Solution 2

Exercise Session: 9.5.2024

Question 2

We know that the node degree in an ER graph is binomially distributed (see Slide 14 in the lecture). We will make use of the formula

$$\mathrm{Var}(X) = \mathrm{E}(X^2) - \mathrm{E}(X)^2. \tag{1}$$

We will abbreviate $q := 1 - p$. We can thus compute

$$\begin{aligned}
\mathrm{E}(\deg(u)^2) &= \sum_{k=0}^{n-1} k^2 \binom{n-1}{k} p^k q^{n-1-k} \\
&= \sum_{k=0}^{n-1} k(n-1)\binom{n-2}{k-1} p^k q^{n-1-k} \quad \left[k\binom{n}{k} = n\binom{n-1}{k-1}\right] \\
&= (n-1)\sum_{k=1}^{n-1} k\binom{n-2}{k-1} p^k q^{n-1-k} \\
&= (n-1)p\sum_{k=1}^{n-1} k\binom{n-2}{k-1} p^{k-1} q^{(n-2)-(k-1)} \\
&= (n-1)p\sum_{j=0}^{m} (j+1)\binom{m-1}{j} p^j q^{m-1-j} \quad [j := k-1, m := n-1] \\
&= (n-1)p\sum_{j=0}^{m-1} (j+1)\binom{m-1}{j} p^j q^{m-1-j} \\
&= (n-1)p\left(\sum_{j=0}^{m-1} j\binom{m-1}{j} p^j q^{m-1-j} + \sum_{j=0}^{m-1}\binom{m-1}{j} p^j q^{m-1-j}\right) \\
&= (n-1)p\left(\sum_{j=0}^{m-1} (m-1)\binom{m-2}{j-1} p^j q^{m-1-j} + \sum_{j=0}^{m-1}\binom{m-1}{j} p^j q^{m-1-j}\right) \quad \left[j\binom{m}{j} = m\binom{m-1}{j-1}\right] \\
&= (n-1)p\left((n-2)p\sum_{j=1}^{m-1}\binom{m-2}{j-1} p^{j-1} q^{(m-2)-(j-1)} + \sum_{j=0}^{m-1}\binom{m-1}{j} p^j q^{m-1-j}\right) \\
&= (n-1)p\left((n-2)p(p+q)^{m-2} + (p+q)^{m-1}\right) \quad \text{[Binomial Theorem]} \\
&= (n-1)p\left((n-2)p+1\right) \quad [p+q=1] \\
&= (n-1)^2p^2 + p(1-p)(n-1).
\end{aligned}$$

Inserting this into expression (1) in combination with the already known expression for the expected node degree yields

$$\begin{aligned}\mathrm{Var}(\deg(u)) &= (n-1)^2p^2 + p(1-p)(n-1) - ((n-1)p)^2 \\ &= p(1-p)(n-1).\end{aligned}$$