CS 245 — Assignment #5
Spring 2006

Due Date: Tuesday, June 20 at 5pm.

Use `makeCover` to produce a cover page for your assignment and hand in your assignment in the CS 245 assignment box. Assignments are to be done individually.

Consider the murder mystery from the previous assignment. Below is the sample solution where each of the clues is expressed in predicate logic.

1. Someone who lives at Wisteria Lodge murdered Aunt Agatha.
   $\exists x \bullet \text{lodge}(x) \wedge \text{murdered}(x, \text{Agatha})$
2. Aunt Agatha, Beatrice, and Charles live at Wisteria Lodge and nobody else lives there.
   $\text{lodge}(\text{Agatha}) \wedge \text{lodge}(\text{Beatrice}) \wedge \text{lodge}(\text{Charles}) \wedge$
   $\forall x \bullet \text{lodge}(x) \Rightarrow (x = \text{Agatha} \vee x = \text{Beatrice} \vee x = \text{Charles})$
3. Beatrice is the only person Aunt Agatha doesn't hate.
   $\forall x \bullet \neg\text{hates}(\text{Agatha}, x) \Rightarrow (x = \text{Beatrice})$
4. Aunt Agatha hates no-one that Charles hates.
   $\forall x \bullet \text{hates}(\text{Charles}, x) \Rightarrow \neg\text{hates}(\text{Agatha}, x)$
5. Beatrice hates everyone unless that person is richer than Aunt Agatha.
   $\forall x \bullet \text{hates}(\text{Beatrice}, x) \vee \text{richer}(x, \text{Agatha})$
6. Beatrice hates everyone Aunt Agatha hates.
   $\forall x \bullet \text{hates}(\text{Agatha}, x) \Rightarrow \text{hates}(\text{Beatrice}, x)$
7. Aunt Agatha and Beatrice are not the same person.
   $\neg(\text{Agatha} = \text{Beatrice})$
8. Everyone has someone that they don't hate.
   $\forall x \bullet \exists y \bullet \neg\text{hates}(x, y)$
9. A murder victim is always hated by their murderer.
   $\forall x \bullet \forall y \bullet \text{murdered}(x, y) \Rightarrow \text{hates}(x, y)$
10. A murderer is never richer than his victim.
    $\forall x \bullet \forall y \bullet \text{murdered}(x, y) \Rightarrow \neg\text{richer}(x, y)$

1. (20 points) Natural deduction proofs. In each of your proofs, do not begin your proof by repeating the premises—just assume they are as given above and start numbering new lines in the proof starting at 11. You may reuse formulas proved in earliers questions in

your proof for a question (or, if you get stuck on a question, just continue on to the next question and use formulas from previous questions as if they had been proved). In your proofs, only use natural deduction inference rules. You will need the inference rules for equality.

(a) Give a natural deduction proof that Charles did not murder Aunt Agatha.
I.e., prove ¬murdered(Charles, Agatha)

11.	$\forall y \bullet \text{murdered}(\text{C}, y) \Rightarrow \text{hates}(\text{C}, y)$	$9, \forall\_\text{E}$
12.	$\text{murdered}(\text{C}, \text{A}) \Rightarrow \text{hates}(\text{C}, \text{A})$	$11, \forall\_\text{E}$
13.	$\text{hates}(\text{C}, \text{A}) \Rightarrow \neg\text{hates}(\text{A}, \text{A})$	$4, \forall\_\text{E}$
14.	$\neg\text{hates}(\text{A}, \text{A}) \Rightarrow (\text{A} = \text{B})$	$3, \forall\_\text{E}$
15.	$\neg\neg\text{hates}(\text{A}, \text{A})$	$7, 14, \Rightarrow\_\text{E}$
16.	$\neg\text{hates}(\text{C}, \text{A})$	$13, 15, \Rightarrow\_\text{E}$
17.	$\neg\text{murdered}(\text{C}, \text{A})$	$12, 16, \Rightarrow\_\text{E}$

(b) Give a natural deduction proof that Beatrice does not hate herself (Hint: Use premise 8). I.e., prove ¬hates(Beatrice, Beatrice)

11.		$\exists y \bullet \neg\text{hates}(\text{B}, y)$	$8, \forall\_\text{E}$
12.	y_u	$\neg\text{hates}(\text{B}, y_u)$	$11, \text{assumption}$
13.		$\text{hates}(\text{A}, y_u) \Rightarrow \text{hates}(\text{B}, y_u)$	$6, \forall\_\text{E}$
14.		$\neg\text{hates}(\text{A}, y_u)$	$12, 13, \Rightarrow\_\text{E}$
15.		$\neg\text{hates}(\text{A}, y_u) \Rightarrow (y_u = \text{B})$	$3, \forall\_\text{E}$
16.		$(y_u = \text{B})$	$14, 15, \Rightarrow\_\text{E}$
17.		$\neg\text{hates}(\text{B}, \text{B})$	$12, 16, =\_\text{E}$
19.		$\neg\text{hates}(\text{B}, \text{B})$	$12 - 17, \exists\_\text{E}$

(c) Give a natural deduction proof that Beatrice is richer than Aunt Agatha.
I.e., prove richer(Beatrice, Agatha)

11.	$\neg\text{hates}(\text{B}, \text{B})$	Proven in (b)
12.	$\text{hates}(\text{B}, \text{B}) \vee \text{richer}(\text{B}, \text{A})$	$5, \forall\_\text{E}$
13.	$\text{richer}(\text{B}, \text{A})$	$11, 12, \vee\_\text{E}$

(d) Give a natural deduction proof that Beatrice did not murder Aunt Agatha.
I.e., prove ¬murdered(Beatrice, Agatha)

$$
\begin{array}{lll}
\left[\begin{array}{lll}
11. & \text{murdered}(\mathrm{B},\mathrm{A}) & \text{assumption} \\
12. & \forall y \bullet \text{murdered}(\mathrm{B}, y) \Rightarrow \neg\text{richer}(\mathrm{B}, y) & 10, \forall\_\mathrm{E} \\
13. & \text{murdered}(\mathrm{B},\mathrm{A}) \Rightarrow \neg\text{richer}(\mathrm{B},\mathrm{A}) & 12, \forall\_\mathrm{E} \\
14. & \neg\text{richer}(\mathrm{B},\mathrm{A}) & 11, 13, \Rightarrow\_\mathrm{E} \\
15. & \text{richer}(\mathrm{B},\mathrm{A}) & \text{Proven in (c)} \\
16. & \textbf{false} & 14, 15, \neg\_\mathrm{E}
\end{array}\right. \\
17. \quad \neg\text{murdered}(\mathrm{B},\mathrm{A}) \quad 11-16, \neg\_\mathrm{I}
\end{array}
$$

(e) Formulate a conclusion about who murdered Aunt Agatha and prove it using natural deduction (Hint: Use premises 1 & 2).

$$
\begin{array}{l}
\left[\begin{array}{llll}
11. & x_u & \text{lodge}(x_u) \wedge \text{murdered}(x_u, \mathrm{A}) & 1, \text{assumption} \\
12. & & \text{lodge}(x_u) & 11, \wedge\_\mathrm{E} \\
13. & & \text{murdered}(x_u, \mathrm{A}) & 11, \wedge\_\mathrm{E} \\
14. & & \forall x \bullet \text{lodge}(x) \Rightarrow (x = \mathrm{A} \vee x = \mathrm{B} \vee x = \mathrm{C}) & 2, \wedge\_\mathrm{E} \\
15. & & \text{lodge}(x_u) \Rightarrow (x_u = \mathrm{A}) \vee (x_u = \mathrm{B}) \vee (x_u = \mathrm{C}) & 14, \forall\_\mathrm{E} \\
16. & & (x_u = \mathrm{A}) \vee (x_u = \mathrm{B}) \vee (x_u = \mathrm{C}) & 12, 15, \Rightarrow\_\mathrm{E} \\
\multicolumn{4}{l}{\left[\begin{array}{lll}
17. & (x_u = \mathrm{B}) & \text{assumption} \\
18. & \text{murdered}(\mathrm{B},\mathrm{A}) & 13, 17, =\_\mathrm{E} \\
19. & \neg\text{murdered}(\mathrm{B},\mathrm{A}) & \text{Proven in (d)} \\
20. & \textbf{false} & 18, 19, \neg\_\mathrm{E}
\end{array}\right.} \\
21. & & \neg(x_u = \mathrm{B}) & 17-20, \neg\_\mathrm{I} \\
22. & & (x_u = \mathrm{A}) \vee (x_u = \mathrm{C}) & 16, 21, \vee\_\mathrm{E} \\
\multicolumn{4}{l}{\left[\begin{array}{lll}
23. & (x_u = \mathrm{C}) & \text{assumption} \\
24. & \text{murdered}(\mathrm{C},\mathrm{A}) & 13, 23, =\_\mathrm{E} \\
25. & \neg\text{murdered}(\mathrm{C},\mathrm{A}) & \text{Proven in (a)} \\
26. & \textbf{false} & 24, 25, \neg\_\mathrm{E}
\end{array}\right.} \\
27. & & \neg(x_u = \mathrm{C}) & 23-26, \neg\_\mathrm{I} \\
28. & & (x_u = \mathrm{A}) & 22, 27, \vee\_\mathrm{E} \\
29. & & \text{murdered}(\mathrm{A},\mathrm{A}) & 13, 28, =\_\mathrm{E}
\end{array}\right. \\
30. \quad \text{murdered}(\mathrm{A},\mathrm{A}) \quad 11-29, \exists\_\mathrm{E}
\end{array}
$$

2. (5 points) Semantic tableaux proofs. As above, do not begin your proof by repeating the premises—just assume they are as given above and start numbering new lines in the proof starting at 11.

(a) Give a semantic tableaux proof that Charles did not murder Aunt Agatha.
I.e., prove ¬murdered(Charles, Agatha)

11. $\neg\neg$ murdered(C, A) negated conclusion

NOT-NOT, 11

12. murdered(C, A)

FOR-ALL, 9

13. $\forall y.$ murdered(C, y) $\Rightarrow$ hates(C, y)

FOR-ALL, 13

14. murdered(C, A) $\Rightarrow$ hates(C, A)

IMPLIES, 14

15. $\neg$murdered(C, A)

CLOSED, 12, 15

16. hates(C, A)

FOR-ALL, 4

17. hates(C, A) $\Rightarrow$ $\neg$hates(A, A)

18. $\neg$hates(C, A)

CLOSED, 16, 18

19. $\neg$hates(A, A)

FOR-ALL, 3

20. $\neg$hates(A, A) $\Rightarrow$ (A = B)

21. $\neg\neg$hates(A, A)

NOT-NOT, 21

23. hates(A, A)

CLOSED, 19, 23

22. (A = B)

CLOSED, 7, 22

Because all of the branches are closed — showing inconsistency — we can conclude that $\neg$murdered(C, A)