Name: Your name here
Due: 2024/10/14

Homework 5

Be sure to submit **both** the .pdf and .qmd file to Canvas by Monday, October 14th at 11:59 pm.

0. [1 pt] With whom did you work on this assignment?

1. [pt] The purpose of this question is to demonstrate that there exist time series structures that Holt-Winters cannot estimate very well. To demonstrate this idea, we are going to simulate data from an AR(2) process with $\alpha_1 = 1.1$ and $\alpha_2 = -.3$.

   a) [1 pt] Write the hypothetical AR(2) model in proper notation.

   > The model is $x_t = 1.1x_{t-1} - .3x_{t-2} + w_t$, where $\{w_t\}$ is white noise.

   b) [1 pt] Express the model from part a in terms of the backshift operator.

   > $(1 - 1.1B + .3B^2)x_t = w_t$

   c) [1 pt] Determine whether the model is stationary.

   > Since the roots both exceed unity in magnitude, the model is stationary.

```r
a <- .3
b <- -1.1
c <- 1

# oldschool
(-b + c(-1,1)*sqrt(b^2 - 4*a*c))/(2*a)
```

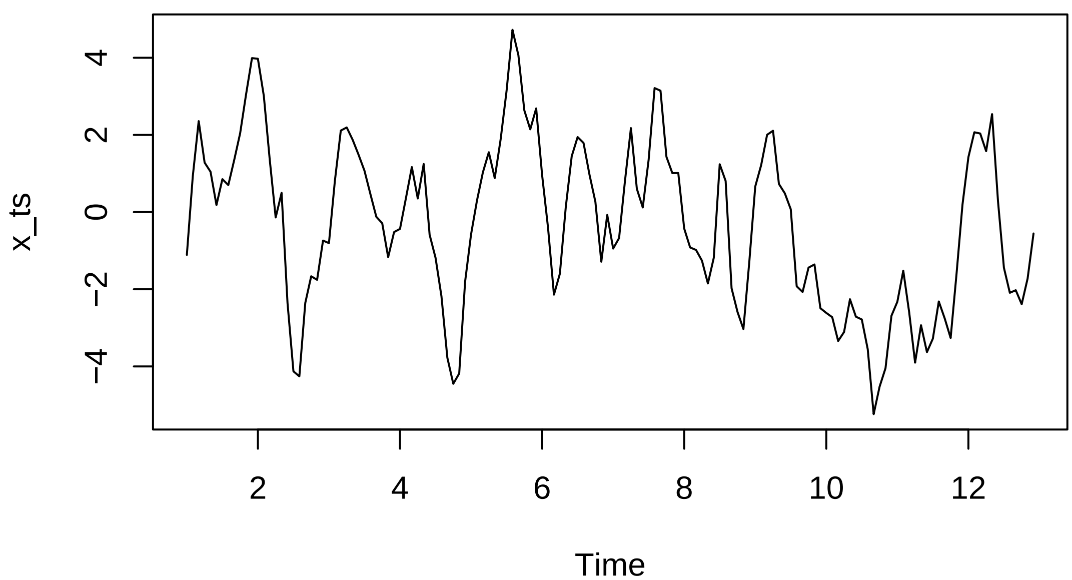
```
[1] 1.666667 2.000000
```

```r
# or
polyroot(c(c,b,a))
```

```
[1] 1.666667+0i 2.000000-0i
```

   d) [3 pt] Simulate 12 years of monthly data from an AR(2) process with $\alpha_1 = .8$ and $\alpha_2 = -.1$. Convert the synthetic data to a `ts` object and plot the resulting series.

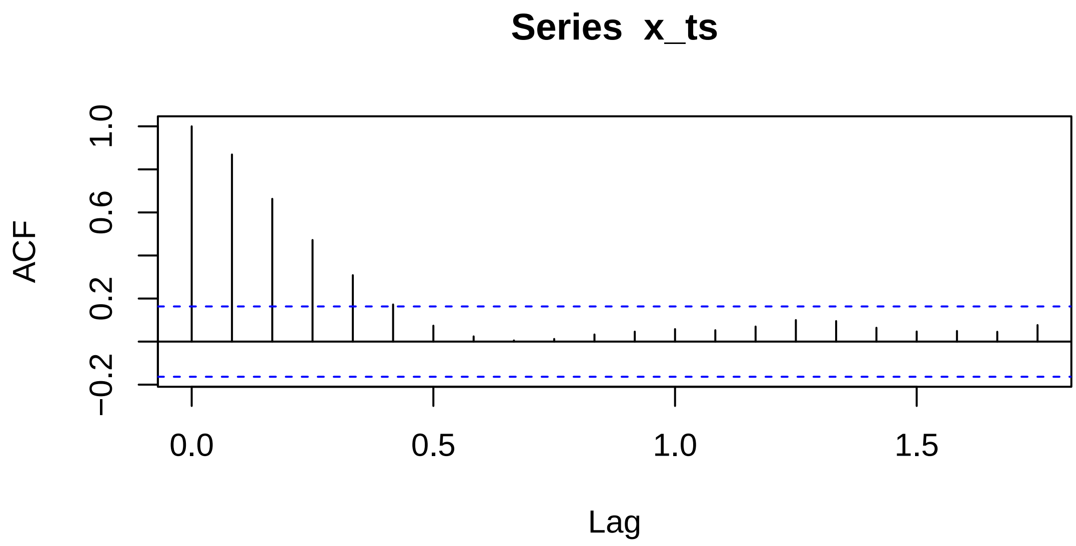
```
# simulate series
n <- 144
x <- w <- rnorm(n)
for(t in 3:n) x[t] <- 1.1*x[t-1] -.3*x[t-2] + w[t]

# create ts object
x_ts <- ts(
  x,
  start = c(1, 1),
  freq = 12
)
plot(x_ts)
```



e) [1 pt] Create an ACF plot of the synthetic series and comment what the plot suggests about the autocorrelation.

> There is obviously severe autocorrelation that decays over time.

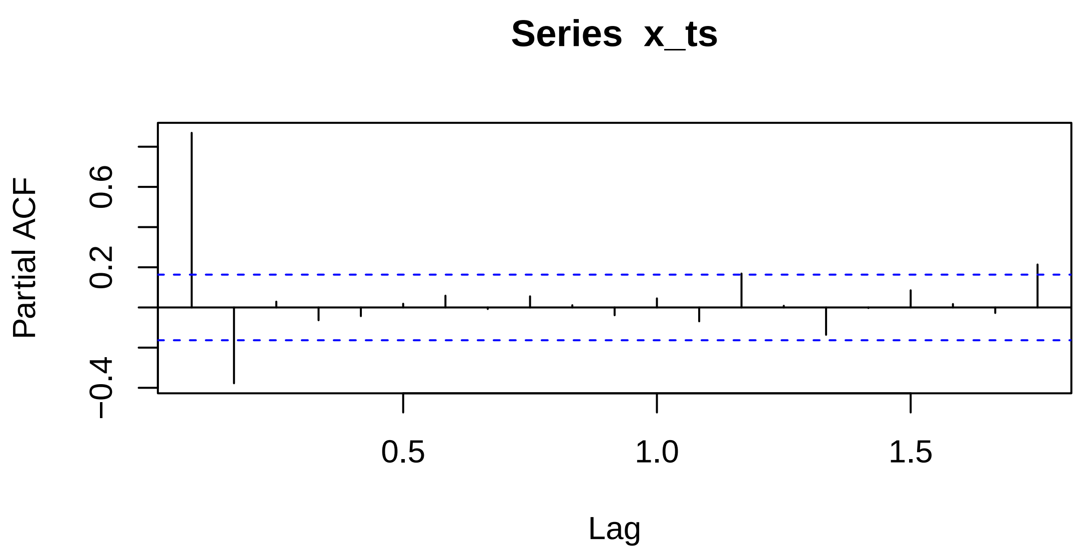
```
acf(x_ts)
```



f) [1 pt] Create an PACF plot of the synthetic series and comment on what the plot suggests about the autocorrelation.

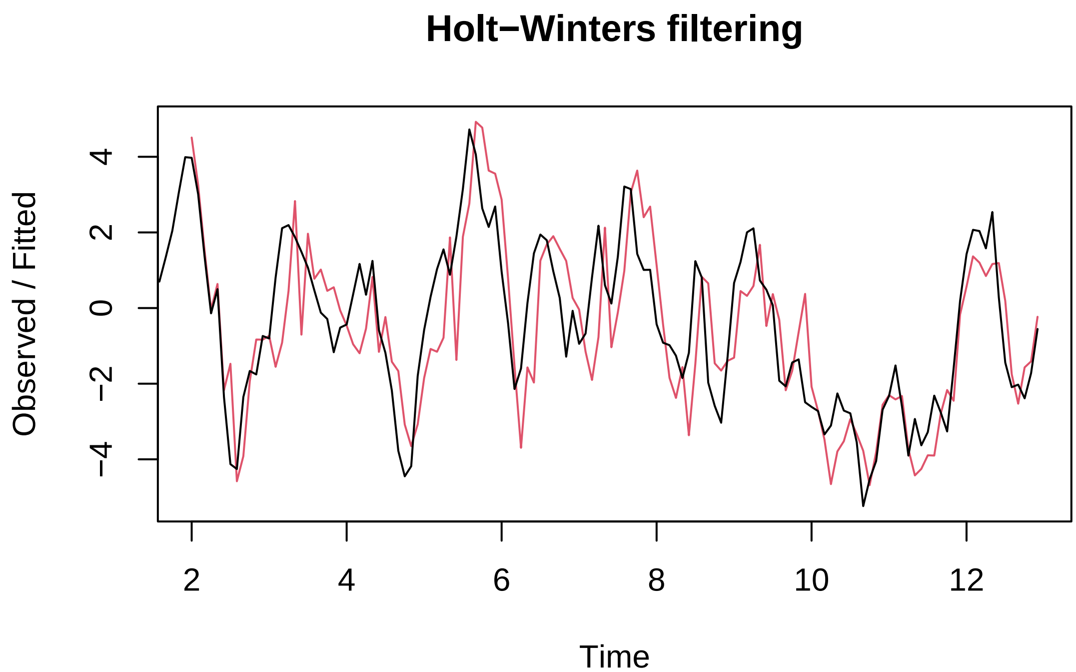
> The PACF plot makes it clear that there is massive autocorrelation at lag 1, and strong negative autocorrelation at lag 2.

```
pacf(x_ts)
```



g) [2 pt] Fit a Holt-Winters model to the synthetic series and plot the results.

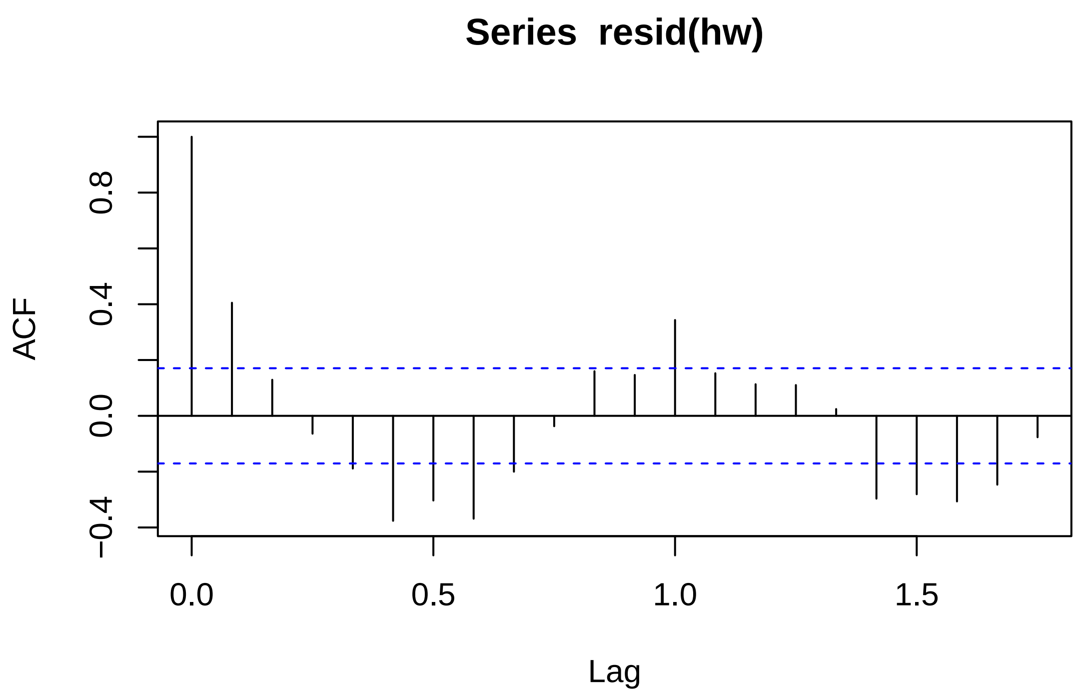
```r
hw <- HoltWinters(x_ts)
plot(hw)
```



h) [2 pt] Create an ACF plot of the residuals from the Holt-Winters model and comment on whether the model has accounted for the residual autocorrelation.

It has not! Lots of autocorrelation left.

```r
acf(resid(hw))
```


Series resid(hw)
ACF
-0.4
0.0
0.4
0.8
0.0
0.5
1.0
1.5
Lag

2. [2 pt] Using the characteristic equation, show that a random walk is not stationary.

Recall that a random walk is given by $x_t = x_{t-1} + w_t$, which can be written as $(1-B)x_t = w_t$. The characteristic equation for this model has root $B = 1$, which does not exceed unity. Therefore, the random walk is not stationary.

3. [3 pt] Determine for what values of α an AR(1) process is stationary.

This is another characteristic equation question. Recall that an AR(1) process is

$$\begin{aligned} x_t &= \alpha x_{t-1} + w_t \\ x_t - \alpha B x_t &= w_t \\ (1 - \alpha B)x_t &= w_t \end{aligned}$$

The characteristic equation for an AR(1) process is

$$1 - \alpha B = 0$$

which has a root of $\frac{1}{\alpha}$. In order for the model to be stationary, the root must exceed unity in magnitude. Therefore $-1 < \alpha < 1$ results in a stationary model.

4. [6 pt] The purpose of this question is to prove one of the second order properties of an AR(1) process.

a) [1 pt] Express an AR(1) process in terms of the backshift operator.

From question 3, $(1 - \alpha B)x_t = w_t$.

b) [3 pt] Using the expression from part a, show that $x_t = \sum_{i=0}^{\infty} \alpha^i w_{t-i}$. It is likely helpful to know that $(1 - B)^{-1} = 1 + B + B^2 + \ldots$ by Binomial expansion.

From question 3, $(1 - \alpha B)x_t = w_t$, which implies that $x_t = (1 - \alpha B)^{-1} w_t$. By the handy hint I provided

$$\begin{aligned} x_t &= (1 - \alpha B)^{-1} w_t \\ &= (1 + \alpha B + \alpha B^2 + \ldots)w_t \\ &= w_t + \alpha B w_t + \alpha B^2 w_t + \ldots \\ &= w_t + \alpha w_{t-1} + \alpha w_{t-2} + \ldots \\ &= \sum_{i=0}^{\infty} \alpha^i w_{t-i} \end{aligned}$$

c) [2 pt] Using the expression from part b, show that the $E(x_t) = 0$. Recall that $E\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} E(X_i)$ and that $E(cX) = cE(X)$ if c is a constant.

We just plug it in.

$$E(x_t) = E\left(\sum_{i=0}^{\infty} \alpha^i w_{t-i}\right) = \sum_{i=0}^{\infty} \alpha^i E(w_{t-i}) = 0$$

The last equality follows from the fact that $\{w_t\}$ is white noise.