Set 04: Pseudo-Random Generators, and Computational Lower Bounds

CS240: Data Structures and Data Management

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Outline

Pseudo-Random Number Generators
- In general
- Linear Congruential Generator

Computational Complexity Lower Bounds
- Motivation
- The Comparison Model
- Lower Bound Techniques: Comparison-based Search
- Application: Comparison-based Sorting

Summary about Algorithmic Analysis

Pseudo-Random Number Generators

- Computers are built to be deterministic
- A Real Random source is not always necessary.
- Independant "Pseudo-Random" sources are often sufficient.

Requirements of a Pseudo-Random source

We should not be able to observe any biases such as:

- Some numbers occur significantly more often than other numbers
- Odd numbers occur significantly more often than even numbers
- The sequence contains arithmetic progressions that are longer or shorted than would be expected by chance
- . . .

Example: Linear Congruential Generator

- Initialize a variable *seed* to an independant value (computer clock).
- Generate pseudo-random numbers as follows:

```
seed = ( a*seed + c ) mod max
return seed
```

Disadvantages of this Generator:

- Values of a, c, and *max* must be set with care.
- Value of *max* must be large.

In practice, more complicated functions are used, but the principle stays the same.

Disadvantages of Randomized Algorithms

- Computer Architecture tries to predict the future behavior of algorithms to optimize its execution.
- Algorithms randomized at the lowest level lose the benefit of this optimisation.
- Randomization usefull only for very hard problems.

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Computational Complexity Lower Bounds: Why?

Given a problem to solve,

- Till when can you improve your algorithm?
- Where can you improve your technique?

For difficult problems where lower bounds are not known, other techniques ⇒ NP-hardness.

Example

- Is $O(n \log n)$ optimal for sorting?
- Is $O(2^n)$ optimal for solving $f(x_1, \ldots, x_n) = 1$?

The Comparison Model

When you don't manage to prove results on all possible algorithms, consider a restricted class of algorithms.

Definition

An algorithm is in the Comparison Model if it access the data only using the following operations:

- comparing two elements.
- moving elements around.

This is a common example of model.
Other models are used for more specific problems.

Example

- Selection Sort is in the Comparison Model:
- Merge Sort is in the Comparison Model:
- Counting Sort is in the Comparison Model:

Comparison-based Search

Consider a sorted array A of n elements, and an element x.
Is x in A?

$$x = 76$$

3	18	$\cdots$	169	453

Which algorithm to use?

How many comparisons are necessary to answer?

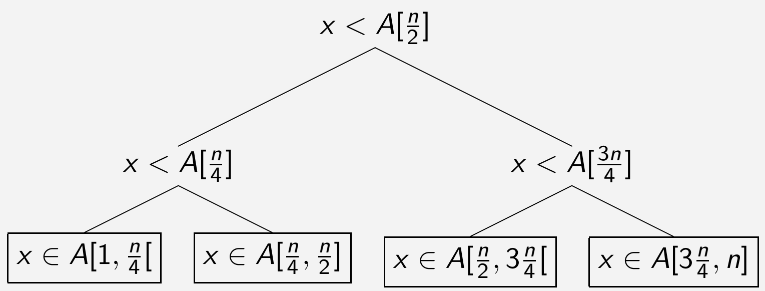
Deterministic Lower Bound: Adversary Strategy

Define a game between the deterministic algorithm A and an adversary E.

- A doesn't see the array, compares x to values of the array.
- E can choose the content as long as it is coherent with its past answers.
- E maintains two positions l and r s.t. $A[l] < x < A[r]$.
- When A asks if $x > A[k]$:
  - E forces A to search in the largest interval:
  - E answers yes if $r - k > k - l$, no otherwise
- A will have to perform $1 + \log_2 n$ comparisons. □

Det/Rand Lower Bound: Decision Tree

Consider the comparisons performed by a binary search:


$x < A[\frac{n}{2}]$
$x < A[\frac{n}{4}]$
$x < A[\frac{3n}{4}]$
$x \in A[1, \frac{n}{4}[$
$x \in A[\frac{n}{4}, \frac{n}{2}]$
$x \in A[\frac{n}{2}, 3\frac{n}{4}[$
$x \in A[3\frac{n}{4}, n]$

They form a Decision Tree: a binary tree where the leaves corresponds to a possible result.
For any such tree:

- height is
- average branch length is

Randomized Lower Bounds

In most interesting problems, the tree is not as nice.

Example

Consider an element x, and k sorted array $A_1, \ldots, A_k$ of n elements each. Is x in $A_1 \cap \ldots A_k$?

$$x = 76$$

A	3	18	$\cdots$	169	453
B	22	37	$\cdots$	102	103
C	1	12	$\cdots$	982	1200
D	5	12	$\cdots$	300	389

This is called the (elementary) intersection problem.
It has application in databases, and search engines.

Minimax Principle, Yao-von Neumann Theorem

Theorem (Yao Principle)

The average complexity of the best randomized algorithm
on the worst instance
is equal to
the complexity of the best deterministic algorithm
on the worst probability distribution.

$\Rightarrow$ Just define the worst probability distribution you can.

Example

A	3	18	$\cdots$	76	$\cdots$	169	453
B	22	37	$\cdots$	76	$\cdots$	102	103
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Lower bound for Comparison-based Sorting

Many Sorting Algorithms:

Sort	Running time	Analysis
Selection Sort		worst-case
Merge Sort		worst-case
Heap Sort		worst-case
Deterministic Quick Sort Randomized Quick Sort		worst-case expected

Theorem

Any correct comparison-based sorting algorithm requires at least comparison operations.

Proof: lower bound

Proof.

- Any algorithm sorting an array of n elements will have $n!$ possible outputs.
- The corresponding decision tree has $n!$ leaves, and height at least $\log(n!)$.
- $\log(n!) \approx n \lg(n) - n + \frac{\ln n}{2} + \frac{\ln 2\pi}{2}$ [Stirling].

Any comparison based algorithm performs $\Omega(n \log n)$ comparisons to sort an array of n elements. □

Similar technique used to prove a lower bound on *randomized* algorithms.

Summary

- Many Techniques to prove Lower Bounds and optimality:

- comparison-based search is .
- comparison-based sort is .

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	Topic	Concept(s)
1	Math Fundamentals	
2	Asymptotic Notations Selection Sort Merge Sort Counting Sort	
3	Average Case Analysis Randomized Algorithms	
4	Simple Lower Bounds Application to Sorting	

Summary about Algorithmic Analysis

	Topic	Concept(s)
1	Math Fundamentals	log, $\sum$
2	Asymptotic Notations Selection Sort Merge Sort Counting Sort	$O(), \Omega(), \Theta(), o(), \omega()$ $O(n^2)$ $O(n \lg n)$ $O(n)$
3	Average Case Analysis Randomized Algorithms	$E(\sum_i X_i) = \sum_i E(X_i)$ prob. dist. on det. algos.
4	Simple Lower Bounds Application to Sorting	Decision Tree Comparison model, $\Omega(n \lg n)$

Summary about Algorithmic Analysis (cont')

- Asymptotic Analysis simplifies the study,
- Worst case and Average case analysis simplify it further.
- Randomized Algorithms is linked to Average Analysis.
- Many Techniques to prove Lower Bounds and optimality.

Reading Materials

	Topic	GT	CLRS
1	Math Fundamentals	4-9, 21-24	5-12, 21-22, 51-56
2	Asymptotic Notations Selection Sort Merge Sort Counting Sort	13-20 - 219-224, 263-273 235-238	15-26, 41-50 15-20 62-66 168-170
3	Average Case Analysis Randomized Algorithms	235-237 238	91-114, 145-164 (same)
4	Simple Lower Bounds Application to Sorting	239-240 (same)	- 165-168

- GT = Algorithm Design, by Goodrich & Tamassia
- CLRS = Introduction to Algorithms, by Cormen, Leisersen, Rivest & Stein