National Technical University of Athens

School of Electrical and Computer Engineering

Advanced Algorithm Topics

Network Algorithms and Complexity

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(NTUA – ALMA)

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1st Series of Exercises

Delivery deadline: 10/5/2024

Exercise 1 (Min-Cut Algorithm, 1.5 units). Solve [1, Exercise **1.24].**

Exercise 2 (Random Sampling, 1 unit). Consider a poll on citizens' attitudes towards a major political-economic change. Citizens answer the poll with “yes” or “no” (for or against the change). If the (actual) percentage of citizens who are in favor of the change is p, we want to calculate an estimate pˆ of p such that IPr[|pˆÿp| ÿ ÿp] > 1 ÿ ÿ, for given ÿ, ÿ ÿ (0, 1). For the poll, we will ask N citizens, chosen equally likely and independently from the total population. Our estimate pˆ will be the percentage of
N citizens who are in favor of the change. Using Chernoff-Hoeffding bounds, calculate (as a function of ÿ, ÿ, and p) the minimum sample size N that we need. Find the value of N for ÿ = 0.02 and ÿ = 0.05, if we know that p ÿ [0.1, 0.7] (and note that this value is independent of the population of the country!). Also calculate the minimum sample size Nÿ (as a function of ÿ and ÿ) such that our estimate pˆ ÿp| ÿ ÿ] > 1ÿÿ. What is the value of Nÿ for IPr[|p to satisfy ÿ = 0.02 and ÿ = 0.05?
Note: This is a variation of [1, Exercise **4.5].**

Exercise 3 (Sparsification, 2 credits). (a) Let a, x ÿ [0, 1]n , with ÿ i xi = 1 (we will say that x is a probability vector in [n] ÿ {1, . . . , n}). Let k(ÿ) = ÿln(2)/(2ÿ ^{2})ÿ + 1. Show that for every ÿ > 0, there is a k(ÿ)-uniform probability vector y on [n] such that |a x ÿ a y| ÿ ÿ. A probability vector y is k- *uniform* if each yi is an integer multiple of 1/k. (b) Let A be an m × n matrix with all its elements in [0, 1] and let x

be a probability vector on [n]. Let k(m, ÿ) = ÿln(2m)/(2ÿ ^{2})ÿ + 1. Show that for every ÿ > 0, there exists a k(m, ÿ)-uniform probability vector y on [n] such that ÿAx ÿ Ayÿÿ ÿ ÿ.

Hint: Use the following Hoeffding bound: Let X1, . . . , Xn be independent random variables on [0, 1], and let X = (ÿn i=1 Xi) /n. For every ÿ > 0, IPr[|X ÿ IExp[X]| > ÿ] ÿ 2e .

Exercise 4 (Capacited Max k-Cut, 1.5 units). In the Max k-Cut problem, a simple undirected graph G(V, E, w), with positive integer weights w : E ÿ INÿ on the edges, is given, and a partition of the vertices into k ÿ 2 sets (S1, . . . , Sk) is requested, with ÿk i=1 Si = V, so as to maximize the total weight of edges with ends in different sets. In particular, if we denote by

$$ÿ(S1, . . . , Sk) = \{ e = \{u, v\} ÿ E : v ÿ Si \;' u ÿ Sj \text{ and } i ÿ= j \}$$

the set of edges in the k-cut (S1, . . . , Sk), we are asked to maximize the

$$w(S1, . . . , Sk) = ÿ_{eÿÿ(S1,...,Sk)} w(e).$$

We consider a variant of the Max k-Cut problem where the desired sizes (c1, . . . , ck), with c1 + · · · + ck = |V | of the sets defining the k-cut are given. We apply the probabilistic algorithm that places each vertex v in the set Si randomly, with probability ci/|V |, and independently. Compute the expected value of w(S1, . . . , Sk)(and investigate when it is maximized). Also compute upper and lower bounds (so that they hold with high probability) for the sizes of the resulting sets.

Exercise 5 (1 unit). We consider the Halving Algorithm (as presented in the slides of the corresponding lecture), which seeks to minimize the number of errors in an online learning environment. We consider the case where the hypothesis class H is finite and the samples are categorized based on a hypothesis f ÿ H (realizability). Show that if the samples (xt , f(xt) come from an arbitrary (unknown, but specified) distribution D and the set of valid hypotheses St does not change for ÿ(log(1/ÿ)/ÿ) consecutive samples, then with probability at least 1 ÿ ÿ, every valid hypothesis h ÿ St achieves loss L(D,f) (h) ÿ ÿ (i.e. we have achieved the PAC Learning guarantee). What is the sampling complexity of the algorithm for a finite hypothesis class H?

Exercise 6 (1 credit). Repeat the analysis of the Weighted Majority Algorithm (WMA, as presented in the slides of the corresponding lecture) for the case where the confidence weight of each expert / hypothesis h ÿ H is multiplied by (1 ÿ ÿ) (instead of 1/2) each time expert h makes a mistake (i.e., we have that for each h with h(xt) ÿ= yt , wt+1(h) = (1 ÿ ÿ)wt(h)).

Exercise 7 (Regret Analysis of the Hedge Algorithm, 2 credits). We consider the following algorithm for online learning / online linear optimization and analyze the regret it achieves.

- **Input:** n actions {1, . . . , n}, time horizon T, w1 = (1, . . . , 1), x1 = (1/n, . . . , 1/n)
- for t = 1 to T do:
  - Select action it ÿ {1, . . . , n} with probability xt(it)
  - Get loss ÿt ÿ [0, 1]n for all actions and incur loss ÿt(it)
  - Update weights wt+1(i) = wt(i)e ÿÿÿt(i) , for all i ÿ {1, . . . , n}
  - Update probabilities xt+1(i) = ÿ $\frac{\text{wt+1(i)}}{\sum_{i=1}^{n}}$ for all i ÿ {1, . . . , n} wt+1(i) ,

(a) We consider the potential function ÿ(t) = ÿn i=1 wt(i) (total trust weight of ÿt(i ÿ) actions at time t). Initially ÿ(1) = n. Show that ÿ(T) ÿ e ÿt(i) is the optimal choice. i ÿÿ ÿT t=1 , where ÿ = arg mini ÿT t=1

(b) Also show that for every t ÿ 1,

$$ÿ(t + 1) ÿ ÿ(t)e^{ÿÿxt·ÿt+ÿ\ 2xtlÿ^2_t},$$

where $ÿ^2_t$ (i) = (ÿt(i))2 , for all i ÿ {1, . . . , n}. Using this relation, find an upper bound for ÿ(T) as a function of ÿ(1) = n. (c) Using the upper and lower bounds for ÿ(T) that you calculated in (a) and (b) and the fact that ÿt ÿ [0, 1]n , for all t ÿ {1, . . . , T}, show that:

$$\text{Regret(T)} = ÿ \sum_{t=1}^{T} ÿt(it)\ ÿ\ ÿ \sum_{t=1}^{T} ÿt(i^{ÿ})\ ÿ\ ÿT + \ln(n)/ÿ$$

What value of ÿ would you choose? What is the Regret(T) that the algorithm achieves for this value of ÿ?

References

[1] M. Mitzenmacher and E. Upfal. *Probability and Computing: Randomized Algorithms and Probabilistic Analysis.* Cambridge University Press, 2005.