Set 12: Ordered Dictionary Abstract Data Types: $(2,4)$ Trees, B Trees

CS240: Data Structures and Data Management

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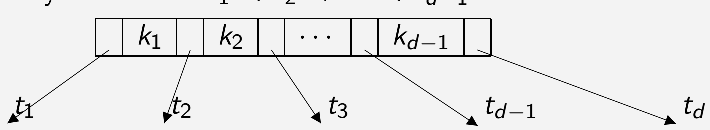
Outline

Multi-way search trees

Definition

A d-node is an internal node with

- d children, $t_1, \ldots, t_d$, and
- $d-1$ keys such that $k_1 < k_2 < \ldots < k_{d-1}$.

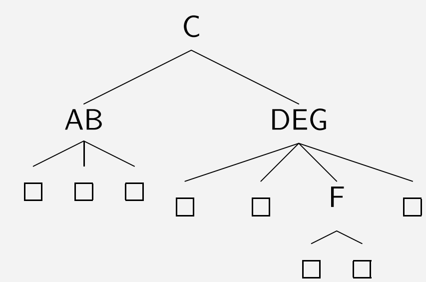


This generalizes binary search trees to larger degrees.

Multi-Way Search Trees

Definition

A Multi-Way Search Tree is an ordered search tree consisting of linked d-nodes, where each node may have a different value for d:



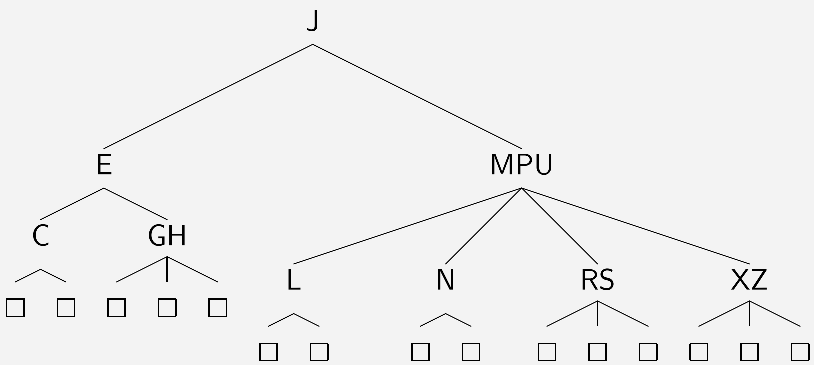
It is searched almost like a binary tree.

$(2,4)$-Trees

Definition

A (2,4)-tree is a multi-way search tree such that

- Every node has between 2 and 4 children
- All external nodes have the same depth



$(2,4)$-Trees

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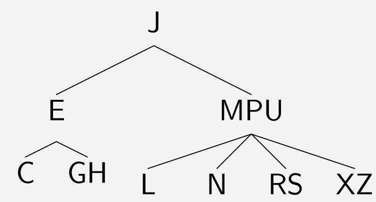


Note: As all external nodes have the same depth, placeholders don't carry much information anymore, and can be omitted.

Properties

Theorem

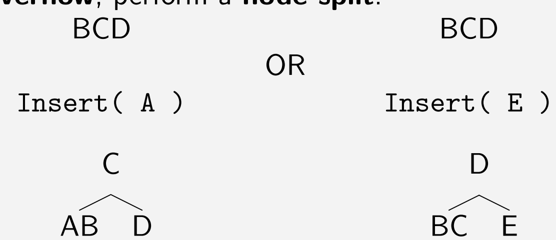
Consider a $(2,4)$*-tree with* n *internal keys:*

1. *The number of external placeholders is* $|E| = n + 1$.
2. *The height* h *is* $\Theta(\log n)$.

Proof: Exercise.

Insertion

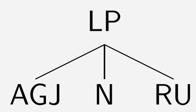
- Find deepest node where the key belongs, and insert.
- If **overflow**, perform a **node split**:



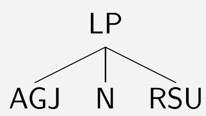
- The third element (counting the new element) moves up to parent, possibly causing a new overflow.

Insertion

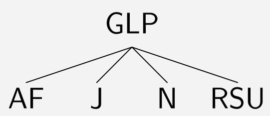
Example



`Insert( S )`

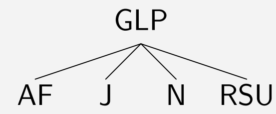


`Insert( F )`

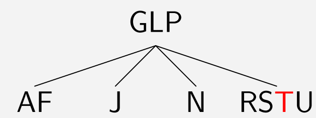


Insertion

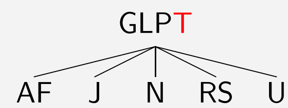
Example (cont)



`Insert( T )`



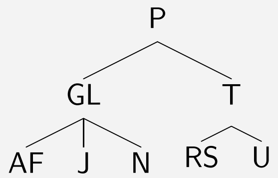
Overflow in RSTU!



Overflow in GLPT!

Insertion

Example (end)

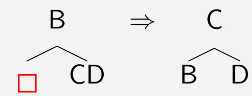


Theorem

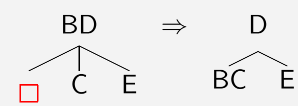
Insertion requires at most $h \in O(\lg n)$ *operations.*

Deletion

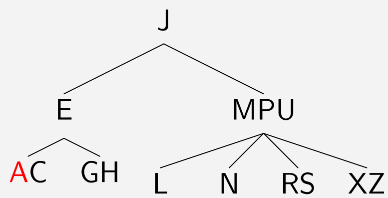
- Search for the key
- As in an AVL:
  - if it has no children, remove it.
  - if it has children, replace it with in-order predecessor or successor.
- If too few keys (**underflow**), then
  - **Transfer** a node from an *immediate sibling* if possible



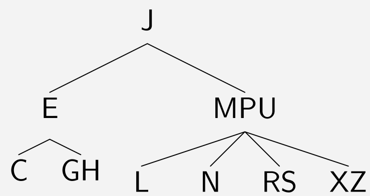
  - Otherwise, **fuse** with an *immediate sibling* and parent element



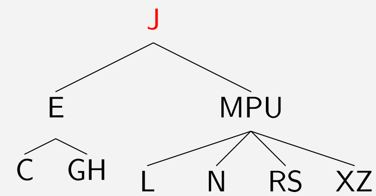
Deletion
Example



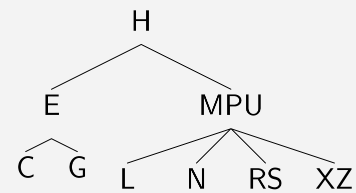
`Delete( A )` Simply remove A



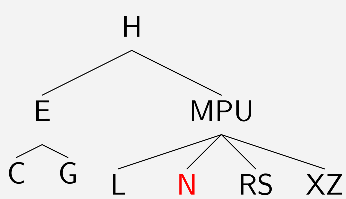
Deletion
Example (cont)



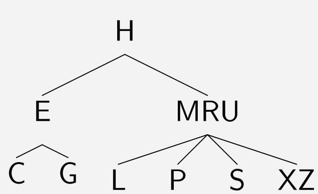
`Delete( J )` Replace J with H



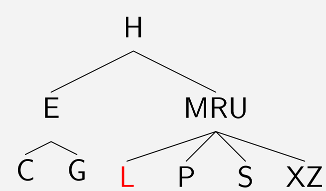
Deletion
Example (cont)



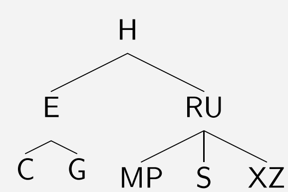
`Delete( N )` Transfer from RS



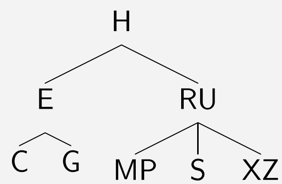
Deletion
Example (cont)



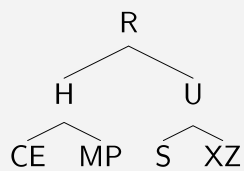
`Delete( L )` Fusion MRP and split



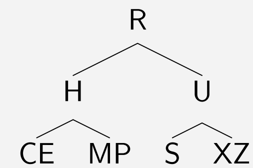
Deletion
Example (cont)



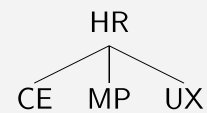
`Delete( G )` Fusion CE + Underflow + transfer from RU



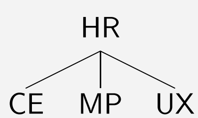
Deletion
Example (cont)



`Delete( Z )` immediate
`Delete( S )` Fusion UX + Underflow + Fusion HR



Deletion
Example (end)



Theorem
Deletion requires at most $h \in O(\lg n)$ operations.

Summary for $(2, 4)$-trees

- search easy
- insert may involve several splittings.
- deletion may involve one transfer, or several fusions.
- h increases only if root is split.
- h decreases only if root's sibling's fuse,
  and root becomes empty.
- $O(\log n)$ since constant amount of work at each node.

Outline

B-Trees

Definition

- A generalization of $(2, 4)$-trees
- A B-Tree of order d ($d \geq 3$) is a multi-way search such that
  - every node has $\leq d$ children
  - every *non-root* node has $\geq \lceil \frac{d}{2} \rceil$ children
  - all the external nodes have the same depth
- Often called an (a, b)-tree where $a = \lceil \frac{d}{2} \rceil$ and $b = d$
- The operations are performed the same as before
  - For overflow we promote element $\lceil \frac{d+1}{2} \rceil$ (counting the new element)

B-Tree of Order 6

Example

Also known as a $(3, 6)$-tree:

BCDEF

- `Insert( A )`

D

ABC EF

- `Delete ( F )`

C

AB DF

- `Delete ( B )`

ACDF

Motivations

External Searching

- What if the dictionary cannot fit in main memory?
- Need to store data in persistent memory (e.g. on disks)
- A single access to the data structure takes much longer
  - RAM Seek – 100 000 memory accesses
  - Disk Seek – for 1 disk access
- A disk access brings in a whole page of data
- Assume we can fit B dictionary elements per page: How many disk accesses would binary search (or an AVL tree) require?

$$\lg n - \lg B$$

B-Trees

- **Problem**: One disk access only cuts range of keys in half
- **Solution**: Use a B-Tree of order d
  - Choose d such that one d-node fills exactly one disk page
  - One disk access narrows search a lot more
- Searching the d-node is still far less expensive than bringing it into memory
- Running time is proportional to the number of blocks read
  - Insert and delete designed to reduce the number of d-nodes examined

Performance

- Suppose $d = 256$
- Minimum and maximum number of keys *found at each depth*:

Depth	Minimum # Keys	Maximum # Keys
0	1	255
1	254	65,280
2	32,512	16,711,680
3	4,161,536	4,278,190,080
4	532,676,608	1.1×10^{12}

Property

Theorem

The height h of a B-tree of order d is

- $\Omega(\ \log_d(n)\ )$
- $O(\ log_{\lceil \frac{d}{2} \rceil}(n)\ )$

Proof.

We know $2\lceil \frac{d}{2} \rceil^{h-1} \leq |E| \leq d^h$ combined with $|E| = n+1$ gives the result. □

Improvements

One-Pass Update

One-Pass Update

- Insert and delete require two passes
  - First pass down tree finds the bottom level node
  - Second pass up tree performs splitting or fusing
- Algorithm can be reworked to perform preemptive splits and fusions on the way down
- See CLRS textbook for details
  - You will never need to perform this in our class

Improvements

B^*-Tree

B*-Tree

- Each non-root *d*-node could have as low as 50% utilization
- On average a node is 69% filled
- We could insist that a non-root node is at least $\frac{2}{3}$ filled
- More difficult to do a node split or fusion
- A B^*-Tree node is 90% filled on average

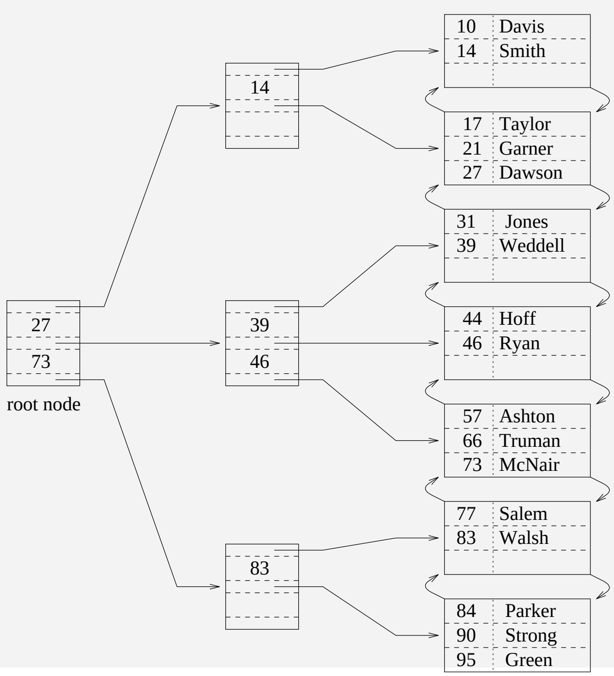
Improvements

B^+-Tree

B$^+$-Tree

- Desire greatest branching possible
- Internal nodes contain only keys (not the corresponding satellite data)
- Bottom level of tree contains the real key-data pair
- We also wish to perform **Range Queries**
  - List all professors with id-numbers between 39 and 75
- Each node has a pointer to the next and previous bottom level page

B$^+$-Tree



Summary for B-trees

- Generalisation of $(2,4)$-trees
- Usefull for large dictionaries, which does not fit in memory.
- Several variants, corresponding to various needs.
- Very important in practical Databases.

Outline

(2, 4) trees
- Definitions
- Properties
- Insertion
- Deletion

B-Trees
- Definition
- Motivations
- Improvements

Conclusion to Ordered Dictionary ADTs
- Concepts
- References

Ordered Dictionary ADTs and their DS

- Array Compact
- Binary Search Tree (BST) Fast for static
- Sequence (Skip Lists) Dynamic
- AVL Fast and Dynamic
- $(2, 4)$ Trees Fast and "amortized" dynamic
- B-Trees extension for memory issues.

Diferent solutions to different problems...

References

	GT	CLRS
Arrays	pp. 140-151	pp. 253-264
BST		
Skiplists	pp. 195-202	Not covered.
AVL	pp. 152-158	pp. 296 (poorly covered)
$(2, 4)$-trees	pp. 159-169	pp, 434-452 (indirectly)
B-Trees	pp. 649-653	pp. 434-452