

Day 9 - White noise and random walks

Introduction

Last week, we introduced the Holt-Winters filter, a model that is capable of estimating a time series with trend and seasonal components. However, sometimes the Holt-Winters model is not able to capture the serial correlation present in the data, resulting in left-over serial correlation in the *residual error series*. In order to capture serial correlation in the residual error series, we need to build more complicated models by combining time series techniques used for the mean and error structure of the series. This week, we begin considering basic stochastic models for time series, including: white noise, random walks, and autoregressive models.

To guide our discussion, we will use a [data set](#) describing the number of visits to a website for a statistics course at Duke University, called Statistical Forecasting. The data set contains daily counts of the number of visits to the website between 2014 and 2020.

```
# packages
library(tidyverse)
library(lubridate)

# read table from the web
web <- read_csv("daily-website-visitors.csv")
```

Review

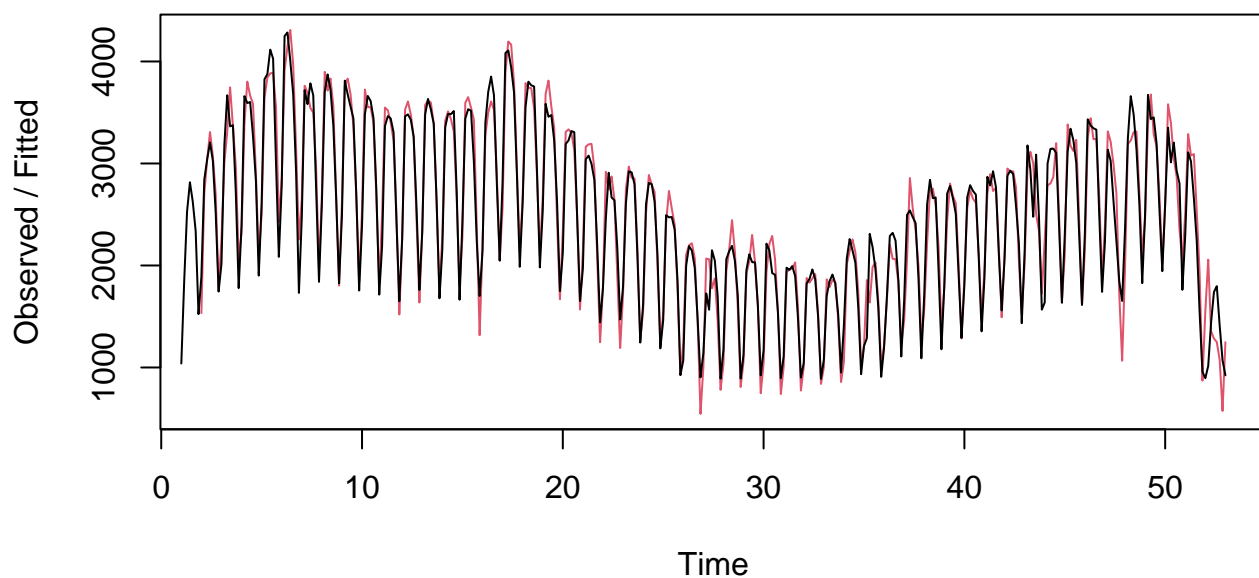
Construct a time series of the daily unique visits to the website in the year of 2017, then fit a Holt-Winters model to the data and plot the resulting model. Then create an ACF plot of the residual error series and comment on what you see.

💡 Tip

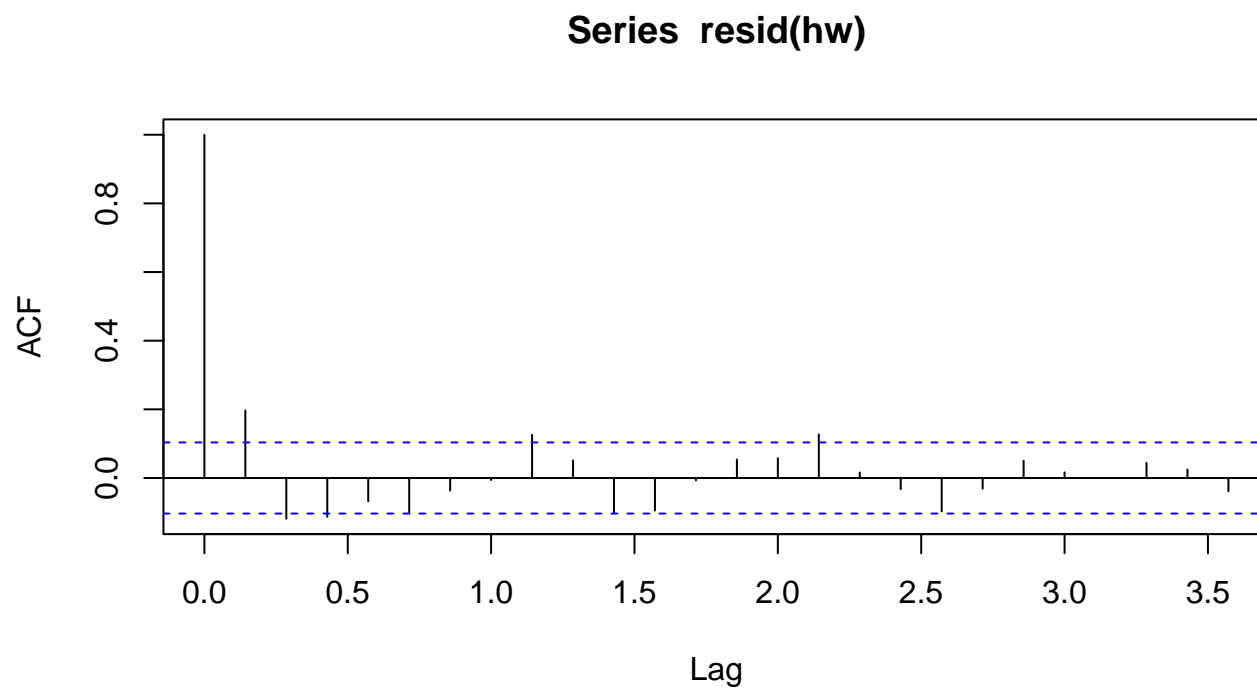
You can use the `resid` function on a Holt-Winters model to obtain the residual error series.

```
web2017 <- web %>%  
  mutate(dt = mdy(Date)) %>%  
  filter(year(dt) == 2017) %>%  
  arrange(dt)  
  
web_ts <- ts(  
  web2017$Unique.Visits,  
  start = c(1,1),  
  freq = 7  
)  
  
hw <- HoltWinters(web_ts)  
plot(hw)
```

Holt-Winters filtering



```
acf(resid(hw))
```



i Note

A time series $\{w_t : t = 1, 2, \dots, n\}$ is called _____ (DWN) if the variables w_1, w_2, \dots, w_n are _____ and _____ distributed with a mean of _____.

If the variables also follow a normal distribution, the series is called _____.

$$w_t \sim N(0, \sigma^2)$$

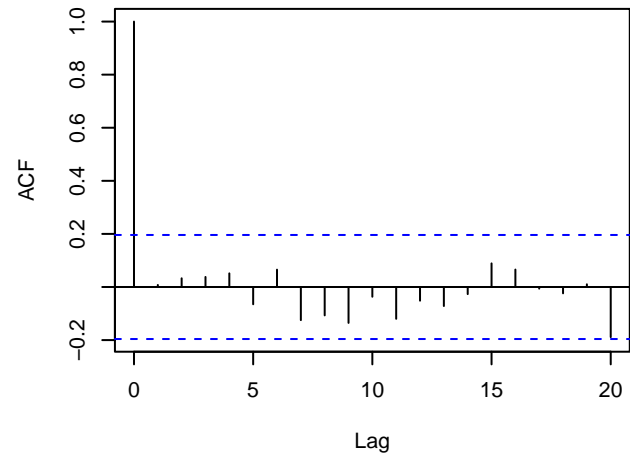
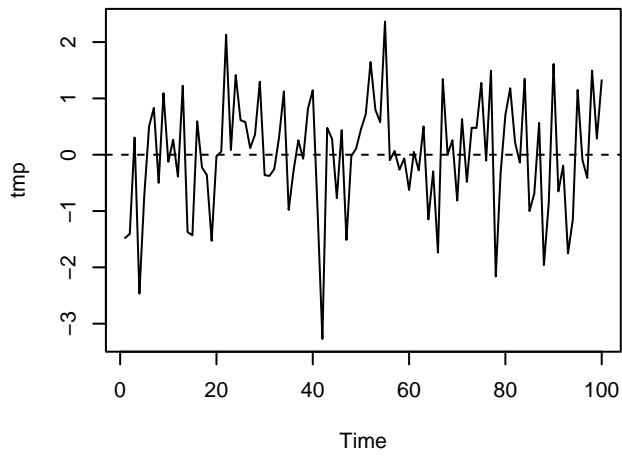
i Second-order properties

$$\begin{aligned}\mu(t) &= 0 \\ \gamma_k = \text{Cov}(w_t, w_{t+k}) &= \begin{cases} \sigma^2 & k = 0 \\ 0 & k \neq 0 \end{cases} \\ \rho_k &= \begin{cases} 1 & k = 0 \\ 0 & k \neq 0 \end{cases}\end{aligned}$$

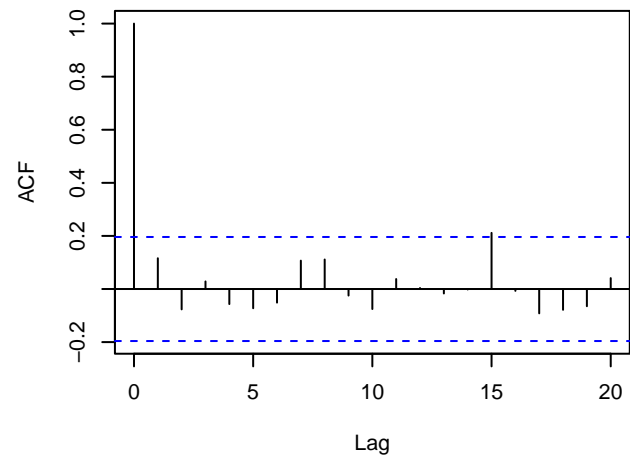
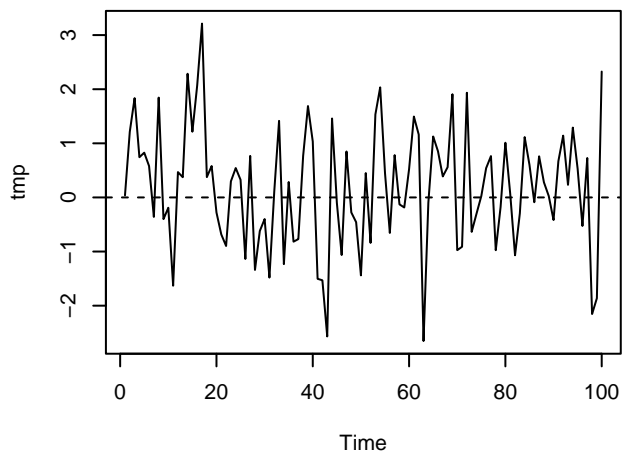
How can we simulate Gaussian white noise?

```
set.seed(10062024)
par(mfrow = c(3, 2))
for(i in 1:3) {
  tmp <- rnorm(100)
  plot(
    tmp, type = "l",
    xlab = "Time"
  )
  abline(h = 0, lty = 2)
  acf(tmp)
}
```

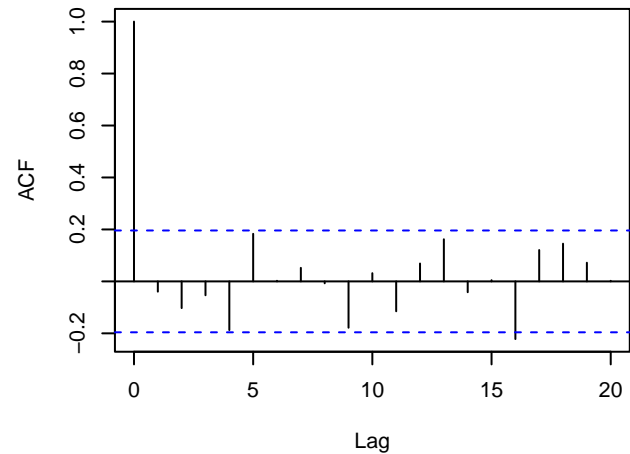
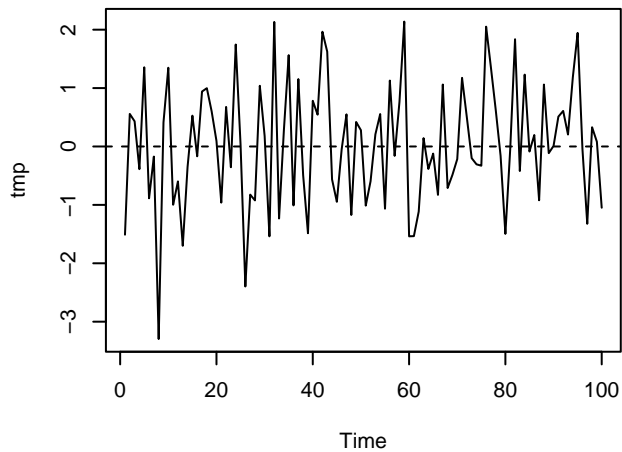
Series tmp



Series tmp



Series tmp



Random walks

i Note

A time series $\{x_t\}$ is a _____ if

$$x_t = x_{t-1} + w_t$$

where w_t is discrete white noise. It can be shown that

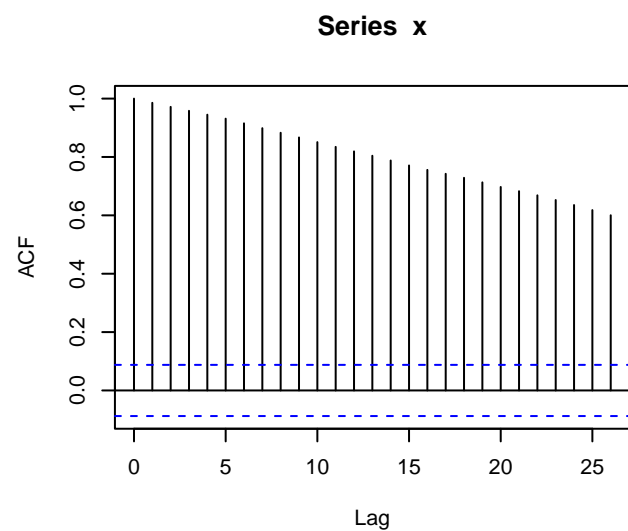
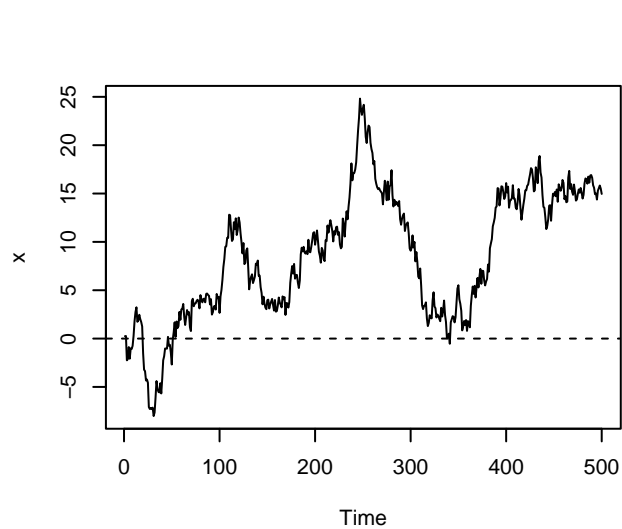
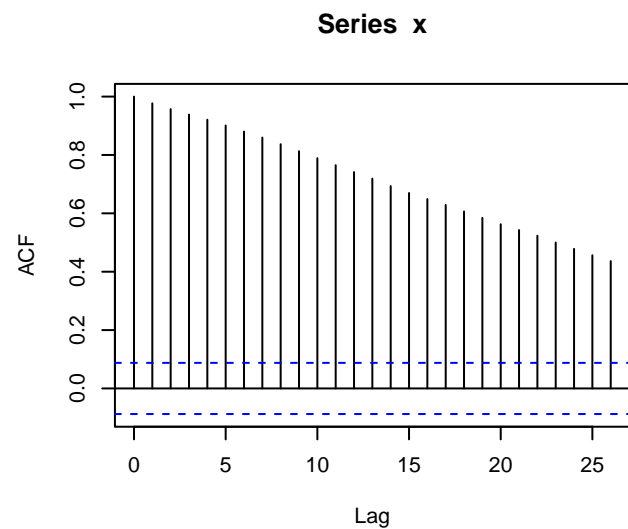
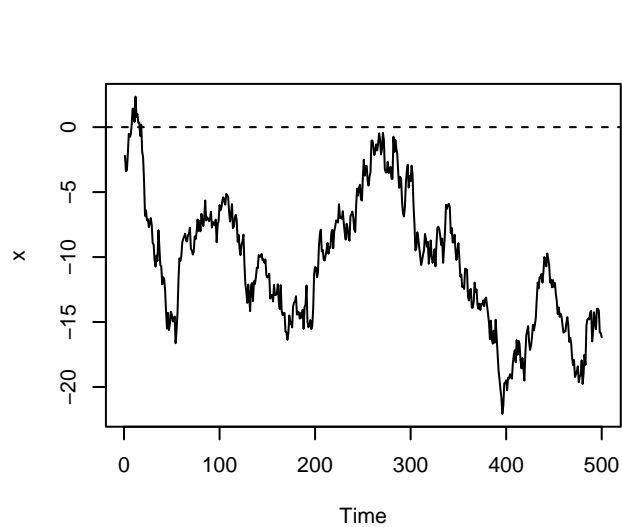
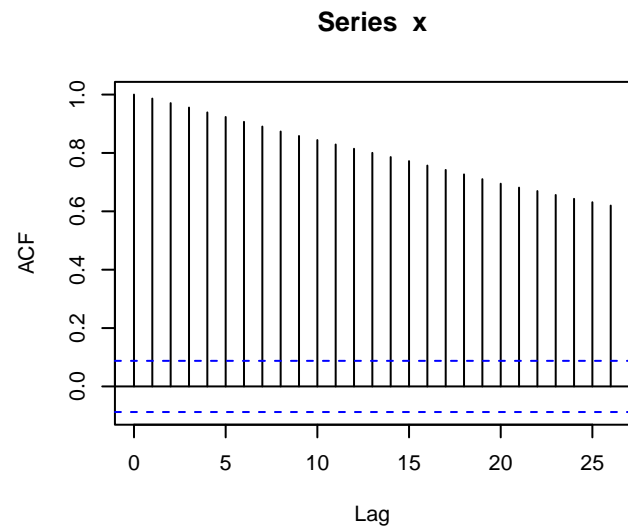
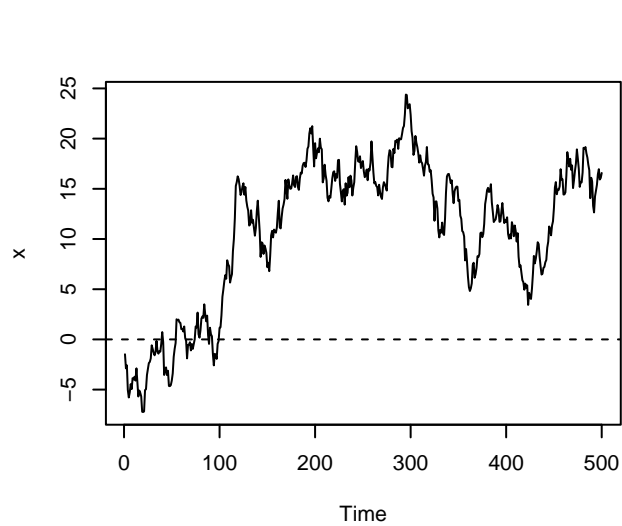
$$x_t = w_t + w_{t-1} + w_{t-2} + \dots$$

i Second-order properties

$$\begin{aligned}\mu(t) &= 0 \\ \gamma_k(t) &= \text{Cov}(x_t, x_{t+k}) = t\sigma^2 \\ \rho_k(t) &= \frac{\text{Cov}(x_t, x_{t+k})}{\sqrt{\text{Var}(x_t)\text{Var}(x_{t+k})}} = \frac{t\sigma^2}{\sqrt{t\sigma^2(t+k)\sigma^2}} = \frac{1}{\sqrt{1+k/t}}\end{aligned}$$

How can we simulate a random walk?

```
set.seed(10062024)
par(mfrow = c(3, 2))
for(i in 1:3) {
  x <- w <- rnorm(500)
  for(t in 2:500) x[t] <- x[t-1] + w[t]
  plot(x, type = "l", xlab = "Time")
  abline(h = 0, lty = 2)
  acf(x)
}
```



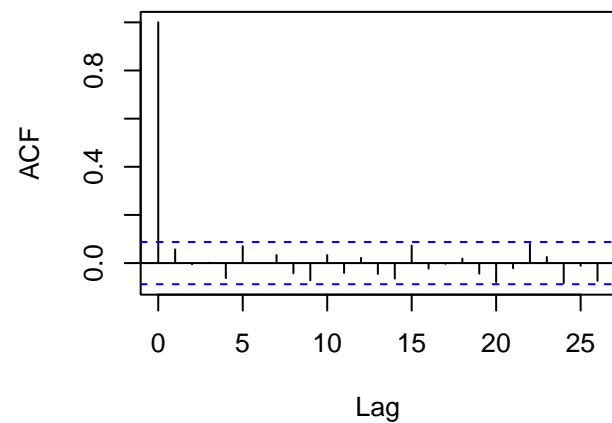
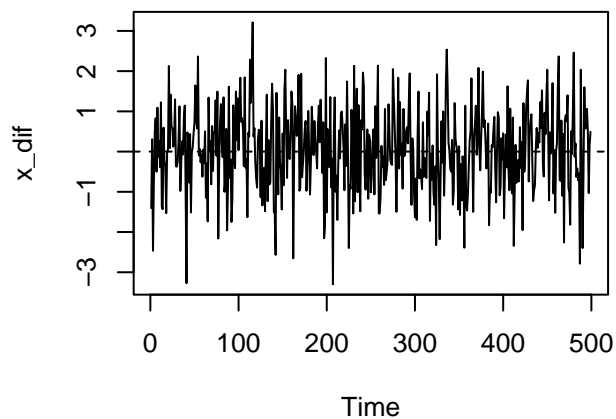
```

set.seed(10062024)
par(mfrow = c(2, 2))
for(i in 1:2) {
  # generate random walk
  x <- w <- rnorm(500)
  for(t in 2:500) x[t] <- x[t-1] + w[t]

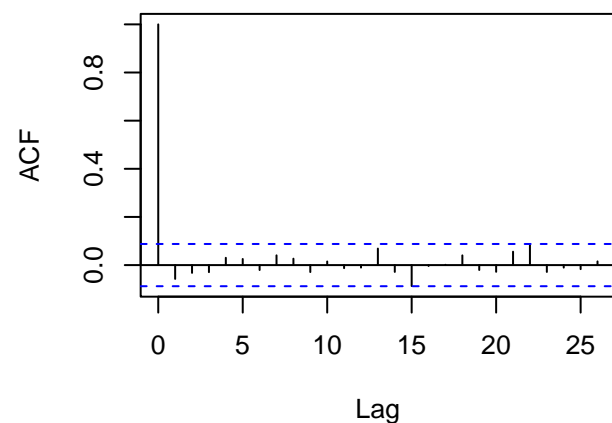
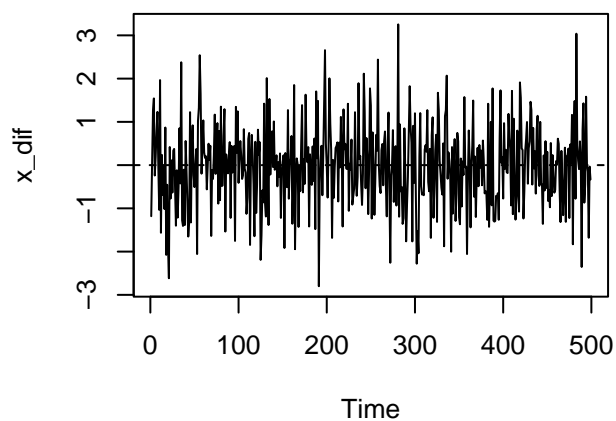
  # first order difference
  x_dif <- diff(x)
  plot(x_dif, type = "l", xlab = "Time")
  abline(h = 0, lty = 2)
  acf(x_dif)
}

```

Series x_dif



Series x_dif



i Backshift operators

The backward shift operator B is defined by

$$Bx_t = x_{t-1}$$

and

$$B^n x_t = x_{t-n}$$

which provides a convenient way to express autoregressive series.

Example: Express the random walk model in terms of the backshift operator.

Random walk with drift

i Note

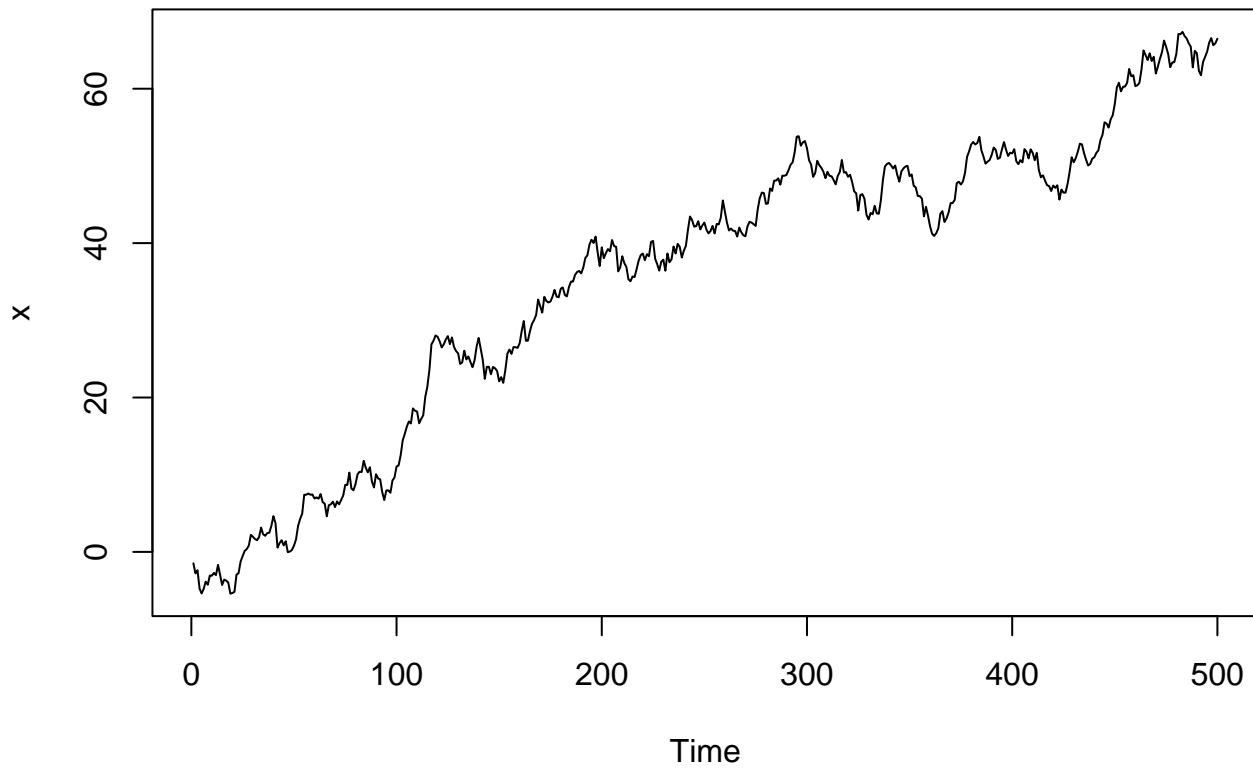
A time series $\{x_t\}$ is a _____ if

$$x_t = x_{t-1} + \delta + w_t$$

where w_t is white noise.

```
set.seed(10062024)
par(mfrow = c(1,1))
# generate random walk with drift
x <- w <- rnorm(500)
for(t in 2:500) x[t] <- x[t-1] + .10 + w[t]

plot(x, xlab = "Time", type = "l")
```



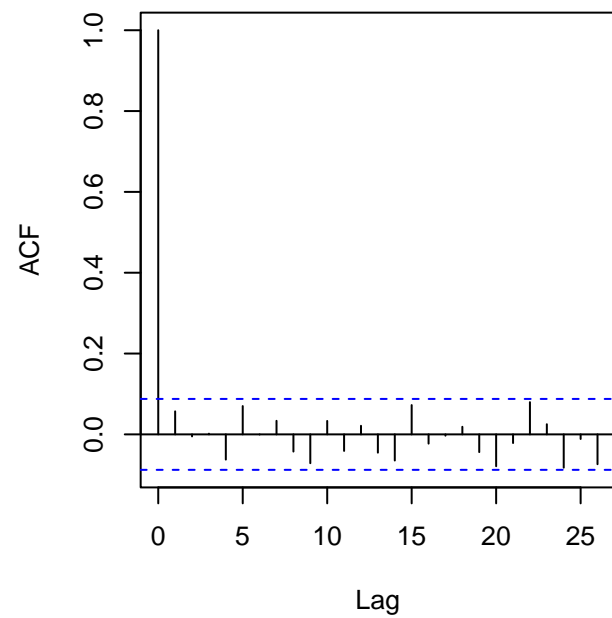
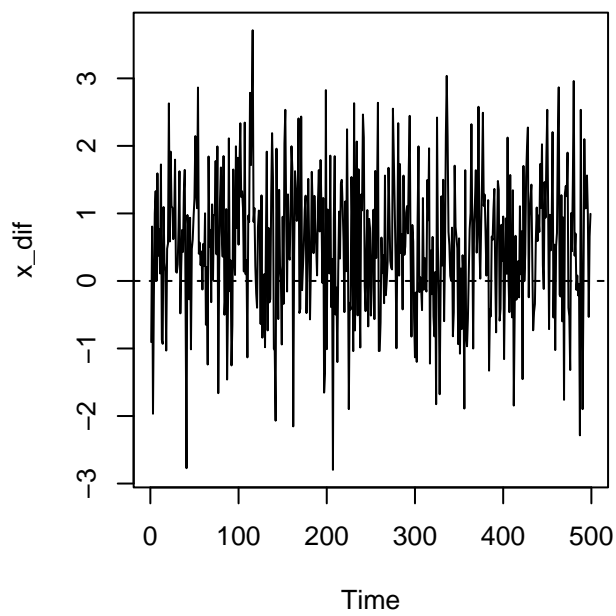
```
set.seed(10062024)
par(mfrow = c(2, 2))
# generate random walk with drift
x <- w <- rnorm(500)
for(t in 2:500) x[t] <- x[t-1] + .5 + w[t]

# first order difference - notice the mean
x_dif <- diff(x)
mean(x_dif)
```

```
[1] 0.5361847
```

```
plot(x_dif, type = "l", xlab = "Time")
abline(h = 0, lty = 2)
acf(x_dif)

# we can fix that
x_dif_centered <- x_dif - mean(x_dif)
plot(x_dif_centered, type = "l", xlab = "Time")
abline(h = 0, lty = 2)
acf(x_dif_centered)
```

Series x_dif**Series x_dif_centered**