Set 06: Trees

CS240: Data Structures and Data Management

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Outline

Simple Tree ADT
- Definitions
- Binary Trees

Tree Encodings
- Separating structure from content
- Structural Encodings

Binary Representation of Ordinal Trees (and vice-versa)
- The Theory
- Exercise
- Operations

Tree ADT

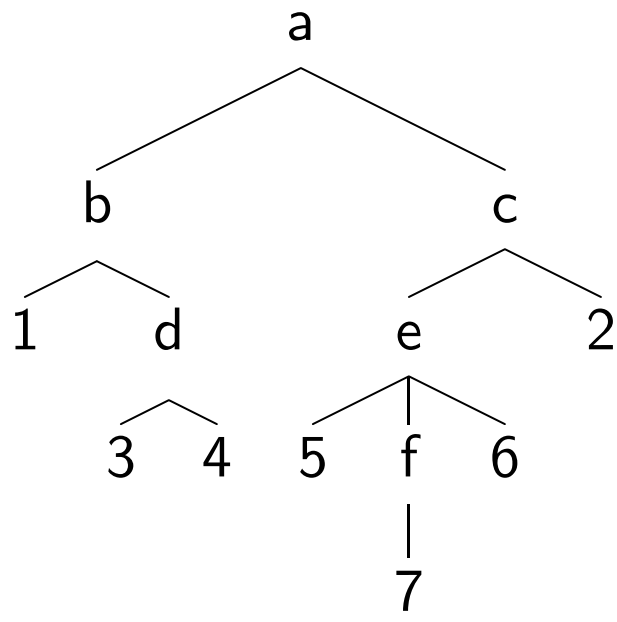
- Operations
  - `root()`, `size()`
  - `isInternal( node )`, `children( node )`, `parent( node )`
  - `attachSubtree( node, tree )`, `detachSubtree( node )`
- Use trees to implement other ADTs

Definitions

- Recursive: A finite collection of nodes (at least one) that is
  1. A single distinguished node called the root or
  2. Partitioned into $k + 1$ subcollections: a designated root node connected together with k trees, $T_1 \ldots T_k$, by an edge
- Graph which is Rooted, connected and acyclic
- List of nodes and oriented edges s.t. all nodes but one have a parent node.

Terminology

- parent, child, sibling, subtree
- ancestor, descendent
  (**note**: a node is its own ancestor and descendent)
- external node (leaf)
- internal node



Applications

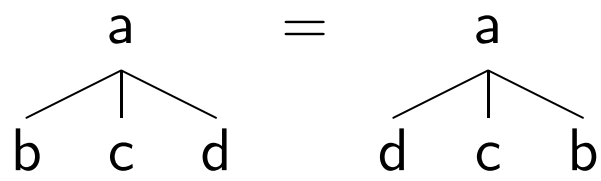
- Representing Hierarhies: Genealogical tree.
- DNS in Networking.
- Modelisation of algorithms (Merge Sort, Comparison based searching).
- Parsing (Arithmetic expression, LaTeX, XML).
- Codes (Huffman).

Depth/Height

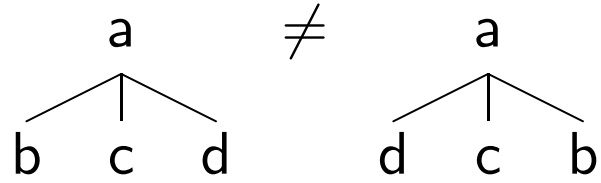
- Node Depth – The number of edges between the node, and the root of the entire tree
- Node Height – The maximum number of edges between the node and any of its descendants
- Note:
  - $\textsc{depth}(\textit{root}) = 0$
  - $\textsc{height}(\textit{leaf}) = 0$
- **Exercise**: How do we compute each of these for a given node?

Tree variants

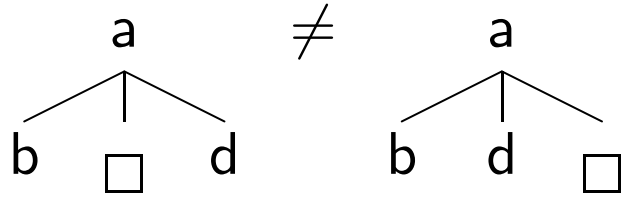
- Unordered – like a graph



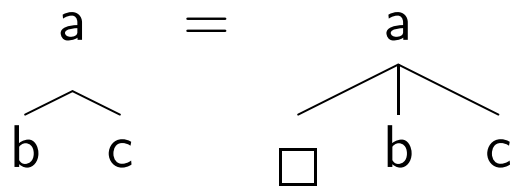
- Ordered – linear ordering on the children (first, second, . . .)



- Cardinal – children identified by their absolute position.



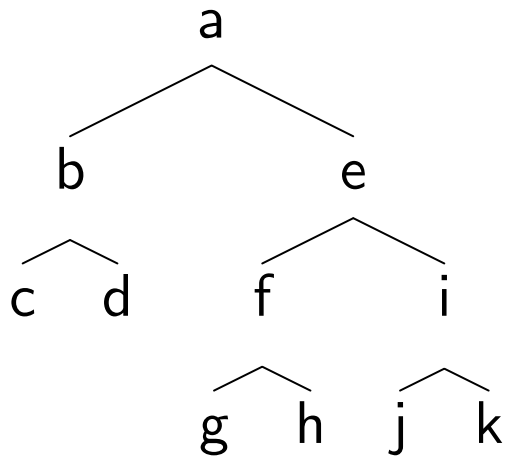
- Ordinal – children identified by their rank.



Tree variants (cont')

- Binary – each node has at most 2 children (cardinal tree)
- Proper Binary – each node has 0 or 2 children
- Full Binary – proper binary tree, all leaves at the same level

Example



This tree is:

- binary True
- Proper Binary True
- Full Binary False

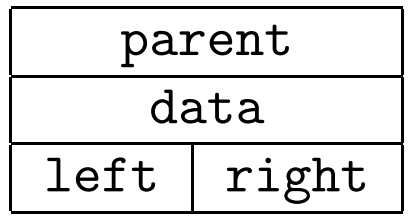
We will study more binary trees with

- Priority Queue ADT (Heaps)
- Ordered Dictionaries ADT (AVL Trees)

Binary Trees

Binary Tree Data Structures

- Linked Structure
  - Tree Node with 4 fields



  - Parent is optional
- Array
  - An array of size 2^{h+1}, from 1 to 2^{h+1}.
  - children of cell i at positions $2i$ and $2i + 1$.
  - Special value indicates no node.
  - More on this with `heaps`.
- There are more sophisticated ones...

Properties of Binary Trees

Theorem

Let $|E|$ and $|I|$ represent the number of external and internal nodes respectively in a proper binary tree. Then

$$|E| = |I| + 1$$

Proof: By Induction on $|I|$

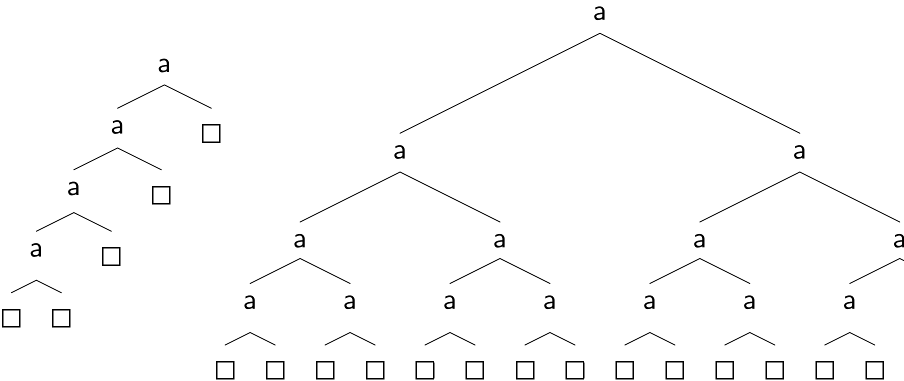
Base Case(s): $|I| = 0$ and $|I| = 1$

Inductive Cases:

1. Root node has one internal child
2. Root node has two internal children

More Properties of proper binary trees

- $h + 1 \leq |E| \leq 2^h$

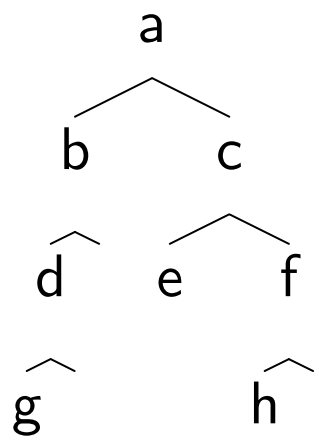


- As $n = |E| + |I|$, this gives bounds on $|I|$, n and h.

Recursive General Traversal

General Traverse(*node*)

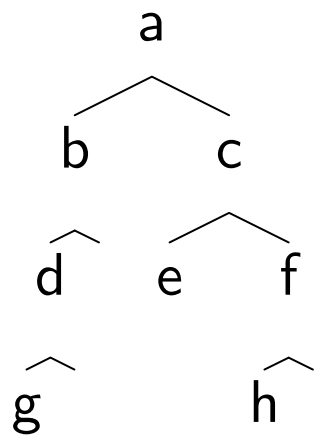
Visit *node*
if *node* has left child **then**
  TRAVERSE(*node.left*)
end if
Visit *node*
if *node* has right child **then**
  TRAVERSE(*node.right*)
end if
Visit *node*



abdgggddbbaceeecfhhhffca

This algorithm is easily modified for other depth first traversals, or for trees of higher degree.

More specific Traversals



- Depth-First Traversal
  - General Traversal: **abdgggddbbaceeecfhhhffca**
  - Pre-Order: **abdgcefh**
  - In-Order: **gdbaechf**
  - Post-Order: **gdbehfca**
- Breadth-First Traversal: **abcdgefh**
  - Level-Order: **abcdefgh**

Representation of a binary tree

How can we prove if a trace identifies a tree?

- When tree is identified: Method to build the tree.
- Otherwise: Two distinct trees with same trace.

Which traversal permit to identify a binary tree by the trace?

1. general: True
2. pre-order: False
3. in-order: False
4. post-order: False
5. breadth-first order: False
6. level-order: False

Summary

- The Tree ADT define
  - operators for navigation and construction;
  - terms: Height, Depth, ...
  - properties
  - with many variants: Cardinal/Ordinal, ...
- The Binary Tree is a particular cardinal variant, which will be studied more in details later.

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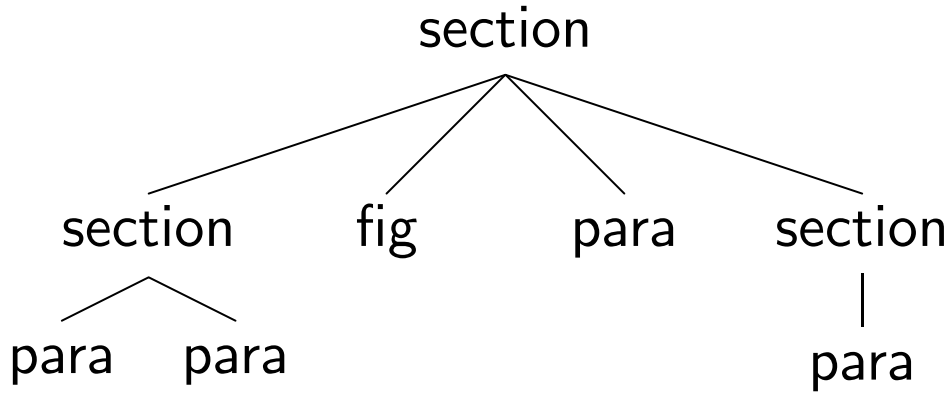
Tree Encodings

Documents structured as a tree

Some trees represent static documents, which must be stored.



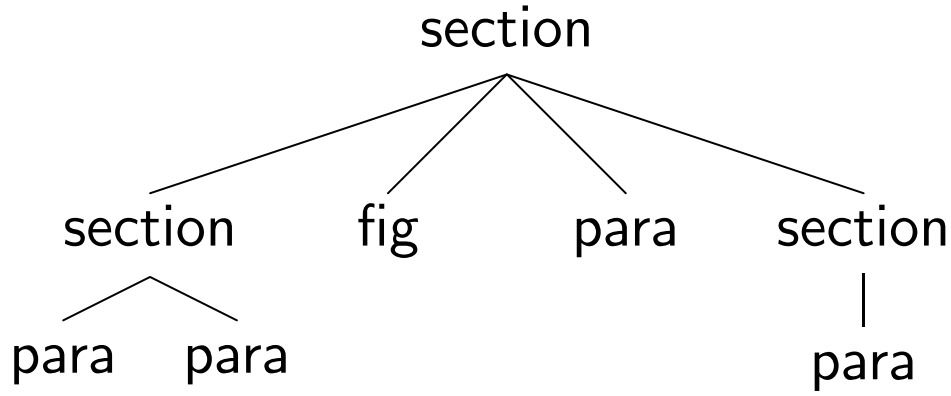
How do we encode a tree?

- Dynamically, with one data array per node
- Dynamically, but with one number per node
- Statically, but how?

XML notation

How to exchange trees between applications.

```
<section>
  <section>
    <para> (...) </para>
    <para> (...) </para>
  </section>
  <fig> (...) </fig>
  <para> (...) </para>
  <section>
    <para> (...) </para>
  </section>
</section>
```



Totally specifies an ordinal tree? True

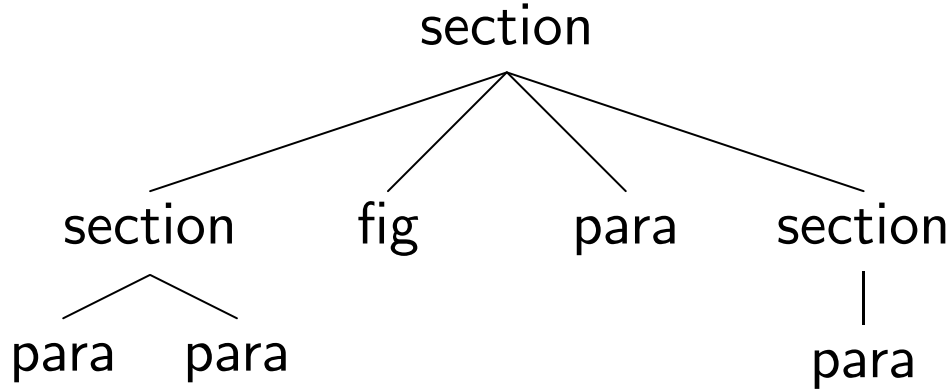
Applications?

XHTML, future standard of the web?

Exchange format between XML and Gnumeric.

\Tree notation

```
\Tree
[ .{section}
  [ .{section}
    {para}
    {para}
  ]
  {fig}
  {para}
  [ .{section}
    {para}
  ]
]
```



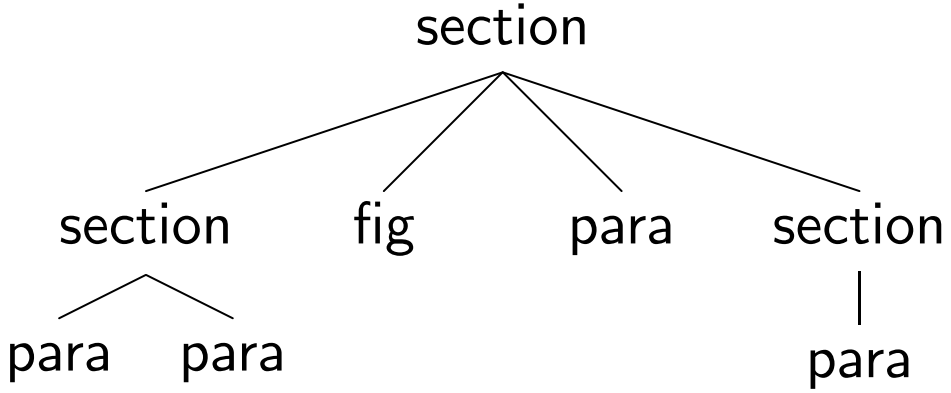
Totally specifies an ordinal tree? True

Applications?

Used in LaTeX to draw trees.

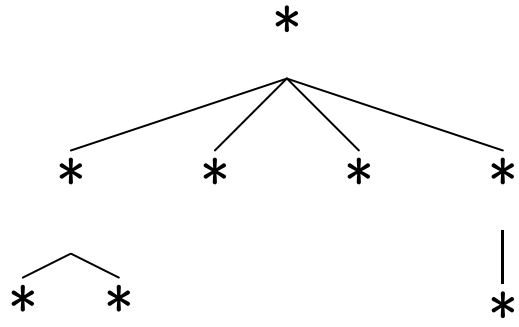
Description of trees as input to project.

Separating structure from content



Encode separately

- the structure



- from the content *ssppfpsp*

How much space do we need to encode each part?

$n \lg \sigma$ bits or n words for the content,
where n is the number of nodes and σ is number of distinct labels.

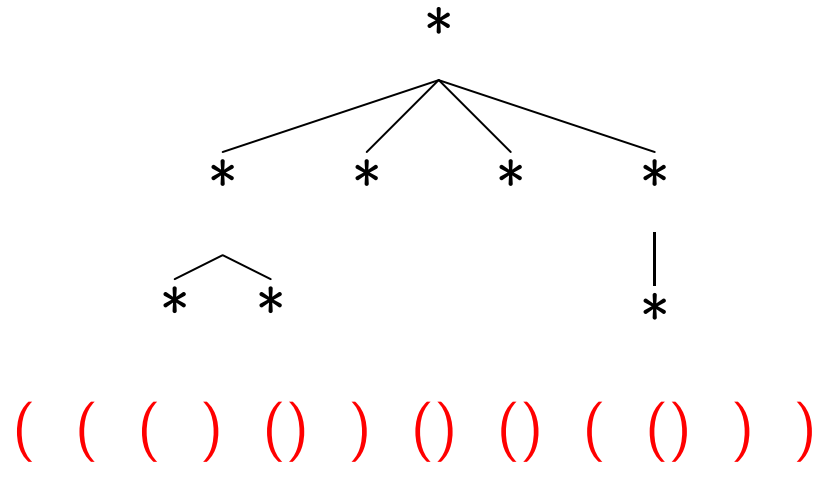
Structural Encoding of Ordinal Trees

Theorem

An ordinal tree of n nodes can be encoded in $2n$ bits.

`Transpose`(x)

print "("
for each child c of x **do**
  `Transpose`(c)
end for
print ")"



Exercises

1. How do we build the tree from the string?
2. Can we do the same for binary (cardinal) trees?

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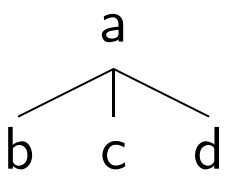
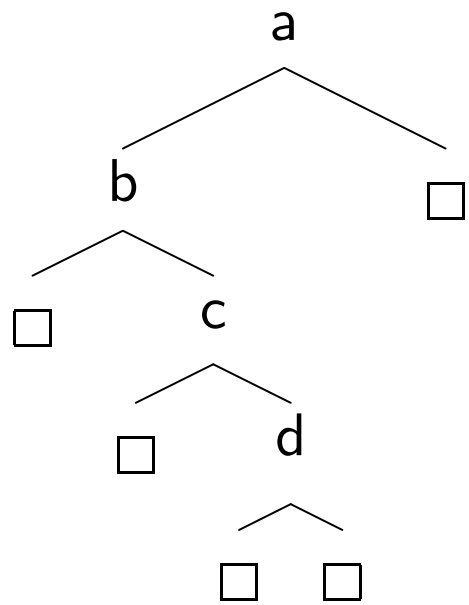
Binary Tree Representation of Ordinal Trees

Theorem

An ordinal tree T can be represented by a (cardinal) binary tree T'.

- For each internal node $v \in T$, an internal node $v' \in T'$
- If v has an immediate sibling w, then w' is the right child of v'
- If v has first child w, then w' is the left child of v'
- Fill all other spots with empty external nodes (i.e. leaves).



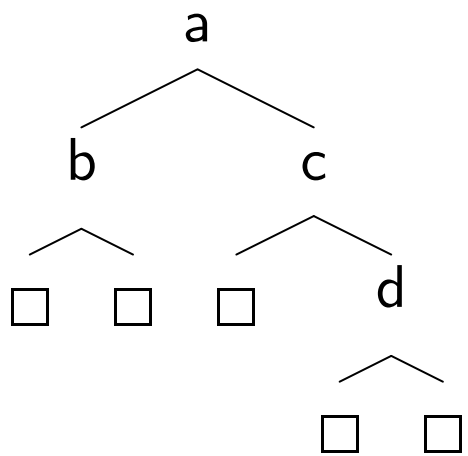


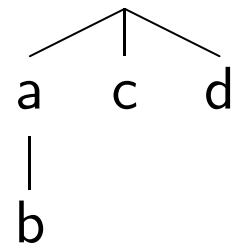
Ordinal Tree Representation of Binary Trees

Theorem

A (cardinal) binary tree T can be represented by forest of ordinal trees.

- Two-by-two correspondance between internal nodes.
- The right child of v is the sibling of v'.
- The left child of v is the first child of v'.
- Ignore empty subtrees,

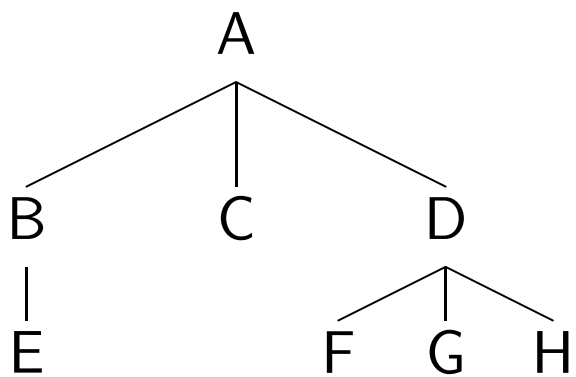




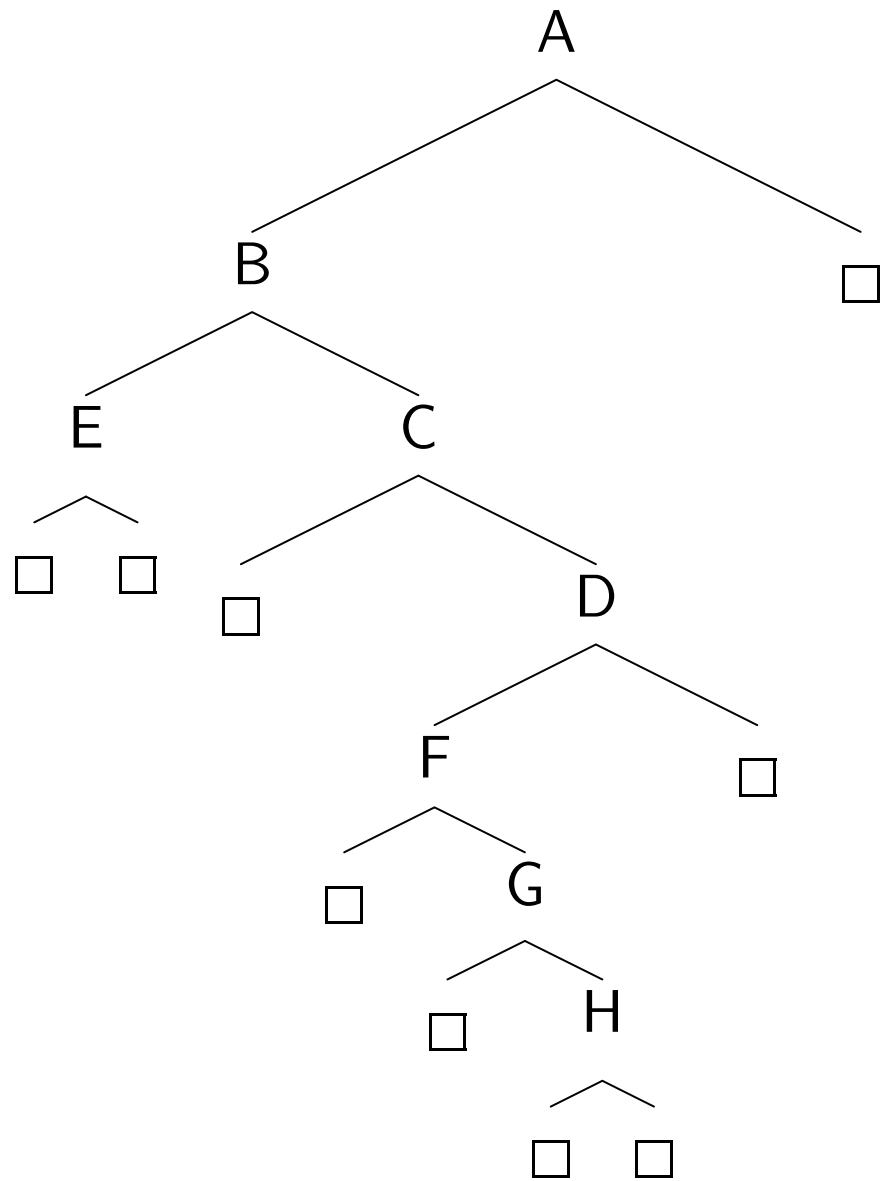
Exercise

Ordinal to Binary

Original Tree T:

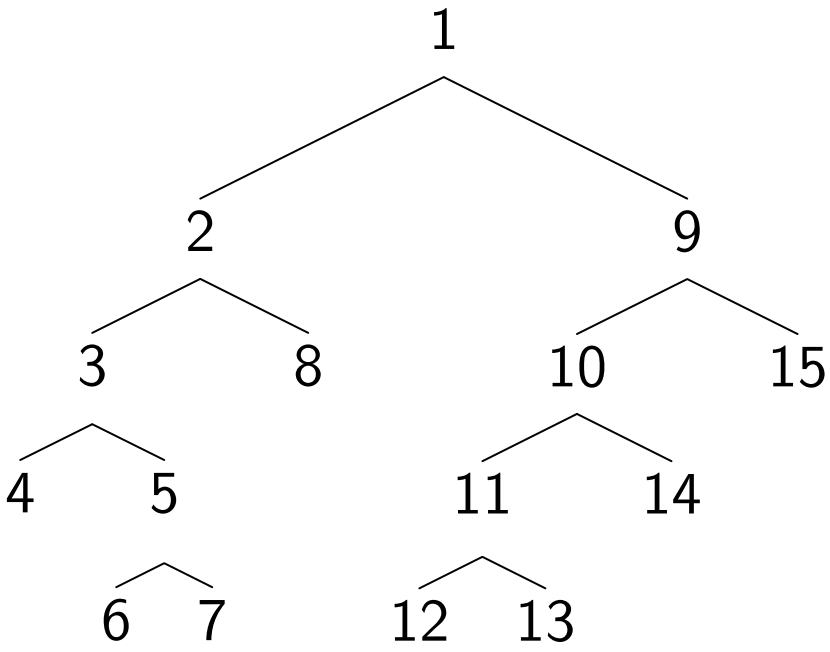


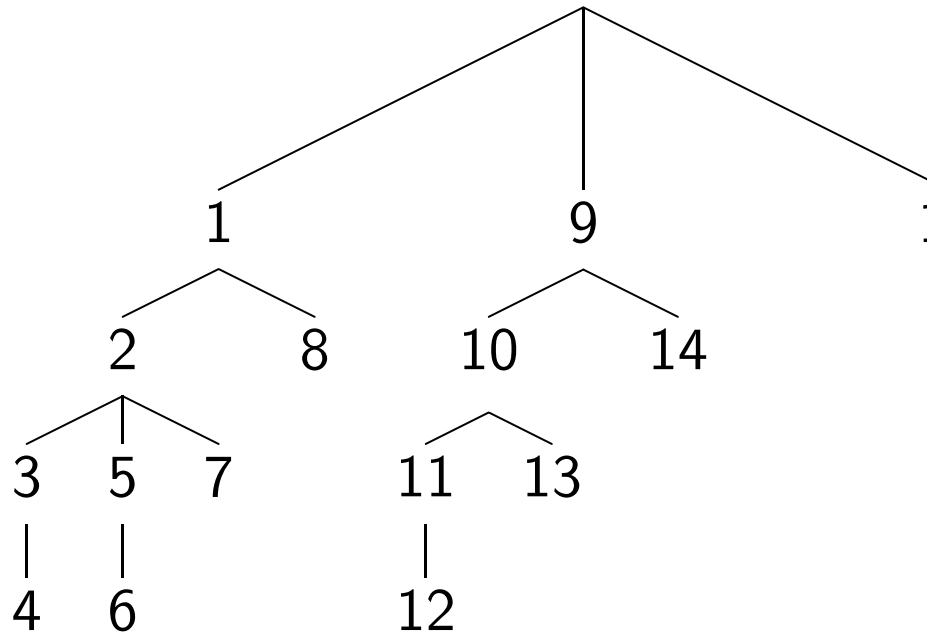
Binary-Tree Representation T':



Exercise

Binary to Ordinal





Operations on this representation

Given an ordinal tree representated in a binary tree:

1. How to compute the height? By counting the maximum number of left edges on a rooted path. Complexity $O(n)$.
2. How to compute the maximum degree? By counting the maximum number of right edges on a rooted path. Complexity $O(n)$.

Exercise:

Given a binary tree representated as a forest of ordinal trees, how to compute the height?