CS 245 — Assignment #6
Spring 2006

Due Date: Tuesday, July 4 at 5pm.

Use `makeCover` to produce a cover page for your assignment and hand in your assignment in the CS 245 assignment box. Assignments are to be done individually.

1. (10 points) For each of the following $p(n)$, prove $\forall n : \mathbb{N} \bullet p(n)$ using mathematical induction.

   (a) Let $p(n)$ be $\exists c \bullet T(n) \le c(n^2 + n)$, for $n \ge 1$, where

   $$\begin{array}{lll} \mathrm{T}(n) & = 1 & \text{if } n = 1 \\ \mathrm{T}(n) & = \mathrm{T}(n-1) + n & \text{if } n > 1 \end{array}$$

   Base case: Prove $p(1)$

   $$\begin{array}{l} \exists c \bullet T(1) \le c \cdot 2 \\ \quad \Longleftrightarrow \ \exists c \bullet 1 \le c \cdot 2 \\ \quad \Longleftrightarrow \ \textbf{true} \end{array}$$

   Inductive case: Prove $p(k) \Rightarrow p(k+1)$
   I.e., prove $(\exists c \bullet T(k) \le c \cdot (k^2 + k)) \Rightarrow (\exists c \bullet T(k+1) \le c \cdot ((k+1)^2 + (k+1)))$

   $$\begin{array}{llll}
   1. & & \exists c \bullet T(k) \le c \cdot (k^2 + k) & \text{assumption (induction hypothesis)} \\
   2. & c_u & T(k) \le c_u \cdot (k^2 + k) & \text{assumption, using 1} \\
   3. & & T(k+1) = T(k+1) & = _\mathrm{I} \\
   4. & & T(k+1) = T(k) + k + 1 & \text{definition} \\
   5. & & T(k+1) \le c_u \cdot (k^2 + k) + k + 1 & 2, 4, = _\mathrm{E} \\
   6. & & T(k+1) \le c_u \cdot (k^2 + 2k + 1) + k + 1 & 5, \text{algebra} \\
   7. & & T(k+1) \le c_u \cdot (k+1)^2 + k + 1 & 6, \text{algebra} \\
   8. & & T(k+1) \le c_u \cdot ((k+1)^2 + k + 1) & 7, \text{algebra} \\
   9. & & \exists c \bullet T(k+1) \le c((k+1)^2 + (k+1)) & 8, \exists_\mathrm{I} \\
   10. & & \exists c \bullet T(k+1) \le c((k+1)^2 + (k+1)) & 2-9, \exists_\mathrm{E} \\
   11. & & p(k) \Rightarrow p(k+1) & 1-10, \Rightarrow_\mathrm{I}
   \end{array}$$

(b) Let $p(n)$ be $\exists b \bullet \exists c \bullet T(2^n) = b2^n - c$, for $n \geq 0$, where

$$
\begin{array}{ll}
\mathrm{T}(2^n) = 1 & \text{if } n = 0 \\
\mathrm{T}(2^n) = \mathrm{T}(2^{n-1}) + \mathrm{T}(2^{n-1}) + 1 & \text{if } n > 0
\end{array}
$$

Base case: Prove $p(2^0)$

$\exists b \bullet \exists c \bullet T(2^0) = b2^0 - c$

$\iff \exists b \bullet \exists c \bullet 1 = b - c$

$\iff$ **true**

Inductive case: Prove $p(k) \Rightarrow p(k+1)$

I.e., prove $(\exists b \bullet \exists c \bullet T(2^k) = b2^k - c) \Rightarrow (\exists b \bullet \exists c \bullet T(2^{k+1}) = b2^{k+1} - c)$

1.		$\exists b \bullet \exists c \bullet T(2^k) = b2^k - c)$	assumption (induction hypothesis)
2.	b_u	$\exists c \bullet T(2^k) = b_u \cdot 2^k - c$	assumption, using 1
3.	c_u	$T(2^k) = b_u \cdot 2^k - c_u$	assumption, using 2
4.		$T(2^{k+1}) = T(2^{k+1})$	$=\_\mathrm{I}$
5.		$T(2^{k+1}) = T(2^k) + T(2^k) + 1$	definition
6.		$T(2^{k+1}) = (b_u \cdot 2^k - c_u) + (b_u \cdot 2^k - c_u) + 1$	$3, 5, =\_\mathrm{E}$
7.		$T(2^{k+1}) = 2 \cdot b_u \cdot 2^k - 2 \cdot c_u + 1$	6, algebra
8.		$T(2^{k+1}) = 2 \cdot b_u \cdot 2^k - (2 \cdot c_u - 1)$	7, algebra
9.		$T(2^{k+1}) = b_u \cdot 2^{k+1} - (2 \cdot c_u - 1)$	8, algebra
10.		$\exists c \bullet T(k+1) = b_u \cdot 2^{k+1} - c$	$9, \exists\_\mathrm{I}$
11.		$\exists c \bullet T(k+1) = b_u \cdot 2^{k+1} - c$	$3 - 10, \exists\_\mathrm{E}$
12.		$\exists b \bullet \exists c \bullet T(k+1) = b \cdot 2^{k+1} - c$	$11, \exists\_\mathrm{I}$
13.		$\exists b \bullet \exists c \bullet T(k+1) = b \cdot 2^{k+1} - c$	$11, \exists\_\mathrm{E}$
14.		$p(k) \Rightarrow p(k+1)$	$1 - 10, \Rightarrow\_\mathrm{I}$

2. (10 points) Prove the following set equalities and subset relations.

(a) $A \cup (A' \cap B) = A \cup B$

$$
\begin{array}{lll}
x \in A \cup (A' \cap B) & & \\
\iff & x \in A \vee x \in (A' \cap B) & \text{Set union} \\
\iff & x \in A \vee (x \in A' \wedge x \in B) & \text{Set intersection} \\
\iff & x \in A \vee (x \in U \wedge \neg(x \in A) \wedge x \in B) & \text{Set complement} \\
\iff & x \in A \vee (\neg(x \in A) \wedge x \in U \wedge x \in B) & \text{Commutativity} \\
\iff & (x \in A \vee \neg(x \in A)) \wedge (x \in A \vee (x \in U \wedge x \in B)) & \text{Distributivity} \\
\iff & \textbf{true} \wedge (x \in A \vee (x \in U \wedge x \in B)) & \text{Excluded Middle} \\
\iff & x \in A \vee (x \in U \wedge x \in B) & \text{Simplification I} \\
\iff & x \in A \vee x \in (U \cap B) & \text{Set intersection} \\
\iff & x \in (A \cup (U \cap B)) & \text{Set union} \\
\iff & x \in (A \cup B) & \text{Universal set identity}
\end{array}
$$

(b) $A \cap (A \cup B) \subseteq A$

Prove: $x \in (A \cap (A \cup B)) \Rightarrow x \in A$

$$
\begin{array}{lll}
\left[\begin{array}{lll}
1. & x \in (A \cap (A \cup B)) & \text{assumption} \\
2. & x \in A \wedge x \in (A \cup B) & 1, \text{set intersection} \\
3. & x \in A & 2, \wedge\text{\_E}
\end{array}\right. & & \\
4. \quad x \in (A \cap (A \cup B)) \Rightarrow x \in A & & 1-3, \Rightarrow \text{\_I}
\end{array}
$$

3. (10 points) Using appropriate sets, formalize the following sentences. Your answers must not contain logical quantifiers.

   (a) Computer Science students are not History students.

   $$CS \cap History = \emptyset$$

   (b) No student is a Psychology student and not an Arts student.

   $$Psychology \subseteq Arts$$

   (c) Any student who is not a Computer Science student is a Psychology student.

   $$CS' \subseteq Psychology$$

   (d) History students are Arts students.

   $$History \subseteq Arts$$

   Prove that (d) logically follows from (a)-(c) using natural deduction. In your proof you may use *any* of the inference rules from propositional and predicate logic and you may use any of the definitions given on the summary sheet for set theory.

1.	$CS \cap History = \emptyset$	premise
2.	$Psychology \subseteq Arts$	premise
3.	$CS' \subseteq Psychology$	premise
4.	$\forall x \bullet \neg(x \in (CS \cap History))$	$1,$ empty set
5.	$\forall x \bullet x \in Psychology \Rightarrow x \in Arts$	$2,$ subset
6.	$\forall x \bullet x \in CS' \Rightarrow x \in Psychology$	$3,$ subset
7.	$\forall x \bullet \neg(x \in CS \wedge x \in History)$	$4,$ intersection
8.	$\forall x \bullet \neg(x \in CS) \vee \neg(x \in History)$	$7,$ De Morgan
9.	$\forall x \bullet (x \in U \wedge \neg(x \in CS)) \Rightarrow x \in Psychology$	$6,$ set complement
10.	x_g	
11.	$x_g \in History$	assumption
12.	$x_g \in Psychology \Rightarrow x_g \in Arts$	$5, \forall$_E
13.	$\neg(x_g \in CS) \vee \neg(x_g \in History)$	$8, \forall$_E
14.	$(x_g \in U \wedge \neg(x_g \in CS)) \Rightarrow x_g \in Psychology$	$9, \forall$_E
15.	$\neg(x_g \in CS)$	$11, 13, \vee$_E
16.	$(x_g \in U)$	universal set, $\forall$_E
17.	$(x_g \in U) \wedge \neg(x_g \in CS)$	$15, 16, \wedge$_I
18.	$x_g \in Psychology$	$14, 17, \Rightarrow$ _E
19.	$x_g \in Arts$	$12, 18, \Rightarrow$ _E
20.	$x_g \in History \Rightarrow x_g \in Arts$	$11 - 19, \Rightarrow$ _I
21.	$\forall x \bullet x \in History \Rightarrow x \in Arts$	$10 - 20, \forall$_I
22.	$History \subseteq Arts$	$21,$ subset