Set 09: Ordered Dictionary Abstract Data Types: Sorted Arrays and BST

CS240: Data Structures and Data Management

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Outline

Introducing Order in Dictionaries

Motivations

- What if we access keys uniformly?
- Use ordering on keys to speed up the `Find` operatior.
- How expensive is it for the operator `Insert`?

Dictionary ADT

- Container of key-element pairs,
  where the keys are totally ordered.
- Required operations (as for general Dictionaries):
  - `insert( k,e )`,
  - `remove( k )`,
  - `find( k )`,
  - `isEmpty()`
- Now also supports:
  - `closestKeyBefore( k )`,
  - `closestElemAfter( k )`

This corresponds to dictionaries in the Comparison Model.

Ordered Dictionary ADTs and their DS

- Array
- Binary Search Tree (BST)
- Sequence (Skip Lists)
- AVL
- $(2,4)$ Trees
- B-Trees

Diferent solutions to different problems.

References:

- Goodrich and Tamassia: pp. 140-151
- Cormen, Leisersen, Rivest, Stein: 253-264

Outline

Sorted Arrays as Dictionaries

Motivations

- Maintain an array sorted by keys:
  This is the most compact representation.
- Use ordering on keys to speed up the `Find` operation:
  We achieve logarithic complexity.
- How expensive is it for the operators `Insert` and `Remove`?
  n in the worst case.

Binary Search

Naive Implementation

$\textsc{Binary Search}(x, A, l, r)$

```
if l > r then
    return false
else
    m ← ⌊(l+r)/2⌋
    if A[m] = x then
        return true
    else if A[m] < x then
        return Binary Search(x, A, m, r)
    else
        return Binary Search(x, A, l, m)
    end if
end if
```

In the worst case,

- each recursive call performs 2 comparisons.
- the entire search performs $2\lceil \lg n \rceil \in \Theta(\log n)$ comparisons.

Binary Search

Better Implementation

$\textsc{Binary Search}(x, A, l, r)$

if $l = r$ **then**
  return $(A[l] = x)$
end if
$m \leftarrow \left\lfloor \frac{l+r}{2} \right\rfloor$
if $A[m] > x$ **then**
  return $\textsc{Binary Search}(x, A, m, r)$
else
  return $\textsc{Binary Search}(x, A, l, m)$
end if

- Note that $l \leq m < r$.
- In the worst case, each recursive call performs 1 comparisons, and the entire search performs only $\lceil \lg n \rceil + 1 \in \Theta(\log n)$ comparisons

Binary Search

Non Recursive Implementation

Binary Search(x, A, l, r)

while $l < r$ **do**
  $m \leftarrow \left\lfloor \frac{l+r}{2} \right\rfloor$
  if $A[m] > x$ **then**
    $l \leftarrow m$
  else
    $r \leftarrow m$
  end if
end while
return $(A[l] = x)$

- Note that $l \leq m < r$.
- Removing recursion makes it faster in practice.
- The Worst Case complexity is still $\lceil \lg n \rceil + 1 \in \Theta(\log n)$.

Average Performance of Binary Search

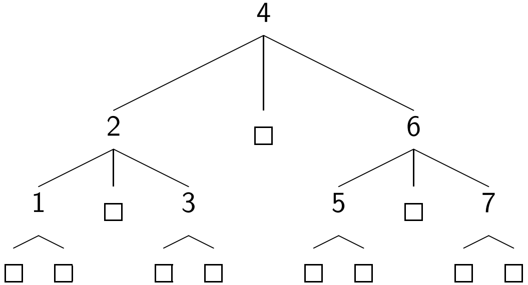
Exercise:

Consider searching in the array

1	2	3	4	5	6	7

. Suppose that x takes a value randomly and uniformly chosen among the elements of A. What is the average number of comparison performed by the first implementation?

Example:



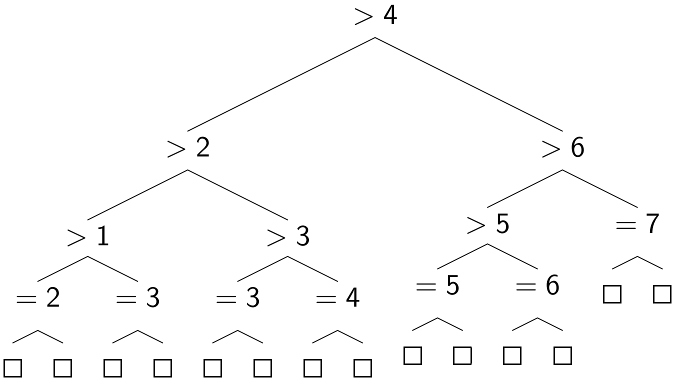
The algorithm performs 2 comparisons per node.

Average Performance of Binary Search (end)

Exercise:

What is the average number of comparison performed by the second implementation (exact, not asymptotic)?

Example:



The algorithm performs 1 comparison per node.

One Sided Binary Search

What if d elements have to be searched in the same sorted array?

- Naive algorithm performs d binary searches and $O(d \lg n)$ comparisons.
- Another possibility is to sort the d elements in $O(d \lg d)$ comparisons and search for them using one sided doubling search (also called gallop, or doubling search).

Example:

Consider searching for 18 and 203 in the array

1	4	5	9	15	16	17	...	369210

Complexity of One Sided Binary Search

Theorem

Given an element x and a sorted array A, One sided binary search finds its insertion rank *p such that $A[p] \leq x < A[p+1]$ after* $2\lceil \lg p \rceil + 1$ *comparisons.*

Proof.

- After the ith galloping comparison,
  - $1 + 2 + 4 + \ldots + 2^{i-1} = 2^i$ elements have been "eliminated".
  - the interval considered is of size 2^i, and a binary search on it would perform $1 + i$ comparisons.
- The algorithm finds p after the $i = \lceil \lg p \rceil$th galloping comparison. $\square$

Use of One Sided Binary Search

Theorem

Given an increasing sequence of elements $x1, \ldots, x_d$ and a sorted array A, there is an algorithm which checks if those elements are in A in only $O(d \lg(n/d))$ comparisons.

Proof.

Let $p_0 = 0$, call p_i the insertion rank of x_i in A, and $q_i = p_i - p_{i-1}$ the distance to the last one . Using one sided binary search, each p_i is found using $2\lceil \lg q_i \rceil + 1$ comparisons, hence a total of

$$2\sum_i (\lceil \lg q_i \rceil + 1) \leq 4d + \sum_i \lg q_i.$$

We can simplify more using the concativity of the function lg:

$$\sum_i \lg q_i \leq d \lg\big((\sum_i q_i)/d\big) \leq d \lg(n/d).$$

Hence the result, as $4d + d \lg(n/d) \in O(d \lg(n/d))$. $\square$

Short Summary

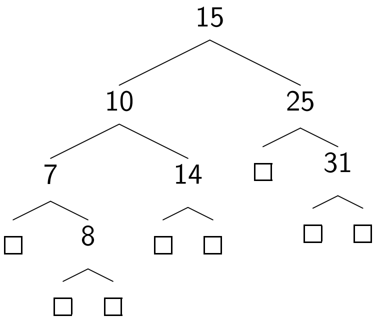
- An order on the elements helps.
- Arrays are not practical for insertion.
- Many variants of Binary Search, and many implementations.

Question: what about interpolation search?

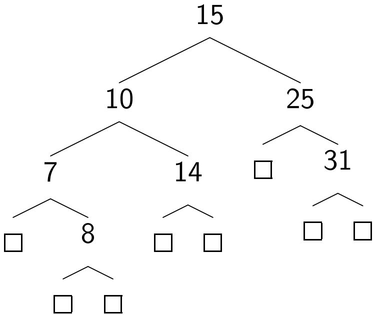
Outline

Binary Search Tree

- A binary tree storing (k, e) pairs at the internal nodes such that
  - All keys in nodes of left subtree are $< k$
  - All keys in nodes of right subtree are $> k$
- A set merely stores the keys (example below)
- External nodes are only placeholders and often not shown
- $\Theta(n)$ additional space usage

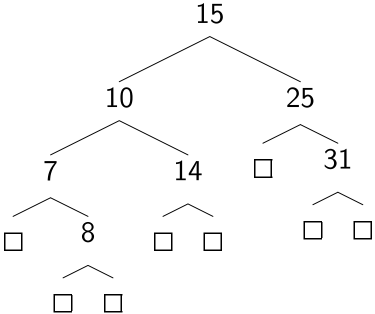


Find



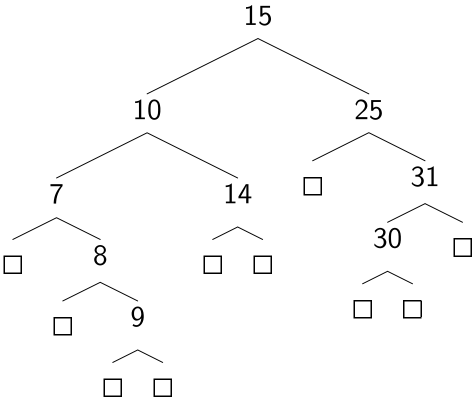
- Compare root's key to look-up key, K, and possibly traverse subtree
- If found return the node (or data associated with it)
- If not found return the external node where it **would** have been found
- Worst-case running time? $O(h)$, i.e. $O(n)$

Insert



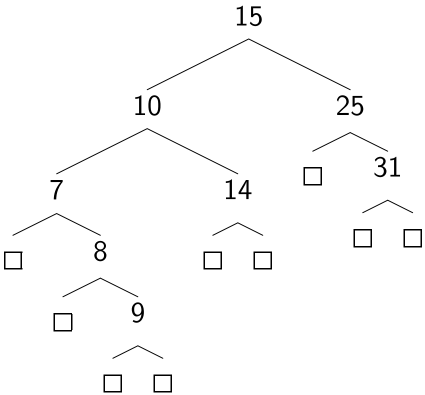
- Perform a search first
- Insert at the external node if one is returned
- **Example**: `Insert( 9 )` and `Insert( 30 )`
- Worst-case running time? $O(h)$, i.e. $O(n)$

Remove



- Perform a search first
- Remove the internal node if one is returned
- Three cases:
  - **2 external children**: `Remove( 30 )`
  - **1 external child**: `Remove( 7 )`

Remove



- If node has two internal children replace contents with in-order predecessor, or in-order successor
- Remove the emptied node of the in-order predecessor, or in-order successor
- **Example**: `Remove( 15 )`
- Worst-case running time? $O(h)$, i.e. $O(n)$

Two Answers