Valued Dictionary Abstract Data Types: Hashing techniques

CS240: Data Structures and Data Management
Slide Set 13

Jérémy Barbay

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Outline

Algorithms using values

Consider other valued algorithms:

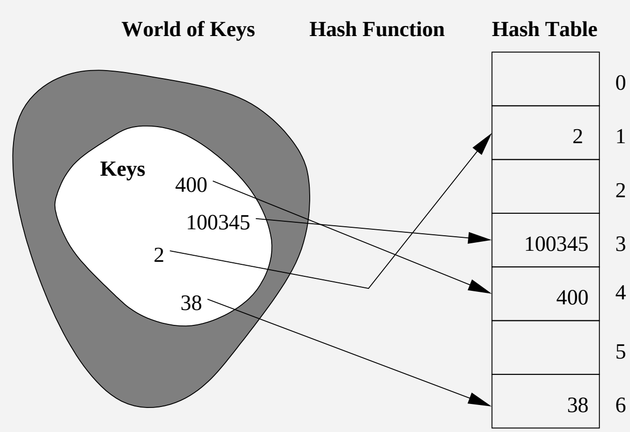
- Counting Sort
- Interpolation Search

Those algorithms are very good on data following a "well-behaved" distribution, in particular on uniformly distributed data, but can behave badly on a bad distribution.

Principle of Hashing

A projection of the data might be more "well-behaved" (e.g. uniform) than the data itself.

Definition
Use a projection of the key to compute an array index.



Terminology

Definition

A Hash Table is an array of size N,
which elements are often referred to as slots or buckets,
in which n keys are inserted.

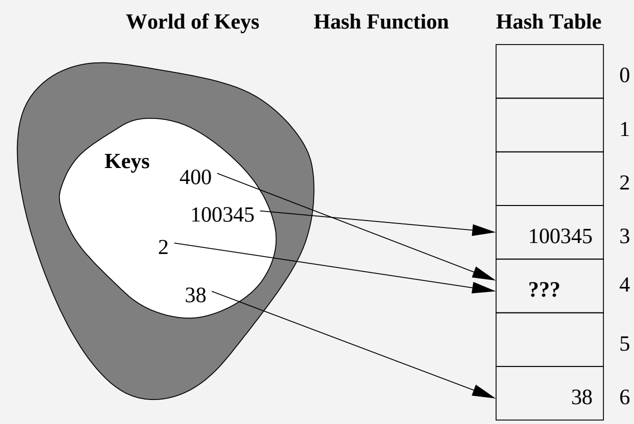
Definition

A Hash Function $h(K)$ maps each key K to an array index $\{0 \dots N-1\}$

A good hash function distributes keys evenly throughout the table:

$$P[\ h(K) = i\ ] = \frac{1}{N} \quad \text{for all keys } K \text{ and buckets } i$$

Collision



- Two different numbers map to the same range value
- Possible solutions?

Birthday Paradox

- How likely is it in a room with n people that 2 have the same birthday?
- What is the probability of all unique birthdays?

$$P(U) = \frac{\frac{364!}{(365-n)!}}{365^{n-1}}$$

- Likelihood of a shared birthday is $P(S) = 1 - P(U)$

n	$P(S)$
10	.12
23	.5
50	.97
100	.9999996

- Collisions are highly probable!

Hashing Techniques

Open Hashing

Associate a linked list with each bucket.
Insert At the begining.

Example

- $h(K) = K \bmod 10$
- Insert Sequence: 52, 18, 70, 22, 44, 38, 62

0 :	70
1 :	
2 :	62, 22, 52
3 :	
4 :	44
5 :	
6 :	
7 :	
8 :	38, 18
9 :	

Open Hashing

Analysis

Assume that h is a uniform hash function,
and let the load factor be $\lambda = \frac{n}{N}$.

Fact

Find and Delete takes time

- *Worst-case:* $O(N)$
- *Average-case*
  - *If search unsuccessful:* $O(\lambda)$
  - *If search successful:* $O(\lambda/2)$

Fact

Insert takes time

- *if no duplicate check* $\Theta(1)$
- *otherwise* $O(\lambda)$ *on average*

Comments

- Advantages
  - Fast on average, especially if $\lambda = \frac{n}{N}$ is small.
  - easy to implement
  - the dictionary can be larger than the table size.
- Disadvantages
  - Random
  - Good hash functions are data-dependant
  - no guarantee on running time
  - should rehash regularly
  - space (pointers)

Generalization and application

- How else could we store the lists/chains?
  - array (sorted/unsorted)
  - list/chain (sorted / self arranging)
  - trees
  - more generally, any dynamic dictionary structure
- Applications
  - Symbol table in Compilators
  - game playing, to check quickly the reoccurence of positions
  - spell checkers
  - In general, all applications where no deletion is needed.

Hashing Techniques

Closed Hashing (i.e. open addressing)

Does everything within the confines of the hash table array:
If there is a collision, probe a different slot, such that the i'th probe examines the slot

$$(h(K) + f(i)) \bmod N$$

Example

- $f(i) = i$ gives Linear Probing
- $f(i) = i^2$ gives Quadratic Probing

Example

Linear Probing

- $f(i) = i$
- Probe sequence: $h(K) \pmod N$, $h(K)+1 \pmod N$, $h(K)+2 \pmod N$, $\dots$
- $h(K) = K \bmod 11$
- Insert Sequence:

K	48,	67,	9,	53,	21,	75
$h(K)$	4,	1,	9,	9,	10,	9

0 :	21
1 :	67
2 :	75
3 :	
4 :	48
5 :	
6 :	
7 :	
8 :	
9 :	9
10 :	53

Search

Analysis

Fact

Average-case running time of search is

- *if search unsuccessful* $\frac{1}{2}\left(1 + \frac{1}{(1-\lambda)^2}\right)$;
- *if search successful* $\frac{1}{2}\left(1 + \frac{1}{(1-\lambda)}\right)$.

(*see Art of Computer Programming Vol. 3*) (Difficult)

Insertion

Analysis

Insert has same cost as unsuccessful search, but suffers from primary clustering (forming of long sequences).
Solution: If table fills too much, rehash

- Double the size of the table, and run all keys through the new hash function.
- If rehashing every τ insertions, the amortized cost is n/τ per insertion.

Deletion

Analysis

Delete is much harder – Lazy Delete

- To delete, simply flag the position;
- To insert, search until empty or flagged position found;
- To search, scan until empty bucket found.
- Rehash threshold counts only insertions.

Example

- $f(i) = i$
- probe $h(K), h(K)+1, \ldots \pmod N$
- example: table mod 11

0	
1	67
2	
3	
4	48
5	
6	
7	84
8	
9	9
10	21

- insert 29 (will try position 7 then 8), 18 (will try 7,8,9,10, 0)

Primary Clustering

Linear Hashing has a tendency to for large consecutive blocks.

0 :	399
1 :	5117
2 :	2409
3 :	9000
4 :	1154
5 :	5472
6 :	
7 :	
8 :	8261
9 :	2903
10 :	
11 :	
12 :	4650
13 :	
14 :	6488
15 :	
16 :	871
17 :	7623
18 :	4344

Solution to Primal Clustering

Quadratic Probing

- Example: $f(i) = i^2$
- More generally, i'th probe is to slot:

$$(h(K) + bi + ai^2) \bmod N$$

- Alleviates primary clustering, which results in much better performance
- May not hit every cell!
- Suffers from secondary clustering
  - If $h(K_1) = h(K_2)$, the probe sequences are the same

Hashing Techniques

Double Hashing

- Utilize second hash function, $h'(K)$
- $f(i) = i \cdot h'(K)$
- For example $h'(K) = K \pmod{10}$
- Insert Sequence:

	48,	67,	9,	53,	21,	75
pos:	4,	1,	9,	9,	10,	9
				3,		5

0 :	
1 :	67
2 :	
3 :	53
4 :	48
5 :	75
6 :	
7 :	
8 :	
9 :	9
10 :	21

$h'(K)$ should be relatively prime to N,
because otherwise we could loop forever missing one bucket.

Hashing Techniques
Ideal Hashing

A hash function generating a seemingly random probe sequence, such that all slots are equally probable to be probed next.

- A probe sequence may hit the same slot twice
- Identical keys follow the same probe sequence
- We will assume no deletions ever take place

Analysis
Ideal Hashing

Theorem

Given an open-address hash table with load factor $\lambda = j/N$, the expected number of probes in an unsuccesful search is at most $1/(1-\lambda)$, assuming uniform hashing.

Proof.

Let u_j be the expected cost of an unsuccessful search with j keys in the table. A slot is empty with probability $1-\lambda$.

$$\begin{aligned} u_j &= 1(1-\lambda) + 2\lambda(1-\lambda) + 3\lambda^2(1-\lambda) + \ldots \\ &= 1 + \lambda + \lambda^2 + \ldots \\ &= \frac{1}{1-\lambda} \end{aligned}$$

$\square$

Analysis
Ideal Hashing

Corollary

Inserting an element into an open-address hash table with load factor λ requires at most $1/(1-\lambda)$ probes on average, assuming uniform hashing.

Proof.

The cost of an insertion corresponds to the cost of an unsuccessful search plus th cost of placing the key in the empty slot found: hence $1/(1-\lambda)$ probes. $\square$

Analysis
Ideal Hashing

Fact

Given an open-address hash table with load factor $\lambda < 1$, the expected number of probes in a succesful search is at most $\frac{1}{\lambda} \ln \frac{1}{1-\lambda}$

Proof.

Complete Proof in CLRS, page 243.
Intuition: The cost of a successful search corresponds to the number of comparisons when the key was inserted, i.e. with a load factor smaller than λ. On average:

$$\frac{1}{\lambda}\int_0^{\lambda} \frac{1}{1-x}dx = \frac{1}{\lambda}\ln\left(\frac{1}{1-\lambda}\right)$$

$\square$

Comments

- Double hashing performs close to ideal if h and h' are uniform
- Quadratic probing also performs reasonably well
- What is the worst-case performance of double hashing?

Summary

- Even when data is not uniformly distributed, a suitable projection of it might be.
- To deal with collisions, use a second hashing function.
- Hashing permits to reach constant time on average, even though worst case is linear.

References:

- Algorithm Design, by Goodrich & Tamassia : pp. 114-124
- Introduction to Algorithms, by Cormen, Leisersen, Rivest & Stein: pp. 221-232, 237-244

Outline

Hash Functions

- Hash functions generally operate on numeric keys
- How do we handle **Strings**?
  - Need to convert an L character string to a number
  - Let c_i by the numeric value of the i'th character

$$K = \sum_{i=0}^{L-1} c_i \cdot r^i$$

Division Hash Functions

- $h(K) = K \pmod{N}$
- Fast
- Generally use a prime table size

Multiplication Hash Functions

- Allows non-prime table sizes
- Choose an A such that $0 < A < 1$ (e.g. $\frac{\sqrt{5}-1}{2}$)
- $h(K) = \lfloor N \cdot frac(K \cdot A) \rfloor$
- $frac(X)$ is the fractional part of a real number X

Universal Hash functions

(see *CLRS*)

- $h(K) = (K \bmod p) \bmod N$
- p is a prime number greater than N
- Universal hashing uses a more sophisticated function:
  - $h(K) = ((aK + b) \bmod p) \bmod N$
  - a is in range $[1..p-1]$
  - b is in range $[0..p-1]$
- a and b chosen randomly at startup
- Also allows non-prime table sizes

Outline

Motivation

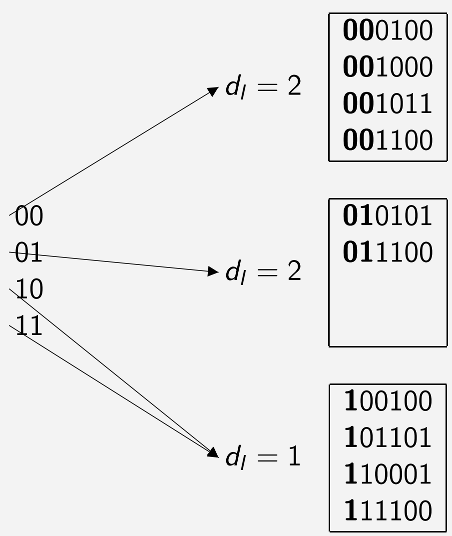
- What if hash table does not fit in main memory?
  Going through the probe seqence needs many page loads.
- Idea: Emulate a B-Tree
  - Organize data using hash values.
  - Keep a directory at the root node.
  - Only use **part** of the hash value to determine subtree.
  - Require the directory to **extend** as more data is inserted.

Description

- B – the nb of items that fit on one page
  Each leaf page stores at most B key-data pairs
- h – hash function which maps keys to range $[0..2^k - 1]$
  (for some $k > 0$, where k the nb of digits in the hash function.)
- D – the global depth where $D \leq k$
  Root has 2^D pointers to leaf pages
- d_l – the local depth of each leaf page l:
  - Hash values in l have leading d_l bits in common
  - There are 2^{D-d_l} pointers to leaf l

Example

$B = 4,\ k = 6,\ D = 2$



The values in the nodes represent the hash value of the key.

Search

Find(K)
  $x \leftarrow$ *first D bits of* $h(K)$
  Read in page referenced by Root$[\ x\ ]$
  Scan page for K

- We naively use the hash function $h(K) = K$
- On the previous slide:
  - `Find( 110001 )`
  - `Find( 001111 )`

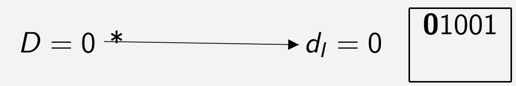
Insert

Read in the leaf page, and three possible cases

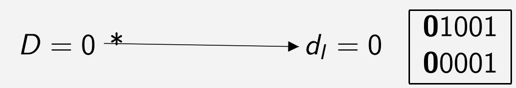
1. There is some room in the leaf page: add the key to it.
2. No room and $d_l < D$
   - Split page (more than once possibly) incrementing d_l
   - Add the key.
3. No room and $d_l = D$
   - Double the size of the directory (incrementing D)
   - Update leaf pointers
   - Split page (more than once possibly) incrementing d_l
   - Add the key.

Example

- We will use $B = 2$ and $k = 5$
- Start with an initially empty extendible hash table
- `Insert( 01001 )`

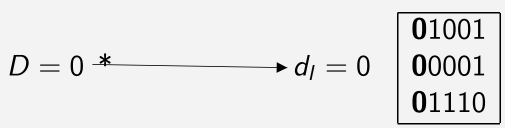


- `Insert( 00001 )`

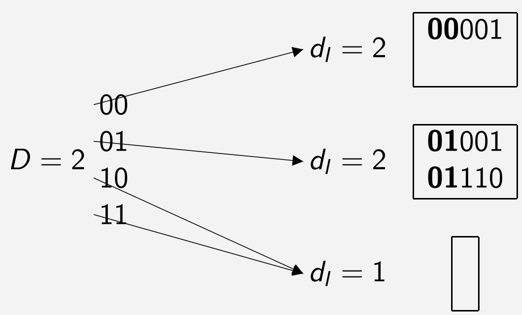


Example

`Insert( 01110 )`

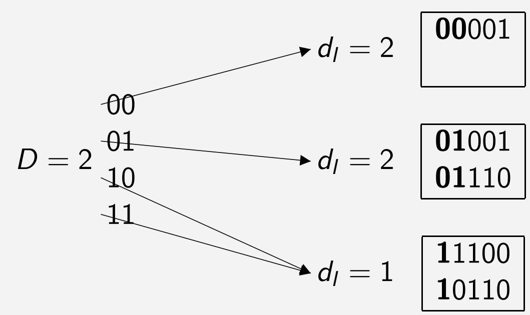


Overflow! Split as many times as needed:



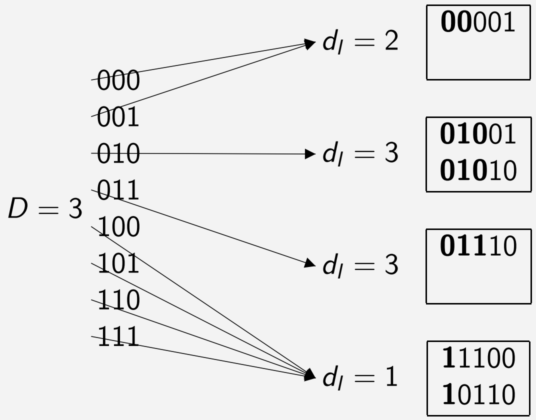
Example

- `Insert( 11100 )`
- `Insert( 10110 )`



Example

- `Insert( 01010 )`

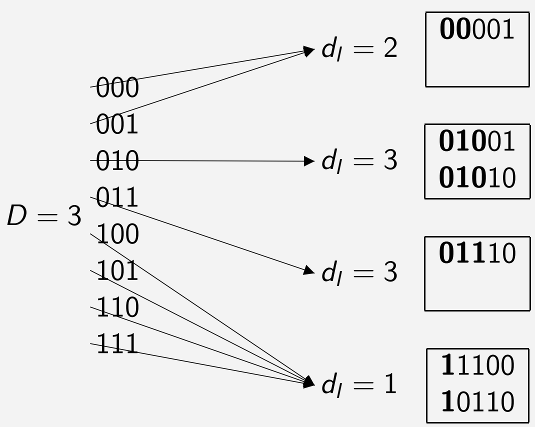


- **Exercise**: `Insert( 11101 )` and then `Insert( 01011 )`

Delete

- Search and remove entry from leaf
- Try and merge with "buddy"
  - Buddy is a leaf with same local depth
  - Buddies agree in first $d_l - 1$ bits
- Several merges may occur
- Shrink root directory if possible

Example

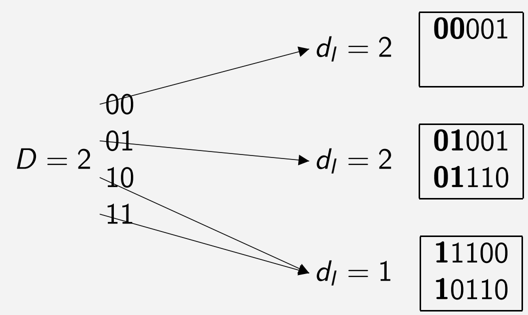


`Delete( 01010 )`

Merge and Reduce Dictionary

Example

After merging and reducing the dictionary:



Analysis

- Search:
  - If the directory fits in one page: One single disk access.
  - Otherwise: Use a B+ Tree to store the dictionary.
- Expected number of pages, P, to store n keys is

$$\frac{n}{B \ln 2} \approx 1.44 \frac{n}{B}$$

Hence pages are about 69% full.

Summary

- Hashing can be combined with other techniques, here B-trees.

References:

- Data Structures and Algorithm Analysis, by Mark Allen Weiss: pp. 204-212

Outline

Counting Sort

We want to sort n integer numbers from interval $[0, k]$.

```
clear array count[0..k];

for i:=1 to n
| count[A[i]]++;

pos[0]:=1;
for i:=1 to k
| pos[i]:=pos[i-1]+count[i-1];
// now pos[i] is the first position where
// integer i will come in the sorted array B

for i:=1 to n
| B[pos[A[i]]]:=A[i];
| pos[A[i]]++;
```

Running time: $\Theta(n + k)$

Radix Sort

We want to sort n integer numbers which have at most d digits (can be straightforwardly adapted for strings that have at most d characters).

```
for i:=1 to d
| use counting sort to sort A[1..n] by
| the d-th least significant digit (i.e. k=10)
| // inv: array is sorted by last i digits
```

The algorithm is demonstrated in Figure 8.3 of CLRS.
Running time: $\Theta(nd)$

Bucket Sort

We want to sort n real numbers that are chosen independently randomly uniformly from interval $[0, 1)$.

```
bucket[0..n-1] is an array of empty lists
for i:=1 to n
| insert A[i] into bucket[floor(A[i]*n)]

for i:=0 to n-1
| sort list bucket[i] by insertion sort

concatenate bucket[0], bucket[1], ..., bucket[n-1]
```

The algorithm is demonstrated in Figure 8.4 in CLRS.

Theorem
Expected running time for bucket sort is $O(n)$.

Summary for Valued based sorting

- The principle of hashing can be applied to algorithms as well.
- Radix sort has worst case complexity $\Theta(nd)$ to sort n integers of d digits.
- Bucket sort has expected complexity $O(n)$ to sort elements of bounded value.

Outline

Specific Dictionary ADTs and their DS

Unordered
- Array
- Sequence

Ordered
- Array
- Sequence (Skip Lists)
- Binary Search Tree (BST)
- AVL
- $(2,4)$ Trees
- B-Trees

Valued
- Hash Tables
- Extendible Hashing

Reading Materials

	Topic	GT	CLRS
Unordered	MTF	114-115, 28-30	Not covered.
Ordered	Arrays	pp. 140-151	pp. 253-264
	BST		
	Skiplists	pp. 195-202	Not covered.
	AVL	pp. 152-158	pp. 296 (poorly covered)
	$(2,4)$-trees	pp. 159-169	pp, 434-452 (indirectly)
	B-Trees	pp. 649-653	pp. 434-452
Valued	Hashing	pp. 114-124	pp. 221-232, 237-244

- GT = Algorithm Design, by Goodrich & Tamassia
- CLRS = Introduction to Algorithms, by Cormen, Leisersen, Rivest & Stein

Additional reference for Extendible Hashing: Data Structures and Algorithm Analysis, by Weiss, pp. 204-212.