. By the continuity of the demand functions, and assumption (c.1), V is a continuous function of q , which therefore attains its maximum on the set $\{q | X(q) \in G\}$.

One criterion for the attainable set to be nonempty follows immediately from the existence of competitive equilibrium in an exchange economy:

THEOREM 2. *If assumptions (a.1)–(a.4) and (b.1) hold, the attainable set is nonempty.*

PROOF:

See Debreu (Section 5.7) for a proof that there exists an equilibrium for the exchange economy with these consumers.

The equilibrium prices result in a feasible demand.

If the production set is taken to be the set of possible production vectors net of government consumption, the assumption that zero production is possible is excessively strong, especially for governments with large military establishments. But it is easy to construct examples of economies not satisfying (b.1) in which there is no attainable point. Consider the one-consumer economy depicted in example c shown in Figure 8.

The boundedness of the attainable set would be implied by the boundedness of the consumption sets, or the boundedness of production, but the following case is more appealing:

THEOREM 3. *If assumptions (a.1) and (b.2)–(b.4) hold, then the attainable set is bounded.⁹*

PROOF:

Suppose the attainable set is not bounded. Then there exists a sequence of attainable vectors x_n such that $\|x_n\|$ is an unbounded increasing sequence of real numbers. There exists an n' such that $\|x_{n'}\| > \|\bar{a}\|$, where \bar{a} is the vector employed in (b.4). Consider the sequence of vectors $(\|x_{n'}\|/\|x_n\|)x_n$ for $n \geq n'$. Each vector is in the convex hull of G (being a convex combination of the origin and x_n). Further the sequence is bounded. Thus there is a limit point, ξ , which is in the convex closure of G and satisfies $\|\xi\| > \|\bar{a}\|$. Let $b = \sum_h a_h$, where a_h are the vectors employed in (a.1). Then $x_n = \sum_h x_n^h \geq \sum_h a_h = b$. Further $(\|x_{n'}\|/\|x_n\|)x_n \geq (\|x_{n'}\|/\|x_n\|)b$. But the latter sequence of vectors converges to zero. Thus $\xi \geq 0$. This is a contradiction.

⁹ The attainable set will also be bounded if (b.2)–(b.4) hold for the true production set, *gross* of government consumption, rather than the net production set, G . Thus the assumption that zero production is possible is not of great consequence.

Finally, we should remark that the strong convexity assumption, (a.3), which was made in Theorem 1 can be changed to convexity without affecting the conclusion. All that is required is to replace the continuous functions of the proof by upper semi-continuous correspondences. On the other hand, one can easily construct examples in which an optimum fails to exist because of the absence of continuity.

Efficiency

The following lemma provides two criteria for optimal production to be on the frontier of the production set. It will be used to deduce a theorem about the case where household preferences are respected.

LEMMA 1: *Assume an optimum, q^* , exists. If aggregate demand functions and the indirect welfare function are continuous in the neighborhood of the optimal prices; and if either*

(1) *for some i , V is a strictly increasing function of q_i in the neighborhood of q^* ; or*

(2) *for some i with $q^* > 0$, V is a strictly decreasing function of q_i in the neighborhood of q^* ,*

then $X(q^)$ is on the frontier of G .*

PROOF:

Let l_i be the vector with all zero components except the i th, which is one. In case 1, for ϵ sufficiently small $V(q^* + \epsilon l_i) > V(q^*)$. Hence $X(q^* + \epsilon l_i)$ is not in G . Letting ϵ decrease to zero, the continuity of X shows that $X(q^*)$ is a limit of points not in G , and therefore belongs to the boundary of G . In case 2, a similar argument can be made using $V(q^* - \epsilon l_i)$.

These conditions are weak. They are, naturally, independent of production possibilities. It may also be noticed that, when V is a differentiable function of prices, the stated conditions are equivalent to assuming that

(29) It is not the case that $V'(q^*) \leq 0$

Here $V'(q)$ is the vector of first derivatives of V with respect to prices. The equivalence of the conditions of the theorem and (29) is clear if we remember that

$$(30) \quad V'(q) \cdot q = \sum \frac{\partial V}{\partial q_k} q_k = 0,$$

since V is homogeneous of degree zero in q . Therefore $V' \leq 0$ if, and only if, $\partial V / \partial q_k = 0$ when $q_k > 0$ and $\partial V / \partial q_k \leq 0$ in any case.

In the following theorem, we strengthen the assumptions in a different way: they remain notably weak.

THEOREM 4. *If (a.1)–(a.4) and (c.1) hold; if social welfare respects individual preferences; and if either*

(1) *for some i , $x_i^h \leq 0$ for all h , and $x_i^h < 0$ for some h' ; or*

(2) *for some i , with $q_i > 0$, $x_i^h \geq 0$ for all h and $x_i^h > 0$ for some h' ;*

Then if an optimum exists, production for the optimum is on the frontier of the feasible set.

PROOF:

Individual demand functions are continuous in the neighborhood of the optimum and thus aggregate demands and the indirect welfare function are continuous. Since social welfare respects preferences, indirect social welfare can be written as an increasing function of indirect utilities. In case 1, indirect utilities are a nondecreasing function of q_i in the neighborhood of q^* for all h while the indirect utility function of h' is strictly increasing in q_i . Thus V increases with q_i . Case 2 follows similarly.

The assumption of strictly convex preferences made in Theorem 4 is required in the theorem as stated.

Example d: Consider an economy with one consumer whose indifference curves have

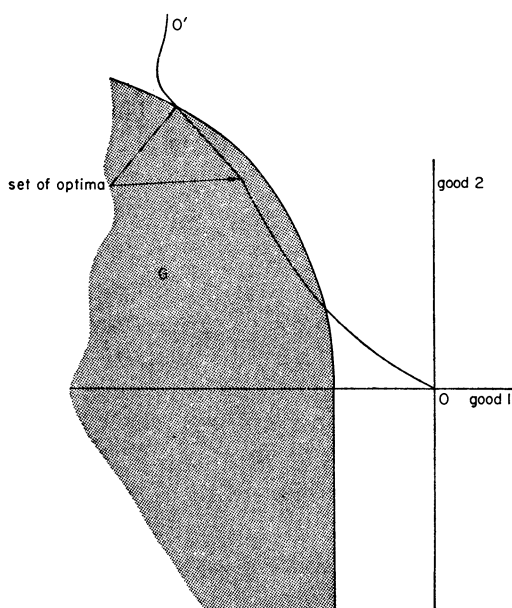


FIGURE 9

a linear section. Then the offer curve may coincide with the linear part of an indifference curve, giving a set of optima, only one of which is on the production frontier. As an illustration, see Figure 9.

The example suggests that we weaken the conclusion of Theorem 4 to say that there exists an optimum on the frontier of G : this generalization is indeed correct if we merely assume convexity of preferences. The proof follows that of Theorem 4, with upper semi-continuity of the demand correspondence replacing continuity of demand functions.

V. Extensions

We can summarize the efficiency result by considering an economy with three sectors—consumers, private producers, public producers. We assumed that only the equilibrium position of the consumer sector enters the welfare function, and that only market transactions take place between sectors, while the government has power to tax any intersector transaction

at any desired rate. One conclusion was that all sectors not containing consumers should be viewed as a single sector, and treated so that aggregate production efficiency is achieved. By regrouping the parts of the economy according to this schematic division, we can extend the efficiency result to several other problems. In each case, we indicate briefly how application of this schematic view shows the relationship of the extension to the basic model.

Intermediate Good Taxation

The model, as presented above, left no scope for intermediate good taxation. If we separate private production possibilities into two (or many) sectors, we introduce the possibility of taxing transactions between firms. In the schematic view presented above, we could consider a consumer sector and two, constant returns to scale, private production sectors. We conclude that we want efficiency for these private production possibilities taken together. Therefore the optimal tax structure includes no intermediate good taxes, since these would prevent efficiency. (Similarly we conclude that government sales to firms should be untaxed while those to consumers are taxed.)

There is a straightforward interpretation of this result, which helps to explain the desirability of production efficiency. In the absence of profits, taxation of intermediate goods must be reflected in changes in final good prices. Therefore, the revenue could have been collected by final good taxation, causing no greater change in final good prices and avoiding production inefficiency. This interpretation highlights the necessity of our assumption of constant returns to scale in privately controlled production.

However, it may well be desirable to tax transactions between consumers or to charge different taxes on producer sales to

different consumers. There are two ways in which we can consider doing this. The country might be geographically partitioned with different consumer prices in different regions. Ignoring migration, and consumers making purchases in neighboring regions, our analysis can be applied to determine taxes region by region. In general the tax structure will vary over the country.

Alternatively, we might consider taxation on all consumer-consumer transactions. Here, too, we would expect to be able to increase social welfare by having these additional tax controls. Neither addition to the available tax structure alters the desirability of production efficiency.

Untaxable Sectors

One problem that arises with a model considering taxation of all transactions is that some transactions may not be taxable, practically or legally. An example of the former might be subsistence agriculture where transactions with consumers are hard to tax while those with firms are not. If the introduction of other taxes (e.g., on land or output) is ruled out, we can accommodate this problem in the model by including subsistence agriculture in the consumer rather than producer sector (or treating it as a second consumer sector). Efficiency would then be desired for the modern and government production sectors taken together; while the tax structure rules would be stated in terms of demand derivatives of the augmented consumer sector rather than of just the true consumers.

Similarly, in an economy without taxes, a public producer subject to a budget constraint is unable to charge different prices to consumers and producers. Lumping together the entire private sector as a single consumer sector, we obtain the conditions for optimal public production of an industry regulated in this manner. This

is the problem considered by Boiteux in the context of costless income redistribution. He also analyzed such an economy with several firms, each limited by a budget constraint.

Foreigners

It is not easy to provide a satisfactory welfare economics for a world of many countries. The study of world welfare maximization is interesting, and, one may hope, "relevant." But it has the serious limitation that its results can seldom be applied to the actions of governments. However altruistic the principles on which a government seeks to act, it has to allow for the actions other governments may take, based on different principles, or for different reasons. (A somewhat analogous problem arises in intertemporal welfare economics.) In the following two subsections, we shall, in order to keep the discussion brief, refer only to the case where the reactions of all other countries are well-defined functions of the actions of the country directly considered. Thus we neglect, reluctantly, those situations that have come to be called "game-theoretic." Also, we shall not consider the problem of formulating a social welfare function in an international setting.

International Trade

So long as we are completely indifferent to the welfare of the rest of the world, and so long as the reactions of other countries are well-defined, international trade simply provides us with additional possibilities for transforming some goods and services into others. The efficiency result then implies that we would want to equate marginal rates of transformation between producing and importing. If there is a monopoly position to be exploited, it should be. If international prices are unaffected by this country's demand, intermediate goods should not be subject to a tariff, but final

good sales direct to consumers should be subject to a tariff equal to the tax on the same sale by a domestic producer.

Sometimes it is not possible to sell goods to foreigners at prices different from those at which they are sold to domestic consumers, although the theory just outlined suggests that foreigners should be treated like producers. As examples, we may cite tourists and commodities covered by special kinds of international agreement. If tourism, say, is an important trading opportunity for the country, and tourists have to be charged the same prices as domestic consumers, this will affect the optimal level of taxes on certain commodities. The general efficiency result is not upset, however. The analysis can be performed by treating tourists as consumers whose income does not affect social welfare.

The authors do not, of course, recommend indifference to the welfare of the rest of the world; although it happens to make the results somewhat neater. International trade provides the country with another set of consumers who can trade with it at prices different from its own consumers: the case (when foreign reactions are well-defined) is similar to the possibility of using different consumer prices in different regions of the same economy. In that case, there is no reason why optimal international trade prices should be the same as producer prices, p , or domestic consumer prices, q .

Migration

In all that has gone before, we have been holding constant the set of consumers in the economy. We can introduce migration in a straightforward manner. Social welfare may be a function of the consumption of every household in the world. Changes in the consumer prices charged in the home country cause migration in one direction or another, and therefore affect wel-

fare in ways we have not previously discussed (such as the effect on the inhabitants of another country of having additional taxpayers join them). But we can still define an indirect welfare function $V(q)$, so long as the reactions of the rest of the world are well-defined. Similarly we can define aggregate demand functions $X(q)$, but these are no longer continuous. For, when a man decides to emigrate, his contribution to aggregate demand changes from x^h to 0.¹⁰ But the number of migrants arising from a small price change may, quite reasonably, be assumed small relative to the population as a whole. We can therefore adequately approximate this situation by considering a continuum of consumers. In this way we can restore continuity to aggregate demand, and to the indirect welfare function. It is to be expected, then, that production efficiency is still desired. Since the derivatives of the demand functions, and possibly also the derivatives of V , will be different when the possibility of migration is allowed for, the optimal tax structure will be changed to reflect the loss of tax revenue when net taxpayers, for example, leave the country. While we do not wish to examine this problem in detail here, we believe that these ideas provide an interesting approach to the analysis.

Consumption Externalities

The schematic view of this problem given above suggests that the basic structure of the results, although not the specific optimal taxes, are unchanged by complications which occur wholly within the consumer sector. Thus, if we introduce consumption externalities that leave aggregate demand continuous we will still obtain production efficiency at the optimum, if we can argue that $V(q)$ has no unconstrained local maximum for finite q .

¹⁰ A similar discontinuity problem arises in the case of tourists' decisions not to visit the country.

The conditions used above are no longer sufficient for this argument since the direct effects of a price change might be offset by the change in the pattern of externalities induced by the price change. Although we have not examined this case in detail, there are a number of cases where arguments similar to those in the no-externality case will be valid.¹¹ Furthermore it seems quite likely to us that efficiency will be desired in realistic settings.

Capital Market Imperfections

While some capital market imperfections affecting firms are complicated to deal with, some imperfections relevant only for consumers can be described as elements solely within the consumer sector. For example, consider the constraint that consumers can lend but not borrow. We must then rewrite consumer utility maximization as subject to a set of budget constraints for the different time periods. In the case of two periods, for example, it would appear as

$$\begin{aligned}
 &\text{Maximize } u(x^1, x^2) \\
 &\text{subject to } q^1x^1 + s \leq 0 \\
 &\qquad\qquad q^2x^2 - s \leq 0 \\
 &\qquad\qquad s \geq 0
 \end{aligned}
 \tag{31}$$

where s represents first period savings. From this consumer problem, we still have utility and demand expressible in terms of

prices. We expect that the efficiency result continues to hold. In calculating the optimal formula, though, it becomes necessary to distinguish the time period of the good in question for there are now two Lagrange multipliers giving the marginal utility of income in each of the two periods. For this consumer we have

$$(32) \quad \frac{\partial v}{\partial q_k^1} = -\alpha^1 x_k^1, \quad \frac{\partial v}{\partial q_k^2} = -\alpha^2 x_k^2$$

Since savings are allowed $\alpha^1 \geq \alpha^2$. If the consumer would borrow if he could, $\alpha^1 > \alpha^2$ and the optimal tax structure is altered by this market limitation.

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¹¹ We have benefited from discussions with Elisha Pazner on this subject.