John Von Neumann
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A Model of General Economic Equilibrium[1]

The subject of this paper is the solution of a typical economic equation system. The system has the following properties:

(1) Goods are produced not only from "natural factors of production," but in the first place from each other. These processes of production may be circular, i.e. good G_1 is produced with the aid of good G_2, and G_2 with the aid of G_1.

(2) There may be more technically possible processes of production than goods and for this reason "counting of equations" is of no avail. The problem is rather to establish which processes will actually be used and which not (being "unprofitable").

In order to be able to discuss (1), (2) quite freely we shall idealise other elements of the situation (see paragraphs 1 and 2). Most of these idealisations are irrelevant, but this question will not be discussed here.

The way in which our questions are put leads of necessity to a system of inequalities (3)—(8′) in paragraph 3 the possibility of a solution of which is not evident, i.e. *it cannot be proved by any qualitative argument*. The mathematical proof is possible only by means of a generalisation of Brouwer's Fix-Point Theorem, i.e. by the use of very fundamental *topological* facts. This generalised fix-point theorem (the "lemma" of paragraph 7) is also interesting in itself.

The connection with topology may be very surprising at first, but the author thinks that it is natural in problems of this kind. The immediate reason for this is the occurrence of a certain "minimum-maximum" problem, familiar from the calculus of variations. In our present question, the minimum-maximum problem has been formulated in paragraph 5. It is closely related to another problem occurring in the theory of games (see footnote 1 in paragraph 6).

A direct interpretation of the function $\phi(X, Y)$ would be highly desirable. Its rôle appears to be similar to that of thermodynamic potentials in phenomenological thermodynamics; it can be surmised that the similarity will persist in its full phenomenological generality (independently of our restrictive idealisations).

Another feature of our theory, so far without interpretation, is the remarkable duality (symmetry) of the monetary variables (prices y_j, interest factor β) and the technical variables (intensities of production x_i, coefficient of expansion of the economy α). This is brought out very clearly in paragraph 3 (3)—(8′) as well as in the minimum-maximum formulation of paragraph 5 (7**)—(8**).

Lastly, attention is drawn to the results of paragraph 11 from which follows, among other things, that the normal price mechanism brings about—if our assumptions are valid—the technically most efficient intensities of production. This seems not unreasonable since we have eliminated all monetary complications.

The present paper was read for the first time in the winter of 1932 at the mathematical seminar of Princeton University. The reason for its publication was an invitation from Mr. K. Menger, to whom the author wishes to express his thanks.

1. Consider the following problem: there are n goods $G_1, \ldots, G_n$ which can be produced by m processes $P_1, \ldots, P_m$. Which processes will be used (as "profitable") and what prices of the goods will obtain? The problem is evidently

[1] This paper was first published in German, under the title *Über ein Ökonomisches Gleichungssystem und eine Verallgemeinerung des Brouwerschen Fixpunktsatzes* in the volume entitled *Ergebuisse eines Mathematischen Seminars*, edited by K. Menger (Vienna, 1938). It was translated into English by G. Morgenstern. A commentary note on this article, by D. G. Champernowne, is printed below.

non-trivial since either of its parts can be answered only after the other one has been answered, i.e. its solution is implicit. We observe in particular:

(*a*) Since it is possible that $m > n$ it cannot be solved through the usual counting of equations.

In order to avoid further complications we assume:

(*b*) That there are constant returns (to scale);

(*c*) That the natural factors of production, including labour, can be expanded in unlimited quantities.

The essential phenomenon that we wish to grasp is this: goods are produced from each other (see equation (7) below) and we want to determine (*i*) which processes will be used; (*ii*) what the relative velocity will be with which the total quantity of goods increases; (*iii*) what prices will obtain; (*iv*) what the rate of interest will be. In order to isolate this phenomenon completely we assume furthermore:

(*d*) Consumption of goods takes place only through the processes of production which include necessities of life consumed by workers and employees.

In other words we assume that all income in excess of necessities of life will be reinvested.

It is obvious to what kind of theoretical models the above assumptions correspond.

2. In each process P_i ($i = 1, \ldots, m$) quantities a_{ij} (expressed in some units) are used up, and quantities b_{ij} are produced, of the respective goods G_j ($j = 1, \ldots, n$). The process can be symbolised in the following way:

$$P_i : \sum_{j=1}^{n} a_{ij} G_j \rightarrow \sum_{j=1}^{n} b_{ij} G_j \qquad \ldots\ldots (1)$$

It is to be noted:

(*e*) Capital goods are to be inserted on both sides of (1); wear and tear of capital goods are to be described by introducing different stages of wear as different goods, using a separate P_i for each of these.

(*f*) Each process to be of unit time duration. Processes of longer duration to be broken down into single processes of unit duration introducing if necessary intermediate products as additional goods.

(*g*) (1) can describe the special case where good G_j can be produced only jointly with certain others, viz. its permanent joint products.

In the actual economy, these processes P_i, $i = 1, \ldots, m$, will be used with certain *intensities* x_i, $i = 1, \ldots, m$. That means that for the total production the quantities of equations (1) must be multiplied by x_i. We write symbolically:

$$E = \sum_{i=1}^{m} x_i P_i \qquad \ldots\ldots (2)$$

$x_i = 0$ means that process P_i is not used.

We are interested in those states where the whole economy expands without change of structure, i.e. where the ratios of the intensities $x_1 : \ldots : x_m$ remain unchanged, although $x_1, \ldots x_m$ themselves may change. In such a case they are multiplied by a common factor α per unit of time. This factor is the *coefficient of expansion of the whole economy*.

3. The numerical unknowns of our problem are: (*i*) the *intensities* $x_1, \ldots, x_m$ of the processes $P_1, \ldots, P_m$; (*ii*) the *coefficient of expansion* of the whole economy α; (*iii*) the *prices* $y_1, \ldots, y_n$ of goods $G_1, \ldots, G_n$; (*iv*) the interest factor β $(= 1 + \frac{z}{100}$, z being the rate of interest in % per unit of time. Obviously:

$$x_i \geqq 0, \ldots\ldots (3) \qquad y_j \geqq 0, \ldots\ldots (4)$$

and since a solution with $x_1 = \ldots = x_m = 0$, or $y_1 = \ldots = y_n = 0$ would be meaningless:

$$\sum_{i=1}^{m} x_i > 0, \qquad (5) \qquad\qquad \sum_{j=1}^{n} y_j > 0, \qquad (6)$$

The economic equations are now:

$$a \sum_{i=1}^{m} a_{ij}\, x_i \leqq \sum_{i=1}^{m} b_{ij}\, x_i, \qquad (7)$$

and if in (7) $<$ applies, $y_j = 0$ .... (7′)

$$\beta \sum_{j=1}^{n} a_{ij}\, y_j \geqq \sum_{j=1}^{n} b_{ij}\, y_j, \qquad (8)$$

and if in (8) $>$ applies, $x_i = 0$ .... (8′)

The meaning of (7), (7′) is: it is impossible to consume more of a good G_j in the total process (2) than is being produced. If, however, less is consumed, i.e. if there is excess production of G_j, G_j becomes a free good and its price $y_j = 0$.

The meaning of (8), (8′) is: in equilibrium no profit can be made on any process P_i (or else prices or the rate of interest would rise—it is clear how this abstraction is to be understood). If there is a loss, however, i.e. if P_i is unprofitable, then P_i will not be used and its intensity $x_i = 0$.

The quantities a_{ij}, b_{ij} are to be taken as given, whereas the x_i, y_j, α, β are unknown. There are, then, $m + n + 2$ unknowns, but since in the case of x_i, y_j only the ratios $x_1 : \ldots : x_m$, $y_1 : \ldots : y_n$ are essential, they are reduced to $m + n$. Against this, there are $m + n$ conditions (7) + (7′) and (8) + (8′). As these, however, are not equations, but rather complicated inequalities, the fact that the number of conditions is equal to the number of unknowns does not constitute a guarantee that the system can be solved.

The dual symmetry of equations (3), (5), (7), (7′) of the variables x_i, α and of the concept "unused process" on the one hand, and of equations (4), (6), (8),)8′) of the variables y_j, β and of the concept "free good" on the other hand seems remarkable.

4. Our task is to solve (3)—(8′). We shall proceed to show:

Solutions of (3)—(8′) *always exist*, although there may be several solutions with different $x_1 : \ldots : x_m$ or with different $y_1 : \ldots : y_n$. The first is possible since we have not even excluded the case where several P_i describe the same process or where several P_i combine to form another. The second is possible since some goods G_j may enter into each process P_i only in a fixed ratio with some others. But even apart from these trivial possibilities there may exist—for less obvious reasons—several solutions $x_1 : \ldots : x_m$, $y_1 : \ldots : y_m$. Against this it is of importance that α, β should have the same value for all solutions; i.e. α, β *are uniquely determined.*

We shall even find that α and β can be directly characterised in a simple manner (see paragraphs 10 and 11).

To simplify our considerations we shall assume that always:

$$a_{ij} + b_{ij} > 0 \qquad (9)$$

(a_{ij}, b_{ij} are clearly always $\geqq 0$). Since the a_{ij}, b_{ij} may be arbitrarily small this restriction is not very far-reaching, although it must be imposed in order to assure uniqueness of α, β as otherwise W might break up into disconnected parts.

Consider now a hypothetical solution x_i, α, y_j, β of (3)—(8′). If we had in (7) always $<$, then we should have always $y_j = 0$ (because of (7′)) in contradiction to (6).

If we had in (8) always $>$ we should have always $x_i = 0$ (because of (8′)) in contradiction to (5). Therefore, in (7) $\leqq$ always applies, but $=$ at least once; in (8) $\geqq$ always applies, but $=$ at least once.

In consequence:

$$\alpha = \underset{j=1,\ldots,n}{\text{Min.}} \left[\frac{\sum_{i=1}^{m} b_{ij} \; x_i}{\sum_{i=1}^{m} a_{ij} \; x_i} \right] \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (10),$$

$$\beta = \underset{i=1,\ldots,m}{\text{Max.}} \frac{\sum_{j=1}^{n} b_{ij} \; y_j}{\sum_{j=1}^{n} a_{ij} \; y_j} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (11).$$

Therefore the x_i, y_j determine uniquely α, β. (The right-hand side of (10), (11) can never assume the meaningless form $\frac{0}{0}$ because of (3)—(6) and (9)). We can therefore state (7) + (7′) and (8) + (8′) as conditions for x_i, y_j only:

$y_j = 0$ for each $j = 1, \ldots, n$, for which:

$$\frac{\sum_{i=1}^{m} b_{ij} \; x_i}{\sum_{i=1}^{m} a_{ij} \; x_i}$$

does not assume its minimum value (for all $j = 1, \ldots, n$) . . . (7*).

$x_i = 0$ for each $i = 1, \ldots, m$, for which:

$$\frac{\sum_{j=1}^{n} b_{ij} \; y_i}{\sum_{j=1}^{n} a_{ij} \; y_i}$$

does not assume its maximum value (for all $i = 1, \ldots, m$) . . . (8*).

The $x_1, \ldots, x_m$ in (7*) and the $y_1, \ldots, y_n$ in (8*) are to be considered as given. We have, therefore, to solve (3)—(6), (7) and (8) for x_i, y_j.

5. Let X' be a set of variables $(x'_1, \ldots, x'_m)$ fulfilling the analoga of (3), (5):

$$x'_i \geqq 0, \ldots\ldots\ldots\ldots (3') \qquad \sum_{i=1}^{m} x'_i > 0, \ldots\ldots\ldots\ldots (5')$$

and let Y' be a series of variables $(y'_1, \ldots, y'_n)$ fulfilling the analoga of (4), (6):

$$y'_j \geqq 0, \ldots\ldots\ldots\ldots (4') \qquad \sum_{j=1}^{n} y'_j > 0, \ldots\ldots\ldots\ldots (6')$$

Let, furthermore,

$$\phi(X'_i, Y'_i) = \frac{\sum_{i=1}^{m} \sum_{j=1}^{n} b_{ij} \; x'_i \; y'_j}{\sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} \; x'_i \; y'_j} \quad \ldots\ldots\ldots\ldots\ldots\ldots (12)$$

Let $X = (x_1, \ldots, x_m)$, $Y = (y_1, \ldots, y_n)$ the (hypothetical) solution, $X' = (x'_1, \ldots, x'_m)$, $Y' = (y'_1, \ldots, y'_n)$ to be freely variable, but in such a way that (3)—(6) and (3')—(6') respectively are fulfilled; then it is easy to verify that (7*) and (8*) can be formulated as follows:

$\phi(X, Y')$ assumes its minimum value for Y' if $Y' = Y$...... (7**).

$\phi(X', Y)$ assumes its maximum value for X' if $X' = X$...... (8**).

The question of a solution of (3)—(8') becomes a question of a solution of (7**), (8**) and can be formulated as follows:

(*) *Consider (X', Y') in the domain bounded by (3')—(6'). To find a saddle point $X' = X$, $Y' = Y$, i.e. where (X, Y') assumes its minimum value for Y', and at the same time (X', Y) its maximum value for Y'.*

From (7), (7*), (10) and (8), (8*), (11) respectively, follows:

$$\alpha = \frac{\sum_{j=1}^{n}\left[\sum_{i=1}^{m} b_{ij}\, x_i\right] y_j}{\sum_{j=1}^{n}\left[\sum_{i=1}^{m} a_{ij}\, x_i\right] y_j} = \phi(x, y) \text{ and } \beta = \frac{\sum_{i=1}^{m}\left[\sum_{j=1}^{n} b_{ij}\, y_j\right] x_i}{\sum_{i=1}^{m}\left[\sum_{j=1}^{n} a_{ij}\, y_j\right] x_i} = \phi(x, y)$$

respectively.

Therefore:

(**) *If our problem can be solved, i.e. if $\phi(X', Y')$ has a saddle point $X' = X$, $Y' = Y$ (see above), then:*

$\alpha = \beta = \phi\,(X,Y) =$ *the value at the saddle point*............ (13)

6. Because of the homogeneity of $\phi\,(X', Y')$ (in X', Y', i.e. in $x', \ldots, x_m'$ and $y_1', \ldots y_m'$) our problem remains unaffected if we substitute the normalisations

$$\sum_{i=1}^{m} x_i = 1, \ldots\ldots\ldots\ldots (5^*) \qquad \sum_{j=1}^{n} y_j = 1, \ldots\ldots\ldots\ldots (6^*)$$

for (5'), (6') and correspondingly for (5), (6). Let S be the X' set described by:

$$x_i' \geqq 0, \ldots\ldots\ldots\ldots (3') \qquad \sum_{i=1}^{m} x_i' = 1, \ldots\ldots\ldots\ldots (5^*)$$

and let T be the Y' set described by:

$$y_j' \geqq 0, \ldots\ldots\ldots\ldots (4') \qquad \sum_{j=1}^{n} y_j' = 1, \ldots\ldots\ldots\ldots (6^*)$$

(S, T are simplices of, respectively, $m - 1$ and $n - 1$ dimensions).

In order to solve[1] we make use of the simpler formulation (7*), (8*) and combine these with (3), (4), (5*), (6*) expressing the fact that $X = (x_1, \ldots, x_m)$ is in S and $Y = (y_1, \ldots, y_n)$ in T.

7. We shall prove a slightly more general lemma: Let R_m be the m-dimensional

[1] The question whether our problem has a solution is oddly connected with that of a problem occurring in the Theory of Games dealt with elsewhere. (Math. Annalen, 100, 1928, pp. 295–320, particularly pp. 305 and 307–311). The problem there is a special case of (*) and is solved here in a new way through our solution of (*) (see below). In fact, if $a_{ij} \equiv 1$, then $\sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij}\, x'_i\, y'_j = 1$ because of (5*), (6*). Therefore $\phi\,(X', Y') = \sum_{i=1}^{m} \sum_{j=1}^{n} b_{ij}\, x'_i\, y'_j$, and thus our (*) coincides with loc. cit., p. 307. (Our $\phi\,(X', Y')$, b_{ij}, x'_i, y'_j, m, n here correspond to $h\,(\xi, \eta)$, a_{pq}, ξ_p, η_q, $M + 1$, $N + 1$ there).

It is, incidentally, remarkable that (*) does not lead—as usual—to a simple maximum or minimum problem, the possibility of a solution of which would be evident, but to a problem of the saddle point or minimum-maximum type, where the question of a possible solution is far more profound.

space of all points $X = (x_1, \ldots, x_m)$, R_n the n-dimensional space of all points $Y = (y_1, \ldots, y_n)$, R_{m+n} the $m + n$ dimensional space of all points $(X, Y) = (x_1, \ldots x_m, y_1, \ldots, y_n)$.

A set (in R_m or R_n or R_{m+n}) which is *not empty, convex closed and bounded* we call a set C.

Let S°, T° be sets C in R_m and R_n respectively and let $S^\circ \times T^\circ$ be the set of all (X, Y) (in R_{m+n}) where the range of X is S° and the range of Y is T°. Let V, W be two closed subsets of $S^\circ \times T^\circ$. For every X in S° let the set $Q\ (X)$ of all Y with (X, Y) in V be a set C; for each Y in T° let the set $P\ (Y)$ of all X with (X, Y) in W be a set C. Then the following lemma applies.

Under the above assumptions, V, W have (at least) one point in common.

Our problem follows by putting $S^\circ = S$, $T^\circ = T$ and $V =$ the set of all $(X, Y) = (x_1, \ldots, x_m, y_1, \ldots, y_n)$ fulfilling (7*), $W =$ the set of all $(X, Y) = (x_1, \ldots, x_m, y_1, \ldots, y_n)$ fulfilling (8*). It can be easily seen that V. W are closed and that the sets $S^\circ = S$, $T^\circ = T$, $Q\ (X)$, $P\ (Y)$ are all simplices, i.e. sets C. The common points of these V, W are, of course, our required solutions $(X, Y) = (x_1, \ldots, x_m, y_1, \ldots, y_m)$.

8. To prove the above lemma let S°, T°, V, W be as described before the lemma. First, consider V. For each X of S° we choose a point $Y^\circ\ (X)$ out of $Q\ (X)$ (e.g. the centre of gravity of this set). It will not be possible, generally, to choose $Y^\circ\ (X)$ as a continuous function of X. Let $\epsilon > 0$; we define:

$$w^\epsilon\ (X, X') = \text{Max. } (0, 1 - \frac{1}{\epsilon} \text{ distance } (X, X')) \quad \ldots\ldots\ldots\ldots \quad (14)$$

Now let $Y^\epsilon\ (X)$ be the centre of gravity of the $Y^\circ\ (X')$ with (relative) weight function $w^\epsilon\ (X, X')$ where the range of X' is S°. I.e. if $Y^\circ\ (X) = (y_1{}^\circ\ (x), \ldots, y_n{}^\circ\ (x))$, $Y^\epsilon\ (X) = (y_1{}^\epsilon\ (x), \ldots, y_n{}^\epsilon\ (x))$, then:

$$y_j{}^\epsilon\ (X) = \int_{S^\circ} w^\epsilon\ (X, X')\ y_j{}^\circ\ (X')\ dX' / \int_{S^\circ} w^\epsilon\ (X, X')\ dX', \ldots\ldots \quad (15)$$

We derive now a number of properties of $Y^\epsilon\ (X)$ (valid for all $\epsilon > 0$):

(*i*) $Y^\epsilon\ (X)$ is in T°. Proof: $Y^\circ\ (X')$ is in $Q\ (X')$ and therefore in T°, and since $Y^\epsilon\ (X)$ is a centre of gravity of points $Y^\circ\ (X')$ and T° is convex, $Y^\epsilon\ (X)$ also is in T°.

(*ii*) $Y^\epsilon\ (X)$ is a continuous function of X (for the whole range of S°). Proof: it is sufficient to prove this for each $y_j{}^\epsilon\ (X)$. Now $w^\epsilon\ (X, X')$ is a continuous function of X, X' throughout; $\int_{S^\circ} w^\epsilon\ (X, X')\ dX'$ is always > 0, and all $y_j{}^\circ\ (X)$ are bounded (being co-ordinates of the bounded set S°). The continuity of the $y_j{}^\epsilon\ (X)$ follows, therefore, from (15).

(*iii*) For each $\delta > 0$ there exists an $\epsilon_0 = \epsilon_0\ (\delta) > 0$ such that the distance of each point $(X, Y^{\epsilon_0}\ (X))$ from V is $< \delta$. Proof: assume the contrary. Then there must exist a $\delta > 0$ and a sequence of $\epsilon_\nu > 0$ with $\lim_{\nu \to \infty} \epsilon_\nu = 0$ such that for every $\nu = 1, 2, \ldots$ there exists a X_ν in S° for which the distance $(X_\nu, Y^{\epsilon_\nu}\ (X_\nu))$ would be $\geqq \delta$. A fortiori $Y^{\epsilon_\nu}\ (X_\nu)$ is at a distance $\geqq \frac{\delta}{2}$ from every $Q\ (X')$, with a distance $(X_\nu, X') \leqq \frac{\delta}{2}$.

All X_ν, $\nu = 1, 2, \ldots$, are in S° and have therefore a point of accumulation X^* in S°; from which follows that there exists a subsequence of X_ν, $\nu = 1, 2, \ldots$, converging towards X^* for which distance $(X_\nu, X^*) \leqq \frac{\delta}{2}$ always applies. Substituting this subsequence for the ϵ_ν, X_ν, we see that we are justified in assuming: $\lim X_\nu = X^*$, distance $(X_\nu, X^*) \leqq \frac{\delta}{2}.$ Therefore we may put $X' = X^*$ for every $\nu = 1, 2, \ldots,$ and in consequence we have always $Y^{\epsilon_\nu}(X_\nu)$ at a distance $\geqq \frac{\delta}{2}$ from $Q(X^*)$.

$Q(X^*)$ being convex, the set of all points with a distance $< \frac{\delta}{2}$ from $(Q(X^*)$ is also convex. Since $Y^{\epsilon_\nu}(X_\nu)$ does not belong to this set, and since it is a centre of gravity of points $Y^\circ(X')$ with distance $(X_\nu, X') \leqq \epsilon_\nu$ (because for distance $(X_\nu, X') > \epsilon_\nu$, $w^{\epsilon_\nu}(X_\nu, X') = 0$ according to (14)), not all of these points belong to the set under discussion. Therefore: there exists a $X' = X_\nu$ for which the distance $(X_\nu, X'_\nu) \leqq \epsilon_\nu$ and where the distance between $Y^\circ(X'_\nu)$ and $Q(X^*)$ is $\geqq \frac{\delta}{2}.$

Lim $X_\nu = X^*$, lim distance $(X_\nu, X'_\nu) = 0$, and therefore lim $X'_\nu = X^*$. All $Y^\circ(Y_\nu)$ belong to T° and have therefore a point of accumulation Y^*. In consequence, (X^*, Y^*) is a point of accumulation of the $(X_\nu, Y^\circ(X_\nu))$ and since they all belong to V, (X^*, Y^*) belongs to V too. Y^* is therefore in $Q(X^*)$. Now the distance of every $Y^\circ(Y_\nu)$ including from $Q(X^*)$ is $\geqq \frac{\delta}{2}.$ This is a contradiction, and the proof is complete.

(*i*)—(*iii*) together assert: for every $\delta > 0$ there exists a continuous mapping $Y_\delta(X)$ of S° on to a subset of T° where the distance of every point $(X, Y_\delta(X))$ from V is $< \delta$. (Put $Y_\delta(X) = Y^\epsilon(X)$ with $\epsilon = \epsilon_0 = \epsilon_0(\delta)$).

9. Interchanging S° and T°, and V and W we obtain now: for every $\delta > 0$ there exists a continuous mapping $X_\delta(Y)$ of T° on to a subset of S° where the distance of every point $(X_\delta(Y), Y)$ from W is $< \delta$.

On putting $f_\delta(X) = X_\delta(Y_\delta(X))$, $f_\delta(X)$ is a continuous mapping of S° on to a subset of S°. Since S° is a set C, and therefore topologically a simplex[1] we can use L. E. J. Brouwer's Fix-point Theorem[2]; $f_\delta(X)$ has a fix-point. I.e., there exists a X^δ in S° for which $X^\delta = f_\delta(X^\delta) = X_\delta(Y_\delta(X^\delta))$. Let $Y^\delta = Y_\delta(X^\delta)$, then we have $X^\delta = X_\delta(Y^\delta)$. Consequently, the distances of the point (X^δ, Y^δ) in R_{m+n} both from V and from W are $< \delta$. The distance of V from W is therefore $< 2\delta$. Since this is valid for every $\delta > 0$, the distance between V and W is $= 0$. Since V, W are closed and bounded, they must have at least one common point. This proves our lemma completely.

10. We have solved (7*), (8*) of paragraph 4 as well as the equivalent problem (*) of paragraph 5 and the original task of paragraph 3: the solution of (3)—(8'). If the x_i, y_j (which were called X, Y in paragraphs 7—9) are determined, α, β follow from (13) in (**) of paragraph 5. In particular, $\alpha = \beta$.

We have emphasised in paragraph 4 already that there may be several solutions x_i, y_j (i.e. X, Y); we shall proceed to show that there exists only one value of α (i.e. of β). In fact, let $X_1, Y_1, \alpha_1, \beta_1$ and $X_2, Y_2, \alpha_2, \beta_2$ be two solutions. From (7**), (8**) and (13) follows:

$$\alpha_1 = \beta_1 = \phi(X_1, Y_1) \leqq \phi(X_1, Y_2),$$
$$\alpha_2 = \beta_2 = \phi(X_2, Y_2) \geqq \phi(X_1, Y_2),$$

therefore $\alpha_1 = \beta_1 \leqq \alpha_2 = \beta_2$. For reasons of symmetry $\alpha_2 = \beta_2 \leqq \alpha_1 = \beta_1$, therefore $\alpha_1 = \beta_1 = \alpha_2 = \beta_2$.

[1] Regarding these as well as other properties of convex sets used in this paper, c.f., e.g. Alexandroff and H. Hopf, *Topologie*, vol. I, J. Springer, Berlin, 1935, pp. 598–609.
[2] Cf., e.g. 1 c, footnote 1, p. 480.

We have shown :

At least one solution X, Y, α, β exists. For all solutions :

$$\alpha = \beta = \phi\ (X,\ Y) \quad \text{(13)}$$

and these have the same numerical value for all solutions, in other words : The interest factor and the coefficient of expansion of the economy are equal and uniquely determined by the technically possible processes $P_1, \ldots, P_m$.

Because of (13), $\alpha > 0$, but may be $\gtreqless 1$. One would expect $\alpha > 1$, but $\alpha \leqq 1$ cannot be excluded in view of the generality of our formulation : processes $P_1, \ldots, P_m$ may really be *unproductive.*

11. In addition, we shall characterise α in two independent ways.

Firstly, let us consider a state of the economy possible on purely technical considerations, expanding with factor α' per unit of time. I.e., for the intensities $x_1, \ldots, x_m$ applies :

$$x_i \geqq 0 \quad \text{(3')} \qquad \sum_{i=1}^{m} x_i' > 0 \quad \text{(5') and}$$

$$\alpha' \sum_{i=1}^{m} a_{ij}\ x_i' \leqq \sum_{i=1}^{m} b_{ij}\ x_i' \quad \text{(7'')}$$

We are neglecting prices here altogether. Let $x_i, y_j, \alpha = \beta$ be a solution of our original problem (3)—(8') in paragraph 3. Multiplying (7'') by y_j and adding $\sum_{j=1}^{n}$ we obtain :

$$\alpha' \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij}\ x_i'\ y_j \leqq \sum_{i=1}^{m} \sum_{j=1}^{n} b_{ij}\ x_i'\ y_j,$$

and therefore $\alpha' \leqq \phi\ (X', Y)$. Because of (8**) and (13) in paragraph 5, we have :

$$\alpha' \leqq \phi\ (X',\ Y) \leqq \phi\ (X,\ Y) = \alpha = \beta \quad \text{(15)}.$$

Secondly, let us consider a system of prices where the interest factor β' allows of no more profits. I.e. for prices $y_1', \ldots y'_n$ applies :

$$y'_j \geqq 0, \quad \text{(4')} \qquad \sum_{j=1}^{n} y'_j > 0, \quad \text{(6') and}$$

$$\beta' \sum_{j=1}^{n} a_{ij}\ y'_j \geqq \sum_{j=1}^{n} b_{ij}\ y'_j \quad \text{(8'')}$$

Hereby we are neglecting intensities of production altogether. Let $x_i, y_j, \alpha = \beta$ as above. Multiplying (8'') by x_i and adding $\sum_{i=1}^{m}$ we obtain :

$$\beta' \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij}\ x_i\ y'_j \leqq \sum_{i=1}^{m} \sum_{j=1}^{n} b_{ij}\ x_i\ y'_j$$

and therefore $\beta' \geqq \phi\ (X,\ Y')$. Because of (7**) and (13) in paragraph 5, we have :

$$\beta' \geqq \phi\ (X,\ Y') \geqq \phi\ (X,\ Y) = \alpha = \beta \quad \text{(16)}$$

These two results can be expressed as follows :

The greatest (purely technically possible) factor of expansion α' of the whole economy is $\alpha' = \alpha = \beta$, neglecting prices.

The lowest interest factor β' at which a profitless system of prices is possible is $\beta' = \alpha = \beta$, neglecting intensities of production.

Note that these characterisations are possible only on the basis of our knowledge that solutions of our original problem exist—without themselves directly referring to this problem. Furthermore, the equality of the maximum in the first form and the minimum in the second can be proved only on the basis of the existence of this solution.

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