

The Foundations of Welfare Economics

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THE FOUNDATIONS OF WELFARE ECONOMICS

By OSCAR LANGE

1. WELFARE ECONOMICS is concerned with the conditions which determine the total economic welfare of a community. In the traditional theory the total welfare of a community was conceived as the *sum of the welfares (utilities) of all constituent individuals*. The problem of maximization of total welfare thus involved the weighing against each other the losses of utility and gains of utility of different individuals. This *implies interpersonal comparability* of utility, as is seen in the dictum about the marginal utility of a dollar for the poor man and for the rich man. Such implication, however, is open to epistemological criticism on the ground of lack of operational significance. In consequence a restatement of the principles of welfare economics is in progress¹ which tries to dispense with the interpersonal comparability of utility. Such restatement, however, implies a restriction of the field of welfare economics. This paper intends to give a precise statement of the basic assumptions and propositions of welfare economics and to discuss their operational significance.

2. In order to dispense with interpersonal comparability of utility the total welfare of a community has to be defined not as the sum of the utilities of the individuals (a scalar quantity) but as a *vector*. The utilities of the individuals are the components of this vector. Let there be θ individuals in the community and let $u^{(i)}$ be the utility of the i th individual. Total welfare is then the vector

$$(2.1) \quad u = (u^{(1)}, u^{(2)}, \dots, u^{(\theta)}).$$

It is convenient for our purpose to order vectors on the basis of the following definition: a vector is said to be greater than another vector *when at least one of its components is greater than the corresponding component of the other vector, and none is less*.² Thus a vector in-

¹ Some of the recent literature: A. P. Lerner, "The Concept of Monopoly and the Measurement of Monopoly Power," *Review of Economic Studies*, June, 1934; A. Burk, "A Reformulation of Certain Aspects of Welfare Economics," *Quarterly Journal of Economics*, February, 1938; H. Hotelling, "The General Welfare in Relation to Problems of Taxation and of Railway and Utility Rates," *ECONOMETRICA*, July, 1938; L. Robbins, "Interpersonal Comparisons of Utility," *Economic Journal*, December, 1938; N. Kaldor, "Welfare Propositions and Interpersonal Comparison of Utility," *Economic Journal*, September, 1939; J. R. Hicks, "The Foundations of Welfare Economics," *Economic Journal*, December, 1939; T. de Seitovszky, "A Note on Welfare Propositions in Economics," *Review of Economic Studies*, November, 1941.

² The ordering of vectors according to this definition must be distinguished from the ordering of vectors according to their length (defined as usual). When

creases when at least one of its components increases and none decreases. According to the definition adopted, a *maximum* of total welfare occurs when conditions cannot be changed so as to increase the vector u , i.e., when it is impossible to increase the utility of any person without decreasing that of others.³ We have, therefore, $u = \max$ when

$$(2.2) \quad u^{(i)} = \max \quad (i = 1, 2, \dots, \theta)$$

subject to

$$(2.3) \quad u^{(j)} = \text{const} \quad (j = 1, 2, \dots, i-1, i+1, \dots, \theta).$$

3. Let the utility of each individual be a function of the commodities in his possession. Denoting by $x_1^{(i)}, x_2^{(i)}, \dots, x_n^{(i)}$ the quantities of n commodities in the possession of the i th individual, his utility is $u^{(i)} = u^{(i)}(x_1^{(i)}, x_2^{(i)}, \dots, x_n^{(i)})$. Denote further by $X_r = \sum_{i=1}^{\theta} x_r^{(i)}$ the total amount of the r th commodity in the community. These amounts are not constant but subject to technological transformation the possibilities of which are circumscribed by a transformation function $F(X_1, X_2, \dots, X_n) = 0$. Our problem is to maximize total welfare subject to the constraint of the transformation function.

We thus have the following maximum problem:

$$u^{(i)}(x_1^{(i)}, x_2^{(i)}, \dots, x_n^{(i)}) = \max \quad (i = 1, 2, \dots, \theta)$$

subject to the side relations

$$(3.1) \quad u^{(j)}(x_1^{(j)}, x_2^{(j)}, \dots, x_n^{(j)}) = \text{const} \\ (j = 1, 2, \dots, i-1, i+1, \dots, \theta),$$

$$(3.2) \quad X_r = \sum_{i=1}^{\theta} x_r^{(i)} \quad (r = 1, 2, \dots, n),$$

$$(3.3) \quad F(X_1, X_2, \dots, X_n) = 0.$$

This is equivalent to maximizing the expression

$$(3.4) \quad \sum_{i=1}^{\theta} \lambda_i u^{(i)}(x_1^{(i)}, x_2^{(i)}, \dots, x_n^{(i)}) + \sum_{r=1}^n \nu_r \left(\sum_{i=1}^{\theta} x_r^{(i)} - X_r \right) \\ + \nu F(X_1, X_2, \dots, X_n),$$

a vector is greater than another in the above sense then its length is also greater than the length of the other vector, but the reverse does not hold true. According to our definition the vectors form a partially ordered system which does not have the "chain" property: given u and v , either $u \geq v$ or $v \geq u$.

³ In the language of the theory of partially ordered systems a maximum of total welfare is a "maximal" element of the set of admissible vectors u . Cf. Garrett Birkhoff, *Lattice Theory*, American Mathematical Society, Colloquium Publications, Vol. XXV, 1940, p. 8. The set of admissible vectors is given by the conditions (3.2) and (3.3) in the text.

where the λ 's and the ν 's are Lagrange multipliers and $\lambda_i \equiv 1$ successively for $i = 1, 2, \dots, \theta$. The result obtained is the same for each i .

The first-order maximum conditions yield, after elimination of the Lagrange multipliers, the $(n-1)\theta$ equations⁴

$$(3.5) \quad \frac{u_r^{(i)}}{u_s^{(i)}} = \frac{F_r}{F_s} \quad (r \text{ and } s = 1, 2, \dots, n; i = 1, 2, \dots, \theta),$$

which together with the equations (3.1) and (3.3) serve to determine the $n\theta$ quantities $x_r^{(i)}$. The equations (3.5) can also be written in the form

$$(3.6) \quad \frac{\partial x_s^{(i)}}{\partial x_r^{(i)}} = \frac{\partial X_s}{\partial X_r} \quad (r \text{ and } s = 1, 2, \dots, n; i = 1, 2, \dots, \theta).$$

The latter form shows clearly the economic interpretation and the operational significance of our maximum conditions. The left-hand side of (3.6) is the marginal rate of substitution of two commodities (the amounts of the remaining commodities being kept constant) which leaves the individual's utility unaffected. The right-hand side is the marginal rate of technological transformation of the two commodities. Thus each individual's marginal rate of substitution of any two commodities must be equal to the marginal rate of transformation of these commodities. Both rates can be determined empirically, the second from the technological conditions of transformation, the first by offering each individual choices between different "bundles" of commodities and adjusting the "bundles" so as to make his choice indifferent.

The derivation of (3.5) or (3.6) does not imply interpersonal comparability of utility. This can be seen also in the following way. From (3.5) we have

$$(3.7) \quad \frac{u_r^{(i)}}{u_r^{(j)}} = \frac{u_s^{(i)}}{u_s^{(j)}} \quad (r \text{ and } s = 1, 2, \dots, n; i \text{ and } j = 1, 2, \dots, \theta; j \neq i).$$

Each side is the ratio of the marginal utilities of different individuals. The numerical value of these ratios is *indeterminate*.

This treatment of the maximum total welfare problem does not imply the measurability of the individuals' utility either. The equations

⁴ The subscripts stand for partial derivatives. Thus, e.g.,

$$u_r^{(i)} = \frac{\partial u^{(i)}}{\partial x_r^{(i)}} \quad \text{and} \quad F_r = \frac{\partial F}{\partial X_r}.$$

(3.5)–(3.7) are invariant with regard to any positive transformation $\phi^{(i)}(u^{(i)})$ (where $\phi^{(i)} > 0$)⁵ of the utility functions of the individuals. Only the *projective* properties of these functions are used. This implies only ordering, not measurement, of each individual's utility.

The equations (3.5) or (3.6) contain *in nuce* most theorems of welfare economics,⁶ e.g., all the propositions in Pigou's *Economics of Welfare*. The only theorems not contained in these equations are those which relate to the optimum distribution of incomes. This limitation and the problem how it can be overcome in a way which is operationally significant will be the subject of the remaining part of this paper.

4. The solution given by (3.5) or (3.6) contains arbitrary parameters, namely the constants of the right-hand side of (3.1). These parameters express the level at which the utilities of all the other individuals are held constant while the utility of the i th individual is being maximized. Thus our solution is *relative* to the values chosen for these parameters. It gives, for instance, the conditions under which the poor man's utility cannot be increased any more without diminishing the rich man's utility (or vice versa), but the level at which the rich man's utility is held constant is arbitrary. Obviously, the poor man's utility corresponding to a situation of maximum total welfare will be different when the level of the rich man's utility is chosen differently.

In an exchange economy the constants on the right-hand side of (3.1) are uniquely related to the money incomes of the respective individuals. This follows from the maximization of the individuals' utility. Let $u^{(i)}(x_1^{(i)}, x_2^{(i)}, \dots, x_n^{(i)}) = \max$ subject to $\sum_{r=1}^n p_r x_r^{(i)} = M^{(i)}$ where $M^{(i)}$ is the individual's income and the p 's are the prices of the commodities. The value of $u_{\max}^{(i)}$ depends on $M^{(i)}$ and on the p 's as parameters. The p 's can be determined from equations which express the equality of demand and supply of each commodity, but $M^{(i)}$ remains arbitrary.⁷ Thus the problem of determining the constants on the right-hand side of (3.1) reduces, in an exchange economy, to that of determining the distribution of incomes. The conditions of maximum total welfare expressed in (3.5) or (3.6) leave this distribution arbitrary.

5. In order to arrive at the optimum determination of the constants on the right-hand side of (3.1) it does not suffice to maximize the vector u . We must be able to choose between different vectors u which cannot

⁵ In fact, they are invariant with respect to any transformation such that $\phi^{(i)'} \neq 0$. But the second-order maximum conditions admit only positive transformations. Negative transformations would change the maximum into a minimum.

⁶ For a somewhat fuller treatment of this point see the Appendix.

⁷ The $M^{(i)}$ must, however, satisfy the relation $\sum_{i=1}^{\theta} M^{(i)} = \sum_{r=1}^{\theta} p_r X_r$, which follows from (3.2) and from the budget equations $\sum_{r=1}^n p_r x_r^{(i)} = M^{(i)}$.

be ordered in the way defined above.⁸ This can be done in two ways. One is to weigh against each other the gains of utility and the losses of utility of different individuals. This need not, however, imply the acceptance of the traditional definition of total welfare as the sum of the utilities of the individuals. The weighting can be based, instead, upon a *social valuation* of the importance of the individuals, the subject exercising the valuation being an agency of the organized community (e.g., Congress).⁹ The other way is to establish *directly* a social valuation of the distribution of commodities or incomes between the individuals, without reference to the individuals' utilities. In the first case the optimum distribution of incomes (and of commodities) is determined by a social valuation of the individuals' utilities. In the second case the utilities of the individuals appear as a more or less accidental by-product of the direct social valuation of the distribution of incomes (or of commodities).

In both cases the social valuation can be expressed in the form of a *scalar function of the vector* u , i.e., $W(u)$, except that in one case the community (or rather its agency) chooses the most preferred vector u and adjusts the distribution of incomes and of commodities among the individuals so as to obtain the desired vector, while in the other case it chooses the most preferred distribution of incomes (or commodities) directly and the vector u adjusts itself to this choice. We shall call the function W the *social value function*.

It is convenient to give names to the different derivatives of this function. We shall call them *marginal social significances*. Let $W_i = \partial W / \partial u^{(i)}$ and call it the marginal social significance of the i th individual. As $u^{(i)} = u^{(i)}(x_1^{(i)}, x_2^{(i)}, \dots, x_n^{(i)})$, we can form the derivative $\partial W / \partial x_r^{(i)}$. It will be called the marginal social significance of the r th commodity in the hands of the i th individual. In the preceding section it was shown that in an exchange economy a unique relation exists between $u^{(i)}$ and the individual's money income $M^{(i)}$. Hence we can form $\partial W / \partial M^{(i)}$ which will be called the marginal social significance of the i th individual's income.

Between these derivatives there are the relations

$$(5.1) \quad \frac{\partial W}{\partial x_r^{(i)}} = W_i u_r^{(i)},$$

$$(5.2) \quad \frac{\partial W}{\partial M^{(i)}} = W_i \mu_i \quad \text{where} \quad \mu_i = \frac{\partial u^{(i)}}{\partial M^{(i)}};$$

⁸ I.e., we need now the "chain" property mentioned in footnote 2 above.

⁹ In a democratically organized community these agencies will have to reflect the valuations of the majority.

μ_i is called the marginal utility of income.¹⁰ We have also

$$\frac{\partial W}{\partial x_r^{(i)}} = \frac{\partial W}{\partial M^{(i)}} \frac{\partial M^{(i)}}{\partial x_r^{(i)}}.$$

But $M^{(i)} = \sum_{r=1}^n p_r x_r^{(i)}$ (*vide* Section 4) and $\partial M^{(i)} / \partial x_r^{(i)} = p_r$. Consequently, we have the relation

$$(5.3) \quad \frac{\partial W}{\partial x_r^{(i)}} = \frac{\partial W}{\partial M^{(i)}} p_r.$$

Our problem is now to maximize W subject to the side relations (3.2) and (3.3). This leads to the maximizing of the following expression

$$(5.4) \quad W(u^{(1)}, u^{(2)}, \dots, u^{(\theta)}) + \sum_{r=1}^n \nu_r \left(\sum_{i=1}^{\theta} x_r^{(i)} - X_r \right) + \nu F(X_1, X_2, \dots, X_n)$$

where the ν 's are Lagrange multipliers.

Eliminating the Lagrange multipliers, we obtain the first-order maximum conditions

$$(5.5) \quad \frac{\partial W}{\partial x_r^{(i)}} \div \frac{\partial W}{\partial x_s^{(j)}} = \frac{F_r}{F_s}$$

(r and $s = 1, 2, \dots, n; i$ and $j = 1, 2, \dots, \theta$).

For $j=i$ and $s \neq r$ these equations become, taking account of (5.1),

$$(5.6) \quad \frac{u_r^{(i)}}{u_s^{(i)}} = \frac{F_r}{F_s};$$

¹⁰ μ_i is also the Lagrange multiplier used in maximizing $u^{(i)}$ subject to $M^{(i)} = \text{const.}$ The first-order maximum conditions are in this case (omitting the superscript i in order to simplify the notation) $u_r = \mu p_r$ ($r = 1, 2, \dots, n$). Write $\partial u / \partial M = \sum_{r=1}^n u_r \partial x_r / \partial M$. It can be shown (cf. J. R. Hicks, *Value and Capital*, Oxford University Press, 1939, p. 308) that

$$\frac{\partial x_r}{\partial M} = \frac{\mu U_r}{U},$$

where

$$U = \begin{vmatrix} 0 & u_1 & \dots & u_n \\ u_1 & u_{11} & \dots & u_{1n} \\ \dots & \dots & \dots & \dots \\ u_n & u_{n1} & \dots & u_{nn} \end{vmatrix}$$

and U_r is the cofactor of the element u_r in the first row. Thus we get

$$\frac{\partial u}{\partial M} = \mu \sum_{r=1}^n \frac{u_r U_r}{U} = \mu.$$

for $j \neq i$ and $s = r$ they turn into

$$(5.7) \quad \frac{\partial W}{\partial x_r^{(i)}} = \frac{\partial W}{\partial x_r^{(j)}}.$$

The conditions (5.6) are identical with (3.5) and have the same economic interpretation. Their operational significance has already been established. The equations (5.7) state that each commodity must have the same marginal social significance in the hands of each individual. The operational significance of this condition requires further inquiry.

6. In virtue of (5.1)–(5.3) the equation (5.7) can be written in the following alternative forms

$$(6.1) \quad \frac{\partial W}{\partial M^{(i)}} = \frac{\partial W}{\partial M^{(j)}},$$

$$(6.2) \quad W_i u_r^{(i)} = W_j u_r^{(j)},$$

$$(6.3) \quad W_i \mu_i = W_j \mu_j.$$

(6.1) states that the marginal social significance of each individual's income must be the same. According to (6.2) the weighted marginal utility of each commodity, and according to (6.3) the weighted marginal utility of income, must be the same for each individual, the marginal social significance of the individual serving as weight.

The operational significance of the maximum conditions obtained depends on which of the two types of social valuation is used. When the communal agency makes its valuation directly in terms of the distribution of commodities or incomes among the individuals, the equations (5.7) and (6.1) can be used. They have, in this case, an immediate operational significance. The communal agency need not bother about the individuals' utilities and it considers W as a direct function of the x 's or of the M 's, i.e., as being in the form $W(x_1^{(1)}, \dots, x_n^{(1)}; \dots; x_1^{(6)}, \dots, x_n^{(6)})$ or $W(M^{(1)}, \dots, M^{(6)})$. A direct valuation in terms of the distribution of commodities is in practice a very complicated affair. It requires a separate evaluation of the marginal social significance of each commodity in the hands of each individual. Therefore, it is rarely fully practiced, except in times of emergency, e.g., during war, when practice comes pretty close to it. A direct valuation in terms of the distribution of incomes does not present the same technical obstacles. It requires only an evaluation of the marginal social significance of each individual's income. This can be done by means of one or a few simple principles and is actually practiced, for instance, in framing income-tax legislation.

When the social valuation is made in terms of weighting the individuals' utilities the equations (6.2) and (6.3) have to be used. This required a knowledge of the marginal utilities of the different individuals. There exists no operational procedure by which such a knowledge can be gained. To that extent (6.2) and (6.3) lack operational significance. This, however, does not make them completely meaningless. It is possible to form certain a priori hypotheses about the relationships between individuals' marginal utilities and to investigate what consequences in terms of the distribution of incomes or of commodities follow. Thus it is possible to control the valuations made directly in terms of incomes or commodities in the light of these hypotheses.

The most interesting of such hypotheses is the hypothesis that the function $\mu_i(M^{(i)})$ which expresses the marginal utility of income is the same for each individual. In this case (6.3) becomes

$$(6.4) \quad W_i \mu(M^{(i)}) = W_j \mu(M^{(j)}) \quad (i \text{ and } j = 1, 2, \dots, \theta),$$

where μ is written without subscript because the function is the same for all individuals. Let us also assume that the community adopts an equalitarian social ideal, i.e., the marginal social significance of each individual is the same. Then $W_i = W_j$ for all i 's and j 's and we obtain from (6.4)

$$(6.5) \quad M^{(i)} = M^{(j)} \quad (i \text{ and } j = 1, 2, \dots, \theta).$$

Each individual has to get the same income.¹¹

In this way it is possible to check up the consistency of the social valuation with the professed ideal of an economic society which, like ours, claims to attach to each individual the same marginal social significance. Upon the hypothesis that the marginal-utility-of-income function is the same for all individuals the inequalities in the distribution of incomes are inconsistent with the equalitarian ideal professed. In a similar way the actual distribution of incomes (or of commodities)

¹¹ This does not imply that each individual's *money earnings* must be the same. Among the goods $x_r^{(i)}$ there are included leisure, safety and attractiveness of the different occupations, social prestige, etc., and prices have to be assigned to them. If an individual prefers, for the reasons indicated, an occupation in which he earns less money than he could earn in some other one, he can be considered as purchasing certain goods associated with the occupation he chooses and as paying a price for them. Thus differences in money earnings which correspond to the individuals' preferences for the various occupations are not in contradiction with the equality of incomes discussed in the text. This takes care of the question of incentives. Cf. on this subject the present writer's essay, *On the Economic Theory of Socialism*, Minneapolis, University of Minnesota Press, 1938, pp. 101-102.

can be checked up with regard to other hypotheses made and with regard to other social valuations of the individuals' utilities.

7. It is seen from (5.5) that the maximum conditions are invariant under a transformation $\phi(W)$ of the social-value function, where $\phi' > 0$.¹² Thus only the projective properties of W are used. Only the ordering, not the measurement, of the social valuations is involved.

The utilities of the individuals need not be measurable either. Let us subject the utility functions of the individuals to the transformation $\phi^{(i)}(u^{(i)})$, where $\phi^{(i)} > 0$ ¹³ and $i=1, 2, \dots, \theta$. We obtain, instead of (6.2),

$$(7.1) \quad \frac{\partial W}{\partial \phi^{(i)}} \phi_r^{(i)} = \frac{\partial W}{\partial \phi^{(i)}} \phi_r^{(i)}.$$

This can be written

$$(7.2) \quad \frac{\partial W}{\partial \phi^{(i)}} \phi^{(i)'} u_r^{(i)} = \frac{\partial W}{\partial \phi^{(i)}} \phi^{(i)'} u_r^{(i)},$$

whence

$$(7.3) \quad \frac{\partial W}{\partial u^{(i)}} u_r^{(i)} = \frac{\partial W}{\partial u^{(i)}} u_r^{(i)},$$

which is identical with (6.2). In a similar way it can be shown that (6.3) is invariant under the transformation $\phi^{(i)}$.

8. Let us restate our conclusions. The propositions of welfare economics can be divided into two parts. One part is based on maximizing the vector u and is concerned with conditions which permit increasing the utility of one individual without diminishing the utility of anybody else. It comprises all propositions of welfare economics except those which relate to the optimum distribution of incomes. These propositions are all operationally significant. The other part requires the setting up of a social value function $W(u)$ which is maximized. The maximum conditions thus obtained may be expressed either directly in terms of the commodities and incomes allowed to different individuals or in terms of the marginal utilities of the individuals. In the first case propositions of immediate operational significance are obtained but each individual's utility is determined quasi-accidentally as a by-product of the valuations made in terms of commodities or incomes. In the other case the optimum distribution of incomes must be derived from certain a priori hypotheses concerning the functions expressing the marginal utility of incomes of the different individuals.

¹² Cf. footnote 5 above.

¹³ Cf. footnote 5 above.

Although these hypotheses have no direct operational significance they lead to definite conclusions as to the appropriate distribution of incomes. They may, therefore, be used as check-ups of a distribution of incomes established by direct valuation.

Neither the social valuations nor the utilities of the individuals need be measurable; it is sufficient that they can be ordered.

APPENDIX

In order to simplify the exposition the transformation function introduced at the beginning of Section 3 is assumed to refer to the whole economy. This is a strong oversimplification of reality admissible only under special circumstances. Actually the technological transformation of commodities is performed by individuals ("firms"; even in a socialist society there would be separate productive establishments) and each individual is confronted with a transformation function of his own. Only when the transformation functions of the individuals are all the same can they be combined in a unique way into a transformation function for the economy as a whole. Otherwise the conditions of transformation in the economy as a whole depend on how the transformation of commodities is distributed among the individuals (i.e., the relation between total "outputs" and total "inputs" depends on how much "output" and "input" is done by each individual). Thus in order to give a better picture of an actual economic system we must assume each individual to be confronted with a separate transformation function.

Denote by $f^{(i)}(y_1^{(i)}, y_2^{(i)}, \dots, y_n^{(i)}) = 0$ the transformation function of the i th individual, where $y_r^{(i)}$ is the quantity of the r th commodity he transforms. Denote, as before, by $x_r^{(i)}$ the quantity of the r th commodity which the i th individual possesses. The amount of a commodity which an individual possesses need not be equal to the amount he obtains or gives up through transformation, for he may acquire commodities or get rid of them by means other than technological transformation (e.g., by exchange or gift). But for the economy as a whole these amounts are equal. We have, therefore, $\sum_{i=1}^{\theta} x_r^{(i)} = \sum_{i=1}^{\theta} y_r^{(i)}$ for $r = 1, 2, \dots, n$.

In place of the maximum problem in Section 3 we now have

$$u^{(i)}(x_1^{(i)}, x_2^{(i)}, \dots, x_n^{(i)}) = \max \quad (i = 1, 2, \dots, \theta),$$

subject to the side relations

$$\begin{aligned} (1) \quad & u^{(j)}(x_1^{(j)}, x_2^{(j)}, \dots, x_n^{(j)}) = \text{const} \\ & (j = 1, 2, \dots, i-1, i+1, \dots, \theta), \\ (2) \quad & f^{(i)}(y_1^{(i)}, y_2^{(i)}, \dots, y_n^{(i)}) = 0 \quad (i = 1, 2, \dots, \theta), \end{aligned}$$

$$(3) \quad \sum_{i=1}^{\theta} x_r^{(i)} = \sum_{i=1}^{\theta} y_r^{(i)} \quad (r = 1, 2, \dots, n).$$

This leads to the expression

$$(4) \quad \sum_{i=1}^{\theta} \lambda_i u^{(i)} + \sum_{i=1}^{\theta} \gamma_i f^{(i)} + \sum_{r=1}^{\theta} \nu_r \left(\sum_{i=1}^{\theta} x_r^{(i)} - \sum_{i=1}^{\theta} y_r^{(i)} \right),$$

where the Greek letters stand for Lagrange multipliers and $\lambda_i \equiv 1$ successively for $i = 1, 2, \dots, \theta$.

Eliminating the Lagrange multipliers, we arrive at the first-order maximum conditions

$$(5) \quad \frac{u_r^{(i)}}{u_s^{(i)}} = \frac{f_r^{(i)}}{f_s^{(i)}} \quad (r \text{ and } s = 1, 2, \dots, n; i \text{ and } j = 1, 2, \dots, \theta),$$

which take the place of (3.5) in the text.

The propositions usually found in the literature on welfare economics are special cases of the conditions (5). We obtain from (5)

$$(6) \quad \frac{f_r^{(i)}}{f_s^{(i)}} = \frac{f_r^{(j)}}{f_s^{(j)}} \quad (i \neq j).$$

$$(7) \quad \frac{u_r^{(i)}}{u_s^{(i)}} = \frac{u_r^{(j)}}{u_s^{(j)}}.$$

The relation (6) states that the marginal rate of transformation of any two commodities must be the same for each individual (i.e., "firm").¹⁴

¹⁴ The relation (6) can be interpreted as the condition of maximum total physical output. In a similar way as total welfare was defined as the vector u , total physical output can be defined as the vector $X = (X_1, X_2, \dots, X_n)$, where $X_s = \sum_{i=1}^{\theta} x_s^{(i)} = \sum_{i=1}^{\theta} y_s^{(i)}$. We have then the problem

$$X_r = \max \quad (r = 1, 2, \dots, n)$$

subject to the side relations

$$(i) \quad X_s = \text{const} \quad (s = 1, 2, \dots, r-1, r+1, \dots, n),$$

$$(ii) \quad X_s = \sum_{i=1}^{\theta} y_s^{(i)} \quad (s = 1, 2, \dots, n),$$

$$(iii) \quad f^{(i)}(y_1^{(i)}, y_2^{(i)}, \dots, y_n^{(i)}) = 0 \quad (i = 1, 2, \dots, \theta),$$

which leads to the conditions (6). The maximum total output is determined purely by the technological transformation possibilities without any reference to utility. Since the relation (6) is part of any maximum-welfare conditions, whether involving the social-value function W or only the vector u , the maximization of total physical output may be considered as the most narrow type of a concept of maximum total welfare. It is concerned only with the possibility of increasing the output of some commodities without diminishing the output of any other commodity, regardless of who is to get the commodities (cf. Lerner,

If the commodities are both factors this means that the ratio of their marginal productivities (in terms of any given product) must be the same in each firm of the economy. If they are both products the ratio of their marginal factor cost (in terms of any given factor) must be the same in all firms. If one is a factor and the other a product the marginal productivity of the factor in terms of that product must be the same in each firm.¹⁵ These are all theorems well known in welfare economics. The relation (7) indicates the well-known theorem that the marginal rate of substitution of any two commodities must be the same for each individual. With these relations in mind, we see that, according to (5), any individual's marginal rate of substitution of two commodities has to be equal to the ratio of the marginal factor costs of these commodities in any firm of the economy. The last is the most widely known theorem of welfare economics.

It was assumed here that each commodity appears as a variable both in the utility functions and in the transformation functions. This need not be the case, however. It may appear only in the utility functions as, for instance, a "gift of nature" which is not produced. Then the relation (7) still applies to it, but the other relations do not. Or, what is of greater practical importance, it may appear in the transformation functions without appearing in the utility functions, i.e., it is a factor of production which has no direct utility. In this case the relation (6) alone applies to it.

Through proper interpretation the relation (5), or (6) and (7) which are derived from it, can be taken as giving the dynamic conditions of maximum total welfare over a period of time. For this purpose we consider the period over which total welfare is maximized as being divided into a finite number of discrete intervals (e.g., "days" or "weeks"); the first of these intervals constitutes the "present," the other ones are in

op. cit., p. 57). We may thus consider the problem of maximum total welfare in three stages (instead of in two, as in the text): (1) maximizing the vector X , (2) maximizing the vector u , (3) maximizing the scalar function W . The maximum conditions in each stage include the maximum conditions of the preceding one.

¹⁵ This condition implies the absence of unemployment. An unemployed factor can be considered as being employed by an "industry" or "firm" where its marginal productivity is nil. Any shift of the factor to an industry or firm where its marginal productivity is positive increases total physical output (as defined in the preceding footnote). The distinction between two types of propositions of welfare economics, one dealing with the allocation of resources and the other dealing with the degree of utilization of resources, which has been recently proposed by Mr. Scitovszky (*op. cit.*, p. 77), while useful pedagogically, is unnecessary from the analytic point of view. All propositions of welfare economics concerned with the degree of utilization of resources can be treated as allocational propositions.

the future.¹⁶ The same physical good in different time intervals is considered to constitute different commodities. The utility functions $u^{(i)}(x_1^{(i)}, x_2^{(i)}, \dots, x_n^{(i)})$ and the transformation functions $f^{(i)}(y_1^{(i)}, y_2^{(i)}, \dots, y_n^{(i)}) = 0$ are taken as covering the whole period of time over which total welfare is maximized. These functions thus contain among their variables commodities in different future time intervals as well as commodities in the "present." The relations (5)–(7) refer then to intertemporal as well as intratemporal substitution and transformation. Condition (5) states, among other things, that the intertemporal marginal rates of substitution must be equal to the corresponding intertemporal marginal rates of transformation.

Thus the condition (5) implicitly determines the rate of capital accumulation which maximizes total welfare over time. The result is pretty much along the lines of the traditional theory. The intertemporal marginal rate of substitution is the marginal rate of time preference [which, according to (7), for any given commodity must be the same for each individual] and the intertemporal marginal rate of transformation is the marginal productivity of waiting [which, according to (6), for any given commodity must be the same for each firm] of the traditional theory.¹⁷ The two must be equal when total welfare is maximized over time. It should be noticed, however, that though for any given commodity and any given two time intervals these rates are the same for each individual (and firm), they need not be the same for different commodities or different pairs of time intervals. We have a separate rate of time preference and of (equal to the former) marginal productivity of waiting for each commodity¹⁸ and for each pair of time intervals. Nor need the time preference and the marginal productivity of waiting be necessarily positive.¹⁹

¹⁶ Cf., for instance, Hicks, *Value and Capital*, Oxford University Press, 1939, pp. 122–127.

¹⁷ Speaking more precisely, the marginal rate of time preference and the marginal productivity of waiting differ by unity from the marginal rate of intertemporal substitution or transformation, respectively. The marginal rate of time preference is usually defined as $u_r^{(i)}/u_s^{(i)} - 1$. Cf. R. G. D. Allen, *Mathematical Analysis for Economists*, London, Macmillan and Co., 1938, p. 344. Correspondingly, the marginal productivity of waiting may be defined as $f_r^{(i)}/f_s^{(i)} - 1$. The subscripts r and s refer here to different time intervals.

¹⁸ Using the terminology of Mr. Keynes, *The General Theory of Employment, Money and Interest*, New York: Harcourt Brace Co., 1937, p. 223, we obtain a system of optimum (from the social point of view) "own rates" of interest.

¹⁹ The proposition made in the traditional treatment of the theory of interest that under conditions of zero capital accumulation these rates are positive rests on empirical assumption, not on theoretical deduction. The empirical assumption is either that the marginal rate of time preference is positive under these conditions and determines a positive value of the marginal productivity of wait-

Our treatment can be generalized further by assuming that the transformation function of each individual (or firm) depends also on the quantities transformed by other individuals (or firms) in the economy. Taking the most general case, the transformation functions are then of the form $f^{(i)}(y_1^{(1)}, \dots, y_n^{(1)}; \dots; y_1^{(\theta)}, \dots, y_n^{(\theta)}) = 0$. The maximum conditions (5) become

$$(8) \quad \frac{u_r^{(i)}}{u_s^{(i)}} = \frac{f_r^{(i)} + \sum_{k \neq j} f_r^{(k)}}{f_s^{(i)} + \sum_{k \neq j} f_s^{(k)}}.$$

The terms under the summation signs represent "external economies" and "external diseconomies" which play such a distinguished role in the analysis of Professor Pigou.

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ing (time-preference theory of interest, or, conversely, that the latter is positive and determines a positive value of the first (marginal-productivity theory of interest). Whether any of these assumptions (and which one) is true is an empirical, not a theoretical question.