Set 06: Trees

CS240: Data Structures and Data Management

Jérémy Barbay

Outline

Tree ADT

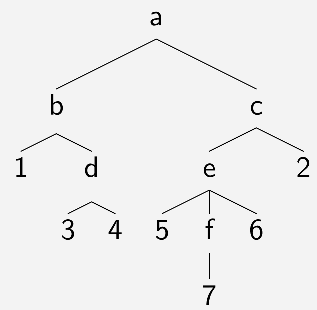
- Operations
  - `root()`, `size()`
  - `isInternal( node )`, `children( node )`, `parent( node )`
  - `attachSubtree( node, tree )`, `detachSubtree( node )`
- Use trees to implement other ADTs

Definitions

- Recursive: A finite collection of nodes (at least one) that is
  1. A single distinguished node called the root or
  2. Partitioned into $k+1$ subcollections: a designated root node connected together with k trees, $T_1 \ldots T_k$, by an edge
- Graph which is Rooted, connected and acyclic
- List of nodes and oriented edges s.t. all nodes but one have a parent node.

Terminology

- parent, child, sibling, subtree
- ancestor, descendent (**note**: a node is its own ancestor and descendent)
- external node (leaf)
- internal node



Applications

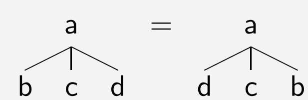
- Representing Hierarhies: Genealogical tree.
- DNS in Networking.
- Modelisation of algorithms (Merge Sort, Comparison based searching).
- Parsing (Arithmetic expression, LaTeX, XML).
- Codes (Huffman).

Depth/Height

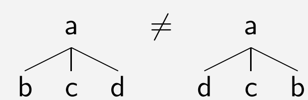
- Node Depth – The number of edges between the node, and the root of the entire tree
- Node Height – The maximum number of edges between the node and any of its descendants
- Note:
  - $\text{DEPTH}(\textit{root}) = 0$
  - $\text{HEIGHT}(\textit{leaf}) = 0$
- **Exercise**: How do we compute each of these for a given node?

Tree variants

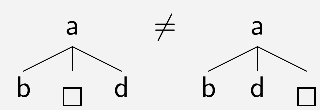
- Unordered – like a graph



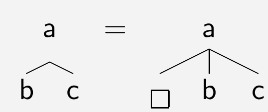
- Ordered – linear ordering on the children (first, second, . . .)



- Cardinal – children identified by their absolute position.



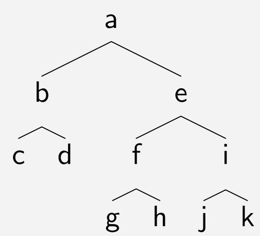
- Ordinal – children identified by their rank.



Tree variants (cont')

- Binary – each node has at most 2 children (cardinal tree)
- Proper Binary – each node has 0 or 2 children
- Full Binary – proper binary tree, all leaves at the same level

Example



This tree is:

- binary True
- Proper Binary True
- Full Binary False

We will study more binary trees with

- Priority Queue ADT (Heaps)
- Ordered Dictionaries ADT (AVL Trees)

Binary Trees

Binary Tree Data Structures

- Linked Structure
  - Tree Node with 4 fields

parent	
data	
`left`	`right`

  - Parent is optional
- Array
  - An array of size 2^{h+1}, from 1 to 2^{h+1}.
  - children of cell i at positions $2i$ and $2i+1$.
  - Special value indicates no node.
  - More on this with `heaps`.
- There are more sophisticated ones...

Properties of Binary Trees

Theorem

Let $|E|$ and $|I|$ represent the number of external and internal nodes respectively in a proper binary tree. Then

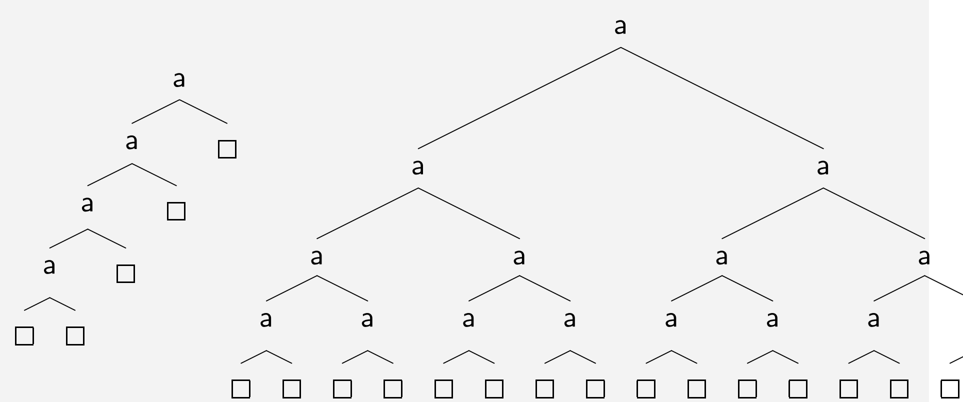
$$|E| = |I| + 1$$

Proof: By Induction on $|I|$
Base Case(s): $|I| = 0$ and $|I| = 1$
Inductive Cases:

1. Root node has one internal child
2. Root node has two internal children

More Properties of proper binary trees

- $h + 1 \leq |E| \leq 2^h$

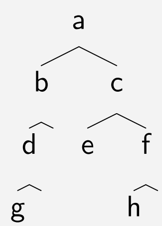


- As $n = |E| + |I|$, this gives bounds on $|I|$, n and h.

Recursive General Traversal

General Traverse(*node*)

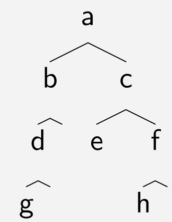
```
Visit node
if node has left child then
   TRAVERSE( node.left )
end if
Visit node
if node has right child then
   TRAVERSE( node.right )
end if
Visit node
```



abdgggddbbaceeecfhhhffca

This algorithm is easily modified for other depth first traversals, or for trees of higher degree.

More specific Traversals



- Depth-First Traversal
  - General Traversal: **abdgggddbbaceeecfhhhffca**
  - Pre-Order: **abdgcefh**
  - In-Order: **gdbaechf**
  - Post-Order: **gdbehfca**
- Breadth-First Traversal: **abcdgefh**
  - Level-Order: **abcdefgh**

Representation of a binary tree

How can we prove if a trace identifies a tree?

- When tree is identified: Method to build the tree.
- Otherwise: Two distinct trees with same trace.

Which traversal permit to identify a binary tree by the trace?

1. general: True
2. pre-order: False
3. in-order: False
4. post-order: False
5. breadth-first order: False
6. level-order: False

Summary

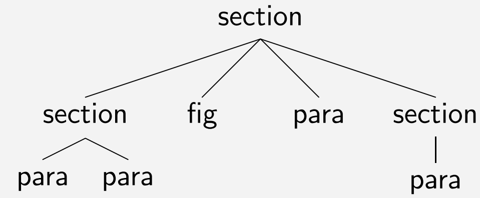
- The Tree ADT define
  - operators for navigation and construction;
  - terms: Height, Depth, ...
  - properties
  - with many variants: Cardinal/Ordinal, ...
- The Binary Tree is a particular cardinal variant, which will be studied more in details later.

Outline

Tree Encodings

Documents structured as a tree

Some trees represent static documents, which must be stored.



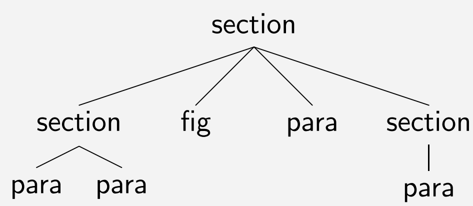
How do we encode a tree?

- Dynamically, with one data array per node
- Dynamically, but with one number per node
- Statically, but how?

XML notation

How to exchange trees between applications.

```
<section>
  <section>
    <para> (...) </para>
    <para> (...) </para>
  </section>
  <fig> (...) </fig>
  <para> (...) </para>
  <section>
    <para> (...) </para>
  </section>
</section>
```



Totally specifies an ordinal tree? True

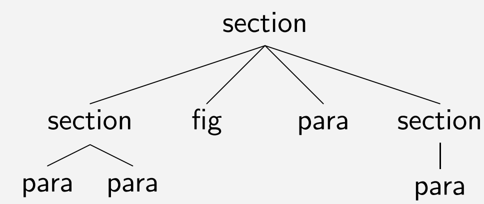
Applications?

XHTML, future standard of the web?

Exchange format between XML and Gnumeric.

\Tree notation

```
\Tree
[ .{section}
  [ .{section}
    {para}
    {para}
  ]
  {fig}
  {para}
  [ .{section}
    {para}
  ]
]
```



Totally specifies an ordinal tree? True

Applications?

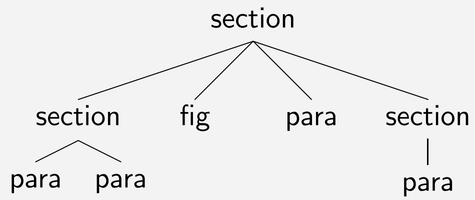
Used in LaTeX to draw trees.

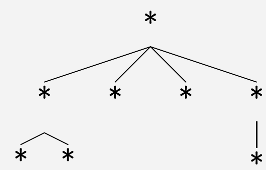
Description of trees as input to project.

Separating structure from content

Encode separately

- the structure





- from the content *ssppfpsp*

How much space do we need to encode each part?

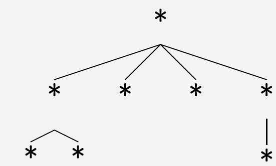
$n \lg \sigma$ bits or n words for the content,
where n is the number of nodes and σ is number of distinct labels.

Structural Encoding of Ordinal Trees

Theorem
An ordinal tree of n nodes can be encoded in $2n$ bits.

Transpose(x)

print "("
for each child c of x **do**
  Transpose(c)
end for
print ")"



((( ) ()) () () (()))

Exercises

1. How do we build the tree from the string?
2. Can we do the same for binary (cardinal) trees?

Outline

Simple Tree ADT
  Definitions
  Binary Trees

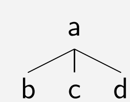
Tree Encodings
  Separating structure from content
  Structural Encodings

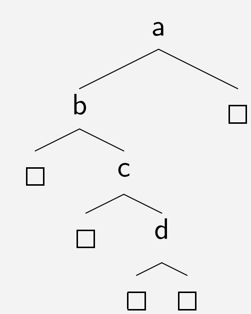
Binary Representation of Ordinal Trees (and vice-versa)
  The Theory
  Exercise
  Operations

Binary Tree Representation of Ordinal Trees

Theorem
An ordinal tree T can be represented by a (cardinal) binary tree T'.

- For each internal node $v \in T$, an internal node $v' \in T'$
- If v has an immediate sibling w, then w' is the right child of v'
- If v has first child w, then w' is the left child of v'
- Fill all other spots with empty external nodes (i.e. leaves).



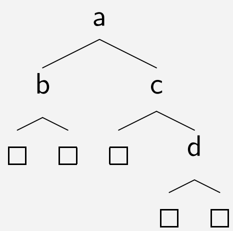


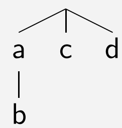
Ordinal Tree Representation of Binary Trees

Theorem

A (cardinal) binary tree T can be represented by forest *of ordinal trees.*

- Two-by-two correspondance between internal nodes.
- The right child of v is the sibling of v'.
- The left child of v is the first child of v'.
- Ignore empty subtrees,

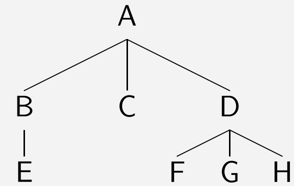




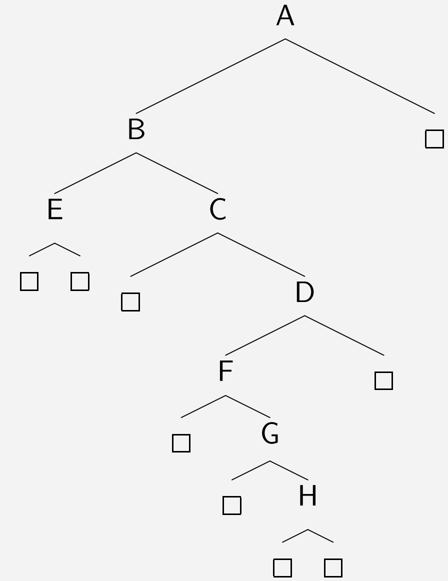
Exercise

Ordinal to Binary

Original Tree T:

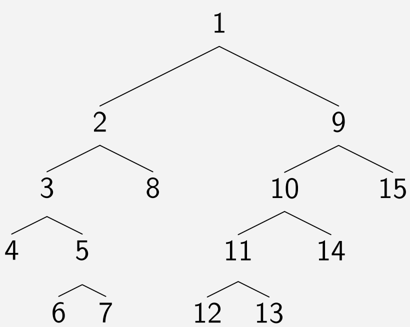


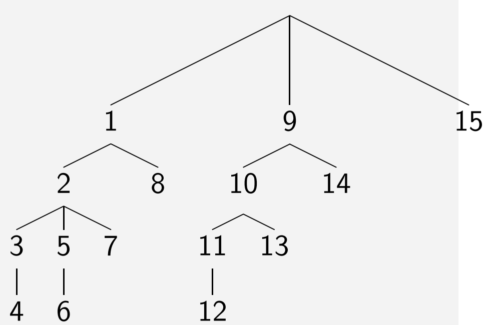
Binary-Tree Representation T':



Exercise

Binary to Ordinal





Operations on this representation

Given an ordinal tree representated in a binary tree:

1. How to compute the height? By counting the maximum number of left edges on a rooted path. Complexity $O(n)$.
2. How to compute the maximum degree? By counting the maximum number of right edges on a rooted path. Complexity $O(n)$.

Exercise:

Given a binary tree representated as a forest of ordinal trees, how to compute the height?