CS 245 — Assignment #2
Spring 2006

Due Date: Tuesday, May 23 at 5pm.

Use `makeCover` to produce a cover page for your assignment and hand in your assignment in the CS 245 assignment box. Assignments are to be done individually.

1. (3 points) Give the formula $\neg(p \wedge (\neg(\neg r \wedge (s \vee p)) \Rightarrow \neg(\neg p \Rightarrow q)))$ in conjunctive normal form (CNF) and prove the equivalence of the two formulas by giving a transformational proof. Be sure to simplify, by using the appropriate logical laws, so that in each clause there are no duplicate literals or contradicting literals.

$$\begin{array}{lll}
\multicolumn{2}{l}{\neg(p \wedge (\neg(\neg r \wedge (s \vee p)) \Rightarrow \neg(\neg p \Rightarrow q)))} & \\
\Longleftrightarrow & \neg(p \wedge (\neg\neg(\neg r \wedge (s \vee p)) \vee \neg(\neg\neg p \vee q))) & \text{Implication } (2\times) \\
\Longleftrightarrow & \neg(p \wedge ((\neg r \wedge (s \vee p)) \vee \neg(p \vee q))) & \text{Negation } (2\times) \\
\Longleftrightarrow & \neg p \vee \neg((\neg r \wedge (s \vee p)) \vee \neg(p \vee q)) & \text{DeMorgan} \\
\Longleftrightarrow & \neg p \vee (\neg(\neg r \wedge (s \vee p)) \wedge \neg\neg(p \vee q)) & \text{DeMorgan} \\
\Longleftrightarrow & \neg p \vee (\neg(\neg r \wedge (s \vee p)) \wedge (p \vee q)) & \text{Negation} \\
\Longleftrightarrow & \neg p \vee ((\neg\neg r \vee \neg(s \vee p)) \wedge (p \vee q)) & \text{DeMorgan} \\
\Longleftrightarrow & \neg p \vee ((r \vee \neg(s \vee p)) \wedge (p \vee q)) & \text{Negation} \\
\Longleftrightarrow & \neg p \vee ((r \vee (\neg s \wedge \neg p)) \wedge (p \vee q)) & \text{DeMorgan} \\
\Longleftrightarrow & \neg p \vee ((r \vee \neg s) \wedge (r \vee \neg p) \wedge (p \vee q)) & \text{Distributive} \\
\Longleftrightarrow & (\neg p \vee r \vee \neg s) \wedge (\neg p \vee r \vee \neg p) \wedge (\neg p \vee p \vee q) & \text{Distributive} \\
\Longleftrightarrow & (\neg p \vee r \vee \neg s) \wedge (\neg p \vee r) \wedge (\neg p \vee p \vee q) & \text{Idempotence} \\
\Longleftrightarrow & (\neg p \vee r \vee \neg s) \wedge (\neg p \vee r) \wedge (\mathbf{true} \vee q) & \text{Excluded Middle} \\
\Longleftrightarrow & (\neg p \vee r \vee \neg s) \wedge (\neg p \vee r) \wedge \mathbf{true} & \text{Simplification I} \\
\Longleftrightarrow & (\neg p \vee r \vee \neg s) \wedge (\neg p \vee r) & \text{Simplification I}
\end{array}$$

2. (10 points) Consider the fragments of code given on the left and the right below, where P1, P2, P3, P4, and P5 are blocks of code.

Fragment #1

```
if( NOT a OR b ) {
   if( NOT a AND NOT b ) {
      P1
   }
   else
   if( b ) {
      P2
   }
   else {
      P3
   }
}
else {
   P4
}

if( NOT a OR b OR (a AND NOT b) ) {
   P5
}
```

Fragment #2

```
if( a AND NOT b ) {
   P4
}
else
if( NOT a AND NOT b ) {
   P1
}
else {
   P2
}

P5
```

(a) For Fragment #1, express in propositional logic the conditions under which each of the blocks of code P1, P2, P3, P4, and P5 will be executed. Do not simplify.

P1: $(\neg a \vee b) \wedge (\neg a \wedge \neg b)$
P2: $(\neg a \vee b) \wedge \neg(\neg a \wedge \neg b) \wedge b$
P3: $(\neg a \vee b) \wedge \neg(\neg a \wedge \neg b) \wedge \neg b$
P4: $\neg(\neg a \vee b)$
P5: $\neg a \vee b \vee (a \wedge \neg b)$

(b) For Fragment #2, express in propositional logic the conditions under which each of the blocks of code P1, P2, P4, and P5 will be executed. Do not simplify.

P1: $\neg(a \wedge \neg b) \wedge (\neg a \wedge \neg b)$
P2: $\neg(a \wedge \neg b) \wedge \neg(\neg a \wedge \neg b)$
P4: $a \wedge \neg b$
P5: **true**

(c) Give transformational proofs to show that Fragment #1 and Fragment #2 have the same behavior. For any unreachable (dead) code, give a transformational proof that the condition under which the code would be executed are a contradiction (equivalent to false). For any reachable code, give a transformational proof that the conditions under which the code would be executed are equivalent in both fragments.

(1) Transformational proof that the conditions under which P1 would be executed are equivalent. I.e., $(\neg a \vee b) \wedge (\neg a \wedge \neg b) \iff \neg(a \wedge \neg b) \wedge (\neg a \wedge \neg b)$.

$\neg(a \wedge \neg b) \wedge (\neg a \wedge \neg b)$		
$\iff$	$(\neg a \vee \neg\neg b) \wedge (\neg a \wedge \neg b)$	De Morgan
$\iff$	$(\neg a \vee b) \wedge (\neg a \wedge \neg b)$	Negation

(2) Transformational proof that the conditions under which P2 would be executed are equivalent. I.e., $(\neg a \vee b) \wedge \neg(\neg a \wedge \neg b) \wedge b \iff \neg(a \wedge \neg b) \wedge \neg(\neg a \wedge \neg b)$.

$(\neg a \vee b) \wedge \neg(\neg a \wedge \neg b) \wedge b$		
$\iff$	$b \wedge (b \vee \neg a) \wedge \neg(\neg a \wedge \neg b)$	Commutative (2×)
$\iff$	$b \wedge \neg(\neg a \wedge \neg b)$	Simplification II
$\iff$	$(b \vee \textbf{false}) \wedge \neg(\neg a \wedge \neg b)$	Simplification I
$\iff$	$(b \vee (a \wedge \neg a) \wedge \neg(\neg a \wedge \neg b)$	Contradiction
$\iff$	$(b \vee a) \wedge (b \vee \neg a) \wedge \neg(\neg a \wedge \neg b)$	Distributive
$\iff$	$(a \vee b) \wedge (\neg a \vee b) \wedge \neg(\neg a \wedge \neg b)$	Commutative (2×)
$\iff$	$(\neg\neg a \vee \neg\neg b) \wedge (\neg\neg\neg a \vee \neg\neg b) \wedge \neg(\neg a \wedge \neg b)$	Negation (4×)
$\iff$	$\neg(\neg a \wedge \neg b) \wedge \neg(\neg\neg a \wedge \neg b) \wedge \neg(\neg a \wedge \neg b)$	De Morgan (2×)
$\iff$	$\neg(\neg a \wedge \neg b) \wedge \neg(a \wedge \neg b) \wedge \neg(\neg a \wedge \neg b)$	Negation
$\iff$	$\neg(a \wedge \neg b) \wedge \neg(\neg a \wedge \neg b)$	Idempotence

An alternative proof of equivalence (and perhaps a simpler approach) is to show that both conditions under which P2 would be executed are equivalent to just b.

(3) Transformational proof that P3 is dead code since the conditions under which P3 would be executed are a contradiction. I.e., $(\neg a \vee b) \wedge \neg(\neg a \wedge \neg b) \wedge \neg b \iff \textbf{false}$.

$(\neg a \vee b) \wedge \neg(\neg a \wedge \neg b) \wedge \neg b$		
$\iff$	$(\neg a \vee b) \wedge (\neg\neg a \vee \neg\neg b) \wedge \neg b$	De Morgan
$\iff$	$(\neg a \vee b) \wedge (\neg\neg a \vee b) \wedge \neg b$	Negation
$\iff$	$(b \vee \neg a) \wedge (b \vee \neg\neg a) \wedge \neg b$	Commutative
$\iff$	$(b \vee (\neg a \wedge \neg\neg a)) \wedge \neg b$	Distributive
$\iff$	$(b \vee \textbf{false}) \wedge \neg b$	Contradiction
$\iff$	$b \wedge \neg b$	Simplification I

$\iff$ **false** Contradiction

(4) Transformational proof that the conditions under which P4 would be executed are equivalent. I.e., $\neg(\neg a \lor b) \iff a \land \neg b$.

$$\begin{array}{lll} \multicolumn{2}{l}{\neg(\neg a \lor b)} & \\ & \iff (\neg\neg a \land \neg b) & \text{De Morgan} \\ & \iff (a \land \neg b) & \text{Negation} \end{array}$$

(5) Transformational proof that the conditions under which P5 would be executed are equivalent. I.e., $\neg a \lor b \lor (a \land \neg b) \iff \textbf{true}$.

$$\begin{array}{lll} \multicolumn{2}{l}{\neg a \lor b \lor (a \land \neg b)} & \\ & \iff \neg a \lor ((b \lor a) \land (b \lor \neg b)) & \text{Distributive} \\ & \iff \neg a \lor ((b \lor a) \land \textbf{true}) & \text{Excluded Middle} \\ & \iff \neg a \lor b \lor a & \text{Simplification I} \\ & \iff \textbf{true} \lor b & \text{Excluded Middle} \\ & \iff \textbf{true} & \text{Simplification I} \end{array}$$

3. (12 points) For each of the following arguments, determine whether the argument is valid or invalid. If the argument is valid, prove it using Natural Deduction. If the argument is invalid, provide a counter example and demonstrate that the argument is invalid.

(a) $p \lor \neg q,\ \neg r \Rightarrow \neg\neg q,\ r \Rightarrow \neg s,\ \neg\neg s \quad \vdash \quad p$

1	$p \lor \neg q$	premise
2	$\neg r \Rightarrow \neg\neg q$	premise
3	$r \Rightarrow \neg s$	premise
4	$\neg\neg s$	premise
5	$\neg r$	$3, 4, \Rightarrow$_E
6	$\neg\neg q$	$2, 5, \Rightarrow$_E
7	p	$1, 6, \lor$_E

(b) $(p \lor q) \Rightarrow r \quad \vdash \quad q \Rightarrow r$

1	$(p \vee q) \Rightarrow r$	premise
2	q	assumption
3	$p \vee q$	$2, \vee$_I
4	r	$1, 7, \Rightarrow$ _E
5	$q \Rightarrow r$	$2 - 4, \Rightarrow$ _I

(c) $p \Rightarrow r,\ q \Rightarrow s \quad \vdash \quad (p \vee q) \Rightarrow (r \vee s)$

1	$p \Rightarrow r$	premise
2	$q \Rightarrow s$	premise
3	$p \vee q$	assumption
4	$\neg(r \vee s)$	assumption
5	r	assumption
6	$r \vee s$	$\vee$_I
7	**false**	$4, 6, \neg$_E
8	$\neg r$	$5 - 7, \neg$_I
9	$\neg p$	$1, 8, \Rightarrow$ _E
10	q	$3, 9, \vee$_E
11	s	$2, 10, \Rightarrow$ _E
12	$r \vee s$	$11, \vee$_I
13	**false**	$4, 12, \neg$_E
14	$\neg\neg(r \vee s)$	$4 - 13, \neg$_I
15	$r \vee s$	$14, \neg$_E
16	$(p \vee q) \Rightarrow (r \vee s)$	$3 - 15, \Rightarrow$ _I

(d) $\neg p \Rightarrow q,\ \neg r \Rightarrow s,\ \neg q \vee s \quad \vdash \quad p \vee r$

This is not a valid argument. Consider the truth assignment where p is false, q is true, r is false, and s is true, All of the premises are true, but the conclusion $p \vee r$ is false. Therefore the premises do not logically imply the conclusion and the arguement is invalid.

(e) $\neg p \Rightarrow q,\ \neg r \Rightarrow s,\ \neg q \vee \neg s \quad \vdash \quad p \vee r$

1	$\neg p \Rightarrow q$	premise
2	$\neg r \Rightarrow s$	premise
3	$\neg q \vee \neg s$	premise
4	$\neg(p \vee r)$	assumption
5	$\neg p$	assumption
6	q	$1, 5, \Rightarrow\_\mathrm{E}$
7	$\neg\neg q$	$6, \neg\_\mathrm{I}$
8	$\neg s$	$3, 7, \vee\_\mathrm{E}$
9	$\neg\neg r$	$2, 8, \Rightarrow\_\mathrm{E}$
10	r	$2, 8, \neg\_\mathrm{E}$
11	$p \vee r$	$9, \vee\_\mathrm{I}$
12	**false**	$4, 11, \neg\_\mathrm{E}$
13	$\neg\neg p$	$5 - 12, \neg\_\mathrm{I}$
14	p	$13, \neg\_\mathrm{E}$
15	$p \vee r$	$14, \vee\_\mathrm{I}$
16	**false**	$4, 15, \neg\_\mathrm{E}$
17	$\neg\neg(p \vee r)$	$4 - 16, \neg\_\mathrm{I}$
18	$(p \vee r)$	$17\neg\_\mathrm{E}$