Set 07: Priority Queue ADT, and Heap DS

CS240: Data Structures and Data Management

Jérémy Barbay

Outline

Min (or Max) Priority Queue ADT

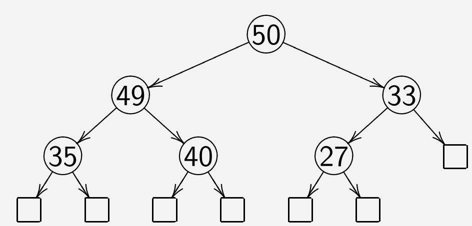
- Container of prioritized elements called keys
- supporting two operations:
  - **insert(x):** Inserts key x into the data structure.
  - **extractMin():** (or **extractMax**) Returns the smallest (largest) key and removes it from the data structure.
  - **isEmpty():** Returns false if the queue contains at least one key.
- In practice, an element is associated to each key.
- Here, we assume keys are distinct and totally ordered.

Min-Heap and Max-Heap

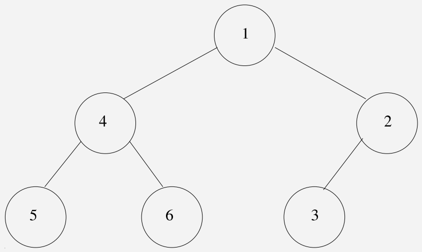
A min-*heap* (resp. max-heap) is a data structure that implements the abstract data type priority queue in a tree such that:

1. Value of key x is smaller (resp. larger) than the value of its descendants.
2. All levels but the last are complete.
3. Last level is filled from left to right.

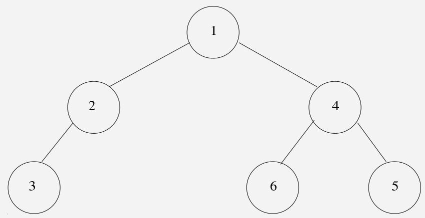
Example:



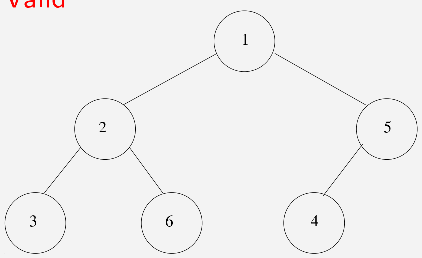
Examples:



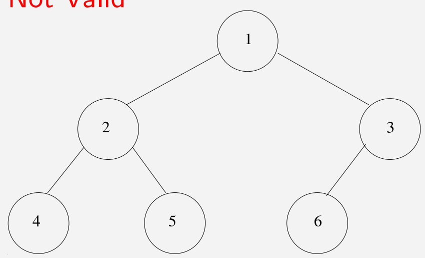
Valid



Not Valid



Not Valid



Valid

Heap Properties

Theorem
A heap with n internal nodes has height

$$h = \lceil \lg(n+1) \rceil.$$

Proof: Note, $x \leq \lceil x \rceil < x + 1$

Corollary
A heap with n internal nodes has height $h \in \Theta(\log n)$.

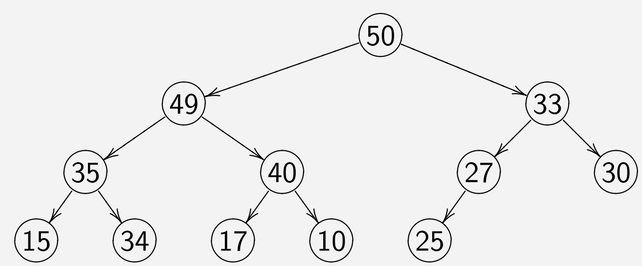
Inserting element into heap.

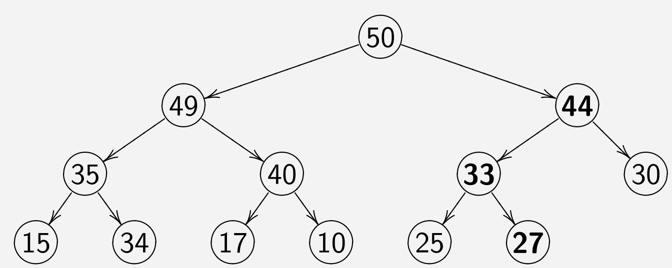
- Add the element to the left-most empty position at the bottom level (or start a new level if the bottom level is full)
- This may lead to violation of the heap property
- "Sift the element up" to restore the heap property

SiftUp(v) for a MaxHeap

if v has a parent **then**
  if the value of v's parent $<$ v's value **then**
    exchange their values;
    SiftUp(parent(v));
  end if
end if

Example: `insert(44)`





How many swaps? $O(\lg n)$

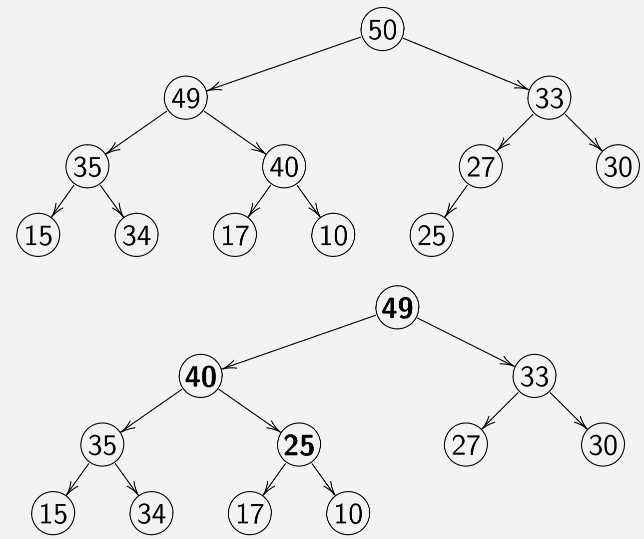
Extracting the minimum element from the heap.

- Minimum is stored in the root of the tree: removeit.
- Fill the gap with the right-most element at the bottom level.
- Restore the heap property by “sinking” the element down.

SiftDown(v) for a MaxHeap

if v has at least a child **then**
   if a child of v has a larger value **then**
      find the child w with the largest value;
      exchange the value of v and w ;
      SiftDown(w);
   end if
end if

Example: `extractMax()`



How many swaps? $O(\lg n)$.

Optimisation

- SiftUp and SiftDown are recursive but particular:
- only one recursive call in the function.
- Remove the recursion with a `while` loop.

Other optimisation: implement the binary tree in an array...

Summary for Heaps

- Priority Queue is an Abstract Data Type
- Heap is a Data Structure based on binary trees.
- Recursive function can be derecursived.

Outline

Priority Queue ADT
- Abstract Data Type
- Data Structure: Heaps

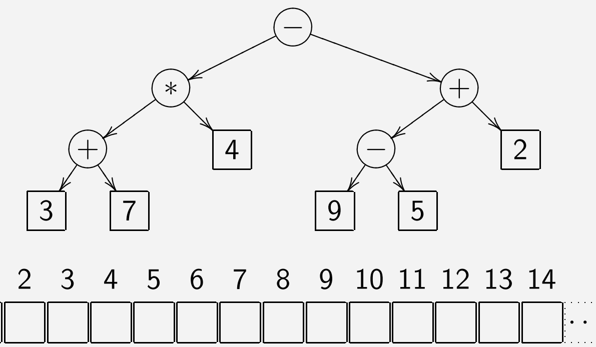
Heaps implemented in array
- Binary Trees in Array
- New operators
- Heapify

Heap Sort
- Sorting with Priority queues
- Sorting with a Heap implemented in an array

Mid-Summary about Abstract Data Types

Binary Trees in Array

- We can represent a binary tree with an array



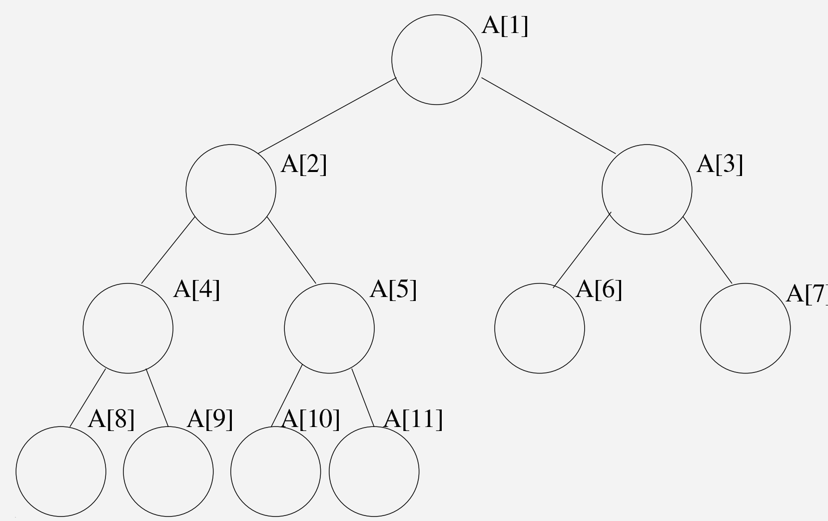
- Given an array index i, what is the index of:
  - left child: $2i$
  - right child: $2i + 1$
  - parent: $\lfloor i/2 \rfloor$

Binary Trees in Array (cont')

- Why do we not generally represent trees with arrays?
  - Only efficient for Complete Trees.
  - Only practical for trees of bounded degrees.
- A heap is a complete tree, though
- We will usually draw heaps as a tree structure
- However, we will implement the heap with
  - An array $A[1..N]$
  - An integer representing the *size* of the current heap

Implement heaps in an array.

Store root in $A[1]$ and continue with elements level-by-level from top to bottom, in each level left-to-right:



SiftDown

To Extract, use SiftDown:

SiftDown($A[1..n]$, i)

if i is not a leaf, and $A[i] < \max\ \{\ A[2i],\ A[2i+1]\}$ **then**
  swap (i, c) where $A[c] = \max\ \{\ A[2i],\ A[2i+1]\}$
  SiftDown(c)
end if

Complexity: $O(\lg n)$

SiftUp

To Insert, use SiftUp:

SiftUp($A[1..n]$, i)

if i is not the root, and $A[i] <$ key$(\lfloor i/2 \rfloor)$ **then**
  swap $(i, \lfloor i/2 \rfloor)$
  SiftUp(i)
end if

Complexity: $O(\lg n)$ To build a heap from n elements, use Insert n times?
It costs $O(n \lg n)$.

Heapify

We can build a heap faster than by n insertions.

BottomUpHeapify($A[1..n]$)

for $i = \lfloor \frac{n}{2} \rfloor$ **down to** 1 **do**
  SiftDown from $A[i]$
end for

Theorem
The complexity of `BottomUpHeapify` *is* $O(n)$

Example:
Heapify this array in a MaxHeap:

1	2	3	4	5	6	7	8	9
7	1	9	4	6	3	2	8	5

As an exercise, Heapify it as a MinHeap.
You should get

1	4	2	5	6	3	9	8	7

Proof of the Complexity of `BottomUpHeapify`

Proof.

- In a heap of n keys, at most $\lceil n/2^{h+1} \rceil$ nodes of height h.
- Each call to `SiftDown` on a subtree of height h is taking at most $2h$ comparisons.
- `BottomUpHeapify` performs at most one `SiftDown` call per sub-tree

Hence the total complexity of BottomUpHeapify is at most

$$\begin{aligned} C(n) &\leq \sum_{h=0}^{\lfloor \lg n \rfloor} 2h \lceil \frac{n}{2^{h+1}} \rceil \\ &= 2n \sum_{h=0}^{\lfloor \lg n \rfloor} \lceil \frac{h}{2^{h+1}} \rceil \\ &\leq 2n \sum_{h=0}^{\infty} \lceil \frac{h}{2^{h+1}} \rceil < 4n \in O(n) \end{aligned}$$

□

How NOT to Heapify

Use `SiftDown`, and not `SiftUp`!

TopDownHeapify($A[1..n]$)

for $i \leftarrow 1$ to n **do**
  SiftUp from $A[i]$
end for

Theorem
The complexity of `TopDownHeapify` *is* $O(n \lg n)$

Exercise:
Heapify this array:

1	2	3	4	
1	9	3	6	...
				...
				...
				...
				...

The difference between `TopDown` and `BottomUp`

- `TopDown` has a few `SiftUp` calls of complexity $O(1)$ and many of complexity $O(\lg n)$;
- `BottomUp` has a few `SiftDown` calls of complexity $O(\lg n)$ and many of complexity $O(1)$.

Summary for Heaps in arrays

Heaps implemented in arrays:

- use less space
- and can be build faster, using `BottomUpHeapify`.
- Do not use the other method!.

Outline

Sorting with Priority queues

PQ-Sort($A[1..n]$)
for $i \leftarrow 1$ *to* n **do**
  PQ.INSERT($A[i]$)
end for
for $i \leftarrow n$ **downto** 1 **do**
  $A[i] \leftarrow$ PQ.EXTRACTMAX()
end for

Heap Sort

HeapSort($A[1..n]$)
for $i \leftarrow 1$ *to* n **do**
  HEAP.INSERT($A[i]$)
end for
for $i \leftarrow n$ **downto** 1 **do**
  $A[i] \leftarrow$ HEAP.EXTRACTMAX()
end for

- Running Time? $O(n \lg n)$
- Space Usage? $O(n)$

HeapSort and MaxHeaps

On arrays, several modifications to improve the space:

- Sort "in place". At step i:
  - the last i elements are sorted.
  - the first $n - i$ elements represent the heap.
- MaxHeap versus MinHeap.

HeapSort "in place"

```
HeapSort:
  for i:=n/2 downto 1
  | SiftUp(i,n)
  for i:=n downto 1
  | Swap(A[1],A[i]);
  | SiftUp(1,i-1);

SiftUp(node,size):
  while (2*node<=size and A[node]<A[2*node])
     or (2*node+1<=size and A[node]<A[2*node+1])
  | if 2*node+1>size or A[2*node]>A[2*node+1]
  | | k:=2*node
  | else
  | | k:=2*node+1
  | swap(A[node],A[k]); node:=k
```

Summary for HeapSort

Heaps implemented in arrays permits to

- sort in time $O(n \lg n)$.
- with space exactly n.

Outline

Mid-Summary about Abstract Data Types

	Topic	Concept(s)
5	What's an ADT?	Abstract the "What" of the "How".
	Stacks (LIFO)	Pile of plates
	Queue (FIFO)	People waiting in line
	Graphs	(non)oriented, (un)weighted
	Adjacency list DS	$O(m)$, good for sparse graphs.
	Adjacency matrix DS	$O(1)$ or $O(n)$, good for dense graphs.
	Algorithms on graphs	Depth First and Breadth First Traversal.
6	Trees	Cardinal/Ordinal, many variants...
	Binary Trees	Can represent other trees.
7	Priority Queue ADT	Find Min or Max.
	Heap DS	Based on Binary Tree.
	Sift up/down	To correct the heap.
	Heapify	To build a heap from an array.
		do it BottomUp!
	Heapsort	Sort in $O(n \lg n)$, and in place.

Reading Materials

	Topic	GT	CLRS
5	What's an ADT?	56-74	200-209
	Graphs	288-306, 313-316	527-552
6	Trees	75-93	214-216
	Binary Trees	(same)	(same)
7	Priority Queue and Heaps	94-112	127-140

- GT = Algorithm Design, by Goodrich & Tamassia
- CLRS = Introduction to Algorithms, by Cormen, Leisersen, Rivest & Stein