$$u_{NN}(x) = \sum_{j=1}^{M} a_i \phi_i(x) = \sum_{j=1}^{M} a_i \underbrace{\sigma(w_i x + b_i)}_{\phi_i(x)}$$

We now select N (collocation) points $x_i \in [0,1]$, $=1,\ldots,N$ at which we evaluate the error/loss.

Then, define the NN loss function as

$$\mathcal{L} = \sum_{i=1}^{N} \| u_{NN}(x_i) - f(x_i) \|_2^2$$

$$= \sum_{i=1}^{N} \left| u_{NN}(x_i) - f(x_i) \right|^2$$

$$= \left\| \begin{bmatrix} u_{NN}(x_i) - f(x_i) \\ \vdots \\ u_{NN}(x_N) - f(x_N) \end{bmatrix} \right\|_2^2$$ by stacking each $(u_{NN}(x_i) - f(x_i))$ into a vector

$$= \left\| \begin{bmatrix} u_{NN}(x_1) \\ \vdots \\ u_{NN}(x_N) \end{bmatrix} - \begin{bmatrix} f(x_1) \\ \vdots \\ f(x_N) \end{bmatrix} \right\|_2^2$$

Now, $u_{NN}(x_i) = \sum_{j=1}^{M} a_j \phi_j(x_i) \left(= \sum_{j=1}^{M} a_j \phi_j(w_j x_i + b_j) \right)$

$$= \begin{bmatrix} \phi_1(x_i) & \phi_2(x_i) & \ldots & \phi_M(x_i) \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_M \end{bmatrix}$$

$$= \begin{bmatrix} \phi_1(x_i) & \phi_2(x_i) & \ldots & \phi_M(x_i) \end{bmatrix} \underline{a}$$

$$\mathcal{L} = \left\| \begin{bmatrix} u_{NN}(x_1) \\ \vdots \\ u_{NN}(x_N) \end{bmatrix} - \begin{bmatrix} f(x_1) \\ \vdots \\ f(x_N) \end{bmatrix} \right\|_2^2$$

and $u_{NN}(x_i) = \begin{bmatrix} \phi_1(x_i) & \phi_2(x_i) & \cdots & \phi_M(x_i) \end{bmatrix} \underline{a}$

so

$$\begin{bmatrix} u_{NN}(x_1) \\ u_{NN}(x_2) \\ \vdots \\ u_{NN}(x_N) \end{bmatrix} = \begin{bmatrix} \phi_1(x_1) & \phi_2(x_1) & \cdots & \phi_M(x_1) \\ \phi_1(x_2) & \phi_2(x_2) & \cdots & \phi_M(x_2) \\ \vdots & \vdots & \ddots & \vdots \\ \phi_1(x_N) & \phi_2(x_N) & \cdots & \phi_M(x_N) \end{bmatrix} \underline{a}$$

$$= \underline{\Phi}\, \underline{a}$$

so $\underline{\Phi}$ here is the $N \times M$ matrix with $\Phi_{ij} = \phi_j(x_i) = \sigma(w_j x_i + b_j)$

Hence, if $\underline{f} = \begin{bmatrix} f(x_1) \\ \vdots \\ f(x_N) \end{bmatrix}$,

$$\mathcal{L} = \| \underline{\Phi}\, \underline{a} - \underline{f} \|_2^2$$

If $M=N$, $\underline{\Phi}$ is square & hopefully invertible. In this case $\underline{a} = \underline{\Phi}^{-1} \underline{f}$

If $M \neq N$, one option is to use the Moore-Penrose pseudoinverse $\underline{\Phi}^{+}$ (numpy.linalg.pinv)

Then $\underline{a} = \underline{\Phi}^{+} \underline{f}$

A second option is to use the normal equations

$$\underline{a} = (\underline{\Phi}^T \underline{\Phi})^{-1} (\underline{\Phi}^T \underline{f})$$