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RISK AVERSION IN THE SMALL AND IN THE LARGE[1]

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This paper concerns utility functions for money. A measure of risk aversion in the small, the risk premium or insurance premium for an arbitrary risk, and a natural concept of decreasing risk aversion are discussed and related to one another. Risks are also considered as a proportion of total assets.

1. SUMMARY AND INTRODUCTION

LET $u(x)$ BE a utility function for money. The function $r(x) = -u''(x)/u'(x)$ will be interpreted in various ways as a measure of local risk aversion (risk aversion in the small); neither $u''(x)$ nor the curvature of the graph of u is an appropriate measure. No simple measure of risk aversion in the large will be introduced. Global risks will, however, be considered, and it will be shown that one decision maker has greater local risk aversion $r(x)$ than another at all x if and only if he is globally more risk-averse in the sense that, for every risk, his cash equivalent (the amount for which he would exchange the risk) is smaller than for the other decision maker. Equivalently, his risk premium (expected monetary value minus cash equivalent) is always larger, and he would be willing to pay more for insurance in any situation. From this it will be shown that a decision maker's local risk aversion $r(x)$ is a decreasing function of x if and only if, for every risk, his cash equivalent is larger the larger his assets, and his risk premium and what he would be willing to pay for insurance are smaller. This condition, which many decision makers would subscribe to, involves the third derivative of u, as $r' \leqq 0$ is equivalent to $u'''u' \geqq u''^2$. It is not satisfied by quadratic utilities in any region. All this means that some natural ways of thinking casually about utility functions may be misleading. Except for one family, convenient utility functions for which $r(x)$ is decreasing are not so very easy to find. Help in this regard is given by some theorems showing that certain combinations of utility functions, in particular linear combinations with positive weights, have decreasing $r(x)$ if all the functions in the combination have decreasing $r(x)$.

The related function $r^*(x) = xr(x)$ will be interpreted as a local measure of aversion to risks measured as a proportion of assets, and monotonicity of $r^*(x)$ will be proved to be equivalent to monotonicity of every risk's cash equivalent measured as a proportion of assets, and similarly for the risk premium and insurance.

These results have both descriptive and normative implications. Utility functions for which $r(x)$ is decreasing are logical candidates to use when trying to describe the behavior of people who, one feels, might generally pay less for insurance against

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a given risk the greater their assets. And consideration of the yield and riskiness per investment dollar of investors' portfolios may suggest, at least in some contexts, description by utility functions for which $r^*(x)$ is first decreasing and then increasing.

Normatively, it seems likely that many decision makers would feel they ought to pay less for insurance against a given risk the greater their assets. Such a decision maker will want to choose a utility function for which $r(x)$ is decreasing, adding this condition to the others he must already consider (consistency and probably concavity) in forging a satisfactory utility from more or less malleable preliminary preferences. He may wish to add a further condition on $r^*(x)$.

We do not assume or assert that utility may not change with time. Strictly speaking, we are concerned with utility at a specified time (when a decision must be made) for money at a (possibly later) specified time. Of course, our results pertain also to behavior at different times if utility does not change with time. For instance, a decision maker whose utility for total assets is unchanging and whose assets are increasing would be willing to pay less and less for insurance against a given risk as time progresses if his $r(x)$ is a decreasing function of x. Notice that his actual expenditure for insurance might nevertheless increase if his risks are increasing along with his assets.

The risk premium, cash equivalent, and insurance premium are defined and related to one another in Section 2. The local risk aversion function $r(x)$ is introduced and interpreted in Sections 3 and 4. In Section 5, inequalities concerning global risks are obtained from inequalities between local risk aversion functions. Section 6 deals with constant risk aversion, and Section 7 demonstrates the equivalence of local and global definitions of decreasing (and increasing) risk aversion. Section 8 shows that certain operations preserve the property of decreasing risk aversion. Some examples are given in Section 9. Aversion to proportional risk is discussed in Sections 10 to 12. Section 13 concerns some related work of Kenneth J. Arrow.[2]

Throughout this paper, the utility $u(x)$ is regarded as a function of total assets rather than of changes which may result from a certain decision, so that $x=0$ is equivalent to ruin, or perhaps to loss of all readily disposable assets. (This is essential only in connection with proportional risk aversion.) The symbol $\sim$ indicates that two functions are equivalent as utilities, that is, $u_1(x) \sim u_2(x)$ means there exist constants a and b (with $b>0$) such that $u_1(x)=a+bu_2(x)$ for all x. The utility functions discussed may, but need not, be bounded. It is assumed, however, that they are sufficiently regular to justify the proofs; generally it is enough that they be twice continuously differentiable with positive first derivative, which is already re-

[2] The importance of the function $r(x)$ was discovered independently by Kenneth J. Arrow and by Robert Schlaifer, in different contexts. The work presented here was, unfortunately, essentially completed before I learned of Arrow's related work. It is, however, a pleasure to acknowledge Schlaifer's stimulation and participation throughout, as well as that of John Bishop at certain points.

quired for $r(x)$ to be defined and continuous. A variable with a tilde over it, such as $\tilde{z}$, is a random variable. The risks $\tilde{z}$ considered may, but need not, have "objective" probability distributions. In formal statements, $\tilde{z}$ refers only to risks which are not degenerate, that is, not constant with probability one, and interval refers only to an interval with more than one point. Also, increasing and decreasing mean nondecreasing and nonincreasing respectively; if we mean strictly increasing or decreasing we will say so.

2. THE RISK PREMIUM

Consider a decision maker with assets x and utility function u. We shall be interested in the *risk premium* π such that he would be indifferent between receiving a risk $\tilde{z}$ and receiving the non-random amount $E(\tilde{z})-\pi$, that is, π less than the actuarial value $E(\tilde{z})$. If u is concave, then $\pi \geqq 0$, but we don't require this. The risk premium depends on x and on the distribution of $\tilde{z}$, and will be denoted $\pi(x,\tilde{z})$. (It is not, as this notation might suggest, a function $\pi(x,z)$ evaluated at a randomly selected value of z, which would be random.) By the properties of utility,

$$u(x+E(\tilde{z})-\pi(x,\tilde{z}))=E\{u(x+\tilde{z})\}\,. \tag{1}$$

We shall consider only situations where $E\{u(x+\tilde{z})\}$ exists and is finite. Then $\pi(x,\tilde{z})$ exists and is uniquely defined by (1), since $u(x+E(\tilde{z})-\pi)$ is a strictly decreasing, continuous function of π ranging over all possible values of u. It follows immediately from (1) that, for any constant μ,

$$\pi(x,\tilde{z})=\pi(x+\mu,\,\tilde{z}-\mu)\,. \tag{2}$$

By choosing $\mu=E(\tilde{z})$ (assuming it exists and is finite), we may thus reduce consideration to a risk $\tilde{z}-\mu$ which is actuarially neutral, that is, $E(\tilde{z}-\mu)=0$.

Since the decision maker is indifferent between receiving the risk $\tilde{z}$ and receiving for sure the amount $\pi_a(x,\tilde{z})=E(\tilde{z})-\pi(x,\tilde{z})$, this amount is sometimes called the cash equivalent or value of $\tilde{z}$. It is also the asking price for $\tilde{z}$, the smallest amount for which the decision maker would willingly sell $\tilde{z}$ if he had it. It is given by

$$u(x+\pi_a(x,\tilde{z}))=E\{u(x+\tilde{z})\}\,. \tag{3a}$$

It is to be distinguished from the bid price $\pi_b(x,\tilde{z})$, the largest amount the decision maker would willingly pay to obtain $\tilde{z}$, which is given by

$$u(x)=E\{u(x+\tilde{z}-\pi_b(x,\tilde{z}))\}\,. \tag{3b}$$

For an unfavorable risk $\tilde{z}$, it is natural to consider the insurance premium $\pi_I(x,\tilde{z})$ such that the decision maker is indifferent between facing the risk $\tilde{z}$ and paying the non-random amount $\pi_I(x,\tilde{z})$. Since paying π_I is equivalent to receiving $-\pi_I$, we have

$$\pi_I(x,\tilde{z})=-\pi_a(x,\tilde{z})=\pi(x,\tilde{z})-E(\tilde{z})\,. \tag{3c}$$

If $\tilde{z}$ is actuarially neutral, the risk premium and insurance premium coincide.

The results of this paper will be stated in terms of the risk premium π, but could equally easily and meaningfully be stated in terms of the cash equivalent or insurance premium.

3. LOCAL RISK AVERSION

To measure a decision maker's local aversion to risk, it is natural to consider his risk premium for a small, actuarially neutral risk $\tilde{z}$. We therefore consider $\pi(x,\tilde{z})$ for a risk $\tilde{z}$ with $E(\tilde{z})=0$ and small variance σ_z^2; that is, we consider the behavior of $\pi(x,\tilde{z})$ as $\sigma_z^2 \to 0$. We assume the third absolute central moment of $\tilde{z}$ is of smaller order than σ_z^2. (Ordinarily it is of order σ_z^3.) Expanding u around x on both sides of (1), we obtain under suitable regularity conditions[3]

$$u(x-\pi)=u(x)-\pi u'(x)+O(\pi^2)\,, \tag{4a}$$

$$\begin{aligned} E\{u(x+\tilde{z})\} &= E\{u(x)+\tilde{z}u'(x)+\tfrac{1}{2}\tilde{z}^2u''(x)+O(\tilde{z}^3)\} \\ &= u(x)+\tfrac{1}{2}\sigma_z^2 u''(x)+o(\sigma_z^2)\,. \end{aligned} \tag{4b}$$

Setting these expressions equal, as required by (1), then gives

$$\pi(x,\tilde{z})=\tfrac{1}{2}\sigma_z^2\, r(x)+o(\sigma_z^2)\,, \tag{5}$$

where

$$r(x) = -\frac{u''(x)}{u'(x)} = -\frac{d}{dx}\log u'(x)\,. \tag{6}$$

Thus the decision maker's risk premium for a small, actuarially neutral risk $\tilde{z}$ is approximately $r(x)$ times half the variance of $\tilde{z}$; that is, $r(x)$ is twice the risk premium per unit of variance for infinitesimal risks. A sufficient regularity condition for (5) is that u have a third derivative which is continuous and bounded over the range of all $\tilde{z}$ under discussion. The theorems to follow will not actually depend on (5), however.

If $\tilde{z}$ is not actuarially neutral, we have by (2), with $\mu=E(\tilde{z})$, and (5):

$$\pi(x,\tilde{z})=\tfrac{1}{2}\sigma_z^2\, r(x+E(\tilde{z}))+o(\sigma_z^2)\,. \tag{7}$$

Thus the risk premium for a risk $\tilde{z}$ with arbitrary mean $E(\tilde{z})$ but small variance is approximately $r(x+E(\tilde{z}))$ times half the variance of $\tilde{z}$. It follows also that the risk premium will just equal and hence offset the actuarial value $E(\tilde{z})$ of a small risk $(\tilde{z})$; that is, the decision maker will be indifferent between having $\tilde{z}$ and not having it when the actuarial value is approximately r(x) times half the variance of $\tilde{z}$. Thus $r(x)$

[3] In expansions, $O(\;)$ means "terms of order at most" and $o(\;)$ means "terms of smaller order than."

may also be interpreted as twice the actuarial value the decision maker requires per unit of variance for infinitesimal risks.

Notice that it is the variance, not the standard deviation, that enters these formulas. To first order any (differentiable) utility is linear in small gambles. In this sense, these are second order formulas.

Still another interpretation of $r(x)$ arises in the special case $\tilde{z}=\pm h$, that is, where the risk is to gain or lose a fixed amount $h>0$. Such a risk is actuarially neutral if $+h$ and $-h$ are equally probable, so $P(\tilde{z}=h)-P(\tilde{z}=-h)$ measures the *probability premium* of $\tilde{z}$. Let $p(x,h)$ be the probability premium such that the decision maker is indifferent between the status quo and a risk $\tilde{z}=\pm h$ with

$$(8) \qquad P(\tilde{z}=h)-P(\tilde{z}=-h)=p(x,h)\,.$$

Then $P(\tilde{z}=h)=\frac{1}{2}[1+p(x,h)]$, $P(\tilde{z}=-h)=\frac{1}{2}[1-p(x,h)]$, and $p(x,h)$ is defined by

$$(9) \qquad u(x)=E\{u(x+\tilde{z})\}=\tfrac{1}{2}[1+p(x,h)]\,u(x+h)+\tfrac{1}{2}[1-p(x,h)]u(x-h)\,.$$

When u is expanded around x as before, (9) becomes

$$(10) \qquad u(x)=u(x)+hp(x,h)u'(x)+\tfrac{1}{2}h^2u''(x)+O(h^3)\,.$$

Solving for $p(x,h)$, we find

$$(11) \qquad p(x,h)=\tfrac{1}{2}hr(x)+O(h^2)\,.$$

Thus for small h the decision maker is indifferent between the status quo and a risk of $\pm h$ with a probability premium of $r(x)$ times $\frac{1}{2}h$; that is, $r(x)$ is twice the probability premium he requires per unit risked for small risks.

In these ways we may interpret $r(x)$ as a measure of the *local risk aversion* or *local propensity to insure* at the point x under the utility function u; $-r(x)$ would measure locally liking for risk or propensity to gamble. Notice that we have not introduced any measure of risk aversion in the large. Aversion to ordinary (as opposed to infinitesimal) risks might be considered measured by $\pi(x,\tilde{z})$, but π is a much more complicated function than r. Despite the absence of any simple measure of risk aversion in the large, we shall see that comparisons of aversion to risk can be made simply in the large as well as in the small.

By (6), integrating $-r(x)$ gives $\log u'(x)+c$; exponentiating and integrating again then gives $e^cu(x)+d$. The constants of integration are immaterial because $e^cu(x)+d\sim u(x)$. (Note $e^c>0$.) Thus we may write

$$(12) \qquad u\sim\int e^{-\int r}\,,$$

and we observe that the local risk aversion function r associated with any utility function u contains all essential information about u while eliminating everything arbitrary about u. However, decisions about ordinary (as opposed to "small") risks are determined by r only through u as given by (12), so it is not convenient entirely to eliminate u from consideration in favor of r.

4. CONCAVITY

The aversion to risk implied by a utility function u seems to be a form of concavity, and one might set out to measure concavity as representing aversion to risk. It is clear from the foregoing that for this purpose $r(x) = -u''(x)/u'(x)$ can be considered a measure of the concavity of u at the point x. A case might perhaps be made for using instead some one-to-one function of $r(x)$, but it should be noted that $u''(x)$ or $-u''(x)$ is not in itself a meaningful measure of concavity in utility theory, nor is the curvature (reciprocal of the signed radius of the tangent circle) $u''(x)(1+[u'(x)]^2)^{-3/2}$. Multiplying u by a positive constant, for example, does not alter behavior but does alter u'' and the curvature.

A more striking and instructive example is provided by the function $u(x) = -e^{-x}$. As x increases, this function approaches the asymptote $u=0$ and looks graphically less and less concave and more and more like a horizontal straight line, in accordance with the fact that $u'(x)=e^{-x}$ and $u''(x) = -e^{-x}$ both approach 0. As a utility function, however, it does not change at all with the level of assets x, that is, the behavior implied by $u(x)$ is the same for all x, since $u(k+x) = -e^{-k-x} \sim u(x)$. In particular, the risk premium $\pi(x,\tilde{z})$ for any risk $\tilde{z}$ and the probability premium $p(x,h)$ for any h remain absolutely constant as x varies. Thus, regardless of the appearance of its graph, $u(x) = -e^{-x}$ is just as far from implying linear behavior at $x=\infty$ as at $x=0$ or $x=-\infty$. All this is duly reflected in $r(x)$, which is constant: $r(x) = -u''(x)/u'(x) = 1$ for all x.

One feature of $u''(x)$ does have a meaning, namely its sign, which equals that of $-r(x)$. A negative (positive) sign at x implies unwillingness (willingness) to accept small, actuarially neutral risks with assets x. Furthermore, a negative (positive) sign for all x implies strict concavity (convexity) and hence unwillingness (willingness) to accept any actuarially neutral risk with any assets. The absolute magnitude of $u''(x)$ does not in itself have any meaning in utility theory, however.

5. COMPARATIVE RISK AVERSION

Let u_1 and u_2 be utility functions with local risk aversion functions r_1 and r_2, respectively. If, at a point x, $r_1(x) > r_2(x)$, then u_1 is locally more risk-averse than u_2 at the point x; that is, the corresponding risk premiums satisfy $\pi_1(x,\tilde{z}) > \pi_2(x,\tilde{z})$ for sufficiently small risks $\tilde{z}$, and the corresponding probability premiums satisfy $p_1(x,h) > p_2(x,h)$ for sufficiently small $h > 0$. The main point of the theorem we are about to prove is that the corresponding global properties also hold. For instance, if $r_1(x) > r_2(x)$ for all x, that is, u_1 has greater local risk aversion than u_2 everywhere, then $\pi_1(x,\tilde{z}) > \pi_2(x,\tilde{z})$ for every risk $\tilde{z}$, so that u_1 is also globally more risk-averse in a natural sense.

It is to be understood in this section that the probability distribution of $\tilde{z}$, which determines $\pi_1(x,\tilde{z})$ and $\pi_2(x,\tilde{z})$, is the same in each. We are comparing the risk premiums for the same probability distribution of risk but for two different utilities. This does not mean that when Theorem 1 is applied to two decision makers, they must have the same personal probability distributions, but only that the notation is imprecise. The theorem could be stated in terms of $\pi_1(x,\tilde{z}_1)$ and $\pi_2(x,\tilde{z}_2)$ where the distribution assigned to $\tilde{z}_1$ by the first decision maker is the same as that assigned to $\tilde{z}_2$ by the second decision maker. This would be less misleading, but also less convenient and less suggestive, especially for later use. More precise notation would be, for instance, $\pi_1(x,F)$ and $\pi_2(x,F)$, where F is a cumulative distribution function.

THEOREM 1: *Let $r_i(x)$, $\pi_i(x,\tilde{z})$, and $p_i(x)$ be the local risk aversion, risk premium, and probability premium corresponding to the utility function u_i, $i=1,2$. Then the following conditions are equivalent, in either the strong form (indicated in brackets), or the weak form (with the bracketed material omitted).*

(a) $r_1(x)\geqq r_2(x)$ *for all x* [*and* $>$ *for at least one x in every interval*].

(b) $\pi_1(x,\tilde{z})\geqq[>]\pi_2(x,\tilde{z})$ *for all x and $\tilde{z}$.*

(c) $p_1(x,h)\geqq[>]p_2(x,h)$ *for all x and all $h>0$.*

(d) $u_1(u_2^{-1}(t))$ *is a* [*strictly*] *concave function of t.*

(e) $\dfrac{u_1(y)-u_1(x)}{u_1(w)-u_1(v)}\leqq[<]\dfrac{u_2(y)-u_2(x)}{u_2(w)-u_2(v)}$ *for all v, w, x, y with $v<w\leqq x<y$.*

The same equivalences hold if attention is restricted throughout to an interval, that is, if the requirement is added that $x, x+\tilde{z}, x+h, x-h, u_2^{-1}(t), v, w$, and y, all lie in a specified interval.

PROOF: We shall prove things in an order indicating somewhat how one might discover that (a) implies (b) and (c).

To show that (b) follows from (d), solve (1) to obtain

$$\pi_i(x,\tilde{z})=x+E(\tilde{z})-u_i^{-1}(E\{u_i(x+\tilde{z})\}) . \tag{13}$$

Then

$$\begin{aligned}\pi_1(x,\tilde{z})-\pi_2(x,\tilde{z})&=u_2^{-1}(E\{u_2(x+\tilde{z})\})-u_1^{-1}(E\{u_1(x+\tilde{z}(\})\\&=u_2^{-1}(E\{\tilde{t}\})-u_1^{-1}(E\{u_1(u_2^{-1}(\tilde{t}))\}) ,\end{aligned} \tag{14}$$

where $\tilde{t}=u_2(x+\tilde{z})$. If $u_1(u_2^{-1}(t))$ is [strictly] concave, then (by Jensen's inequality)

$$E\{u_1(u_2^{-1}(\tilde{t}))\}\leqq[<]\,u_1(u_2^{-1}(E\{\tilde{t}\})) . \tag{15}$$

Substituting (15) in (14), we obtain (b).

To show that (a) implies (d), note that

$$\text{(16)} \qquad \frac{d}{dt} u_1(u_2^{-1}(t)) = \frac{u_1'(u_2^{-1}(t))}{u_2'(u_2^{-1}(t))},$$

which is [strictly] decreasing if (and only if) $\log u_1'(x)/u_2'(x)$ is. The latter follows from (a) and

$$\text{(17)} \qquad \frac{d}{dx} \log \frac{u_1'(x)}{u_2'(x)} = r_2(x) - r_1(x).$$

That (c) is implied by (e) follows immediately upon writing (9) in the form

$$\text{(18)} \qquad \frac{1-p_i(x,h)}{1+p_i(x,h)} = \frac{u_i(x+h)-u_i(x)}{u_i(x)-u_i(x-h)}.$$

To show that (a) implies (e), integrate (a) from w to x, obtaining

$$\text{(19)} \qquad -\log \frac{u_1'(x)}{u_1'(w)} \geqq [>] -\log \frac{u_2'(x)}{u_2'(w)} \quad \text{for} \quad w<x,$$

which is equivalent to

$$\text{(20)} \qquad \frac{u_1'(x)}{u_1'(w)} \leqq [<] \frac{u_2'(x)}{u_2'(w)} \quad \text{for} \quad w<x.$$

This implies

$$\text{(21)} \qquad \frac{u_1(y)-u_1(x)}{u_1'(w)} \leqq [<] \frac{u_2(y)-u_2(x)}{u_2'(w)} \quad \text{for} \quad w \leqq x < y,$$

as may be seen by applying the Mean Value Theorem of differential calculus to the difference of the two sides of (21) regarded as a function of y. Condition (e) follows from (21) upon application of the Mean Value Theorem to the difference of the reciprocals of the two sides of (e) regarded as a function of w.

We have now proved that (a) implies (d) implies (b), and (a) implies (e) implies (c). The equivalence of (a)–(e) will follow if we can prove that (b) implies (a), and (c) implies (a), or equivalently that not (a) implies not (b) and not (c). But this follows from what has already been proved, for if the weak [strong] form of (a) does not hold, then the strong [weak] form of (a) holds on some interval with u_1 and u_2 interchanged. Then the strong [weak] forms of (b) and (c) also hold on this interval with u_1 and u_2 interchanged, so the weak [strong] forms of (b) and (c) do not hold. This completes the proof.

We observe that (e) is equivalent to (20), (21), and

$$\text{(22)} \qquad \frac{u_1(w)-u_1(v)}{u_1'(x)} \geqq [>] \frac{u_2(w)-u_2(v)}{u_2'(x)} \quad \text{for} \quad v<w \leqq x.$$

6. CONSTANT RISK AVERSION

If the local risk aversion function is constant, say $r(x)=c$, then by (12):

(23) $u(x) \sim x$ if $r(x)=0$;

(24) $u(x) \sim -e^{-cx}$ if $r(x)=c>0$;

(25) $u(x) \sim e^{-cx}$ if $r(x)=c<0$.

These utilities are, respectively, linear, strictly concave, and strictly convex.

If the risk aversion is constant locally, then it is also constant globally, that is, a change in assets makes no change in preference among risks. In fact, for any k, $u(k+x) \sim u(x)$ in each of the cases above, as is easily verified. Therefore it makes sense to speak of "constant risk aversion" without the qualification "local" or "global."

Similar remarks apply to constant risk aversion on an interval, except that global consideration must be restricted to assets x and risks $\tilde{z}$ such that $x+\tilde{z}$ is certain to stay within the interval.

7. INCREASING AND DECREASING RISK AVERSION

Consider a decision maker who (i) attaches a positive risk premium to any risk, but (ii) attaches a smaller risk premium to any given risk the greater his assets x. Formally this means

(i) $\pi(x,\tilde{z})>0$ for all x and $\tilde{z}$;
(ii) $\pi(x,\tilde{z})$ is a strictly decreasing function of x for all $\tilde{z}$.

Restricting $\tilde{z}$ to be actuarially neutral would not affect (i) or (ii), by (2) with $\mu=E(\tilde{z})$.

We shall call a utility function (or a decision maker possessing it) *risk-averse* if the weak form of (i) holds, that is, if $\pi(x,\tilde{z}) \geqq 0$ for all x and $\tilde{z}$; it is well known that this is equivalent to concavity of u, and hence to $u'' \leqq 0$ and to $r \geqq 0$. A utility function is *strictly risk-averse* if (i) holds as stated; this is equivalent to strict concavity of u and hence to the existence in every interval of at least one point where $u''<0, r>0$.

We turn now to (ii). Notice that it amounts to a definition of strictly decreasing risk aversion in a global (as opposed to local) sense. On would hope that decreasing global risk aversion would be equivalent to decreasing local risk aversion $r(x)$. The following theorem asserts that this is indeed so. Therefore it makes sense to speak of "decreasing risk aversion" without the qualification "local" or "global." What is nontrivial is that $r(x)$ decreasing implies $\pi(x,\tilde{z})$ decreasing, inasmuch as $r(x)$ pertains directly only to infinitesimal gambles. Similar considerations apply to the probability premium $p(x,h)$.

THEOREM 2: *The following conditions are equivalent.*
(a′) *The local risk aversion function* $r(x)$ *is* [*strictly*] *decreasing.*

(b′) *The risk premium* $\pi(x,\tilde{z})$ *is a* [*strictly*] *decreasing function of* x *for all* $\tilde{z}$.

(c′) *The probability premium* $p(x,h)$ *is a* [*strictly*] *decreasing function of* x *for all* $h>0$.

The same equivalences hold if "increasing" is substituted for "decreasing" throughout and/or attention is restricted throughout to an interval, that is, the requirement is added that x, $x+\tilde{z}$, $x+h$, *and* $x-h$ *all lie in a specified interval.*

PROOF: This theorem follows upon application of Theorem 1 to $u_1(x)=u(x)$ and $u_2(x)=u(x+k)$ for arbitrary x and k.

It is easily verified that (a′) and hence also (b′) and (c′) are equivalent to

(d′) $u'(u^{-1}(t))$ is a [strictly] convex function of t.

This corresponds to (d) of Theorem 1. Corresponding to (e) of Theorem 1 and (20)–(22) is

(e′) $u'(x)u'''(x)\geqq(u''(x))^2$ [and $>$ for at least one x in every interval].

The equivalence of this to (a′)–(c′) follows from the fact that the sign of $r'(x)$ is the same as that of $(u''(x))^2-u'(x)u'''(x)$. Theorem 2 can be and originally was proved by way of (d′) and (e′), essentially as Theorem 1 is proved in the present paper.

8. OPERATIONS WHICH PRESERVE DECREASING RISK AVERSION

We have just seen that a utility function evinces decreasing risk aversion in a global sense if an only if its local risk aversion function $r(x)$ is decreasing. Such a utility function seems of interest mainly if it is also risk-averse (concave, $r\geqq 0$). Accordingly, we shall now formally define a utility function to be [*strictly*] *decreasingly risk-averse* if its local risk aversion function r is [strictly] decreasing and nonnegative. Then by Theorem 2, conditions (i) and (ii) of Section 7 are equivalent to the utility's being strictly decreasingly risk-averse.

In this section we shall show that certain operations yield decreasingly risk-averse utility functions if applied to such functions. This facilitates proving that functions are decreasingly risk-averse and finding functions which have this property and also have reasonably simple formulas. In the proofs, $r(x)$, $r_1(x)$, etc., are the local risk aversion functions belonging to $u(x)$, $u_1(x)$, etc.

THEOREM 3: *Suppose* $a>0$: $u_1(x)=u(ax+b)$ *is* [*strictly*] *decreasingly risk-averse for* $x_0\leqq x\leqq x_1$ *if and only if* $u(x)$ *is* [*strictly*] *decreasingly risk-averse for* $ax_0+b\leqq x\leqq ax_1+b$.

PROOF: This follows directly from the easily verified formula:

$$r_1(x)=ar(ax+b)\,. \tag{26}$$

THEOREM 4: *If* $u_1(x)$ *is decreasingly risk-averse for* $x_0\leqq x\leqq x_1$, *and* $u_2(x)$ *is decreasingly risk-averse for* $u_1(x_0)\leqq x\leqq u_1(x_1)$, *then* $u(x)=u_2(u_1(x))$ *is decreasingly* *risk-averse for* $x_0 \leqq x \leqq x_1$, *and strictly so unless one of* u_1 *and* u_2 *is linear from some* x *on and the other has constant risk aversion in some interval.*

PROOF: We have $\log u'(x) = \log u_2'(u_1'(x)) + \log u_1'(x)$, and therefore

$$r(x) = r_2(u_1(x))u_1'(x) + r_1(x)\,. \tag{27}$$

The functions $r_2(u_1(x))$, $u_1'(x)$, and $r_1(x)$ are $\geqq 0$ and decreasing, and therefore so is $r(x)$. Furthermore, $u_1'(x)$ is strictly decreasing as long as $r_1(x) > 0$, so $r(x)$ is strictly decreasing as long as $r_1(x)$ and $r_2(u_1(x))$ are both >0. If one of them is 0 for some x, then it is 0 for all larger x, but if the other is strictly decreasing, then so is r.

THEOREM 5: *If* $u_1, \ldots, u_n$ *are decreasingly risk-averse on an interval* $[x_0, x_1]$, *and* $c_1, \ldots, c_n$ *are positive constants, then* $u = \Sigma_1^n c_i u_i$ *is decreasingly risk-averse on* $[x_0, x_1]$, *and strictly so except on subintervals (if any) where all* u_i *have equal and constant risk aversion.*

PROOF: The general statement follows from the case $u = u_1 + u_2$. For this case

$$r = -\frac{u_1'' + u_2''}{u_1' + u_2'} = \frac{u_1'}{u_1' + u_2'} r_1 + \frac{u_2'}{u_1' + u_2'} r_2\,; \tag{28}$$

$$\begin{aligned} r' &= \frac{u_1'}{u_1' + u_2'} r_1' + \frac{u_2'}{u_1' + u_2'} r_2' + \frac{u_1'' u_2' - u_1' u_2''}{(u_1' + u_2')^2}(r_1 - r_2) \\ &= \frac{u_1' r_1' + u_2' r_2'}{u_1' + u_2'} - \frac{u_1' u_2'}{(u_1' + u_2')^2}(r_1 - r_2)^2\,. \end{aligned} \tag{29}$$

We have $u_1' > 0$, $u_2' > 0$, $r_1' \leqq 0$, and $r_2' \leqq 0$. Therefore $r' \leqq 0$, and $r' < 0$ unless $r_1 = r_2$ and $r_1' = r_2' = 0$. The conclusion follows.

9. EXAMPLES

9.1. *Example 1.* The utility $u(x) = -(b-x)^c$ for $x \leqq b$ and $c > 1$ is strictly increasing and strictly concave, but it also has strictly *increasing* risk aversion: $r(x) = (c-1)/(b-x)$. Notice that the most general concave quadratic utility $u(x) = \alpha + \beta x - \gamma x^2$, $\beta > 0$, $\gamma > 0$, is equivalent as a utility to $-(b-x)^c$ with $c = 2$ and $b = \frac{1}{2}\beta/\gamma$. Therefore a quadratic utility cannot be decreasingly risk-averse on any interval whatever. This severely limits the usefulness of quadratic utility, however nice it would be to have expected utility depend only on the mean and variance of the probability distribution. Arguing "in the small" is no help: decreasing risk aversion is a local property as well as a global one.

9.2. *Example 2.* If

$$(30)\qquad u'(x)=(x^a+b)^{-c} \quad \text{with} \quad a>0,\ c>0\,,$$

then $u(x)$ is strictly decreasingly risk-averse in the region

$$(31)\qquad x>[\max\{0,\ -b,\ b(a-1)\}]^{1/a}\,.$$

To prove this, note

$$(32)\qquad r(x)= -\frac{d}{dx}\log u'(x) = \frac{ac}{x+bx^{1-a}}\,,$$

which is $\geqq 0$ and strictly decreasing in the region where the denominator $x+bx^{1-a}$ is $\geqq 0$ and strictly increasing, which is the region (30). (The condition $x\geqq 0$ is included to insure that x^a is defined; for $a\geqq 1$ it follows from the other conditions.)

By Theorem 3, one can obtain a utility function that is strictly decreasingly risk-averse for $x>0$ by substituting $x+d$ for x above, where d is at least the right-hand side of (31). Multiplying x by a positive factor, as in Theorem 3, is equivalent to multiplying b by a positive factor.

Given below are all the strictly decreasingly risk-averse utility functions $u(x)$ on $x>0$ which can be obtained by applying Theorem 3 to (30) with the indicated choices of the parameters a and c:

$$(33)\qquad a=1,\ 0<c<1: \quad u(x)\sim(x+d)^q \quad \text{with} \quad d\geqq 0,\ 0<q<1\,;$$

$$(34)\qquad a=1,\ c=1: \quad u(x)\sim\log(x+d) \quad \text{with} \quad d\geqq 0\,;$$

$$(35)\qquad a=1,\ c>1: \quad u(x)\sim -(x+d)^{-q} \quad \text{with} \quad d\geqq 0,\ q>0\,;$$

$$(36)\qquad a=2,\ c=.5: \quad u(x)\sim\log(x+d+[(x+d)^2+b]) \quad \text{with} \quad d\geqq |b|^{\frac{1}{2}}\,;$$

$$(37)\qquad a=2,\ c=1: \quad u(x)\sim\arctan(\alpha x+\beta) \text{ or } \log(1-(\alpha x+\beta)^{-1}) \quad \text{with} \quad \alpha>0,\ \beta\geqq 1\,;$$

$$(38)\qquad a=2,\ c=1.5: \quad u(x)\sim[1+(\alpha x+\beta)^{-2}]^{-\frac{1}{2}} \text{ or } -[1-(\alpha x+\beta)^{-2}]^{-\frac{1}{2}} \quad \text{with} \quad \alpha>0,\ \beta\geqq 1\,.$$

9.3. *Example 3.* Applying Theorems 4 and 5 to the utilities of Example 2 and Section 6 gives a very wide class of utilities which are strictly decreasingly risk-averse for $x>0$, such as

$$(39)\qquad u(x)\sim -c_1e^{-cx}-c_2e^{-dx} \quad \text{with} \quad c_1>0,\ c_2>0,\ c>0,\ d>0\,.$$

$$(40)\qquad u(x)\sim\log(d_1+\log(x+d_2)) \quad \text{with} \quad d_1\geqq 0,\ d_2\geqq 0,\ d_1+\log d_2\geqq 0\,.$$

10. PROPORTIONAL RISK AVERSION

So far we have been concerned with risks that remained fixed while assets varied. Let us now view everything as a proportion of assets. Specifically, let $\pi^*(x,\tilde{z})$ be the *proportional risk premium* corresponding to a proportional risk $\tilde{z}$; that is, a decision maker with assets x and utility function u would be indifferent between receiving a risk $x\tilde{z}$ and receiving the non-random amount $E(x\tilde{z})-x\pi^*(x,\tilde{z})$. Then $x\pi^*(x,\tilde{z})$ equals the risk premium $\pi(x,x\tilde{z})$, so

$$\pi^*(x,\tilde{z}) = \frac{1}{x}\,\pi(x,x\tilde{z})\;. \tag{41}$$

For a small, actuarially neutral, proportional risk $\tilde{z}$ we have, by (5),

$$\pi^*(x,\tilde{z}) = \tfrac{1}{2}\sigma_z^2\, r^*(x) + o(\sigma_z^2)\,, \tag{42}$$

where

$$r^*(x) = x r(x)\;. \tag{43}$$

If $\tilde{z}$ is not actuarially neutral, we have, by (7),

$$\pi^*(x,\tilde{z}) = \tfrac{1}{2}\sigma_z^2\, r^*(x + xE(\tilde{z})) + o(\sigma_z^2)\;. \tag{44}$$

We will call r^* the *local proportional risk aversion* at the point x under the utility function u. Its interpretation by (42) and (44) is like that of r by (5) and (7).

Similarly, we may define the *proportional probability premium* $p^*(x,h)$, corresponding to a risk of gaining or losing a proportional amount h, namely

$$p^*(x,h) = p(x,xh)\;. \tag{45}$$

Then another interpretation of $r^*(x)$ is provided by

$$p^*(x,h) = \tfrac{1}{2}h r^*(x) + O(h^2)\,, \tag{46}$$

which follows from (45) and (11).

11. CONSTANT PROPORTIONAL RISK AVERSION

If the local proportional risk aversion function is constant, say $r^*(x)=c$, then $r(x)=c/x$, so the utility is strictly decreasingly risk-averse for $c>0$ and has negative, strictly increasing risk aversion for $c<0$. By (12), the possibilities are:

$$u(x) \sim x^{1-c} \quad \text{if} \quad r^*(x)=c<1\,, \tag{47}$$

$$u(x) \sim \log x \quad \text{if} \quad r^*(x)=1\,, \tag{48}$$

$$u(x) \sim -x^{-(c-1)} \quad \text{if} \quad r^*(x)=c>1\;. \tag{49}$$

If the proportional risk aversion is constant locally, then it is constant globally, that is, a change in assets makes no change in preferences among proportional risks. This follows immediately from the fact that $u(kx)\sim u(x)$ in each of the cases above. Therefore it makes sense to speak of "constant proportional risk aversion" without the qualification "local" or "global." Similar remarks apply to constant proportional risk aversion on an interval.

12. INCREASING AND DECREASING PROPORTIONAL RISK AVERSION

We will call a utility function [strictly] increasingly or decreasingly proportionally risk-averse if it has a [strictly] increasing or decreasing local proportional risk aversion function. Again the corresponding local and global properties are equivalent, as the next theorem states.

THEOREM 6: *The following conditions are equivalent.*

(a″) *The local proportional risk aversion function* $r^*(x)$ *is* [*strictly*] *decreasing.*

(b″) *The proportional risk premium* $\pi^*(x,\tilde{z})$ *is a* [*strictly*] *decreasing function of* x *for all* $\tilde{z}$.

(c″) *The proportional probability premium* $p^*(x,h)$ *is a* [*strictly*] *decreasing function of* x *for all* $h>0$.

The same equivalences hold if "increasing" is substituted for "decreasing" throughout and/or attention is restricted throughout to an interval, that is, if the requirement is added that x, $x+x\tilde{z}$, $x+xh$, *and* $x-xh$ *all lie in a specified interval.*

PROOF: This theorem follows upon application of Theorem 1 to $u_1(x)=u(x)$ and $u_2(x)=u(kx)$ for arbitrary x and k.

A decreasingly risk-averse utility function may be increasingly or decreasingly proportionally risk-averse or neither. For instance, $u(x)\sim -\exp[-q^{-1}(x+b)^q]$, with $b\geqq 0$, $q<1$, $q\neq 0$, is strictly decreasingly risk-averse for $x>0$ while its local proportional risk aversion function $r^*(x)=x(x+b)^{-1}[(x+b)^q+1-q]$ is strictly increasing if $0<q<1$, strictly decreasing if $q<0$ and $b=0$, and neither if $q<0$ and $b>0$.

13. RELATED WORK OF ARROW

Arrow[4] has discussed the optimum amount to invest when part of the assets x are to be held as cash and the rest invested in a specified, actuarially favorable risk. If $\tilde{\imath}$ is the return per unit invested, then investing the amount a will result in assets $x+a\tilde{\imath}$. Suppose $a(x,\tilde{\imath})$ is the optimum amount to invest, that is $a(x,\tilde{\imath})$ maximizes $E\{u(x+a\tilde{\imath})\}$. Arrow proves that if $r(x)$ is [strictly] decreasing, increasing, or constant for all x, then $a(x,\tilde{\imath})$ is [strictly] increasing, decreasing, or constant, respectively, except that $a(x,\tilde{\imath})=x$ for all x below a certain value (depending on $\tilde{\imath}$). He also proves a theorem about the asset elasticity of the demand for cash which is equivalent to the statement that if $r^*(x)$ is [strictly] decreasing, increasing, or constant for all x, then the optimum proportional investment $a^*(x,\tilde{\imath})=a(x,\tilde{\imath})/x$ is [strictly] increasing, decreasing, or constant, respectively, except that $a^*(x,\tilde{\imath})=1$ for all x below a certain value. In the present framework it is natural to deduce these re-

[4] Kenneth J. Arrow, "Liquidity Preference," Lecture VI in "Lecture Notes for Economics 285, The Economics of Uncertainty," pp. 33-53, undated, Stanford University.

sults from the following theorem, whose proof bears essentially the same relation to Arrow's proofs as the proof of Theorem 1 to direct proofs of Theorems 2 and 6. For convenience we assume that $a_1(x, \tilde{\imath})$ and $a_2(x, \tilde{\imath})$ are unique.

THEOREM 7: *Condition* (a) *of Theorem* 1 *is equivalent to*
(f) $a_1(x,\tilde{\imath}) \leqq a_2(x,\tilde{\imath})$ *for all* x *and* $\tilde{\imath}$ [*and* $<$ *if* $0 < a_1(x,\tilde{\imath}) < x$].
The same equivalence holds if attention is restricted throughout to an interval, that is, if the requirement is added that x *and* $x+\tilde{\imath}x$ *lie in a specified interval.*

PROOF: To show that (a) implies (f), note that $a_j(x,\tilde{\imath})$ maximizes

$$v_j(a) = \frac{1}{u_j'(x)} E\{u_j(x+a\tilde{\imath})\}, \qquad j=1,2. \tag{50}$$

Therefore (f) follows from

$$\frac{d}{da}\{v_1(a)-v_2(a)\} = E\left\{\tilde{\imath}\left(\frac{u_1'(x+a\tilde{\imath})}{u_1'(x)} - \frac{u_2'(x+a\tilde{\imath})}{u_2'(x)}\right)\right\} \leqq [<] 0, \tag{51}$$

which follows from (a) by (20).

If, conversely, the weak [strong] form of (a) does not hold, then its strong [weak] form holds on some interval with u_1 and u_2 interchanged, in which case the weak [strong] form of (f) cannot hold, so (f) implies (a). (The fact must be used that the strong form of (f) is actually stronger than the weak form, even when x and $x+\tilde{\imath}x$ are restricted to a specified interval. This is easily shown.)

Assuming u is bounded, Arrow proves that (i) it is impossible that $r^*(x) \leqq 1$ for all $x > x_0$, and he implies that (ii) $r^*(0) \leqq 1$. It follows, as he points out, that if u is bounded and r^* is monotonic, then r^* is increasing. (i) and (ii) can be deduced naturally from the following theorem, which is an immediate consequence of Theorem 1 (a) and (e).

THEOREM 8: *If* $r_1(x) \geqq r_2(x)$ *for all* $x > x_0$ *and* $u_1(\infty)=\infty$, *then* $u_2(\infty)=\infty$. *If* $r_1(x) \geqq r_2(x)$ *for all* $x<\varepsilon$, $\varepsilon>0$, *and* $u_2(0)=-\infty$, *then* $u_1(0)=-\infty$.

This gives (i) when $r_1(x)=1/x$, $r_2(x)=r(x)$, $u_1(x)=\log x$, $u_2(x)=u(x)$. It gives (ii) when $r_1(x)=r(x)$, $r_2(x)=c/x$, $c>1$, $u_1(x)=u(x)$, $u_2(x)=-x^{1-c}$.

This section is not intended to summarize Arrow's work,[4] but only to indicate its relation to the present paper. The main points of overlap are that Arrow introduces essentially the functions r and r^* (actually their negatives) and uses them in significant ways, in particular those mentioned already, and that he introduces essentially $p^*(x,h)$, proves an equation like (46) in order to interpret decreasing r^*, and mentions the possibility of a similar analysis for r.

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